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On Guaranteed Parameter Estimation of Stochastic Delay Differential Equations by Noisy Observations

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Additional information is available at the end of the chapter

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1. Introduction

Assume $(\Omega, \mathcal{F}, (\mathcal{F}(t), t \geq 0), P)$ is a given filtered probability space and $W = (W(t),\ t \geq 0)$, $V = (V(t),\ t \geq 0)$ are real-valued standard Wiener processes on $(\Omega, \mathcal{F}, (\mathcal{F}(t), t \geq 0), P)$, adapted to $(\mathcal{F}(t))$ and mutually independent. Further assume that $X_0 = (X_0(t),\ t \in [-1, 0])$ and Y_0 are a real-valued cadlag process and a real-valued random variable on $(\Omega, \mathcal{F}, (\mathcal{F}(t), t \geq 0), P)$ respectively with

$$E \int_{-1}^{0} X_0^2(s)ds < \infty \text{ and } EY_0^2 < \infty.$$

Assume Y_0 and $X_0(s)$ are $\mathcal{F}_0-$measurable, $s \in [-1, 0]$ and the quantities $W,\ V,\ X_0$ and Y_0 are mutually independent.

Consider a two–dimensional random process $(X, Y) = (X(t), Y(t), t \geq 0)$ described by the system of stochastic differential equations

$$dX(t) = aX(t)dt + bX(t-1)dt + dW(t), \tag{1}$$

$$dY(t) = X(t)dt + dV(t),\ t \geq 0 \tag{2}$$

with the initial conditions $X(t) = X_0(t),\ t \in [-1, 0]$, and $Y(0) = Y_0$. The process X is supposed to be hidden, i.e., unobservable, and the process Y is observed. Such models are used in applied problems connected with control, filtering and prediction of stochastic processes (see, for example, [1, 4, 17–20] among others).

The parameter $\vartheta = (a, b)' \in \Theta$ is assumed to be unknown and shall be estimated based on continuous observation of Y, Θ is a subset of $\mathcal{R}^2$ ($(a, b)'$ denotes the transposed (a, b)). Equations (1) and (2) together with the initial values $X_0(\cdot)$ and Y_0 respectively have uniquely solutions $X(\cdot)$ and $Y(\cdot)$, for details see [19].

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Equation (1) is a very special case of stochastic differential equations with time delay, see [5, 6] and [20] for example.

To estimate the true parameter ϑ with a prescribed least square accuracy ε we shall construct a sequential plan $(T^*(\varepsilon), \vartheta^*(\varepsilon))$ working for all $\vartheta \in \overline{\Theta}$. Here $T^*(\varepsilon)$ is the duration of observations which is a special chosen stopping time and $\vartheta^*(\varepsilon)$ is an estimator of ϑ. The set $\overline{\Theta}$ is defined to be the intersection of the set Θ with an arbitrary but fixed ball $\mathcal{B}_{0,R} \subset \mathcal{R}^2$. Sequential estimation problem has been solved for sets Θ of a different structure in [7]-[9], [11, 13, 14, 16] by observations of the process (1) and in [10, 12, 15] – by noisy observations (2).

In this chapter the set Θ of parameters consists of all $(a, b)'$ from $\mathcal{R}^2$ which do not belong to lines L_1 or L_2 defined in Section 2 below and having Lebesgue measure zero.

This sequential plan is a composition of several different plans which follow the regions to which the unknown true parameter $\vartheta = (a, b)'$ may belong to. Each individual plan is based on a weighted correlation estimator, where the weight matrices are chosen in such a way that this estimator has an appropriate asymptotic behaviour being typical for the corresponding region to which ϑ belongs to. Due to the fact that this behaviour is very connected with the asymptotic properties of the so-called fundamental solution $x_0(\cdot)$ of the deterministic delay differential equation corresponding to (1) (see Section 2 for details), we have to treat different regions of $\Theta = \mathcal{R}^2 \setminus L$, $L = L_1 \cup L_2$, separately. If the true parameter ϑ belongs to L, the weighted correlation estimator under consideration converges weakly only, and thus the assertions of Theorem 3.1 below cannot be derived by means of such estimators. In general, the exception of the set L does not disturb applications of the results below in adaptive filtration, control theory and other applications because of its Lebesgue zero measure.

In the papers [10, 12] the problem described above was solved for the two special sets of parameters Θ_I (a straight line) and Θ_{II} (where $X(\cdot)$ satisfies (1) is stable or periodic (unstable)) respectively. The general sequential estimation problem for all $\vartheta = (a, b)'$ from $\mathcal{R}^2$ except of two lines was solved in [13, 14, 16] for the equation (1) based on the observations of $X(\cdot)$.

In this chapter the sequential estimation method developed in [10, 12] for the system (1), (2) is extended to the case, considered by [13, 14, 16] for the equation (1) (as already mentioned, for all ϑ from $\mathcal{R}^2$ except of two lines for the observations without noises).

A related result in such problem statement was published first for estimators of an another structure and without proofs in [15].

A similar problem for partially observed stochastic dynamic systems without time-delay was solved in [22, 23].

The organization of this chapter is as follows. Section 2 presents some preliminary facts needed for the further studies about we have spoken. In Section 3 we shall present the main result, mentioned above. In Section 4 all proofs are given. Section 5 includes conclusions.

2. Preliminaries

To construct sequential plans for estimation of the parameter ϑ we need some preparation. At first we shall summarize some known facts about the equation (1). For details the reader is referred to [3]. Together with the mentioned initial condition the equation (1) has a uniquely determined solution X which can be represented for $t \geq 0$ as follows:

$$X(t) = x_0(t)X_0(t) + b\int_{-1}^{0} x_0(t-s-1)X_0(s)ds + \int_0^t x_0(t-s)dW(s). \tag{3}$$

Here $x_0 = (x_0(t), t \geq -1)$ denotes the fundamental solution of the deterministic equation

$$x_0(t) = 1 + \int_0^t (ax_0(s) + bx_0(s-1))ds, \quad t \geq 0, \tag{4}$$

corresponding to (1) with $x_0(t) = 0, t \in [-1, 0),\ x_0(0) = 1$.

The solution X has the property $E\int_0^T X^2(s)ds < \infty$ for every $T > 0$.

From (3) it is clear, that the limit behaviour for $t \to \infty$ of X very depends on the limit behaviour of $x_0(\cdot)$. The asymptotic properties of $x_0(\cdot)$ can be studied by the Laplace-transform of x_0, which equals $(\lambda - a - b\mathrm{e}^{-\lambda})^{-1}$, λ any complex number.

Let $s = u(r)$ $(r < 1)$ and $s = w(r)$ $(r \in \mathcal{R}^1)$ be the functions given by the following parametric representation $(r(\xi), s(\xi))$ in $\mathcal{R}^2$:

$$r(\xi) = \xi \cot \xi,\ s(\xi) = -\xi / \sin \xi$$

with $\xi \in (0, \pi)$ and $\xi \in (\pi, 2\pi)$ respectively.

Now we define the parameter set Θ to be the plane $\mathcal{R}^2$ without the lines $L_1 = (a, u(a))_{a \leq 1}$ and $L_2 = (a, w(a))_{a \in \mathcal{R}^1}$ such that $\mathcal{R}^2 = \Theta \cup L_1 \cup L_2$.

It seems not to be possible to construct a general simple sequential procedure which has the desired properties under P_ϑ for all $\vartheta \in \Theta$. Therefore we are going to divide the set Θ into some appropriate smaller regions where it is possible to do. This decomposition is very connected with the structure of the set Λ of all (real or complex) roots of the so-called characteristic equation of (4):

$$\lambda - a - b\mathrm{e}^{-\lambda} = 0.$$

Put $v_0 = v_0(\vartheta) = \max\{Re\lambda | \lambda \in \Lambda\}$, $v_1 = v_1(\vartheta) = \max\{Re\lambda | \lambda \in \Lambda,\ Re\lambda < v_0\}$. Beside of the case $b = 0$ it holds $-\infty < v_1 < v_0 < \infty$. By $m(\lambda)$ we denote the multiplicity of the solution $\lambda \in \Lambda$. Note that $m(\lambda) = 1$ for all $\lambda \in \Lambda$ beside of $(a, b) \in \mathcal{R}^2$ with $b = -e^a$. In this cases we have $\lambda = a - 1 \in \Lambda$ and $m(a - 1) = 2$. The values $v_0(\vartheta)$ and $v_1(\vartheta)$ determine the asymptotic behaviour of $x_0(t)$ as $t \to \infty$ (see [3] for details).

Now we are able to divide Θ into some appropriate for our purposes regions. Note, that this decomposition is very related to the classification used in [3]. There the plane $\mathcal{R}^2$ was decomposed into eleven subsets. Here we use another notation.

Definition (Θ). The set Θ of parameters is decomposed as

$$\Theta = \Theta_1 \cup \Theta_2 \cup \Theta_3 \cup \Theta_4,$$

where $\Theta_1 = \Theta_{11} \cup \Theta_{12} \cup \Theta_{13}$, $\Theta_2 = \Theta_{21} \cup \Theta_{22}$, $\Theta_3 = \Theta_{31}$, $\Theta_4 = \Theta_{41} \cup \Theta_{42}$ with

$$\Theta_{11} = \{\vartheta \in \mathcal{R}^2 |\ v_0(\vartheta) < 0\},$$

$$\Theta_{12} = \{\vartheta \in \mathcal{R}^2 |\ v_0(\vartheta) > 0 \text{ and } v_0(\vartheta) \notin \Lambda\},$$

$$\Theta_{13} = \{\vartheta \in \mathcal{R}^2 |\ v_0(\vartheta) > 0;\ v_0(\vartheta) \in \Lambda,\ m(v_0) = 2\},$$
$$\Theta_{21} = \{\vartheta \in \mathcal{R}^2 |\ v_0(\vartheta) > 0, v_0(\vartheta) \in \Lambda,\ m(v_0) = 1,\ \ v_1(\vartheta) > 0 \text{ and } \ v_1(\vartheta) \in \Lambda\},$$
$$\Theta_{22} = \{\vartheta \in \mathcal{R}^2 |\ v_0(\vartheta) > 0, v_0(\vartheta) \in \Lambda,\ m(v_0) = 1,\ \ v_1(\vartheta) > 0 \text{ and } \ v_1(\vartheta) \notin \Lambda\},$$
$$\Theta_{31} = \{\vartheta \in \mathcal{R}^2 |\ v_0(\vartheta) > 0,\ \ v_0(\vartheta) \in \Lambda,\ \ m(v_0) = 1 \text{ and } \ v_1(\vartheta) < 0\},$$
$$\Theta_{41} = \{\vartheta \in \mathcal{R}^2 |\ \ v_0(\vartheta) = 0,\ v_0(\vartheta) \in \Lambda,\ m(v_0) = 1\},$$
$$\Theta_{42} = \{\vartheta \in \mathcal{R}^2 |\ \ v_0(\vartheta) > 0,\ v_0(\vartheta) \in \Lambda,\ m(v_0) = 1,\ \ v_1(\vartheta) = 0 \text{ and } v_1(\vartheta) \in \Lambda\}.$$

It should be noted, that the cases $(Q2 \cup Q3)$ and (Q5) considered in [3] correspond to our exceptional lines L_1 and L_2 respectively.

Here are some comments concerning the Θ subsets.

The unions $\Theta_1, \ldots, \Theta_4$ are marked out, because the Fisher information matrix and related design matrices which will be considered below, have similar asymptotic properties for all ϑ throughout every Θ_i $(i = 1, \ldots, 4)$.

Obviously, all sets $\Theta_{11}, \ldots, \Theta_{42}$ are pairwise disjoint, the closure of Θ equals to $\mathcal{R}^2$ and the exceptional set $L_1 \cup L_2$ has Lebesgue measure zero.

The set Θ_{11} is the set of parameters ϑ for which there exists a stationary solution of (1).

Note that the one-parametric set Θ_4 is a part of the boundaries of the following regions: Θ_{11}, $\Theta_{12}, \Theta_{21}, \Theta_3$. In this case $b = -a$ holds and (1) can be written as a differential equation with only one parameter and being linear in the parameter.

We shall use a truncation of all the introduced sets. First chose an arbitrary but fixed positive R. Define the set $\overline{\Theta} = \{\vartheta \in \Theta |\ ||\vartheta|| \leq \mathrm{R}\}$ and in a similar way the subsets $\overline{\Theta}_{11}, \ldots, \overline{\Theta}_{42}$.

Sequential estimators of ϑ with a prescribed least square accuracy we have already constructed in [10, 12]. But in these articles the set of possible parameters ϑ were restricted to $\overline{\Theta}_{11} \cup \overline{\Theta}_{12} \cup \{\overline{\Theta}_{41} \setminus \{(0,0)\}\} \cup \overline{\Theta}_{42}$.

To construct a sequential plan for estimating ϑ based on the observation of $Y(\cdot)$ we follow the line of [10, 12]. We shall use a single equation for Y of the form:

$$dY(t) = \vartheta' A(t)dt + \xi(t)dt + dV(t), \tag{5}$$

where $A(t) = (Y(t), Y(t-1))'$,

$$\xi(t) = X(0) - aY(0) - bY(0) + b\int_{-1}^{0} X_0(s)ds - aV(t) - bV(t-1) + W(t).$$

The random variables $A(t)$ and $\xi(t)$ are $\mathcal{F}(t)$-measurable for every fixed $t \geq 1$ and a short calculation shows that all conditions of type (7) in [12], consisting of

$$E\int_1^T (|Y(t)| + |\xi(t)|)dt < \infty \text{ for all } T > 1,$$

$$E[\tilde{\Delta}\xi(t)|\mathcal{F}(t-2)] = 0,\ E[(\tilde{\Delta}\xi(t))^2|\mathcal{F}(t-2)] \leq 1 + R^2$$

hold in our case. Here $\tilde{\Delta}$ denotes the difference operator defined by $\tilde{\Delta}f(t) = f(t) - f(t-1)$.

