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Geometrical Derivation of Equilibrium Distributions in Some Stochastic Systems

Ricardo López-Ruiz and Jaime Sañudo

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1. Introduction

Classical statistical physics deals with statistical systems in equilibrium. The ensemble theory offers a useful framework that allows to characterize and to work out the properties of this type of systems [1]. Two fundamental distributions to describe situations in equilibrium are the Boltzmann-Gibbs (exponential) distribution and the Maxwellian (Gaussian) distribution. The first one represents the distribution of the energy states of a system and the second one fits the distribution of velocities in an ideal gas. They can be explained from different perspectives. In the physics of equilibrium, they are usually obtained from the principle of maximum entropy [2]. In the physics out of equilibrium, there have recently been proposed two nonlinear models that explain the decay of any initial distribution to these asymptotic equilibria [3, 4].

In this chapter, these distributions are alternatively obtained from a geometrical interpretation of different multi-agent systems evolving in phase space under the hypothesis of equiprobability. Concretely, an economic context is used to illustrate this derivation. Thus, from a macroscopic point of view, we consider that markets have an intrinsic stochastic ingredient as a consequence of the interaction of an undetermined ensemble of agents that trade and perform an undetermined number of commercial transactions at each moment. A kind of models considering this unknowledge associated to markets are the gas-like models [5, 6]. These random models interpret economic exchanges of money between agents similarly to collisions in a gas where particles share their energy. In order to explain the two before mentioned statistical behaviors, the Boltzmann-Gibbs and Maxwellian distributions, we will not suppose any type of interaction between the agents. The geometrical constraints and the hypothesis of equiprobability will be enough to explain these distributions in a situation of statistical equilibrium.

Thus, the Boltzmann-Gibbs (exponential) distribution is derived in Section 2 from the geometrical properties of the volume of an N-dimensional pyramid or from the properties of the surface of an N-dimensional hyperplane [7, 8]. In both cases, the motivation will be a

multi-agent economic system with an open or closed economy, respectively. The Maxwellian (Gaussian) distribution is derived in Section 3 from geometrical arguments over the volume or the surface of an N-sphere [7, 9]. In this case, the motivation will be a multi-particle gas system in contact with a heat reservoir (non-isolated or open system) or with a fixed energy (isolated or closed system), respectively. And finally, in Section 4, the general equilibrium distribution for a set of many identical interacting agents obeying a global additive constraint is also derived [7]. This distribution will be related with the Gamma-like distributions found in several multi-agent economic models. Other two geometrical collateral results, namely the formula for the volume of high-dimensional symmetrical bodies and an alternative image of the canonical ensemble, are proposed in Section 5. And last Section embodies the conclusions.

2. Derivation of the Boltzmann-Gibbs distribution

We proceed to derive here the Boltzmann-Gibbs distribution in two different physical situations with an economic inspiration. The first one considers an ensemble of economic agents that share a variable amount of money (open systems) and the second one deals with the conservative case where the total wealth is fixed (closed systems).

2.1. Multi-agent economic open systems

Here we assume N agents, each one with coordinate x_i, $i = 1, \ldots, N$, with $x_i \geq 0$ representing the wealth or money of the agent i, and a total available amount of money E:

$$x_1 + x_2 + \cdots + x_{N-1} + x_N \leq E. \tag{1}$$

Under random or deterministic evolution rules for the exchanging of money among agents, let us suppose that this system evolves in the interior of the N-dimensional pyramid given by Eq. (1). The role of a heat reservoir, that in this model supplies money instead of energy, could be played by the state or by the bank system in western societies. The formula for the volume $V_N(E)$ of an equilateral N-dimensional pyramid formed by $N + 1$ vertices linked by N perpendicular sides of length E is

$$V_N(E) = \frac{E^N}{N!}. \tag{2}$$

We suppose that each point on the N-dimensional pyramid is equiprobable, then the probability $f(x_i)dx_i$ of finding the agent i with money x_i is proportional to the volume formed by all the points into the $(N-1)$-dimensional pyramid having the ith-coordinate equal to x_i. We show now that $f(x_i)$ is the Boltzmann factor (or the Maxwell-Boltzmann distribution), with the normalization condition

$$\int_0^E f(x_i)dx_i = 1. \tag{3}$$

If the ith agent has coordinate x_i, the $N-1$ remaining agents share, at most, the money $E - x_i$ on the $(N-1)$-dimensional pyramid

$$x_1 + x_2 \cdots + x_{i-1} + x_{i+1} \cdots + x_N \leq E - x_i, \tag{4}$$

whose volume is $V_{N-1}(E - x_i)$. It can be easily proved that

$$V_N(E) = \int_0^E V_{N-1}(E - x_i)dx_i. \tag{5}$$

Hence, the volume of the N-dimensional pyramid for which the ith coordinate is between x_i and $x_i + dx_i$ is $V_{N-1}(E - x_i)dx_i$. We normalize it to satisfy Eq. (3), and obtain

$$f(x_i) = \frac{V_{N-1}(E - x_i)}{V_N(E)},\qquad(6)$$

whose final form, after some calculation is

$$f(x_i) = NE^{-1}\left(1 - \frac{x_i}{E}\right)^{N-1},\qquad(7)$$

If we call ϵ the mean wealth per agent, $E = N\epsilon$, then in the limit of large N we have

$$\lim_{N \gg 1}\left(1 - \frac{x_i}{E}\right)^{N-1} \simeq e^{-x_i/\epsilon}.\qquad(8)$$

The Boltzmann factor $e^{-x_i/\epsilon}$ is found when $N \gg 1$ but, even for small N, it can be a good approximation for agents with low wealth. After substituting Eq. (8) into Eq. (7), we obtain the Maxwell-Boltzmann distribution in the asymptotic regime $N \to \infty$ (which also implies $E \to \infty$):

$$f(x)dx = \frac{1}{\epsilon}e^{-x/\epsilon}dx,\qquad(9)$$

where the index i has been removed because the distribution is the same for each agent, and thus the wealth distribution can be obtained by averaging over all the agents. This distribution has been found to fit the real distribution of incomes in western societies [10, 11].