Using this operator and (5) we obtain the following equation:

$$d\tilde{\Delta}Y(t) = a\tilde{\Delta}Y(t)dt + b\tilde{\Delta}Y(t-1)dt + \tilde{\Delta}\xi(t)dt + d\tilde{\Delta}V(t) \tag{6}$$

with the initial condition $\tilde{\Delta}Y(1) = Y(1) - Y_0$.

Thus we have reduced the system (1), (2) to a single differential equation for the observed process $(\tilde{\Delta}Y(t), t \geq 2)$ depending on the unknown parameters a and b.

3. Construction of sequential estimation plans

In this section we shall construct the sequential estimation procedure for each of the cases $\overline{\Theta}_1 \ldots, \overline{\Theta}_4$ separately. Then we shall define, similar to [11, 13, 14, 16], the final sequential estimation plan, which works in $\overline{\Theta}$ as a sequential plan with the smallest duration of observations.

We shall construct the sequential estimation procedure of the parameter ϑ on the basis of the correlation method in the cases $\overline{\Theta}_1, \overline{\Theta}_4$ (similar to [12, 14, 15]) and on the basis of correlation estimators with weights in the cases $\overline{\Theta}_2 \cup \overline{\Theta}_3$. The last cases and $\overline{\Theta}_{13}$ are new. It should be noted, that the sequential plan, constructed e.g. in [2] does not work for $\overline{\Theta}_3$ here, even in the case if we observe $(X(\cdot))$ instead of $(Y(\cdot))$.

3.1. Sequential estimation procedure for $\vartheta \in \overline{\Theta}_1$

Consider the problem of estimating $\vartheta \in \overline{\Theta}_1$. We will use some modification of the estimation procedure from [12], constructed for the Case II thereon. It can be easily shown, that Proposition 3.1 below can be proved for the cases $\overline{\Theta}_{11} \cup \overline{\Theta}_{12}$ similarly to [12]. Presented below modified procedure is oriented, similar to [16] on all parameter sets $\overline{\Theta}_{11}$, $\overline{\Theta}_{12}$, $\overline{\Theta}_{13}$. Thus we will prove Proposition 3.1 in detail for the case $\overline{\Theta}_{13}$ only. The proofs for cases $\overline{\Theta}_{11} \cup \overline{\Theta}_{12}$ are very similar.

For the construction of the estimation procedure we assume h_{10} is a real number in $(0, 1/5)$ and h_1 is a random variable with values in $[h_{10}, 1/5]$ only, $\mathcal{F}(0)$-measurable and having a known continuous distribution function.

Assume $(c_n)_{n\geq 1}$ is a given unboundedly increasing sequence of positive numbers satisfying the following condition:

$$\sum_{n\geq 1} \frac{1}{c_n} < \infty. \tag{7}$$

This construction follows principally the line of [14, 16] (see [12] as well), for which the reader is referred for details.

We introduce for every $\varepsilon > 0$ and every $s \geq 0$ several quantities:
– the functions

$$\Psi_s(t) = \begin{cases} (\tilde{\Delta}Y(t), \tilde{\Delta}Y(t-s))' & \text{for} \quad t \geq 1+s, \\ (0,0)' & \text{for} \quad t < 1+s; \end{cases}$$

– the sequence of stopping times

$$\tau_1(n,\varepsilon) = h_1 \inf\{k \geq 1 : \int_0^{kh_1} ||\Psi_{h_1}(t-2-5h_1)||^2 dt \geq \varepsilon^{-1} c_n\} \quad \text{for} \quad n \geq 1;$$

– the matrices

$$G_1(T,s)=\int_0^T \Psi_s(t-2-5s)\Psi_1'(t)dt,\quad \Phi_1(T,s)=\int_0^T \Psi_s(t-2-5s)d\tilde{\Delta}Y(t),$$

$$G_1(n,k,\varepsilon)=G_1(\tau_1(n,\varepsilon)-kh_1,h_1),\ \ \Phi_1(n,k,\varepsilon)=\Phi_1(\tau_1(n,\varepsilon)-kh_1,h_1);$$

– the times

$$k_1(n)=\arg\min_{k=\overline{1,5}}||G_1^{-1}(n,k,\varepsilon)||,\ n\geq 1;$$

– the estimators

$$\vartheta_1(n,\varepsilon)=G_1^{-1}(n,\varepsilon)\Phi_1(n,\varepsilon),\ n\geq 1,\quad G_1(n,\varepsilon)=G_1(n,k_1(n),\varepsilon),\ \Phi_1(n,\varepsilon)=\Phi_1(n,k_1(n),\varepsilon);$$

– the stopping time

$$\sigma_1(\varepsilon)=\inf\{N\geq 1:\ S_1(N)>(\rho_1\delta_1^{-1})^{1/2}\},\tag{8}$$

where $S_1(N)=\sum\limits_{n=1}^{N}\beta_1^2(n,\varepsilon)$,

$$\beta_1(n,\varepsilon)=||\tilde{G}_1^{-1}(n,\varepsilon)||,\quad \tilde{G}_1(n,\varepsilon)=(\varepsilon^{-1}c_n)^{-1}G_1(n,k_1(n),\varepsilon)$$

and $\delta_1\in(0,1)$ is some fixed chosen number,

$$\rho_1=15(3+\mathrm{R}^2)\sum_{n\geq 1}\frac{1}{c_n}.$$

The deviation of the 'first-step estimators' $\vartheta_1(n,\varepsilon)$ has the form:

$$\vartheta_1(n,\varepsilon)-\vartheta=(\varepsilon^{-1}c_n)^{-1/2}\tilde{G}_1^{-1}(n,\varepsilon)\tilde{\zeta}_1(n,\varepsilon),\ n\geq 1,\tag{9}$$

$$\tilde{\zeta}_1(n,\varepsilon)=(\varepsilon^{-1}c_n)^{-1/2}\int_0^{\tau_1(n,\varepsilon)-k_1(n)h_1}\Psi_{h_1}(t-2-5h_1)(\tilde{\Delta}\xi(t)dt+dV(t)-dV(t-1)).$$

By the definition of stopping times $\tau_1(n,\varepsilon)-k_1(n)h_1$ we can control the noise $\tilde{\zeta}_1(n,\varepsilon)$:

$$E_\vartheta||\tilde{\zeta}_1(n,\varepsilon)||^2\leq 15(3+\mathrm{R}^2),\ \ n\geq 1,\ \ \varepsilon>0$$

and by the definition of the stopping time $\sigma_1(\varepsilon)$ - the first factor $\tilde{G}_1^{-1}(n,\varepsilon)$ in the representation of the deviation (9).

Define the sequential estimation plan of ϑ by

$$T_1(\varepsilon)=\tau_1(\sigma_1(\varepsilon),\varepsilon),\ \ \vartheta_1(\varepsilon)=\frac{1}{S(\sigma_1(\varepsilon))}\sum_{n=1}^{\sigma_1(\varepsilon)}\beta_1^2(n,\varepsilon)\vartheta_1(n,\varepsilon).\tag{10}$$

We can see that the construction of the sequential estimator $\vartheta_1(\varepsilon)$ is based on the family of estimators $\vartheta(T,s)=G_1^{-1}(T,s)\Phi(T,s)$, $s\geq 0$. We have taken the discretization step h_1 as above, because for $\vartheta\in\overline{\Theta}_{12}$ the functions

$$f(T,s)=e^{2v_0T}\,G_1^{-1}(T,s)$$

for every $s \geq 0$ have some periodic matrix functions as a limit on T almost surely. These limit matrix functions are finite and may be infinite on the norm only for four values of their argument T on every interval of periodicity of the length $\Delta > 1$ (see the proof of Theorem 3.2 in [10, 12]).

In the sequel limits of the type $\lim\limits_{n\to\infty} a(n,\varepsilon)$ or $\lim\limits_{\varepsilon\to 0} a(n,\varepsilon)$ will be used. To avoid repetitions of similar expressions we shall use, similar to [12, 14, 16], the unifying notation $\lim\limits_{n\vee\varepsilon} a(n,\varepsilon)$ for both of those limits if their meaning is obvious.

We state the results concerning the estimation of the parameter $\vartheta \in \overline{\Theta}_1$ in the following proposition.

Proposition 3.1. *Assume that the condition* (7) *on the sequence* (c_n) *holds and let the parameter* $\vartheta = (a,b)'$ *in* (1) *be such that* $\vartheta \in \overline{\Theta}_1$.

Then:

I. For any $\varepsilon > 0$ *and every* $\vartheta \in \overline{\Theta}_1$ *the sequential plan* $(T_1(\varepsilon), \vartheta_1(\varepsilon))$ *defined by* (10) *is closed* $(T_1(\varepsilon) < \infty \ P_\vartheta - a.s.)$ *and possesses the following properties:*

$$1^\circ. \quad \sup_{\vartheta\in\overline{\Theta}_1} E_\vartheta ||\vartheta_1(\varepsilon) - \vartheta||^2 \leq \delta_1 \varepsilon;$$

2°. *the inequalities below are valid:*

– for $\vartheta \in \overline{\Theta}_{11}$

$$0 < \varliminf_{\varepsilon\to 0} \varepsilon \cdot T_1(\varepsilon) \ \leq \ \varlimsup_{\varepsilon\to 0} \varepsilon \cdot T_1(\varepsilon) \ < \ \infty \quad P_\vartheta - a.s.,$$

– for $\vartheta \in \overline{\Theta}_{12}$

$$0 < \varliminf_{\varepsilon\to 0} [T_1(\varepsilon) - \frac{1}{2v_0} \ln \varepsilon^{-1}] \ \leq \ \varlimsup_{\varepsilon\to 0} [T_1(\varepsilon) - \frac{1}{2v_0} \ln \varepsilon^{-1}] \ < \ \infty \quad P_\vartheta - a.s.,$$

– for $\vartheta \in \overline{\Theta}_{13}$

$$0 < \varliminf_{\varepsilon\to 0} [T_1(\varepsilon) + \frac{1}{v_0} \ln T_1(\varepsilon) - \Psi'_{13}(\varepsilon)], \qquad \varlimsup_{\varepsilon\to 0} [T_1(\varepsilon) + \frac{1}{v_0} \ln T_1(\varepsilon) - \Psi''_{13}(\varepsilon)] \ < \ \infty \quad P_\vartheta - a.s.,$$

the functions $\Psi'_{13}(\varepsilon)$ *and* $\Psi''_{13}(\varepsilon)$ *are defined in (30).*

II. For every $\vartheta \in \overline{\Theta}_1$ *the estimator* $\vartheta_1(n,\varepsilon)$ *is strongly consistent:*

$$\lim_{n\vee\varepsilon} \vartheta_1(n,\varepsilon) = \vartheta \ \ P_\vartheta - a.s.$$

3.2. Sequential estimation procedure for $\vartheta \in \overline{\Theta}_2$

Assume $(c_n)_{n\geq 1}$ is an unboundedly increasing sequence of positive numbers satisfying the condition (7).

We introduce for every $\varepsilon > 0$ several quantities:

– the parameter $\lambda = \mathrm{e}^{v_0}$ and its estimator

$$\lambda_t = \frac{\int\limits_2^t \tilde{\Delta}Y(s)\tilde{\Delta}Y(s-1)ds}{\int\limits_2^t (\tilde{\Delta}Y(s-1))^2 ds}, \quad t > 2, \quad \lambda_t = 0 \text{ otherwise;} \tag{11}$$

– the functions

$$Z(t) = \begin{cases} \tilde{\Delta}Y(t) - \lambda\tilde{\Delta}Y(t-1) & \text{for} \quad t \geq 2, \\ 0 \quad \text{for} \quad t < 2; \end{cases}$$

$$\tilde{Z}(t) = \begin{cases} \tilde{\Delta}Y(t) - \lambda_t\tilde{\Delta}Y(t-1) & \text{for} \quad t \geq 2, \\ 0 \quad \text{for} \quad t < 2, \end{cases}$$

$$\Psi(t) = \begin{cases} (\tilde{\Delta}Y(t), \tilde{\Delta}Y(t-1))' & \text{for} \quad t \geq 2, \\ (0,0)' \quad \text{for} \quad t < 2, \end{cases}$$

$$\tilde{\Psi}(t) = \begin{cases} (\tilde{Z}(t), \tilde{\Delta}Y(t))' & \text{for} \quad t \geq 2, \\ (0,0)' \quad \text{for} \quad t < 2; \end{cases}$$

– the parameter $\alpha = v_0/v_1$ and its estimator

$$\alpha_2(n,\varepsilon) = \frac{\ln \int\limits_4^{\nu_2(n,\varepsilon)} (\tilde{\Delta}Y(t-3))^2 dt}{\delta \ln \varepsilon^{-1} c_n}, \tag{12}$$

where

$$\nu_2(n,\varepsilon) = \inf\{T > 4 : \int_4^T \tilde{Z}^2(t-3)dt = (\varepsilon^{-1}c_n)^\delta\}, \tag{13}$$

$\delta \in (0,1)$ is a given number;
– the sequence of stopping times

$$\tau_2(n,\varepsilon) = h_2 \inf\{k > h_2^{-1}\nu_2(n,\varepsilon) : \int\limits_{\nu_2(n,\varepsilon)}^{kh_2} ||\Psi_2^{-1/2}(n,\varepsilon)\tilde{\Psi}(t-3)||^2 dt \geq 1\},$$

where suppose $h_2 = 1/5$ and

$$\Psi_2(n,\varepsilon) = \operatorname{diag}\{\varepsilon^{-1}c_n, (\varepsilon^{-1}c_n)^{\alpha_2(n,\varepsilon)}\};$$

– the matrices

$$G_2(S,T) = \int_S^T \tilde{\Psi}(t-3)\Psi'(t)dt, \quad \Phi_2(S,T) = \int_S^T \tilde{\Psi}(t-3)d\tilde{\Delta}Y(t),$$

$$G_2(n,k,\varepsilon) = G_2(\nu_2(n,\varepsilon), \tau_2(n,\varepsilon) - kh_2), \quad \Phi_2(n,k,\varepsilon) = \Phi_2(\nu_2(n,\varepsilon), \tau_2(n,\varepsilon) - kh_2);$$

– the times

$$k_2(n) = \arg\min_{k=\overline{1,5}} ||G_2^{-1}(n,k,\varepsilon)||, \; n \geq 1;$$

– the estimators

$$\vartheta_2(n,\varepsilon) = G_2^{-1}(n,\varepsilon)\Phi_2(n,\varepsilon), \; n \geq 1,$$

where

$$G_2(n,\varepsilon) = G_2(n,k_2(n),\varepsilon), \quad \Phi_2(n,\varepsilon) = \Phi_2(n,k_2(n),\varepsilon);$$

– the stopping time

$$\sigma_2(\varepsilon) = \inf\{n \geq 1 :\ S_2(N) > (\rho_2\delta_2^{-1})^{1/2}\}, \tag{14}$$

where $S_2(N) = \sum\limits_{n=1}^{N} \beta_2^2(n,\varepsilon)$, $\quad \rho_2 = \rho_1, \quad \delta_2 \in (0,1)$ is some fixed chosen number,

$$\beta_2(n,\varepsilon) = ||\tilde{G}_2^{-1}(n,\varepsilon)||, \quad \tilde{G}_2(n,\varepsilon) = (\varepsilon^{-1}c_n)^{-1/2}\Psi_2^{-1/2}(n,\varepsilon)G_2(n,\varepsilon).$$

In this case we write the deviation of $\vartheta_2(n,\varepsilon)$ in the form

$$\vartheta_2(n,\varepsilon) - \vartheta = (\varepsilon^{-1}c_n)^{-1/2}\tilde{G}_2^{-1}(n,\varepsilon)\tilde{\zeta}_2(n,\varepsilon),\ n \geq 1,$$

where

$$\tilde{\zeta}_2(n,\varepsilon) = \Psi_2^{-1/2}(n,\varepsilon) \int\limits_{\nu_2(n,\varepsilon)}^{\tau_2(n,\varepsilon)-k_2(n)h_2} \tilde{\Psi}(t-3)(\tilde{\Delta}\xi(t)dt + dV(t) - dV(t-1))$$

and we have

$$E_\vartheta||\tilde{\zeta}_2(n,\varepsilon)||^2 \leq 15(3+\mathrm{R}^2),\ \ n \geq 1,\ \ \varepsilon > 0.$$

Define the sequential estimation plan of ϑ by

$$T_2(\varepsilon) = \tau_2(\sigma_2(\varepsilon),\varepsilon),\ \ \vartheta_2(\varepsilon) = \vartheta_2(\sigma_2(\varepsilon),\varepsilon). \tag{15}$$

The construction of the sequential estimator $\vartheta_2(\varepsilon)$ is based on the family of estimators $\vartheta_2(S,T) = G_2^{-1}(S,T)\Phi_2(S,T) = \mathrm{e}^{-v_1T}\tilde{G}_2(S,T)\tilde{\Phi}_2(S,T),\ T > S \geq 0$, where

$$\tilde{G}_2(S,T) = \mathrm{e}^{-v_1T}\Psi_2^{-1/2}(T)G_2(S,T), \quad \tilde{\Phi}_2(S,T) = \Psi_2^{-1/2}(T)\Phi_2(S,T)$$

and $\Psi_2(T) = \mathrm{diag}\{\mathrm{e}^{v_1T}, \mathrm{e}^{v_0T}\}$. We have taken the discretization step h as above, because for $\vartheta \in \overline{\Theta}_{22}$, similar to the case $\vartheta \in \overline{\Theta}_{12}$, the function

$$f_2(S,T) = \tilde{G}_2^{-1}(S,T)$$

has some periodic (with the period $\Delta > 1$) matrix function as a limit almost surely (see (35)). This limit matrix function may have an infinite norm only for four values of their argument T on every interval of periodicity of the length Δ.

We state the results concerning the estimation of the parameter $\vartheta \in \overline{\Theta}_2$ in the following proposition.

Proposition 3.2. *Assume that the condition* (7) *on the sequence* (c_n) *holds as well as the parameter* $\vartheta = (a,b)'$ *in* (1) *be such that* $\vartheta \in \overline{\Theta}_2$. *Then:*

I. For any $\varepsilon > 0$ *and every* $\vartheta \in \overline{\Theta}_2$ *the sequential plan* $(T_2(\varepsilon), \vartheta_2(\varepsilon))$ *defined by* (15) *is closed and possesses the following properties:*

$$1^\circ. \quad \sup_{\vartheta \in \overline{\Theta}_2} E_\vartheta||\vartheta_2(\varepsilon) - \vartheta||^2 \leq \delta_2\varepsilon;$$

2°. *the inequalities below are valid:*

$$0 < \underline{\lim}_{\varepsilon\to 0}\,[T_2(\varepsilon) - \frac{1}{2v_1}\ln\varepsilon^{-1}] \;\le\; \overline{\lim}_{\varepsilon\to 0}\,[T_2(\varepsilon) - \frac{1}{2v_1}\ln\varepsilon^{-1}] \;<\; \infty \;\; P_\vartheta - a.s.;$$

II. For every $\vartheta \in \overline{\Theta}_2$ *the estimator* $\vartheta_2(n,\varepsilon)$ *is strongly consistent:*

$$\lim_{n\vee\varepsilon} \vartheta_2(n,\varepsilon) = \vartheta \;\; P_\vartheta - a.s.$$

3.3. Sequential estimation procedure for $\vartheta \in \overline{\Theta}_3$

We shall use the notation, introduced in the previous paragraph for the parameter $\lambda = \mathrm{e}^{v_0}$ and its estimator λ_t as well as for the functions $Z(t), \tilde{Z}(t), \Psi(t)$ and $\tilde{\Psi}(t)$.

Chose the non-random functions $\nu_3(n,\varepsilon),\ n \ge 1,\ \varepsilon > 0$, satisfying the following conditions as $\varepsilon \to 0$ or $n \to \infty$:

$$\nu_3(n,\varepsilon) = o(\varepsilon^{-1}c_n), \quad \frac{\log^{1/2}\nu_3(n,\varepsilon)}{\mathrm{e}^{v_0\nu_3(n,\varepsilon)}}\varepsilon^{-1}c_n = o(1). \tag{16}$$

Example: $\nu_3(n,\varepsilon) = \log^2\varepsilon^{-1}c_n$.

We introduce several quantities:
– the parameter $\alpha_3 = v_0$ and its estimator

$$\alpha_3(n,\varepsilon) = \ln|\lambda_{\nu_3(n,\varepsilon)}|,$$

where λ_t is defined in (11);
– the sequences of stopping times

$$\tau_{31}(n,\varepsilon) = \inf\{T > 0 : \int_{\nu_3(n,\varepsilon)}^{T} \tilde{Z}^2(t-3)dt = \varepsilon^{-1}c_n\}, \tag{17}$$

$$\tau_{32}(n,\varepsilon) = \inf\{T > 0 : \int_{\nu_3(n,\varepsilon)}^{T} (\tilde{\Delta}Y(t-3))^2 dt = \mathrm{e}^{2\alpha_3(n,\varepsilon)\varepsilon^{-1}c_n}\}, \tag{18}$$

$$\tau_{min}(n,\varepsilon) = \min\{\tau_{31}(n,\varepsilon), \tau_{32}(n,\varepsilon)\}, \quad \tau_{max}(n,\varepsilon) = \max\{\tau_{31}(n,\varepsilon), \tau_{32}(n,\varepsilon)\},$$

– the matrices

$$\begin{aligned} G_3(S,T) &= \int_S^T \tilde{\Psi}(t)\Psi(t)dt, \\ \Phi_3(S,T) &= \int_S^T \tilde{\Psi}(t)d\tilde{\Delta}Y(t), \\ G_3(n,\varepsilon) &= G_3(\nu_3(n,\varepsilon), \tau_{min}(n,\varepsilon)), \\ \Phi_3(n,\varepsilon) &= \Phi_3(\nu_3(n,\varepsilon), \tau_{min}(n,\varepsilon)); \end{aligned}$$

– the estimators

$$\vartheta_3(n,\varepsilon) = G_3^{-1}(n,\varepsilon)\Phi_3(n,\varepsilon),\ n \ge 1,\ \varepsilon > 0;$$

– the stopping time

$$\sigma_3(\varepsilon) = \inf\{n \geq 1 : \ S_3(N) > (\rho_3 \delta_3^{-1})^{1/2}\}, \tag{19}$$

where $S_3(N) = \sum\limits_{n=1}^{N} \beta_3^2(n,\varepsilon)$, $\delta_3 \in (0,1)$ is some fixed chosen number,

$$\beta_3(n,\varepsilon) = ||\tilde{G}_3^{-1}(n,\varepsilon)||, \quad \rho_3 = 6(3+R^2)\sum_{n\geq 1}\frac{1}{c_n},$$

$$\tilde{G}_3(n,\varepsilon) = (\varepsilon^{-1}c_n)^{-1/2}\Psi_3^{-1/2}(n,\varepsilon)G_3(n,\varepsilon), \quad \Psi_3(n,\varepsilon) = \mathrm{diag}\{\varepsilon^{-1}c_n, \mathrm{e}^{2\alpha_3(n,\varepsilon)\varepsilon^{-1}c_n}\}.$$

In this case we write the deviation of $\vartheta_3(n,\varepsilon)$ in the form

$$\vartheta_3(n,\varepsilon) - \vartheta = (\varepsilon^{-1}c_n)^{-1/2}\tilde{G}_3^{-1}(n,\varepsilon)\tilde{\zeta}_3(n,\varepsilon), \ n \geq 1,$$

where

$$\tilde{\zeta}_3(n,\varepsilon) = \Psi_3^{-1/2}(n,\varepsilon)\int\limits_{\nu_3(n,\varepsilon)}^{\tau_{min}(n,\varepsilon)} \tilde{\Psi}(t-3)(\tilde{\Delta}\xi(t)dt + dV(t) - dV(t-1))$$

and we have

$$E_\vartheta||\tilde{\zeta}_3(n,\varepsilon)||^2 \leq 6(3+R^2), \ \ n \geq 1, \ \ \varepsilon > 0.$$

Define the sequential estimation plan of ϑ by

$$T_3(\varepsilon) = \tau_{max}(\sigma_3(\varepsilon),\varepsilon), \ \ \vartheta_3(\varepsilon) = \vartheta_3(\sigma_3(\varepsilon),\varepsilon). \tag{20}$$

Proposition 3.3. *Assume that the condition* (7) *on the sequence* (c_n) *holds and let the parameter* $\vartheta = (a,b)'$ *in* (1) *be such that* $\vartheta \in \overline{\Theta}_3$. *Then:*
I. For every $\vartheta \in \overline{\Theta}_3$ *the sequential plan* $(T_3(\varepsilon), \vartheta_3(\varepsilon))$ *defined in* (20) *is closed and possesses the following properties:*

1°. *for any* $\varepsilon > 0$

$$\sup_{\vartheta\in\overline{\Theta}_3} E_\vartheta||\vartheta_3(\varepsilon) - \vartheta||^2 \leq \delta_3\varepsilon;$$

2°. *the following inequalities are valid:*

$$0 < \varliminf_{\varepsilon\to 0} \ \varepsilon T_3(\varepsilon) \leq \overline{\lim_{\varepsilon\to 0}} \ \varepsilon T_3(\varepsilon) < \infty \ \ P_\vartheta - a.s.;$$

II. For every $\vartheta \in \Theta_3$ *the estimator* $\vartheta_3(n,\varepsilon)$ *is strongly consistent:*

$$\lim_{n\vee\varepsilon} \ \vartheta_3(n,\varepsilon) = \vartheta \ \ P_\vartheta - a.s.$$

3.4. Sequential estimation procedure for $\vartheta \in \overline{\Theta}_4$

In this case $b = -a$ and (6) is the differential equation of the first order:

$$d\tilde{\Delta}Y(t) = aZ^*(t)dt + \tilde{\Delta}\xi(t)dt + dV(t) - dV(t-1), \ t \geq 2,$$

where

$$Z^*(t) = \begin{cases} \tilde{\Delta}Y(t) - \tilde{\Delta}Y(t-1) & \text{for} \quad t \geq 2, \\ 0 \quad \text{for} \quad t < 2. \end{cases}$$

We shall construct sequential plan $(T_4(\varepsilon), \vartheta_4(\varepsilon))$ for estimation of the vector parameter $\vartheta = a(1,-1)'$ with the $(\delta_4\varepsilon)$-accuracy in the sense of the L_2-norm for every $\varepsilon > 0$ and fixed chosen $\delta_4 \in (0,1)$.