This geometrical image of the volume-based statistical ensemble [7] allows us to recover the same result than that obtained from the microcanonical ensemble [8] that we show in the next section.

2.2. Multi-agent economic closed systems

Here, we derive the Boltzmann-Gibbs distribution by considering the system in isolation, that is, a closed economy. Without loss of generality, let us assume N interacting economic agents, each one with coordinate x_i, $i = 1, \ldots, N$, with $x_i \geq 0$, and where x_i represents an amount of money. If we suppose now that the total amount of money E is conserved,

$$x_1 + x_2 + \cdots + x_{N-1} + x_N = E,\qquad(10)$$

then this isolated system evolves on the positive part of an equilateral N-hyperplane. The surface area $S_N(E)$ of an equilateral N-hyperplane of side E is given by

$$S_N(E) = \frac{\sqrt{N}}{(N-1)!}E^{N-1}.\qquad(11)$$

Different rules, deterministic or random, for the exchange of money between agents can be given. Depending on these rules, the system can visit the N-hyperplane in an equiprobable manner or not. If the ergodic hypothesis is assumed, each point on the N-hyperplane is equiprobable. Then the probability $f(x_i)dx_i$ of finding agent i with money x_i is proportional to the surface area formed by all the points on the N-hyperplane having the ith-coordinate

equal to x_i. We show that $f(x_i)$ is the Boltzmann-Gibbs distribution (the Boltzmann factor), with the normalization condition (3).

If the ith agent has coordinate x_i, the $N-1$ remaining agents share the money $E-x_i$ on the $(N-1)$-hyperplane

$$x_1 + x_2 \cdots + x_{i-1} + x_{i+1} \cdots + x_N = E - x_i, \tag{12}$$

whose surface area is $S_{N-1}(E-x_i)$. If we define the coordinate θ_N as satisfying

$$\sin \theta_N = \sqrt{\frac{N-1}{N}}, \tag{13}$$

it can be easily shown that

$$S_N(E) = \int_0^E S_{N-1}(E - x_i) \frac{dx_i}{\sin \theta_N}. \tag{14}$$

Hence, the surface area of the N-hyperplane for which the ith coordinate is between x_i and $x_i + dx_i$ is proportional to $S_{N-1}(E - x_i)dx_i / \sin \theta_N$. If we take into account the normalization condition (3), we obtain

$$f(x_i) = \frac{1}{S_N(E)} \frac{S_{N-1}(E - x_i)}{\sin \theta_N}, \tag{15}$$

whose form after some calculation is

$$f(x_i) = (N-1)E^{-1}\left(1 - \frac{x_i}{E}\right)^{N-2}, \tag{16}$$

If we call ϵ the mean wealth per agent, $E = N\epsilon$, then in the limit of large N we have

$$\lim_{N \gg 1} \left(1 - \frac{x_i}{E}\right)^{N-2} \simeq e^{-x_i/\epsilon}. \tag{17}$$

As in the former section, the Boltzmann factor $e^{-x_i/\epsilon}$ is found when $N \gg 1$ but, even for small N, it can be a good approximation for agents with low wealth. After substituting Eq. (1) into Eq. (16), we obtain the Boltzmann distribution (9) in the limit $N \to \infty$ (which also implies $E \to \infty$). This asymptotic result reproduces the distribution of real economic data [10] and also the results obtained in several models of economic agents with deterministic, random or chaotic exchange interactions [6, 12, 13].

Depending on the physical situation, the mean wealth per agent ϵ takes different expressions and interpretations. For instance, we can calculate the dependence of ϵ on the temperature, which in the microcanonical ensemble is defined by the derivative of the entropy with respect to the energy. The entropy can be written as $S = -kN \int_0^\infty f(x) \ln f(x)\, dx$, where $f(x)$ is given by Eq. (9) and k is Boltzmann's constant. If we recall that $\epsilon = E/N$, we obtain

$$S(E) = kN \ln \frac{E}{N} + kN. \tag{18}$$

The calculation of the temperature T gives

$$T^{-1} = \left(\frac{\partial S}{\partial E}\right)_N = \frac{kN}{E} = \frac{k}{\epsilon}. \tag{19}$$

Thus $\epsilon = kT$, and the Boltzmann-Gibbs distribution is obtained in its usual form:

$$f(x)dx = \frac{1}{kT} e^{-x/kT} dx. \tag{20}$$

3. Derivation of the Maxwellian distribution

We proceed to derive here the Maxwellian distribution in two different physical situations with inspiration in the theory of ideal gases. The first one considers an ideal gas with a variable energy (open systems) and the second one deals with the case of a gas with a fixed energy (closed systems).

3.1. Multi-particle open systems

Let us suppose a one-dimensional ideal gas of N non-identical classical particles with masses m_i, with $i = 1, \ldots, N$, and total maximum energy E. If particle i has a momentum $m_i v_i$, we define a kinetic energy:

$$K_i \equiv p_i^2 \equiv \frac{1}{2} m_i v_i^2, \tag{21}$$

where p_i is the square root of the kinetic energy K_i. If the total maximum energy is defined as $E \equiv R^2$, we have

$$p_1^2 + p_2^2 + \cdots + p_{N-1}^2 + p_N^2 \leq R^2. \tag{22}$$

We see that the system has accessible states with different energy, which can be supplied by a heat reservoir. These states are all those enclosed into the volume of the N-sphere given by Eq. (22). The formula for the volume $V_N(R)$ of an N-sphere of radius R is

$$V_N(R) = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2} + 1)} R^N, \tag{23}$$

where $\Gamma(\cdot)$ is the gamma function. If we suppose that each point into the N-sphere is equiprobable, then the probability $f(p_i)dp_i$ of finding the particle i with coordinate p_i (energy p_i^2) is proportional to the volume formed by all the points on the N-sphere having the ith-coordinate equal to p_i. We proceed to show that $f(p_i)$ is the Maxwellian distribution, with the normalization condition