First define the sequential estimation plans for the scalar parameter a on the bases of correlation estimators which are generalized least squares estimators:

$$a_4(T) = G_4^{-1}(T)\Phi_4(T),$$
$$G_4(T) = \int_0^T Z^*(t-2)Z^*(t)dt,$$
$$\Phi_4(T) = \int_0^T Z^*(t-2)d\tilde{\Delta}Y(t),\ T > 0.$$

Let $(c_n,\ n \geq 1)$ be an unboundedly increasing sequence of positive numbers, satisfying the condition (7).

We shall define

– the sequence of stopping times $(\tau_4(n,\varepsilon), n \geq 1)$ as

$$\tau_4(n,\varepsilon) = \inf\{T > 2 :\ \int_0^T (Z^*(t-2))^2 dt = \varepsilon^{-1}c_n\},\ n \geq 1;$$

– the sequence of estimators

$$a_4(n,\varepsilon) = a_4(\tau_4(n,\varepsilon)) = G_4^{-1}(\tau_4(n,\varepsilon))\Phi_4(\tau_4(n,\varepsilon));$$

– the stopping time

$$\sigma_4(\varepsilon) = \inf\{n \geq 1 :\ S_4(N) > (\rho_4\delta_4^{-1})^{1/2}\}, \tag{21}$$

where $S_4(N) = \sum\limits_{n=1}^{N} \tilde{G}_4^{-2}(n,\varepsilon)$, $\rho_4 = \rho_3$, $\tilde{G}_4(n,\varepsilon) = (\varepsilon^{-1}c_n)^{-1}G_4(\tau_4(n,\varepsilon))$. The deviation of $a_4(n,\varepsilon)$ has the form

$$a_4(n,\varepsilon) - a = (\varepsilon^{-1}c_n)^{-1/2}\tilde{G}_4^{-1}(n,\varepsilon)\tilde{\zeta}_4(n,\varepsilon),\ n \geq 1,$$

where

$$\tilde{\zeta}_4(n,\varepsilon) = (\varepsilon^{-1}c_n)^{-1/2} \int_0^{\tau_4(n,\varepsilon)} Z^*(t-2)(\tilde{\Delta}\xi(t)dt + dV(t) - dV(t-1))$$

and we have

$$E_\vartheta||\tilde{\zeta}_4(n,\varepsilon)||^2 \leq 3(3+\mathrm{R}^2),\ \ n \geq 1,\ \ \varepsilon > 0.$$

We define the sequential plan $(T_4(\varepsilon), \vartheta_4(\varepsilon))$ for the estimation of ϑ as

$$T_4(\varepsilon) = \tau_4(\sigma_4(\varepsilon),\varepsilon),\ \vartheta_4(\varepsilon) = a_4(\sigma_4(\varepsilon),\varepsilon)(1,-1)'. \tag{22}$$

The following proposition presents the conditions under which $T_4(\varepsilon)$ and $\vartheta_4(\varepsilon)$ are well-defined and have the desired property of preassigned mean square accuracy.

Proposition 3.4. *Assume that the sequence* (c_n) *defined above satisfy the condition* (7). *Then we obtain the following result:*

I. For any $\varepsilon > 0$ *and every* $\vartheta \in \overline{\Theta}_4$ *the sequential plan* $(T_4(\varepsilon), \vartheta_4(\varepsilon))$ *defined by* (22) *is closed and has the following properties:*

$$1^\circ. \quad \sup_{\vartheta \in \overline{\Theta}_4} E_\vartheta ||\vartheta_4(\varepsilon) - \vartheta||^2 \le \delta_4 \varepsilon;$$

2°. *the following relations hold:*

– if $\vartheta \in \overline{\Theta}_{41}$ *then*

$$0 < \varliminf_{\varepsilon \to 0} \varepsilon \cdot T_4(\varepsilon) \ \le \ \varlimsup_{\varepsilon \to 0} \varepsilon \cdot T_4(\varepsilon) \ < \ \infty \ P_\vartheta - a.s.,$$

– if $\vartheta \in \overline{\Theta}_{42}$ *then*

$$0 < \varliminf_{\varepsilon \to 0} [T_4(\varepsilon) - \frac{1}{2v_0} \ln \varepsilon^{-1}] \ \le \ \varlimsup_{\varepsilon \to 0} [T_4(\varepsilon) - \frac{1}{2v_0} \ln \varepsilon^{-1}] \ < \ \infty \ P_\vartheta - a.s.;$$

II. For every $\vartheta \in \overline{\Theta}_4$ *the estimator* $\vartheta_4(n, \varepsilon)$ *is strongly consistent:*

$$\lim_{n \vee \varepsilon} \vartheta_4(n, \varepsilon) = \vartheta \ P_\vartheta - a.s.$$

3.5. General sequential estimation procedure of the time-delayed process

In this paragraph we construct the sequential estimation procedure for the parameters a and b of the process (1) on the bases of the estimators, presented in subsections 3.1-3.4.

Denote $j^\star = \arg\min_{j=1,4} T_j(\varepsilon)$. We define the sequential plan $(T^\star(\varepsilon), \ \vartheta^\star(\varepsilon))$ of estimation $\vartheta \in \overline{\Theta}$ on the bases of all constructed above estimators by the formulae

$$\mathrm{SEP}^\star(\varepsilon) = (T^\star(\varepsilon), \vartheta^*(\varepsilon)), \quad T^\star(\varepsilon) = T_{j^\star}(\varepsilon), \quad \vartheta^*(\varepsilon) = \vartheta_{j^\star}(\varepsilon).$$

The following theorem is valid.

Theorem 3.1. *Assume that the underlying processes* $(X(t))$ *and* $(Y(t))$ *satisfy the equations* (1), (2), *the parameter* ϑ *to be estimated belongs to the region* $\overline{\Theta}$ *and for the numbers* $\delta_1, \ldots, \delta_4$ *in the definitions* (10), (15), (20) *and* (22) *of sequential plans the condition* $\sum_{j=1}^{4} \delta_j = 1$ *is fulfilled.*

Then the sequential estimation plan $(T^\star(\varepsilon), \ \vartheta^\star(\varepsilon))$ *possess the following properties:*

1°. *for any* $\varepsilon > 0$ *and for every* $\vartheta \in \overline{\Theta}$

$$T^\star(\varepsilon) < \infty \ P_\vartheta - a.s.;$$

2°. *for any* $\varepsilon > 0$

$$\sup_{\vartheta \in \overline{\Theta}} E_\vartheta \|\vartheta^\star(\varepsilon) - \vartheta\|^2 \le \varepsilon;$$

3°. *the following relations hold with* P_ϑ *– probability one:*

– for $\vartheta \in \overline{\Theta}_{11} \cup \overline{\Theta}_3 \cup \overline{\Theta}_{41}$

$$\varlimsup_{\varepsilon \to 0} \varepsilon \cdot T^*(\varepsilon) \ < \ \infty;$$

– *for* $\vartheta \in \overline{\Theta}_{12} \cup \overline{\Theta}_{42}$

$$\overline{\lim_{\varepsilon\to 0}}\,[T^*(\varepsilon) - \frac{1}{2v_0}\ln\varepsilon^{-1}] \;<\; \infty;$$

– *for* $\vartheta \in \overline{\Theta}_{13}$

$$\overline{\lim_{\varepsilon\to 0}}\,[T^*(\varepsilon) + \frac{1}{v_0}\ln T_1(\varepsilon) - \Psi''_{13}(\varepsilon)] \;<\; \infty,$$

the function $\Psi''_{13}(\varepsilon)$ *is defined in (30);*

– *for* $\vartheta \in \overline{\Theta}_2$

$$\overline{\lim_{\varepsilon\to 0}}\,[T^*(\varepsilon) - \frac{1}{2v_1}\ln\varepsilon^{-1}] \;<\; \infty.$$

4. Proofs

Proof of Proposition 3.1. The closeness of the sequential estimation plan, as well as assertions I.2 and II of Proposition 3.1 for the cases $\overline{\Theta}_{11} \cup \overline{\Theta}_{12}$ can be easily verified similar to [10, 12, 14, 16]. Now we verify the finiteness of the stopping time $T_1(\varepsilon)$ in the new case $\overline{\Theta}_{13}$.

By the definition of $\tilde{\Delta}Y(t)$ we have:

$$\tilde{\Delta}Y(t) = \tilde{X}(t) + \tilde{\Delta}V(t), \quad t \geq 1,$$

where

$$\tilde{X}(t) = \int_{t-1}^{t} X(t)dt.$$

It is easy to show that the process $(\tilde{X}(\cdot))$ has the following representation:

$$\tilde{X}(t) = \tilde{x}_0(t)X_0(0) + b\int_{-1}^{0} \tilde{x}_0(t-s-1)X_0(s)ds + \int_0^t \tilde{x}_0(t-s)dW(s)$$

for $t \geq 1$, $\tilde{X}(t) = \int_{t-1}^{0} X_0(s)ds + \int_0^t X(s)ds$ for $t \in [0,1)$ and $\tilde{X}(t) = 0$ for $t \in [-1,0)$. Based on the representation above for the function $x_0(\cdot)$, the subsequent properties of $x_0(t)$ the function $\tilde{x}_0(t) = \int_{t-1}^{t} x_0(s)ds$ can be easily shown to fulfill $\tilde{x}_0(t) = 0,\ t \in [-1,0]$ and as $t \to \infty$

$$\tilde{x}_0(t) = \begin{cases} o(e^{\gamma t}), & \gamma < 0, \quad \vartheta \in \Theta_{11}, \\ \tilde{\phi}_0(t)e^{v_0 t} + o(e^{\gamma_0 t}), & \gamma_0 < v_0,\ \vartheta \in \Theta_{12}, \\ \frac{2}{v_0}[(1-e^{-v_0})t + e^{-v_0} - \frac{1-e^{-v_0}}{v_0}]e^{v_0 t} + o(e^{\gamma_0 t}), & \gamma_0 < v_0,\ \vartheta \in \Theta_{13}, \\ \frac{1-e^{-v_0}}{v_0(v_0-a+1)}e^{v_0 t} + \frac{1-e^{-v_1}}{v_1(a-v_1-1)}e^{v_1 t} + o(e^{\gamma_1 t}), & \gamma_1 < v_1,\ \vartheta \in \Theta_{21}, \\ \frac{1-e^{-v_0}}{v_0(v_0-a+1)}e^{v_0 t} + \tilde{\phi}_1(t)e^{v_1 t} + o(e^{\gamma_1 t}), & \gamma_1 < v_1,\ \vartheta \in \Theta_{22}, \\ \frac{1-e^{-v_0}}{v_0(v_0-a+1)}e^{v_0 t} + o(e^{\gamma t}), & \gamma < 0, \quad \vartheta \in \Theta_3, \\ \frac{1}{1-a} + o(e^{\gamma t}), & \gamma < 0, \quad \vartheta \in \Theta_{41}, \\ \frac{1-e^{-v_0}}{v_0(v_0-a+1)}e^{v_0 t} - \frac{1}{a-1} + o(e^{\gamma t}), & \gamma < 0, \quad \vartheta \in \Theta_{42}, \end{cases}$$

where

$$\tilde{\phi}_i(t) = \tilde{A}_i \cos \xi_i t + \tilde{B}_i \sin \xi_i t$$

and $\tilde{A}_i, \tilde{B}_i, \xi_i$ are some constants (see [10, 12]).

The processes $\tilde{X}(t)$ and $\tilde{\Delta}V(t)$ are mutually independent and the process $\tilde{X}(t)$ has the representation similar to (3). Then, after some algebra similar to those in [10, 12] we get for the processes $\tilde{X}(t)$, $\tilde{Y}(t) = \tilde{X}(t) - \lambda \tilde{X}(t-1)$, $\lambda = \mathrm{e}^{v_0}$, $\tilde{\Delta}Y(t)$ and

$$Z(t) = \begin{cases} \tilde{\Delta}Y(t) - \lambda \tilde{\Delta}Y(t-1) & \text{for} \quad t \geq 2, \\ 0 \quad \text{for} \quad t < 2 \end{cases}$$

in the case $\overline{\Theta}_{13}$ the following limits:

$$\lim_{t\to\infty} t^{-1}\mathrm{e}^{-v_0 t}\tilde{\Delta}Y(t) = \lim_{t\to\infty} t^{-1}\mathrm{e}^{-v_0 t}\tilde{X}(t) = \tilde{C}_X \quad P_\vartheta - \text{a.s.}, \tag{23}$$

$$\lim_{t\to\infty} \mathrm{e}^{-v_0 t}\tilde{Y}(t) = C_Y, \qquad \lim_{t\to\infty} \mathrm{e}^{-v_0 t} Z(t) = \tilde{C}_Z \quad P_\vartheta - \text{a.s.},$$

and, as follows, for $u \geq 0$

$$\lim_{T\to\infty} \mid T^{-2}\mathrm{e}^{-2v_0 T} \int_1^T \tilde{\Delta}Y(t-u)\tilde{\Delta}Y(t)dt - \frac{\tilde{C}_X^2}{2v_0}\left[1 - \frac{u}{T}\right]\mathrm{e}^{-uv_0} \mid = 0 \quad P_\vartheta - \text{a.s.}, \tag{24}$$

$$\lim_{T\to\infty} \mid T^{-1}\mathrm{e}^{-2v_0 T} \int_1^T \tilde{\Delta}Y(t-u)Z(t)dt - \frac{\tilde{C}_X\tilde{C}_Z}{2v_0}\left[1 - \frac{u}{T}\right]\mathrm{e}^{-uv_0} \mid = 0 \quad P_\vartheta - \text{a.s.},$$

where $\tilde{C}_x, C_Y$ and $\tilde{C}_Z$ are some nonzero constants, which can be found from [10, 12]. From (24) we obtain the limits:

$$\lim_{T\to\infty} \frac{1}{T^2\mathrm{e}^{2v_0 T}} G_1(T,s) = G_{13}(s), \quad \lim_{T\to\infty} T^{-1}\mathrm{e}^{-4v_0 T}|G_1(T,s)| = G_{13}\mathrm{e}^{-(3+11s)v_0} \quad P_\vartheta - a.s.,$$

$$G_{13}(s) = \frac{\tilde{C}_X^2}{2v_0}\begin{pmatrix} \mathrm{e}^{-(2+5s)v_0} & \mathrm{e}^{-(1+5s)v_0} \\ \mathrm{e}^{-2(1+3s)v_0} & \mathrm{e}^{-(1+6s)v_0} \end{pmatrix}, \qquad G_{13} = \frac{s\tilde{C}_X^3\tilde{C}_Z}{4v_0^2}$$

and, as follows, we can find

$$\lim_{T\to\infty} T^{-1}\mathrm{e}^{2v_0 T} G_1^{-1}(T,s) = \tilde{G}_{13}(s) \ P_\vartheta - a.s.,$$

$$\tilde{G}_{13}(s) = \frac{2v_0\mathrm{e}^{(3+11s)v_0}}{s\tilde{C}_X\tilde{C}_Z}\begin{pmatrix} \mathrm{e}^{-(1+6s)v_0} & -\mathrm{e}^{-(1+5s)v_0} \\ -\mathrm{e}^{-2(1+3s)v_0} & \mathrm{e}^{-(2+5s)v_0} \end{pmatrix}$$

is a non-random matrix function.

From (23) and by the definition of the stopping times $\tau_1(n,\varepsilon)$ we have

$$\lim_{n\vee\varepsilon} \frac{\tau_1^2(n,\varepsilon)\mathrm{e}^{2\tau_1(n,\varepsilon)v_0}}{\varepsilon^{-1}c_n} = g_{13}^* \quad P_\vartheta - \text{a.s.}, \tag{25}$$

where $g_{13}^{*} = 2v_0\tilde{C}_X^{-2}\left(\mathrm{e}^{-2v_0(2+5h_1)} + \mathrm{e}^{-4v_0(1+3h_1)}\right)^{-1}$ and, as follows,

$$\lim_{n\vee\varepsilon}[\tau_1(n,\varepsilon) + \frac{1}{v_0}\ln\tau_1(n,\varepsilon) - \frac{1}{2v_0}\ln\varepsilon^{-1}c_n] = \frac{1}{2v_0}\ln g_{13}^{*} \quad P_\vartheta - \text{a.s.}, \tag{26}$$

$$\lim_{n\vee\varepsilon}\frac{\tau_1(n,\varepsilon)}{\ln\varepsilon^{-1}c_n} = \frac{1}{2v_0} \quad P_\vartheta - \text{a.s.}, \tag{27}$$

$$\lim_{n\vee\varepsilon}[\frac{1}{\ln^3\varepsilon^{-1}c_n}\tilde{G}_1^{-1}(n,\varepsilon) - [(2v_0)^3 g_{13}^{*}]^{-1}\mathrm{e}^{-2v_0k_1(n)h_1}\tilde{G}_{13}(h_1)] = 0 \quad P_\vartheta - \text{a.s.} \tag{28}$$

From (8) and (28) it follows the $P_\vartheta - $a.s. finiteness of the stopping time $\sigma_1(\varepsilon)$ for every $\varepsilon > 0$.

The proof of the assertion I.1 of Proposition 3.1 for the case $\overline{\Theta}_{13}$ is similar e.g. to the proof of corresponding assertion in [14, 16]:

$$E_\vartheta||\vartheta_1(\varepsilon) - \vartheta||^2 = E_\vartheta\frac{1}{S^2(\sigma_1(\varepsilon))}||\sum_{n=1}^{\sigma_1(\varepsilon)}\beta_1^2(n,\varepsilon)(\vartheta_1(n,\varepsilon) - \vartheta)||^2 \leq$$

$$\leq \varepsilon\frac{\delta_1}{\rho_1}E_\vartheta\sum_{n=1}^{\sigma_1(\varepsilon)}\frac{1}{c_n}\cdot\beta_1^2(n,\varepsilon)\cdot||\tilde{G}_1^{-1}(n,\varepsilon)||^2\cdot||\tilde{\zeta}_1(n,\varepsilon)||^2 \leq$$

$$\leq \frac{\varepsilon\delta_1}{\rho_1}\sum_{n\geq 1}\frac{1}{c_n}E_\vartheta||\tilde{\zeta}_1(n,\varepsilon)||^2 \leq \varepsilon\delta_1\frac{15(3+R^2)}{\rho_1}\sum_{n\geq 1}\frac{1}{c_n} = \varepsilon\delta_1.$$

Now we prove the assertion I.2 for $\vartheta \in \overline{\Theta}_{13}$. Denote the number

$$\tilde{g}_{13} = [(2v_0)^3 g_{13}^{*}]^2\rho_1^{-1}\delta_1||\tilde{G}_{13}(h_1)||^{-2}$$

and the times

$$\overline{\sigma}'_{13}(\varepsilon) = \inf\{n \geq 1 : \sum_{n=1}^{N}\ln^6\varepsilon^{-1}c_n > \tilde{g}_{13}\mathrm{e}^{4v_0h_1}\},$$

$$\overline{\sigma}''_{13}(\varepsilon) = \inf\{n \geq 1 : \sum_{n=1}^{N}\ln^6\varepsilon^{-1}c_n > \tilde{g}_{13}\mathrm{e}^{20v_0h_1}\}.$$

From (8) and (28) it follows, that for ε small enough

$$\overline{\sigma}'_{13}(\varepsilon) \leq \sigma_1(\varepsilon) \leq \overline{\sigma}''_{13}(\varepsilon) \quad P_\vartheta - \text{a.s.} \tag{29}$$

Denote

$$\Psi'_{13}(\varepsilon) = \frac{1}{2v_0}\ln(\varepsilon^{-1}c_{\overline{\sigma}'_{13}(\varepsilon)}), \quad \Psi''_{13}(\varepsilon) = \frac{1}{2v_0}\ln(\varepsilon^{-1}c_{\overline{\sigma}''_{13}(\varepsilon)}). \tag{30}$$

Then, from (8), (26) and (29) we obtain finally the assertion I.2 of Proposition 3.1:

$$\underline{\lim}_{\varepsilon\to 0}[T_1(\varepsilon) + \frac{1}{v_0}\ln T_1(\varepsilon) - \Psi'_{13}(\varepsilon)] \geq \frac{1}{2v_0}\ln g_{13}^{*} \quad P_\vartheta - \text{a.s.},$$

$$\overline{\lim_{\varepsilon\to 0}}\,[T_1(\varepsilon)+\frac{1}{v_0}\ln T_1(\varepsilon)-\Psi''_{13}(\varepsilon)]\le\frac{1}{2v_0}\ln g^*_{13}\quad P_\vartheta-\text{a.s.}$$

For the proof of the assertion II of Proposition 3.1 we will use the representation (9) for the deviation

$$\vartheta_1(n,\varepsilon)-\vartheta=\frac{1}{\ln^3\varepsilon^{-1}c_n}\tilde{G}_1^{-1}(n,\varepsilon)\cdot\frac{\tau_1^2(n,\varepsilon)\mathrm{e}^{2\tau_1(n,\varepsilon)v_0}}{\varepsilon^{-1}c_n}\left(\cdot\frac{\ln\varepsilon^{-1}c_n}{\tau_1(n,\varepsilon)}\right)^3\cdot\frac{1}{\tau_1^{-1}(n,\varepsilon)\mathrm{e}^{2\tau_1(n,\varepsilon)v_0}}\zeta_1(n,\varepsilon),$$

where

$$\zeta_1(n,\varepsilon)=\zeta_1(\tau_1(n,\varepsilon)-k_1(n)h_1,h_1),$$

$$\zeta_1(T,s)=\int_0^T\Psi_s(t-2-5s)(\tilde{\Delta}\xi(t)dt+dV(t)-dV(t-1)).$$

According to (25), (27) and (28) first three factors in the right-hand side of this equality have $P_\vartheta-$a.s. positive finite limits. The last factor vanishes in $P_\vartheta-$a.s. sense by the properties of the square integrable martingales $\zeta_1(T,s)$:

$$\lim_{n\vee\varepsilon}\frac{\zeta_1(n,\varepsilon)}{\tau_1^{-1}(n,\varepsilon)\mathrm{e}^{2\tau_1(n,\varepsilon)v_0}}=\lim_{T\to\infty}\frac{\zeta_1(T,h_1)}{T^{-1}\mathrm{e}^{2v_0T}}=0\quad P_\vartheta-\text{a.s.}$$

Then the estimators $\vartheta_1(n,\varepsilon)$ are strongly consistent as $\varepsilon\to 0$ or $n\to\infty$ and we obtain the assertion II of Proposition 3.1.

Hence Proposition 3.1 is valid.

Proof of Proposition 3.2.
Similar to the proof of Proposition 3.1 and [7]–[16] we can get the following asymptotic as $t\to\infty$ relations for the processes $\tilde{\Delta}Y(t)$, $Z(t)$ and $\tilde{Z}(t)$:

– for $\vartheta\in\Theta_{21}$

$$\tilde{\Delta}Y(t)=C_Ye^{v_0t}+C_{Y1}e^{v_1t}+o(e^{\gamma t})\quad P_\vartheta-\text{a.s.},$$

$$Z(t)=C_Ze^{v_1t}+o(e^{\gamma t})\quad P_\vartheta-\text{a.s.},$$

$$\lambda_t-\lambda=\frac{2v_0\mathrm{e}^{v_0}}{v_0+v_1}C_ZC_Y^{-1}e^{-(v_0-v_1)t}+o(e^{-(v_0-v_1+\gamma)t})\quad P_\vartheta-\text{a.s.},$$

$$\tilde{Z}(t)=\tilde{C}_Ze^{v_1t}+o(e^{\gamma t})\quad P_\vartheta-\text{a.s.};$$

– for $\vartheta\in\Theta_{22}$

$$|\tilde{\Delta}Y(t)-C_Ye^{v_0t}-C_{Y1}(t)e^{v_1t}|=o(e^{\gamma t})\quad P_\vartheta-\text{a.s.}$$

$$|Z(t)-C_Z(t)e^{v_1t}|=o(e^{\gamma t})\quad P_\vartheta-\text{a.s.},$$

$$\lambda_t-\lambda=2v_0\mathrm{e}^{v_0}C_Y^{-1}U_Z(t)e^{-(v_0-v_1)t}+o(e^{-(v_0-v_1+\gamma)t})\quad P_\vartheta-\text{a.s.},$$

$$|\tilde{Z}(t)-\tilde{C}_Z(t)e^{v_1t}|=o(e^{\gamma t})\quad P_\vartheta-\text{a.s.},$$

where C_Y and C_{Y1} are some non-zero constants, $0<\gamma<v_1$, $C_Z=C_{Y1}(1-\mathrm{e}^{v_0-v_1})$, $\tilde{C}_Z=C_Z\frac{v_1-v_0}{v_1+v_0}$; $C_Z(t)$, $U_Z(t)=\int_0^\infty C_Z(t-u)\mathrm{e}^{-(v_0+v_1)u}du$ and $\tilde{C}_Z(t)=C_Z(t)-2v_0U_Z(t)$ are the periodic (with the period $\Delta>1$) functions.

Denote

$$U_{\tilde{Z}}(T) = \int_0^\infty \tilde{C}_Z(T-u)\mathrm{e}^{-(v_0+v_1)u}du,$$

$$U_{\tilde{Z}Z}(S,T) = \int_0^\infty \tilde{C}_Z(T-u)C_Z(S-u)\mathrm{e}^{-2v_1u}du, \quad \tilde{U}_Z(T) = U_{\tilde{Z}\tilde{Z}}(T,T).$$

It should be noted that the functions $C_Z(t)$, $U_Z(t)$, $\tilde{C}_Z(t)$ and $U_{\tilde{Z}}(T)$ have at most two roots on each interval from $[0,\infty)$ of the length Δ. At the same time the function $U_{\tilde{Z}Z}(S,T)$ - at most four roots.