$$\int_{-R}^{R} f(p_i)dp_i = 1. \tag{24}$$

If the ith particle has coordinate p_i, the $(N-1)$ remaining particles share an energy less than the maximum energy $R^2 - p_i^2$ on the $(N-1)$-sphere

$$p_1^2 + p_2^2 \cdots + p_{i-1}^2 + p_{i+1}^2 \cdots + p_N^2 \leq R^2 - p_i^2, \tag{25}$$

whose volume is $V_{N-1}(\sqrt{R^2 - p_i^2})$. It can be easily proved that

$$V_N(R) = \int_{-R}^{R} V_{N-1}(\sqrt{R^2 - p_i^2})dp_i. \tag{26}$$

Hence, the volume of the N-sphere for which the ith coordinate is between p_i and $p_i + dp_i$ is $V_{N-1}(\sqrt{R^2 - p_i^2})dp_i$. We normalize it to satisfy Eq. (24), and obtain

$$f(p_i) = \frac{V_{N-1}(\sqrt{R^2 - p_i^2})}{V_N(R)}, \tag{27}$$

whose final form, after some calculation is

$$f(p_i) = C_N R^{-1} \left(1 - \frac{p_i^2}{R^2}\right)^{\frac{N-1}{2}},$$

(28)

with

$$C_N = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{N+2}{2})}{\Gamma(\frac{N+1}{2})}.$$

(29)

For $N \gg 1$, Stirling's approximation can be applied to Eq. (29), leading to

$$\lim_{N \gg 1} C_N \simeq \frac{1}{\sqrt{\pi}} \sqrt{\frac{N}{2}}.$$

(30)

If we call ϵ the mean energy per particle, $E = R^2 = N\epsilon$, then in the limit of large N we have

$$\lim_{N \gg 1} \left(1 - \frac{p_i^2}{R^2}\right)^{\frac{N-1}{2}} \simeq e^{-p_i^2/2\epsilon}.$$

(31)

The factor $e^{-p_i^2/2\epsilon}$ is found when $N \gg 1$ but, even for small N, it can be a good approximation for particles with low energies. After substituting Eqs. (30)–(31) into Eq. (28), we obtain the Maxwellian distribution in the asymptotic regime $N \to \infty$ (which also implies $E \to \infty$):

$$f(p)dp = \sqrt{\frac{1}{2\pi\epsilon}} e^{-p^2/2\epsilon} dp,$$

(32)

where the index i has been removed because the distribution is the same for each particle, and thus the velocity distribution can be obtained by averaging over all the particles.

This newly shows that the geometrical image of the volume-based statistical ensemble [7] allows us to recover the same result than that obtained from the microcanonical ensemble [9] that it is presented in the next section.

3.2. Multi-particle closed systems

We start by assuming a one-dimensional ideal gas of N non-identical classical particles with masses m_i, with $i = 1, \ldots, N$, and total energy E. If particle i has a momentum $m_i v_i$, newly we define a kinetic energy K_i given by Eq. (21), where p_i is the square root of K_i. If the total energy is defined as $E \equiv R^2$, we have

$$p_1^2 + p_2^2 + \cdots + p_{N-1}^2 + p_N^2 = R^2.$$

(33)

We see that the isolated system evolves on the surface of an N-sphere. The formula for the surface area $S_N(R)$ of an N-sphere of radius R is

$$S_N(R) = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} R^{N-1},$$

(34)

where $\Gamma(\cdot)$ is the gamma function. If the ergodic hypothesis is assumed, that is, each point on the N-sphere is equiprobable, then the probability $f(p_i)dp_i$ of finding the particle i with

coordinate p_i (energy p_i^2) is proportional to the surface area formed by all the points on the N-sphere having the ith-coordinate equal to p_i. Our objective is to show that $f(p_i)$ is the Maxwellian distribution, with the normalization condition (24).

If the ith particle has coordinate p_i, the $(N-1)$ remaining particles share the energy $R^2 - p_i^2$ on the $(N-1)$-sphere

$$p_1^2 + p_2^2 \cdots + p_{i-1}^2 + p_{i+1}^2 \cdots + p_N^2 = R^2 - p_i^2, \tag{35}$$

whose surface area is $S_{N-1}(\sqrt{R^2 - p_i^2})$. If we define the coordinate θ as satisfying

$$R^2 \cos^2 \theta = R^2 - p_i^2, \tag{36}$$

then

$$R d\theta = \frac{dp_i}{(1 - \frac{p_i^2}{R^2})^{1/2}}. \tag{37}$$

It can be easily proved that

$$S_N(R) = \int_{-\pi/2}^{\pi/2} S_{N-1}(R \cos \theta) R d\theta. \tag{38}$$

Hence, the surface area of the N-sphere for which the ith coordinate is between p_i and $p_i + dp_i$ is $S_{N-1}(R \cos \theta) R d\theta$. We rewrite the surface area as a function of p_i, normalize it to satisfy Eq. (24), and obtain

$$f(p_i) = \frac{1}{S_N(R)} \frac{S_{N-1}(\sqrt{R^2 - p_i^2})}{(1 - \frac{p_i^2}{R^2})^{1/2}}, \tag{39}$$

whose final form, after some calculation is

$$f(p_i) = C_N R^{-1} \left(1 - \frac{p_i^2}{R^2}\right)^{\frac{N-3}{2}}, \tag{40}$$

with

$$C_N = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2})}. \tag{41}$$

For $N \gg 1$, Stirling's approximation can be applied to Eq. (41), leading to

$$\lim_{N \gg 1} C_N \simeq \frac{1}{\sqrt{\pi}} \sqrt{\frac{N}{2}}. \tag{42}$$

If we call ϵ the mean energy per particle, $E = R^2 = N\epsilon$, then in the limit of large N we have

$$\lim_{N \gg 1} \left(1 - \frac{p_i^2}{R^2}\right)^{\frac{N-3}{2}} \simeq e^{-p_i^2/2\epsilon}. \tag{43}$$

As in the former section, the Boltzmann factor $e^{-p_i^2/2\epsilon}$ is found when $N \gg 1$ but, even for small N, it can be a good approximation for particles with low energies. After substituting

Eqs. (42)–(43) into Eq. (40), we obtain the Maxwellian distribution (32) in the asymptotic regime $N \to \infty$ (which also implies $E \to \infty$).