With P_ϑ-probability one we have:

– for $\vartheta \in \Theta_2$

$$\lim_{T-S\to\infty} \mathrm{e}^{-2v_0T}\int_S^T(\tilde{\Delta}Y(t-3))^2dt = \frac{C_Y^2}{2v_0}\mathrm{e}^{-6v_0}, \tag{31}$$

– for $\vartheta \in \Theta_{21}$

$$\lim_{T-S\to\infty} \mathrm{e}^{-2v_1T}\int_S^T\tilde{Z}^2(t-3)dt = \frac{\tilde{C}_Z^2}{2v_1}\mathrm{e}^{-6v_1}, \tag{32}$$

$$\lim_{T-S\to\infty} \tilde{G}_2^{-1}(S,T) = \tilde{G}_{21}, \tag{33}$$

where

$$\tilde{G}_{21} = \begin{pmatrix} \frac{2v_1(v_1+v_0)^2}{C_Z\tilde{C}_Z(v_1-v_0)^2}\mathrm{e}^{3v_1} & -\frac{4v_0v_1(v_1+v_0)}{C_ZC_Y(v_1-v_0)^2}\mathrm{e}^{3v_0} \\ -\frac{2v_1(v_1+v_0)^2}{C_Z\tilde{C}_Z(v_1-v_0)^2}\mathrm{e}^{v_0+3v_1} & \frac{4v_0v_1(v_1+v_0)}{C_ZC_Y(v_1-v_0)^2}\mathrm{e}^{4v_0} \end{pmatrix},$$

– for $\vartheta \in \Theta_{22}$

$$\lim_{T-S\to\infty}\left|\mathrm{e}^{-2v_1T}\int_S^T\tilde{Z}^2(t-3)dt - \mathrm{e}^{-6v_1}\tilde{U}_Z(T-3)\right| = 0, \tag{34}$$

$$\lim_{T-S\to\infty}\left|\tilde{G}_2^{-1}(S,T) - \tilde{G}_{22}(T)\right| = 0, \tag{35}$$

where

$$\tilde{G}_{22}(T) = \left[\frac{1}{2v_0}U_{\tilde{Z}Z}(T,T-3) - U_Z(T-3)U_{\tilde{Z}}(T)\right]^{-1}\cdot\begin{pmatrix} \frac{\mathrm{e}^{3v_1}}{2v_0} & -\frac{\mathrm{e}^{3v_0}}{C_Y}U_{\tilde{Z}}(T) \\ -\frac{\mathrm{e}^{v_0+3v_1}}{2v_0} & \frac{\mathrm{e}^{4v_0}}{C_Y}U_{\tilde{Z}}(T) \end{pmatrix}.$$

The matrix $\tilde{G}_{21}$ is constant and non-zero and $\tilde{G}_{22}(T)$ is the periodic matrix function with the period $\Delta > 1$ (see [3], [10, 12, 14]) and may have infinite norm for four points on each interval of periodicity only.

The next step of the proof is the investigation of the asymptotic behaviour of the stopping times $\nu_2(n,\varepsilon)$, $\tau_2(n,\varepsilon)$ and the estimators $\alpha_2(n,\varepsilon)$.

Denote

$$C_{\nu 1} = \mathrm{e}^{-6v_1} \min\left\{\frac{\tilde{C}_Z^2}{2v_1},\ \inf_{T>0} \tilde{U}_Z(T)\right\}, \quad C_{\nu 2} = \mathrm{e}^{-6v_1} \max\left\{\frac{\tilde{C}_Z^2}{2v_1},\ \sup_{T>0} \tilde{U}_Z(T)\right\}.$$

Then for $\vartheta \in \Theta_2$

$$C_{\nu 1} \le \varliminf_{T-S\to\infty} \mathrm{e}^{-2v_1 T} \int_S^T \tilde{Z}^2(t-3)dt \le$$

$$\le \varlimsup_{T-S\to\infty} \mathrm{e}^{-2v_1 T} \int_S^T \tilde{Z}^2(t-3)dt \le C_{\nu 2} \quad P_\vartheta - \text{a.s.} \tag{36}$$

and from the definition (13) of $\nu_2(n,\varepsilon)$ and (32), (34) we have

$$C_{\nu 2}^{-1} \le \varliminf_{n\vee\varepsilon} \frac{\mathrm{e}^{2v_1\nu_2(n,\varepsilon)}}{(\varepsilon^{-1}c_n)^\delta} \le \varlimsup_{n\vee\varepsilon} \frac{\mathrm{e}^{2v_1\nu_2(n,\varepsilon)}}{(\varepsilon^{-1}c_n)^\delta} \le C_{\nu 1}^{-1} \quad P_\vartheta - \text{a.s.}$$

and thus

$$\frac{1}{2v_1}\ln C_{\nu 2}^{-1} \le \varliminf_{n\vee\varepsilon} [\nu_2(n,\varepsilon) - \frac{\delta}{2v_1}\ln\varepsilon^{-1}c_n] \le$$

$$\le \varlimsup_{n\vee\varepsilon} [\nu_2(n,\varepsilon) - \frac{\delta}{2v_1}\ln\varepsilon^{-1}c_n] \le \frac{1}{2v_1}\ln C_{\nu 1}^{-1} \quad P_\vartheta - \text{a.s.} \tag{37}$$

By the definition (12) of $\alpha_2(n,\varepsilon)$ we find the following normalized representation for the deviation $\alpha_2(n,\varepsilon) - \alpha$:

$$\nu_2(n,\varepsilon)(\alpha_2(n,\varepsilon)-\alpha) = \nu_2(n,\varepsilon)\left(\frac{\ln\int_0^{\nu_2(n,\varepsilon)}(\tilde{\Delta}Y(t-3))^2dt}{\ln\int_0^{\nu_2(n,\varepsilon)}\tilde{Z}^2(t-3)dt} - \frac{v_0}{v_1}\right) =$$

$$= \nu_2(n,\varepsilon)\left(\frac{2v_0\nu_2(n,\varepsilon) + \ln\mathrm{e}^{-2v_0\nu_2(n,\varepsilon)}\int_0^{\nu_2(n,\varepsilon)}(\tilde{\Delta}Y(t-3))^2dt}{2v_1\nu_2(n,\varepsilon) + \ln\mathrm{e}^{-2v_1\nu_2(n,\varepsilon)}\int_0^{\nu_2(n,\varepsilon)}\tilde{Z}^2(t-3)dt} - \frac{v_0}{v_1}\right) =$$

$$= \nu_2(n,\varepsilon)\frac{v_1\ln\mathrm{e}^{-2v_0\nu_2(n,\varepsilon)}\int_0^{\nu_2(n,\varepsilon)}(\tilde{\Delta}Y(t-3))^2dt - v_0\ln\mathrm{e}^{-2v_1\nu_2(n,\varepsilon)}\int_0^{\nu_2(n,\varepsilon)}\tilde{Z}^2(t-3)dt}{2v_1^2\nu_2(n,\varepsilon) + v_1\ln\mathrm{e}^{-2v_1\nu_2(n,\varepsilon)}\int_0^{\nu_2(n,\varepsilon)}\tilde{Z}^2(t-3)dt}$$

and using the limit relations (31), (36) and (37) we obtain

$$\alpha_1 \le \varliminf_{n\vee\varepsilon} (\ln\varepsilon^{-1}c_n)\cdot(\alpha-\alpha_2(n,\varepsilon)) \le \varlimsup_{n\vee\varepsilon} (\ln\varepsilon^{-1}c_n)\cdot(\alpha-\alpha_2(n,\varepsilon)) \le \alpha_2 \quad P_\vartheta - \text{a.s.},$$

where $\alpha_i = \dfrac{1}{\delta v_1}[v_0\ln C_{\nu i} - v_1\ln\dfrac{C_Y^2}{2v_0}\mathrm{e}^{-6v_0}], \quad i = 1,2.$

Thus for $\vartheta \in \Theta_2$

$$e^{\alpha_1} \le \lim_{n\vee\varepsilon} (\varepsilon^{-1}c_n)^{(\alpha-\alpha_2(n,\varepsilon))} \le \lim_{n\vee\varepsilon} (\varepsilon^{-1}c_n)^{(\alpha-\alpha_2(n,\varepsilon))} \le e^{\alpha_2} \quad P_\vartheta - \text{a.s.} \tag{38}$$

Let s_1 and s_2 be the positive roots of the following equations

$$C_{v2}\cdot s + \frac{C_Y^2}{2v_0} e^{-6v_0}\cdot e^{\alpha_2}\cdot s^\alpha = 1 \quad \text{and} \quad C_{v1}\cdot s + \frac{C_Y^2}{2v_0} e^{-6v_0}\cdot e^{\alpha_1}\cdot s^\alpha = 1$$

respectively. It is clear that $0 < s_1 \le s_2 < \infty$.

By the definition of stopping times $\tau_2(n,\varepsilon)$ we have

$$\lim_{n\vee\varepsilon}\left[\frac{1}{\varepsilon^{-1}c_n}\int_{\nu_2(n,\varepsilon)}^{\tau_2(n,\varepsilon)} \tilde{Z}^2(t-3)dt + \frac{1}{(\varepsilon^{-1}c_n)^{\alpha_2(n,\varepsilon)}}\int_{\nu_2(n,\varepsilon)}^{\tau_2(n,\varepsilon)} (\tilde{\Delta}Y(t-3))^2 dt\right] =$$

$$= \lim_{n\vee\varepsilon}\left[\frac{1}{e^{2v_1\tau_2(n,\varepsilon)}}\int_0^{\tau_2(n,\varepsilon)} \tilde{Z}^2(t-3)dt\cdot\frac{e^{2v_1\tau_2(n,\varepsilon)}}{\varepsilon^{-1}c_n} + \right.$$

$$\left. + \frac{1}{e^{2v_0\tau_2(n,\varepsilon)}}\int_0^{\tau_2(n,\varepsilon)} (\tilde{\Delta}Y(t-3))^2 dt\cdot(\varepsilon^{-1}c_n)^{(\alpha-\alpha_2(n,\varepsilon))}\cdot\left(\frac{e^{2v_1\tau_2(n,\varepsilon)}}{\varepsilon^{-1}c_n}\right)^\alpha\right] = 1.$$

Then, using (38), for $\vartheta \in \Theta_2$ we have

$$s_1 \le \underline{\lim_{n\vee\varepsilon}}\ \frac{e^{2v_1\tau_2(n,\varepsilon)}}{\varepsilon^{-1}c_n} \le \overline{\lim_{n\vee\varepsilon}}\ \frac{e^{2v_1\tau_2(n,\varepsilon)}}{\varepsilon^{-1}c_n} \le s_2 \quad P_\vartheta - \text{a.s.} \tag{39}$$

and thus

$$\frac{1}{2v_1}\ln s_1 \le \underline{\lim_{n\vee\varepsilon}}\,[\tau_2(n,\varepsilon) - \frac{1}{2v_1}\ln\varepsilon^{-1}c_n] \le$$

$$\le \overline{\lim_{n\vee\varepsilon}}\,[\tau_2(n,\varepsilon) - \frac{1}{2v_1}\ln\varepsilon^{-1}c_n] \le \frac{1}{2v_1}\ln s_2 \quad P_\vartheta - \text{a.s.} \tag{40}$$

From (37) and (40) it follows, in particular, that

$$\lim_{n\vee\varepsilon}\,[\tau_2(n,\varepsilon) - \nu_2(n,\varepsilon)] = \infty \quad P_\vartheta - \text{a.s.} \tag{41}$$

By the definition of $\tilde{G}_2(n,\varepsilon)$, the following limit relation can be proved

$$\underline{\lim_{n\vee\varepsilon}}\,[||\tilde{G}_2^{-1}(n,\varepsilon)||^2 - (1+e^{2v_0})\{\left(\frac{e^{2v_1\tau_2(n,\varepsilon)}}{\varepsilon^{-1}c_n}\right)^{-2}(<\tilde{G}_2^{-1}(0,\tau_2(n,\varepsilon)-k_2(n)h_2)>_{11})^2 +$$

$$+\left(\frac{e^{2v_1\tau_2(n,\varepsilon)}}{\varepsilon^{-1}c_n}\right)^{-(1+\alpha)}(\varepsilon^{-1}c_n)^{\alpha_2(n,\varepsilon)-\alpha}(<\tilde{G}_2^{-1}(0,\tau_2(n,\varepsilon)-k_2(n)h_2)>_{12})^2\}] = 0 \quad P_\vartheta - \text{a.s.},$$

where $< G >_{ij}$ is the ij-th element of the matrix G.

Then, using (33), (35), (38), (39) and (41) we can find, similar to [12, 14], the lower and upper bounds for the limits with P_ϑ-probability one:

$$\tilde{g}_{21} \le \varliminf_{n\vee\varepsilon} ||\tilde{G}_2^{-1}(n,\varepsilon)|| \le \varlimsup_{n\vee\varepsilon} ||\tilde{G}_2^{-1}(n,\varepsilon)|| \le \tilde{g}_{22}, \tag{42}$$

where $\tilde{g}_{21}$ and $\tilde{g}_{22}$ are positive finite numbers.

Thus, by the definition (14) of the stopping time $\sigma_2(\varepsilon)$ and from (42) we have

$$\sigma_{21} \le \varliminf_{\varepsilon\to 0} \sigma_2(\varepsilon) \le \varlimsup_{\varepsilon\to 0} \sigma_2(\varepsilon) \le \sigma_{22} \quad P_\vartheta - \text{a.s.}, \tag{43}$$

where

$$\sigma_{21} = \inf\{n \ge 1 :\ N > \tilde{g}_{22}^{-1}(\rho_2\delta_2^{-1})^{1/2}\}, \qquad \sigma_{22} = \inf\{n \ge 1 :\ N > \tilde{g}_{21}^{-1}(\rho_2\delta_2^{-1})^{1/2}\}$$

and from (40) and (43) we obtain the second property of the assertion I in Proposition 3.2:

$$\frac{1}{2v_1}\ln s_1\sigma_{21} \le \varliminf_{\varepsilon\to 0}\,[T_2(\varepsilon) - \frac{1}{2v_1}\ln\varepsilon^{-1}] \le \varlimsup_{\varepsilon\to 0}\,[T_2(\varepsilon) - \frac{1}{2v_1}\ln\varepsilon^{-1}] \le \frac{1}{2v_1}\ln s_2\sigma_{22} \quad P_\vartheta - \text{a.s.}$$

The assertions I.1 and II of Proposition 3.2 can be proved similar to the proof of the corresponding statement of Proposition 3.1.

Hence Proposition 3.2 is proven.

Proof of Proposition 3.3.
Similar to the proof of Propositions 3.1, 3.2 and [7]–[16] we get for $\vartheta \in \Theta_3$ the needed asymptotic as $t \to \infty$ relations for the processes $\tilde{\Delta}Y(t)$, $Z(t)$ and $\tilde{Z}(t)$. To this end we introduce the following notation:

$$Z_1(t) = \int_{-\infty}^{t} \tilde{y}_0(t-s)dW(s), \ \ \tilde{y}_0(s) = \tilde{x}_0(s) - \lambda\tilde{x}_0(s-1),$$

$$Z_V(t) = \int_{-\infty}^{t} [\tilde{\Delta}V(s) - \lambda\tilde{\Delta}V(s-1)]e^{-v_0(t-s)}ds, \quad Z_2(t) = Z_V(t) + Z_3(t),$$

$$Z_3(t) = \int_{-\infty}^{t} Z_1(s)e^{-v_0(t-s)}ds, \quad \tilde{Z}_1(t) = Z_1(t) - 2v_0Z_2(t-1),$$

$$\tilde{Z}_2(t) = Z_V(t) + \tilde{Z}_3(t), \quad \tilde{Z}_3(t) = \int_{-\infty}^{t} \tilde{Z}_1(s)e^{-v_0(t-s)}ds,$$

$$\tilde{C}_Z = 1 + \lambda^2 + 4[\lambda - v_0^{-1}(\lambda-1)] + E_\vartheta\tilde{Z}_1^2(0),$$

$$C_{\tilde{Z}Z} = 1 + \lambda^2 + 2[\lambda - v_0^{-1}(\lambda-1)] + E_\vartheta Z_1^2(0) - E_\vartheta Z_1(0)Z_3(-1).$$

It should be noted that in the considered case Θ_3 all the introduced processes $Z_1(\cdot), \ldots, \tilde{Z}_3(\cdot)$ are stationary Gaussian processes, continuous in probability, having a spectral density and, as follows, ergodic, see [21].

According to the definition of the set Θ_3 as $t \to \infty$ we have:

$$\tilde{\Delta}Y(t) = C_Y e^{v_0 t} + o(e^{\gamma t}) \;\; P_\vartheta - \text{a.s.},$$

$$|Z(t) - [\tilde{\Delta}V(t) - \lambda\tilde{\Delta}V(t-1)] - Z_1(t)| = o(1) \;\; P_\vartheta - \text{a.s.},$$

where C_Y and $\gamma < v_0$ are some constants.

Using this properties and the representation for the deviation

$$\lambda_t - \lambda = \frac{\int\limits_0^t Z(s)\tilde{\Delta}Y(s-1)ds}{\int\limits_0^t (\tilde{\Delta}Y(s-1))^2 ds}$$

of the estimator λ_t defined in (11), it is easy to obtain with P_ϑ-probability one the following limit relations:

$$\lim_{T\to\infty} \frac{1}{e^{2v_0 T}} \int\limits_0^T \tilde{\Delta}Y(t-u)\tilde{\Delta}Y(t-s)dt = \frac{C_Y^2}{2v_0} e^{-v_0(u+s)}, \;\; u, s \geq 0, \tag{44}$$

$$\lim_{T\to\infty} \left| \frac{1}{e^{v_0 T}} \int\limits_0^T Z(t)\tilde{\Delta}Y(t-u)dt - C_Y e^{-v_0 u} Z_2(T) \right| = 0, \;\; u \geq 0, \tag{45}$$

$$\lim_{t\to\infty} |e^{v_0 t}(\lambda_t - \lambda) - 2v_0 e^{v_0} C_Y^{-1} Z_2(t)| = 0, \tag{46}$$

$$\lim_{t\to\infty} |\tilde{Z}(t) - (\tilde{\Delta}V(t) - \lambda\tilde{\Delta}V(t-1)) - \tilde{Z}_1(t)| = 0,$$

$$\lim_{T\to\infty} \left| \frac{1}{e^{v_0 T}} \int\limits_0^T \tilde{Z}(t)\tilde{\Delta}Y(t)dt - C_Y \tilde{Z}_2(T) \right| = 0, \tag{47}$$

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T \tilde{Z}(t)Z(t)dt = C_{\tilde{Z}Z}, \tag{48}$$

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T \tilde{Z}^2(t)dt = \tilde{C}_Z. \tag{49}$$

For the investigation of the asymptotic properties of the components of sequential plan we will use Propositions 2 and 3 from [14]. According to these propositions the processes $Z_i(\cdot), \tilde{Z}_i(\cdot), \; i = \overline{1,3}$ and $Z_V(\cdot)$ defined above are $O((\log t)^{\frac{1}{2}})$ as $t \to \infty \; P_\vartheta -$ a.s.

Denote

$$Q = \begin{pmatrix} 1 & 1 \\ -\lambda & 0 \end{pmatrix}, \qquad \varphi(T) = \mathrm{diag}\{T, \mathrm{e}^{2v_0 T}\}.$$

From (44), (45), (47), (48) with P_ϑ-probability one holds

$$\lim_{T\to\infty} \varphi^{-1/2}(T) \cdot G_3(0,T) \cdot Q \cdot \varphi^{-1/2}(T) = \mathrm{diag}\{C_{\tilde{Z}Z}, \frac{C_Y^2}{2v_0}\}$$

and, as follows,

$$\lim_{T\to\infty} T \cdot G_3^{-1}(0,T) = C_{\tilde{Z}Z}^{-1} \cdot \begin{pmatrix} 1 & 0 \\ -\lambda & 0 \end{pmatrix} \quad P_\vartheta - \text{a.s.} \tag{50}$$

Further, by the definition (17) of stopping times $\tau_{31}(n,\varepsilon)$, first condition in (16) on the function $\nu_3(n,\varepsilon)$ and from (49) we find

$$\lim_{n\vee\varepsilon} \frac{\tau_{31}(n,\varepsilon)}{\varepsilon^{-1}c_n} = \tilde{C}_Z^{-1} \quad P_\vartheta - \text{a.s.} \tag{51}$$

For the investigation of asymptotic properties of stopping times $\tau_{32}(n,\varepsilon)$ with P_ϑ-probability one we show, using the second condition in (16) on the function $\nu_3(n,\varepsilon)$ and (46), that

$$\lim_{n\vee\varepsilon} \ln \frac{\mathrm{e}^{2\alpha_3(n,\varepsilon)\varepsilon^{-1}c_n}}{\mathrm{e}^{2\alpha_3\varepsilon^{-1}c_n}} = \lim_{n\vee\varepsilon} 2(\alpha_3(n,\varepsilon) - \alpha)\varepsilon^{-1}c_n = \lim_{n\vee\varepsilon} 2\lambda^{-1}(\lambda_{\nu_3(n,\varepsilon)} - \lambda)\varepsilon^{-1}c_n =$$

$$= 2C_3^{-1} \lim_{n\vee\varepsilon} \frac{Z_2(\nu_3(n,\varepsilon))\varepsilon^{-1}c_n}{\mathrm{e}^{v_0\nu_3(n,\varepsilon)}} = 2C_3^{-1} \lim_{n\vee\varepsilon} \frac{Z_2(\nu_3(n,\varepsilon))}{\log^{1/2}\nu_3(n,\varepsilon)} \cdot \frac{\log^{1/2}\nu_3(n,\varepsilon)}{\mathrm{e}^{v_0\nu_3(n,\varepsilon)}} \cdot \varepsilon^{-1}c_n = 0$$

and then

$$\lim_{n\vee\varepsilon} \frac{\mathrm{e}^{2\alpha_3(n,\varepsilon)\varepsilon^{-1}c_n}}{\mathrm{e}^{2v_0\varepsilon^{-1}c_n}} = 1 \quad P_\vartheta - \text{a.s.}$$

Thus, by the definition (18) of stopping times $\tau_{32}(n,\varepsilon)$ and from (44) we find

$$\lim_{n\vee\varepsilon} [\tau_{32}(n,\varepsilon) - \varepsilon^{-1}c_n] = \frac{1}{2v_0} \ln \frac{2v_0\mathrm{e}^{6v_0}}{C_Y^2} \quad P_\vartheta - \text{a.s.} \tag{52}$$

Then, from (50)–(52) with P_ϑ-probability one we obtain

$$\lim_{n\vee\varepsilon} \tilde{G}_3^{-1}(n,\varepsilon) = \{\tilde{C}_Z \vee 1\} C_{\tilde{Z}Z}^{-1} \cdot \begin{pmatrix} 1 & 0 \\ -\lambda & 0 \end{pmatrix},$$

where $a \vee b = \max(a,b)$ and, by the definition (19) of the stopping time $\sigma_3(\varepsilon)$, for ε small enough it follows

$$\sigma_3(\varepsilon) = \overline{\sigma}_3 \quad P_\vartheta - \text{a.s.},$$

where

$$\overline{\sigma}_3 = \inf\{n \geq 1 : \ N > g_3^{-1}(\rho_3\delta_3^{-1})^{1/2}\}$$

and $g_3 = \{\tilde{C}_Z \vee 1\}^2 C_{\tilde{Z}Z}^{-2}(1+\lambda^2)$.

Thus we obtain the P_ϑ-finiteness of the stopping time $T_3(\varepsilon)$ and the assertion II.2 of Proposition 3.3:

$$\lim_{\varepsilon\to 0} \varepsilon T_3(\varepsilon) = \{\tilde{C}_Z^{-1} \vee 1\} c_{\overline{\sigma}_3} \quad P_\vartheta - \text{a.s.}$$

The assertions I.1 and II of Proposition 3.3 can be proved similar to the proofs of Propositions 3.1 and 3.2.

Hence Proposition 3.3 is proven.

Proof of Proposition 3.4.
This case is a scalar analogue of the case $\Theta_{11} \cup \Theta_{12}$.

By the definition,

$$Z^*(t) = \tilde{X}(t) - \tilde{X}(t-1) + \tilde{\Delta}V(t) - \tilde{\Delta}V(t-1).$$

According to the asymptotic properties of the process $(\tilde{X}(t))$, for $u = 0, 2$ we have:

– for $\vartheta \in \Theta_{41}$:

exist the positive constant limits

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T Z^*(t) Z^*(t-u) dt = C_{41}^*(u) \;\; P_\vartheta - \text{a.s.}; \tag{53}$$

– for $\vartheta \in \Theta_{42}$:

$$\begin{aligned} &e^{-v_0 t} Z^*(t) = C_{42}^* + o(e^{-(v_0-\gamma)t})) \;\; \text{as } t \to \infty \quad P_\vartheta - \text{a.s.}, \\ &\lim_{T\to\infty} \frac{1}{e^{2v_0 T}} \int_0^T Z^*(t) Z^*(t-u) dt = \frac{(C_{42}^*)^2 e^{-v_0 u}}{2v_0} \quad P_\vartheta - \text{a.s.}, \end{aligned} \tag{54}$$

where

$$C_{42}^* = \frac{1 - e^{v_0}}{v_0(v_0 - a + 1)}.$$

Assertions I.1 and II of Proposition 3.4 can be proved similar to Proposition 3.1.

Now we prove the closeness of the plan (22) and assertion I.2 of Proposition 3.4. To this end we shall investigate the asymptotic properties of the stopping times $\tau_4(n,\varepsilon)$ and $\sigma_4(\varepsilon)$.

From the definition of $\tau_4(n,\varepsilon)$ and (53), (54) we have

– for $\vartheta \in \Theta_{41}$:

$$\lim_{n\vee\varepsilon} \frac{\tau_4(n,\varepsilon)}{\varepsilon^{-1} c_n} = (C_{41}^*(0))^{-1} \;\; P_\vartheta - \text{a.s.}; \tag{55}$$

– for $\vartheta \in \Theta_{42}$:

$$\lim_{n\vee\varepsilon}\frac{\mathrm{e}^{2v_0\tau_4(n,\varepsilon)}}{\varepsilon^{-1}c_n}=\left(\frac{2v_0}{C^*_{42}}\right)^2\quad P_\vartheta-\text{a.s.},\tag{56}$$

$$\lim_{n\vee\varepsilon}[\tau_4(n,\varepsilon)-\frac{1}{2v_0}\ln\varepsilon^{-1}c_n]=\frac{1}{v_0}\ln\frac{2v_0}{C^*_{42}}\quad P_\vartheta-\text{a.s.}\tag{57}$$

As follows, the stopping times $\tau_4(n,\varepsilon)$ are $P_\vartheta-$ a.s. finite for all $\vartheta\in\Theta_4$.