Depending on the physical situation the mean energy per particle ϵ takes different expressions. For an isolated one-dimensional gas we can calculate the dependence of ϵ on the temperature, which in the microcanonical ensemble is defined by differentiating the entropy with respect to the energy. The entropy can be written as $S = -kN \int_{-\infty}^{\infty} f(p) \ln f(p) \, dp$, where $f(p)$ is given by Eq. (32) and k is the Boltzmann constant. If we recall that $\epsilon = E/N$, we obtain

$$S(E) = \frac{1}{2}kN \ln \left(\frac{E}{N} \right) + \frac{1}{2}kN(\ln(2\pi) - 1). \tag{44}$$

The calculation of the temperature T gives

$$T^{-1} = \left(\frac{\partial S}{\partial E} \right)_N = \frac{kN}{2E} = \frac{k}{2\epsilon}. \tag{45}$$

Thus $\epsilon = kT/2$, consistent with the equipartition theorem. If p^2 is replaced by $\frac{1}{2}mv^2$, the Maxwellian distribution is a function of particle velocity, as it is usually given in the literature:

$$g(v)dv = \sqrt{\frac{m}{2\pi kT}} \, e^{-mv^2/2kT} dv. \tag{46}$$

4. General derivation of the equilibrium distribution

In this section, we are interested in the same problem above presented but in a general way. We address this question in the volume-based statistical framework.

Let b be a positive real constant (cases $b = 1, 2$ have been indicated in the former sections). If we have a set of positive variables $(x_1, x_2, \ldots, x_N)$ verifying the constraint

$$x_1^b + x_2^b + \cdots + x_{N-1}^b + x_N^b \leq E \tag{47}$$

with an adequate mechanism assuring the equiprobability of all the possible states $(x_1, x_2, \ldots, x_N)$ into the volume given by expression (47), will we have for the generic variable x the distribution

$$f(x)dx \sim \epsilon^{-1/b} \, e^{-x^b/b\epsilon} dx, \tag{48}$$

when we average over the ensemble in the limit $N, E \to \infty$, with $E = N\epsilon$, and constant ϵ?. Now it is shown that the answer is affirmative. Similarly, we claim that if the weak inequality (47) is transformed in equality the result will be the same, as it has been proved for the cases $b = 1, 2$ in Refs. [8, 9].

From the cases $b = 1, 2$, (see Eqs. (6) and (27)), we can extrapolate the general formula that will give us the statistical behavior $f(x)$ of the generic variable x, when the system runs equiprobably into the volume defined by a constraint of type (47). The probability $f(x)dx$ of finding an agent with generic coordinate x is proportional to the volume $V_{N-1}((E - x^b)^{1/b})$ formed by all the points into the $(N - 1)$-dimensional symmetrical body limited by the constraint $(E - x^b)$. Thus, the N-dimensional volume can be written as

$$V_N(E^{1/b}) = \int_0^{E^{1/b}} V_{N-1}((E - x^b)^{1/b}) \, dx. \tag{49}$$

Taking into account the normalization condition $\int_0^{E^{1/b}} f(x)dx = 1$, the expression for $f(x)$ is obtained:

$$f(x) = \frac{V_{N-1}((E - x^b)^{1/b})}{V_N(E^{1/b})}. \tag{50}$$

The N-dimensional volume, $V_N(b,\rho)$, of a b-symmetrical body with side of length ρ is proportional to the term ρ^N and to a coefficient $g_b(N)$ that depends on N:

$$V_N(b,\rho) = g_b(N)\,\rho^N. \tag{51}$$

The parameter b indicates the original equation (47) that defines the boundaries of the volume $V_N(b,\rho)$. Thus, for instance, from Eq. (2), we have $g_{b=1}(N) = 1/N!$.

Coming back to Eq. (50), we can manipulate $V_N((E - x^b)^{1/b})$ to obtain (the index b is omitted in the formule of V_N):

$$V_N((E - x^b)^{1/b}) = g_b(N)\left[(E - x^b)^{1/b}\right]^N = g_b(N)\,E^{\frac{N}{b}}\left(1 - \frac{x^b}{E}\right)^{\frac{N}{b}}. \tag{52}$$

If we suppose $E = N\epsilon$, then ϵ represents the mean value of x^b in the collectivity, that is, $\epsilon = < x^b >$. If N tends toward infinity, it results:

$$\lim_{N \gg 1}\left(1 - \frac{x^b}{E}\right)^{\frac{N}{b}} = e^{-x^b/b\epsilon}. \tag{53}$$

Thus,

$$V_N((E - x^b)^{1/b}) = V_N(E^{1/b})\,e^{-x^b/b\epsilon}. \tag{54}$$

Substituting this last expression in formula (50), the exact form for $f(x)$ is found in the thermodynamic limit ($N, E \to \infty$):

$$f(x)dx = c_b\,\epsilon^{-1/b}\,e^{-x^b/b\epsilon}dx, \tag{55}$$

with c_b given by

$$c_b = \frac{g_b(N-1)}{g_b(N)\,N^{1/b}}. \tag{56}$$

Hence, the conjecture (48) is proved.

Doing a thermodinamical simile, we can calculate the dependence of ϵ on the temperature by differentiating the entropy with respect to the energy. The entropy can be written as $S = -kN\int_0^\infty f(x)\ln f(x)\,dx$, where $f(x)$ is given by Eq. (55) and k is the Boltzmann constant. If we recall that $\epsilon = E/N$, we obtain

$$S(E) = \frac{kN}{b}\ln\left(\frac{E}{N}\right) + \frac{kN}{b}(1 - b\ln c_b), \tag{57}$$

where it has been used that $\epsilon = < x^b > = \int_0^\infty x^b f(x)dx$.