Denote

$$\sigma_{41}=\inf\{n\geq 1:\ N>(\rho_4\delta_4^{-1})^{1/2}g_{41}^{-1}\},\qquad \sigma_{42}=\inf\{n\geq 1:\ N>(\rho_4\delta_4^{-1})^{1/2}g_{42}^{-1}\},$$

where $g_{41}=[C^*_{41}(2)(C^*_{41}(0))^{-1}\vee 2v_0\mathrm{e}^{-2v_0}]$, $g_{42}=[C^*_{41}(2)(C^*_{41}(0))^{-1}\wedge 2v_0\mathrm{e}^{-2v_0}]$, where $a\wedge b=\min(a,b)$ and by the definition of the stopping time $\sigma_4(\varepsilon)$ (21) as well as from (53)–(56) follows the $P_\vartheta-$ a.s. finiteness of $\sigma_4(\varepsilon)$ and the following inequalities, which hold with P_ϑ-probability one for ε small enough:

$$\sigma_{41}\leq\sigma_4(\varepsilon)\leq\sigma_{42}\quad P_\vartheta-\text{a.s.}\tag{58}$$

Then we obtain the finiteness of the stopping time $T_4(\varepsilon)$ and the assertion I.2 of Proposition 3.4, which follows from (55), (57) and (58):

– for $\vartheta\in\Theta_{41}$:

$$c_{\sigma_{41}}(C^*_{41}(0))^{-1}\leq\underline{\lim_{n\vee\varepsilon}}\ \varepsilon T_4(\varepsilon)\leq\overline{\lim_{n\vee\varepsilon}}\ \varepsilon T_4(\varepsilon)\leq c_{\sigma_{42}}(C^*_{41}(0))^{-1}\ \ P_\vartheta-\text{a.s.};$$

– for $\vartheta\in\Theta_{42}$:

$$\frac{1}{2v_0}\ln c_{\sigma_{41}}\left(\frac{2v_0}{C^*_{42}(0)}\right)^2\leq\underline{\lim_{n\vee\varepsilon}}\,[T_4(\varepsilon)-\frac{1}{2v_0}\ln\varepsilon^{-1}]\leq$$

$$\leq\overline{\lim_{n\vee\varepsilon}}\,[T_4(\varepsilon)-\frac{1}{2v_0}\ln\varepsilon^{-1}]\leq\frac{1}{2v_0}\ln c_{\sigma_{42}}\left(\frac{2v_0}{C^*_{42}(0)}\right)^2\quad P_\vartheta-\text{a.s.}$$

Hence Proposition 3.4 is valid.

Proof of Theorem 3.1. The closeness of the sequential estimation plan $\mathrm{SEP}^\star(\varepsilon)$ (assertion 1) and assertion 3 of Theorem 3.1 follow from Propositions 3.1-3.4 directly.

Now we prove the assertion 2. To this end we show first, that all the stopping times $\tau_i(n,\varepsilon)$, $i=1,2,4$ and $\tau_{3i}(n,\varepsilon)$, $i=1,2$ are $P_\vartheta-$ a.s.-finite for every $\vartheta\in\overline{\Theta}$. It should be noted, that the integrals

$$\int_0^\infty(\tilde{\Delta}Y(t))^2dt=\infty\quad\text{and}\quad\int_0^\infty(Z^*(t))^2dt=\infty\quad P_\vartheta-\text{a.s.}\tag{59}$$

in all the cases $\Theta_1,\ldots,\Theta_4$ and, as follows, all the stopping times $\tau_i(n,\varepsilon)$, $i=1,2,4$ and $\tau_{32}(n,\varepsilon)$ are P_ϑ-a.s.-finite for every $\varepsilon>0$ and all $n\geq 1$. The properties (59) can be established by using the asymptotic properties of the process $(X(t),Y(t))$ (see proofs of Propositions 3.1–3.4 and [3], [7]–[16]).

The stopping times $\tau_{31}(n,\varepsilon)$ are finite in the region $\overline{\Theta}_2\cup\overline{\Theta}_3$ according to Propositions 3.2, 3.3. As follows, it remind only to verify the finiteness of the stopping time $\tau_{31}(n,\varepsilon)$ for $\vartheta\in\overline{\Theta}_1\cup\overline{\Theta}_4$.

According to the definition (17) of these stopping times it is enough to show the divergence of the following integral

$$\int_0^\infty \tilde{Z}^2(t)dt = \infty \quad P_\vartheta - \text{a.s.},$$

where $\tilde{Z}(t) = \tilde{\Delta}Y(t) - \lambda_t \tilde{\Delta}Y(t-1)$.

This property follows from the following facts:

– for $\vartheta \in \Theta_{11}$

$$\lim_{t\to\infty} \lambda_t = \tilde{\lambda}, \quad P_\vartheta - \text{a.s.},$$

where $\tilde{\lambda}$ is some constant and the process $\tilde{Z}(t)$ can be approximated, similar to the case $\overline{\Theta}_3$ (see the proof of Proposition 3.3) by a Gaussian stationary process;

– for $\vartheta \in \Theta_{12}$

$$\lim_{t\to\infty} |\lambda_t - C_1(t)| = 0 \quad P_\vartheta - \text{a.s.},$$

and then

$$\lim_{t\to\infty} |\mathrm{e}^{-v_0 t}\tilde{Z}(t) - C_2(t)| = 0 \quad P_\vartheta - \text{a.s.},$$

where $C_1(t)$ and $C_2(t)$ are some periodic bounded functions;

– for Θ_{13}

$$\lim_{t\to\infty} t(\lambda_t - \lambda) = C_3 \quad P_\vartheta - \text{a.s.}$$

and

$$\lim_{t\to\infty} \mathrm{e}^{-v_0 t}\tilde{Z}(t) = C_4 \quad P_\vartheta - \text{a.s.},$$

where C_3 and C_4 are some non-zero constants;

– for Θ_{41}

$$\lim_{t\to\infty} \left| \tilde{Z}(t) - \frac{1-\lambda}{1-a}\left(X_0(0) + b\int_{-1}^{0} X_0(s)ds\right) - \right.$$

$$\left. - \frac{1}{1-a}(W(t) - \lambda W(t-1)) - (\tilde{\Delta}V(t) - \lambda\tilde{\Delta}V(t-1)) \right| = 0 \quad P_\vartheta - \text{a.s.};$$

– for Θ_{42}

$$\lim_{t\to\infty} \mathrm{e}^{-v_0 t}\tilde{Z}(t) = C_5 \quad P_\vartheta - \text{a.s.},$$

where C_5 is some non-zero constant.

Denote $\mu_1 = \mu_2 = 1$, $\mu_3 = \mu_4 = 2/5$.

Now we can verify the second property of the estimator $\vartheta(\varepsilon)$. By the definition of stopping times $\sigma_j(\varepsilon)$, $j = \overline{1,4}$, we get

$$\sup_{\vartheta\in\overline{\Theta}} E_\vartheta ||\vartheta(\varepsilon) - \vartheta||^2 \le \varepsilon \sup_{\vartheta\in\overline{\Theta}} E_\vartheta \rho_{j^\star}^{-1}\delta_{j^\star} \sum_{n=1}^{\sigma_{j^*}(\varepsilon)} \frac{1}{c_n}\beta_{j^*}^2(n,\varepsilon)\cdot||\tilde{G}_{j^*}^{-1}(n,\varepsilon)||^2 \cdot ||\tilde{\zeta}_{j^\star}(n,\varepsilon)||^2 \le$$

$$\le \varepsilon \sup_{\vartheta\in\overline{\Theta}} E_\vartheta \rho_{j^\star}^{-1}\delta_{j^\star} \sum_{n\ge 1} \frac{1}{c_n}||\tilde{\zeta}_{j^\star}(n,\varepsilon)||^2.$$

Due to the obtained finiteness properties of all the stopping times in these sums all the mathematical expectations are well-defined and we can estimate finally

$$\sup_{\vartheta\in\overline{\Theta}} E_\vartheta||\vartheta(\varepsilon)-\vartheta||^2 \le \varepsilon\sum_{j=1}^{4}\rho_j^{-1}\delta_j\sum_{n\ge1}\frac{1}{c_n}\sup_{\vartheta\in\overline{\Theta}} E_\vartheta||\tilde{\zeta}_j(n,\varepsilon)||^2 \le$$

$$\le 15(3+R^2)\varepsilon\sum_{j=1}^{4}\rho_j^{-1}\delta_j\mu_j\sum_{n\ge1}\frac{1}{c_n} = \varepsilon\sum_{j=1}^{4}\delta_j = \varepsilon.$$

Hence Theorem 3.1 is proven.

5. Conclusion

This chapter presents a sequential approach to the guaranteed parameter estimation problem of a linear stochastic continuous-time system. We consider a concrete stochastic delay differential equation driven by an additive Wiener process with noisy observations.

At the same time for the construction of the sequential estimation plans we used mainly the structure and the asymptotic behaviour of the solution of the system. Analogously, the presented method can be used for the guaranteed accuracy parameter estimation problem of the linear ordinary and delay stochastic differential equations of an arbitrary order with and without noises in observations.

The obtained estimation procedure can be easily generalized, similar to [9, 11, 13, 16], to estimate the unknown parameters with preassigned accuracy in the sense of the L_q-norm $(q \ge 2)$. The estimators with such properties may be used in various adaptive procedures (control, prediction, filtration).

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6. References

[1] Arato, M. [1982]. *Linear Stochastic Systems with Constant Coefficients. A Statistical Approach*, Springer Verlag, Berlin, Heidelberg, New York.

[2] Galtchouk, L. & Konev, V. [2001]. On sequential estimation of parameters in semimartingale regression models with continuous time parameter, *The Annals of Statistics* Vol. (29), 5: 1508–1536.

[3] Gushchin, A. A. & Küchler, U. [1999]. Asymptotic inference for a linear stochastic differential equation with time delay, *Bernoulli* Vol. (5), 6: 1059–1098.

[4] Kallianpur, G. [1987]. *Stochastic Filtering Theory*, Springer Verlag, New York, Heidelberg, Berlin.

[5] Kolmanovskii, V. & A. Myshkis. [1992]. *Applied Theory of Functional Differential Equations*, Kluwer Acad. Pabl.

[6] Küchler, U. & Sørensen, M. [1997]. *Exponential Families of Stochastic Processes*, Springer Verlag, New York, Heidelberg.
[7] Küchler, U. & Vasiliev, V. [2001]. On sequential parameter estimation for some linear stochastic differential equations with time delay, *Sequential Analysis* Vol. (20), 3: 117–146.
[8] Küchler, U. & Vasiliev, V. [2003]. On sequential identification of a diffusion type process with memory, *Proceedings Symp. Int. Fed. Autom. Contr. SYSID-2003*, Rotterdam, Holland, 27-29 August, pp. 1217–1221.
[9] Küchler, U. & Vasiliev, V. [2005a]. Sequential identification of linear dynamic systems with memory, *Statist. Inference for Stochastic Processes* Vol. (8): 1–24.
[10] Küchler, U. & Vasiliev, V. [2005b]. On parameter estimation of stochastic delay differential equations with guaranteed accuracy by noisy observations, *Preprint–15 of the Math. Inst. of Humboldt University*, Berlin, pp. 1–24.
[11] Küchler, U. & Vasiliev, V. [2006]. On Sequential Estimators for an Affine Stochastic Delay Differential Equations, *In Proceedengs of the 5th International Conference, A. Iske, J. Levesley editors, Algorithms for Approximation*, Chester, July 2005, Springer-Verlag, Berlin, Heidelberg, pp. 287–296.
[12] Küchler, U. & Vasiliev, V. [2007]. On parameter estimation of stochastic delay differential equations with guaranteed accuracy by noisy observations, *Journal of Statistical Planning and Inferences*, Elsevier Science Vol. (137): 3007-Ű3023, doi: 10/1016j.jspi.2006.12.001.
[13] Küchler, U. & Vasiliev, V. [2008]. On sequential parameter estimation of a linear regression process, *Proceedings of the 17TH World Congress the Int. Fed. Autom. Contr.*, Seoul, Korea, 5-9 Juli, pp. 10230–10235.
[14] Küchler, U. & Vasiliev, V. [2009a]. On guaranteed parameter estimation of a multiparameter linear regression process, *Preprint–7 of the Math. Inst. of Humboldt University*, Berlin, pp. 1–60.
[15] Küchler, U. & Vasiliev, V. [2009b]. On sequential parameter estimation of stochastic delay differential equations by noisy observations, *Preprints of the 15th IFAC Symposium on System Identification*, Saint-Malo, France, July 6-8, pp. 892–897.
[16] Küchler, U. & Vasiliev, V. [2010a]. On guaranteed parameter estimation of a multiparameter linear regression process, *Automatica, Journal of IFAC*, Elsevier Vol. (46) 4: 637–646.
[17] Küchler, U. & Vasiliev, V. [2010b]. On adaptive control problems of continuous-time stochastic systems, *The 1-st Virtual Control Conference* (VCC-2010, Aalborg University, Denmark), pages 1-5, http://www.vcc-10.org/index_files/papers/p107.pdf.
[18] Liptzer, R. S. & Shiryaev, A. N. [1977]. *Statistics of Random Processes*, Springer-Verlag, New York, Heidelberg.
[19] Mao, X. [1997]. *Stochastic Differential Equations and Application*, Harwood Publishing Chichester, Second Edition.
[20] Mohammed, S. E-A. [1996]. *Stochastic differential systems with memory: Theory, examples and applications*, Probability and Statistics, Birkhäuser Vol. (42): 1–77.
[21] Rozanov, Yu.A. [1967]. *Stationary Gaussian Processes*, Holden Day: San Francisco, CA.
[22] Vasiliev, V.A. & Konev, V.V. [1987]. On sequential identification of linear dynamic systems in continuous time by noisy observations, *Probl. of Contr. and Information Theory* Vol. (16) 2: 101–112.
[23] Vasiliev, V.A. & Konev, V.V. [1991]. Fixed Accuracy Identification of Partly Observable Dynamic Systems, *Proceedings of the 11th World Congress the Int. Fed. Autom. Contr.*, Tallinn, 13-17 Aug., 1990, 2, Oxford etc., pp. 179–183.