The calculation of the temperature T gives

$$T^{-1} = \left(\frac{\partial S}{\partial E}\right)_N = \frac{kN}{bE} = \frac{k}{b\epsilon}. \tag{58}$$

Thus $\epsilon = kT/b$, a result that recovers the theorem of equipartition of energy for the quadratic case $b = 2$. The distribution for all b is finally obtained:

$$f(x)dx = c_b \left(\frac{b}{kT} \right)^{1/b} e^{-x^b/kT} dx. \tag{59}$$

4.1. General relationship between geometry and economic gas models

If we perform the change of variables $y = \epsilon^{-1/b} x$ in the normalization condition of $f(x)$, $\int_0^\infty f(x)dx = 1$, where $f(x)$ is expressed in (55), we find that

$$c_b = \left[\int_0^\infty e^{-y^b/b} \, dy \right]^{-1}. \tag{60}$$

If we introduce the new variable $z = y^b/b$, the distribution $f(x)$ as function of z reads:

$$f(z)dz = \frac{c_b}{b^{1-\frac{1}{b}}} z^{\frac{1}{b}-1} e^{-z} dz. \tag{61}$$

Let us observe that the Gamma function appears in the normalization condition,

$$\int_0^\infty f(z)dz = \frac{c_b}{b^{1-\frac{1}{b}}} \int_0^\infty z^{\frac{1}{b}-1} e^{-z} dz = \frac{c_b}{b^{1-\frac{1}{b}}} \Gamma\left(\frac{1}{b}\right) = 1. \tag{62}$$

This implies that

$$c_b = \frac{b^{1-\frac{1}{b}}}{\Gamma\left(\frac{1}{b}\right)}. \tag{63}$$

By using Mathematica the positive constant c_b is plotted versus b in Fig. 1. We see that $\lim_{b \to 0} c_b = \infty$, and that $\lim_{b \to \infty} c_b = 1$. The minimum of c_b is reached for $b = 3.1605$, taking the value $c_b = 0.7762$. Still further, we can calculate from Eq. (63) the asymptotic dependence of c_b on b:

$$\lim_{b \to 0} c_b = \sqrt{\frac{1}{2\pi}} \sqrt{b} \, e^{1/b} \left(1 - \frac{b}{12} + \cdots \right), \tag{64}$$

$$\lim_{b \to \infty} c_b = b^{-1/b} \left(1 + \frac{\gamma}{b} + \cdots \right), \tag{65}$$

where γ is the Euler constant, $\gamma = 0.5772$. The asymptotic function (64) is obtained after substituting in (63) the value of $\Gamma(1/b)$ by $(1/b - 1)!$, and performing the Stirling approximation on this last expression, knowing that $1/b \to \infty$. The function (65) is found after looking for the first Taylor expansion terms of the Gamma function around the origin $x = 0$. They can be derived from the Euler's reflection formula, $\Gamma(x)\Gamma(1 - x) = \pi/\sin(\pi x)$. We obtain $\Gamma(x \to 0) = x^{-1} + \Gamma'(1) + \cdots$. From here, recalling that $\Gamma'(1) = -\gamma$, we get $\Gamma(1/b) = b - \gamma + \cdots$, when $b \to \infty$. Although this last term of the Taylor expansion, $-\gamma$, is negligible we maintain it in expression (65). The only minimum of c_b is reached for the solution $b = 3.1605$ of the equation $\psi(1/b) + \log b + b - 1 = 0$, where $\psi(\cdot)$ is the digamma function (see Fig. 1).

Figure 1. Normalization constant c_b versus b, calculated from Eq. (63). The asymptotic behavior is: $\lim_{b\to 0} c_b = \infty$, and $\lim_{b\to\infty} c_b = 1$. This last asymptote is represented by the dotted line. The minimum of c_b is reached for $b = 3.1605$, taking the value $c_b = 0.7762$.
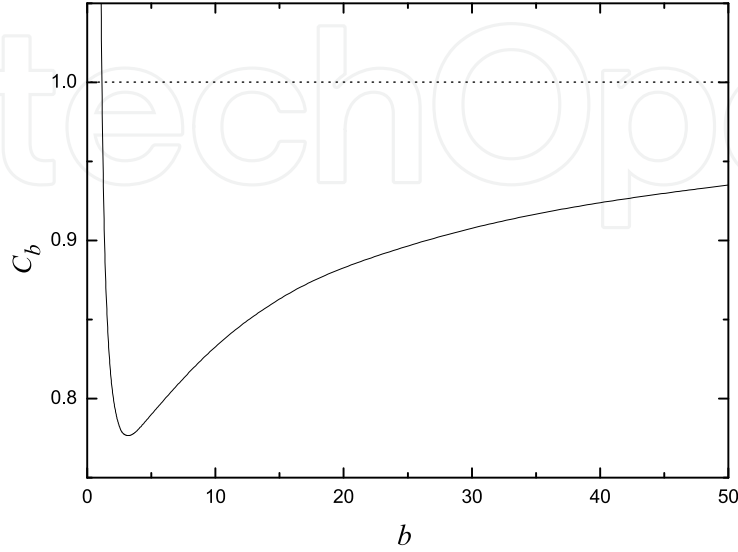
Let us now recall two interesting statistical economic models that display a statistical behavior given by distributions nearly to the form (61), that is, the standard Gamma distributions with shape parameter $1/b$,

$$f(z)dz = \frac{1}{\Gamma(\frac{1}{b})} z^{\frac{1}{b}-1} e^{-z} dz. \tag{66}$$

ECONOMIC MODEL A: The first one is the saving propensity model introduced by Chakraborti and Chakrabarti [11]. In this model a set of N economic agents, having each agent i (with $i = 1, 2, \cdots, N$) an amount of money, u_i, exchanges it under random binary (i, j) interactions, $(u_i, u_j) \to (u'_i, u'_j)$, by the following the exchange rule:

$$u'_i = \lambda u_i + \epsilon(1-\lambda)(u_i + u_j), \tag{67}$$
$$u'_j = \lambda u_j + \bar{\epsilon}(1-\lambda)(u_i + u_j), \tag{68}$$

with $\bar{\epsilon} = (1-\epsilon)$, and ϵ a random number in the interval $(0, 1)$. The parameter λ, with $0 < \lambda < 1$, is fixed, and represents the fraction of money saved before carrying out the transaction. Let us observe that money is conserved, i.e., $u_i + u_j = u'_i + u'_j$, hence in this model the economy is closed. Defining the parameter $n(\lambda)$ as

$$n(\lambda) = \frac{1+2\lambda}{1-\lambda}, \tag{69}$$

and scaling the wealth of the agents as $\bar{z} = nu/<u>$, with $<u>$ representing the average money over the ensemble of agents, it is found that the asymptotic wealth distribution in this

system is nearly obeying the standard Gamma distribution [14, 15]

$$f(\bar{z})d\bar{z} = \frac{1}{\Gamma(n)} \, \bar{z}^{n-1} \, e^{-\bar{z}} \, d\bar{z}. \tag{70}$$

The case $n = 1$, which means a null saving propensity, $\lambda = 0$, recovers the model of Dragulescu and Yakovenko [10] in which the Gibbs distribution is observed. If we compare Eqs. (70) and (66), a close relationship between this economic model and the geometrical problem solved in the former section can be established. It is enough to make

$$n = 1/b, \tag{71}$$
$$\bar{z} = z, \tag{72}$$

to have two equivalent systems. This means that, from Eq. (71), we can calculate b from the saving parameter λ with the formula

$$b = \frac{1-\lambda}{1+2\lambda}. \tag{73}$$

As λ takes its values in the interval $(0,1)$, then the parameter b also runs in the same interval $(0,1)$. On the other hand, recalling that $z = x^b/b\epsilon$, we can get the equivalent variable x from Eq. (72),

$$x = \left[\frac{\epsilon}{<u>} u \right]^{1/b}, \tag{74}$$

where ϵ is a free parameter that determines the mean value of x^b in the equivalent geometrical system. Formula (74) means to perform the change of variables $u_i \to x_i$, with $i = 1, 2, \cdots, N$, for all the particles/agents of the ensemble. Then, we conjecture that the economic system represented by the generic pair (λ, u), when it is transformed in the geometrical system given by the generic pair (b, x), as indicated by the rules (73) and (74), runs in an equiprobable form on the surface defined by the relationship (47), where the inequality has been transformed in equality. This last detail is due to the fact the economic system is closed, and then it conserves the total money, whose equivalent quantity in the geometrical problem is E. If the economic system were open, with an upper limit in the wealth, then the transformed system would evolve in an equiprobable way over the volume defined by the inequality (47), although its statistical behavior would continue to be the same as it has been proved for the cases $b = 1, 2$ in Refs. [8, 9].

ECONOMIC MODEL B: The second one is a model introduced in [16]. In this model a set of N economic agents, having each agent i (with $i = 1, 2, \cdots, N$) an amount of money, u_i, exchanges it under random binary (i, j) interactions, $(u_i, u_j) \to (u'_i, u'_j)$, by the following the exchange rule:

$$u'_i = u_i - \Delta u, \tag{75}$$
$$u'_j = u_j + \Delta u, \tag{76}$$

where

$$\Delta u = \eta(x_i - x_j) \, \epsilon \omega x_i - [1 - \eta(x_i - x_j)] \, \epsilon \omega x_j, \tag{77}$$

with ϵ a continuous uniform random number in the interval $(0,1)$. When this variable is transformed in a Bernouilli variable, i.e. a discrete uniform random variable taking on the

values 0 or 1, we have the model studied by Angle [17], that gives very different asymptotic results. The exchange parameter, ω, represents the maximum fraction of wealth lost by one of the two interacting agents ($0 < \omega < 1$). Whether the agent who is going to loose part of the money is the i-th or the j-th agent, depends nonlinearly on ($x_i - x_j$), and this is decided by the random dichotomous function $\eta(t)$: $\eta(t > 0) = 1$ (with additional probability 1/2) and $\eta(t < 0) = 0$ (with additional probability 1/2). Hence, when $x_i > x_j$, the value $\eta = 1$ produces a wealth transfer from agent i to agent j with probability 1/2, and when $x_i < x_j$, the value $\eta = 0$ produces a wealth transfer from agent j to agent i with probability 1/2. Defining in this case the parameter $n(\omega)$ as

$$n(\omega) = \frac{3 - 2\omega}{2\omega}, \tag{78}$$

and scaling the wealth of the agents as $\bar{z} = nu/ < u >$, with $< u >$ representing the average money over the ensemble of agents, it is found that the asymptotic wealth distribution in this system is nearly to fit the standard Gamma distribution [15, 16]

$$f(\bar{z})d\bar{z} = \frac{1}{\Gamma(n)} \bar{z}^{n-1} e^{-\bar{z}} d\bar{z}. \tag{79}$$

The case $n = 1$, which means an exchange parameter $\omega = 3/4$, recovers the model of Dragulescu and Yakovenko [10] in which the Gibbs distribution is observed. If we compare Eqs. (79) and (66), a close relationship between this economic model and the geometrical problem solved in the last section can be established. It is enough to make

$$n = 1/b, \tag{80}$$

$$\bar{z} = z, \tag{81}$$

to have two equivalent systems. This means that, from Eq. (80), we can calculate b from the exchange parameter ω with the formula

$$b = \frac{2\omega}{3 - 2\omega}. \tag{82}$$

As ω takes its values in the interval $(0, 1)$, then the parameter b runs in the interval $(0, 2)$. It is curious to observe that in this model the interval $\omega \in (3/4, 1)$ maps on $b \in (1, 2)$, a fact that does not occur in MODEL A. On the other hand, recalling that $z = x^b/b\epsilon$, we can get the equivalent variable x from Eq. (81),

$$x = \left[\frac{\epsilon}{< u >} u \right]^{1/b}. \tag{83}$$

where ϵ is a free parameter that determines the mean value of x^b in the equivalent geometrical system. Formula (83) means to perform the change of variables $u_i \to x_i$, with $i = 1, 2, \cdots, N$, for all the particles/agents of the ensemble. Then, also in this case, we conjecture that the economic system represented by the generic pair (λ, u), when it is transformed in the geometrical system given by the generic pair (b, x), as indicated by the rules (82) and (83), runs in an equiprobable form on the surface defined by the relationship (47), where the inequality has been transformed in equality. As explained above, this last detail is due to the fact the economic system is closed, and then it conserves the total money, whose equivalent quantity in the geometrical problem is E. If the economic system were open, with an upper limit in the wealth, then the transformed system would evolve in an equiprobable way over the volume defined by the inequality (47), although its statistical behavior would continue to be the same as it has been proved for the cases $b = 1, 2$ in Refs. [8, 9].

5. Other additional geometrical questions

As two collateral results, we address two additional problems in this section. The first one presents the finding of the general formula for the volume of a high-dimensional symmetrical body and the second one offers an alternative presentation of the canonical ensemble.

5.1. Formula for the volume of a high-dimensional body

We are concerned now with the asymptotic formula ($N \to \infty$) for the volume of the N-dimensional symmetrical body enclosed by the surface

$$x_1^b + x_2^b + \cdots + x_{N-1}^b + x_N^b = E. \tag{84}$$

The linear dimension ρ of this volume, i.e., the length of one of its sides verifies $\rho \sim E^{1/b}$. As argued in Eq. (51), the N-dimensional volume, $V_N(b, \rho)$, is proportional to the term ρ^N and to a coefficient $g_b(N)$ that depends on N. Thus,

$$V_N(b, \rho) = g_b(N) \rho^N, \tag{85}$$

where the characteristic b indicates the particular boundary given by equation (84).

For instance, from Equation (2), we can write in a formal way:

$$g_{b=1}(N) = \frac{1^{\frac{N}{1}}}{\Gamma(\frac{N}{1} + 1)}. \tag{86}$$

From Eq. (23), if we take the diameter, $\rho = 2R$, as the linear dimension of the N-sphere, we obtain:

$$g_{b=2}(N) = \frac{\left(\frac{\pi}{4}\right)^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2} + 1\right)}. \tag{87}$$

These expressions (86) and (87) suggest a possible general formula for the factor $g_b(N)$, let us say

$$g_b(N) = \frac{a^{\frac{N}{b}}}{\Gamma\left(\frac{N}{b} + 1\right)}, \tag{88}$$

where a is a b-dependent constant to be determined. For example, $a = 1$ for $b = 1$ and $a = \pi/4$ for $b = 2$.

In order to find the dependence of a on the parameter b, the regime $N \to \infty$ is supposed. Applying Stirling approximation for the factorial $(\frac{N}{b})!$ in the denominator of expression (88), and inserting it in expression (56), it is straightforward to find out the relationship:

$$c_b = (ab)^{-1/b}. \tag{89}$$

From here and formula (63), we get:

$$a = \left[\Gamma\left(\frac{1}{b} + 1\right)\right]^b, \tag{90}$$

Figure 2. The factor $g_b(N)$ versus b for $N = 10, 40, 100$, calculated from Eq. (91). Observe that $g_b(N) = 0$ for $b = 0$, and $\lim_{b \to \infty} g_b(N) = 1$.
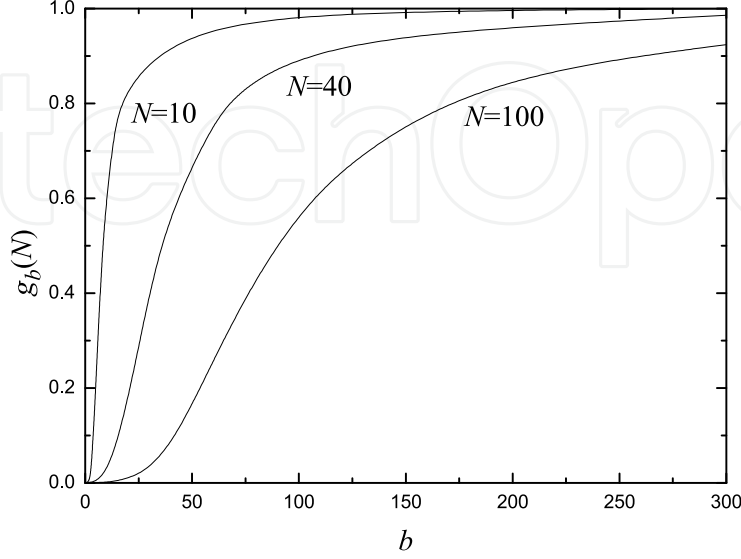
that recovers the exact results for $b = 1, 2$. The behavior of a is monotonous decreasing when b is varied from $b = 0$, where a diverges as $a \sim 1/b + \cdots$, up to the limit $b \to \infty$, where a decays asymptotically toward the value $a_\infty = e^{-\gamma} = 0.5614$.

Hence, the formula for $g_b(N)$ is obtained:

$$g_b(N) = \frac{\Gamma\left(\frac{1}{b} + 1\right)^N}{\Gamma\left(\frac{N}{b} + 1\right)}, \tag{91}$$

It would be also possible to multiply this last expression (91) by a general polynomial $K(N)$ in the variable N, and all the derivation done from Eq. (88) would continue to be correct. We omit this possibility in our calculations. For a fixed N, we have that $g_b(N)$ increases monotonously from $g_b(N) = 0$, for $b = 0$, up to $g_b(N) = 1$, in the limit $b \to \infty$ (see Fig. 2). For a fixed b, we have that $g_b(N)$ decreases monotonously from $g_b(N) = 1$, for $N = 1$, up to $g_b(N) = 0$, in the limit $N \to \infty$ (see Fig. 3).

The final result, that has been shown to be valid for any N [18], for the volume of an N-dimensional symmetrical body of characteristic b given by the boundary (84) reads:

$$V_N(b, \rho) = \frac{\Gamma\left(\frac{1}{b} + 1\right)^N}{\Gamma\left(\frac{N}{b} + 1\right)} \rho^N, \tag{92}$$

with $\rho \sim E^{1/b}$.

Figure 3. The factor $g_b(N)$ versus N for $b = 10, 40, 100$, calculated from Eq. (91). Observe that $g_b(N) = 1$ for $N = 1$, and $\lim_{N \to \infty} g_b(N) = 0$.
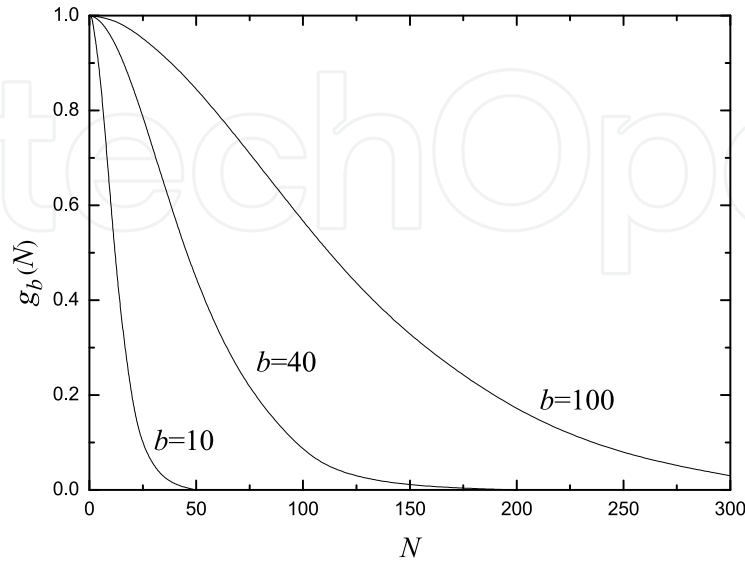
5.2. A microcanonical image of the canonical ensemble

From Section 2, here we offer a different image of the usual presentation that can be found in the literature [1] about the canonical ensemble.

Let us suppose that a system with mean energy $\bar{E}$, and in thermal equilibrium with a heat reservoir, is observed during a very long period τ of time. Let E_i be the energy of the system at time i. Then we have:

$$E_1 + E_2 + \cdots + E_{\tau-1} + E_\tau = \tau \cdot \bar{E}. \tag{93}$$

If we repeat this process of observation a huge number (toward infinity) of times, the different vectors of measurements, $(E_1, E_2, \ldots, E_{\tau-1}, E_\tau)$, with $0 \le E_i \le \tau \cdot \bar{E}$, will finish by covering equiprobably the whole surface of the τ-dimensional hyperplane given by Eq. (93). If it is now taken the limit $\tau \to \infty$, the asymptotic probability $p(E)$ of finding the system with an energy E (where the index i has been removed),

$$p(E) \sim e^{-E/\bar{E}}, \tag{94}$$

is found by means of the geometrical arguments exposed in Section 2 [8]. Doing a thermodynamic simile, the temperature T can also be calculated. It is obtained that

$$\bar{E} = kT. \tag{95}$$

The *stamp* of the canonical ensemble, namely, the Boltzmann factor,

$$p(E) \sim e^{-E/kT}, \tag{96}$$

is finally recovered from this new image of the canonical ensemble.

6. Conclusion

In summary, this work has presented a straightforward geometrical argument that in a certain way recalls us the equivalence between the canonical and the microcanonical ensembles in the thermodynamic limit for the particular context of physical sciences. In the more general context of homogeneous multi-agent systems, we conclude by highlighting the statistical equivalence of the volume-based and surface-based calculations in this type of systems.

Thus, we have shown that the Boltzmann factor or the Maxwellian distribution describe the general statistical behavior of each small part of a multi-component system in equilibrium whose components or parts are given by a set of random linear or quadratic variables, respectively, that satisfy an additive constraint, in the form of a conservation law (closed systems) or in the form of an upper limit (open systems), and that reach the equiprobability when they decay to equilibrium.

Let us remark that these calculations do not need the knowledge of the exact or microscopic randomization mechanisms of the multi-agent system in order to attain the equiprobability. In some cases, it can be reached by random forces [6], in other cases by chaotic [13, 19] or deterministic [12] causes. Evidently, the proof that these mechanisms generate equiprobability is not a trivial task and it remains as a typical challenge in this kind of problems.

The derivation of the equilibrium distribution for open systems in a general context has also been presented by considering a general multi-agent system verifying an additive constraint. Its statistical behavior has been derived from geometrical arguments. Thus, the Maxwellian and the Boltzmann-Gibbs distributions are particular cases of this type of systems. Also, other multi-agent economy models, such as the Dragalescu and Yakovenko's model [10], the Chakraborti and Chakrabarti's model [11] and the modified Angle's model [16], show similar statistical behaviors than our general geometrical system. This fact has fostered our particular geometrical interpretation of all those models.

We hope that this framework can be useful to establish other possible relationships between the statistics of multi-agent systems and the geometry associated to such systems in equilibrium.

Author details

López-Ruiz Ricardo
Department of Computer Science, Faculty of Science, Universidad de Zaragoza, Zaragoza, Spain
Also at BIFI, Institute for Biocomputation and Physics of Complex Systems, Universidad de Zaragoza, Zaragoza, Spain

Sañudo Jaime
Department of Physics, Faculty of Science, Universidad de Extremadura, Badajoz, Spain
Also at BIFI, Institute for Biocomputation and Physics of Complex Systems, Universidad de Zaragoza, Zaragoza, Spain

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