Chapter 1

Mathematical Model of the Three-Phase Induction Machine for the Study of Steady-State and Transient Duty Under Balanced and Unbalanced States

Alecsandru Simion, Leonard Livadaru and Adrian Munteanu

Additional information is available at the end of the chapter

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1. Introduction

A proper study of the induction machine operation, especially when it comes to transients and unbalanced duties, requires effective mathematical models above all. The mathematical model of an electric machine represents all the equations that describe the relationships between electromagnetic torque and the main electrical and mechanical quantities.

The theory of electrical machines, and particularly of induction machine, has mathematical models with *distributed* parameters and with *concentrated* parameters respectively. The first mentioned models start with the cognition of the magnetic field of the machine components. Their most important advantages consist in the high generality degree and accuracy. However, two major disadvantages have to be mentioned. On one hand, the computing time is rather high, which somehow discountenance their use for the real-time control. On the other hand, the distributed parameters models do not take into consideration the influence of the temperature variation or mechanical processing upon the material properties, which can vary up to 25% in comparison to the initial state. Moreover, particular constructive details (for example slots or air-gap dimensions), which essentially affects the parameters evaluation, cannot be always realized from technological point of view.

The mathematical models with concentrated parameters are the most popular and consequently employed both in scientific literature and practice. The equations stand on resistances and inductances, which can be used further for defining magnetic fluxes, electromagnetic torque, and et.al. These models offer results, which are globally acceptable but cannot detect important information concerning local effects (Ahmad, 2010; Chiasson, 2005; Krause et al., 2002; Ong, 1998; Sul, 2011).

The family of mathematical models with concentrated parameters comprises different approaches but two of them are more popular: *the phase coordinate* model and the *orthogonal (dq)* model (Ahmad, 2010; Bose, 2006; Chiasson, 2005; De Doncker et al., 2011; Krause et al., 2002; Marino et al., 2010; Ong, 1998; Sul, 2011; Wach, 2011).

The first category works with the real machine. The equations include, among other parameters, the mutual stator-rotor inductances with variable values according to the rotor position. As consequence, the model becomes non-linear and complicates the study of dynamic processes (Bose, 2006; Marino et al., 2010; Wach, 2011).

The orthogonal (dq) model has begun with Park's theory nine decades ago. These models use parameters that are often independent to rotor position. The result is a significant simplification of the calculus, which became more convenient with the defining of the *space phasor* concept (Boldea & Tutelea, 2010; Marino et al., 2010; Sul, 2011).

Starting with the "classic" theory we deduce in this contribution a mathematical model that exclude the presence of the currents and angular velocity in voltage equations and uses total fluxes alone. Based on this approach, we take into discussion two control strategies of induction motor by principle of constant total flux of the stator and rotor, respectively.

The most consistent part of this work is dedicated to the study of unbalanced duties generated by supply asymmetries. It is presented a comparative analysis, which confronts a balanced duty with two unbalanced duties of different unbalance degrees. The study uses as working tool the Matlab-Simulink environment and provides variation characteristics of the electric, magnetic and mechanical quantities under transient operation.

2. The equations of the three-phase induction machine in phase coordinates

The structure of the analyzed induction machine contains: 3 identical phase windings placed on the stator in an 120 electric degrees angle of phase difference configuration; 3 identical phase windings placed on the rotor with a similar difference of phase; a constant air-gap (close slots in an ideal approach); an unsaturated (linear) magnetic circuit that allow to each winding to be characterized by a main and a leakage inductance. Each phase winding has W_S turns on stator and W_R turns on rotor and a harmonic distribution. All inductances are considered constant. The schematic view of the machine is presented in Fig. 1a.

The voltage equations that describe the 3+3 circuits are:

$$u_{as} = R_s i_{as} + \frac{d\psi_{as}}{dt}, \quad u_{bs} = R_s i_{bs} + \frac{d\psi_{bs}}{dt}, \quad u_{cs} = R_s i_{cs} + \frac{d\psi_{cs}}{dt} \tag{1}$$

$$u_{AR} = R_R i_{AR} + \frac{d\psi_{AR}}{dt}, \quad u_{BR} = R_R i_{BR} + \frac{d\psi_{BR}}{dt}, \quad u_{CR} = R_R i_{CR} + \frac{d\psi_{CR}}{dt} \tag{2}$$

In a matrix form, the equations become:

$$\left[u_{abcs}\right]=\left[R_s\right]\left[i_{abcs}\right]+\frac{d\left[\psi_{abcs}\right]}{dt} \tag{3}$$

$$\left[u_{ABCR}\right]=\left[R_R\right]\left[i_{ABCR}\right]+\frac{d\left[\psi_{ABCR}\right]}{dt} \tag{4}$$

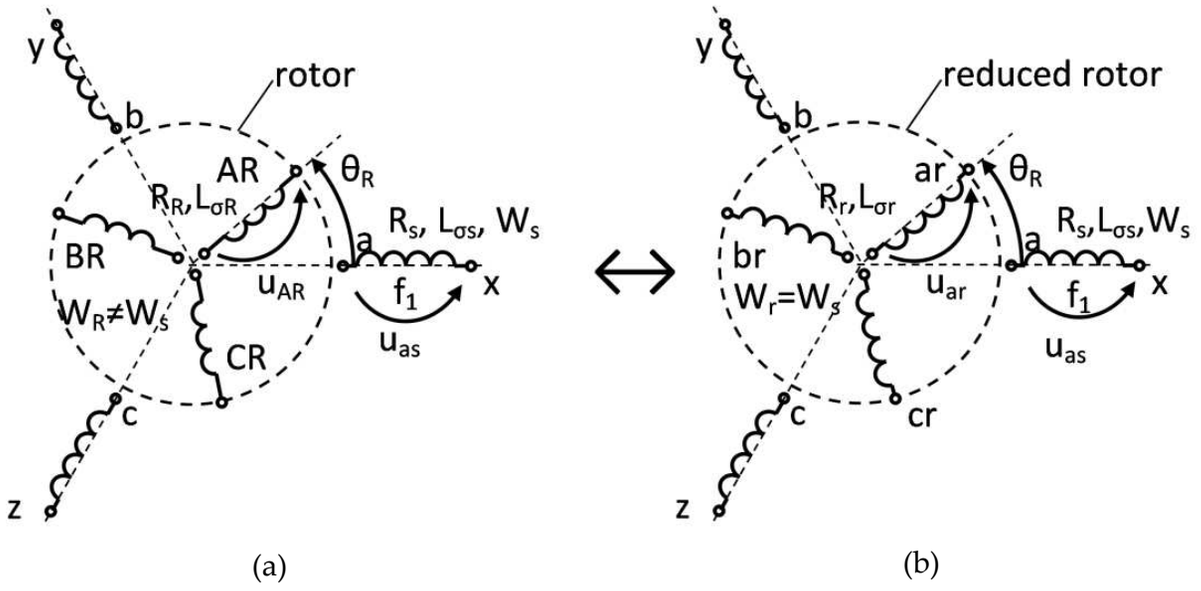

Figure 1. Schematic model of three-phase induction machine: a. real; b. reduced rotor

The quantities in brackets represent the matrices of voltages, currents, resistances and total flux linkages for the stator and rotor. Obviously, the total fluxes include both main and mutual components. Further, we define the self-phase inductances, which have a leakage and a main component: $L_{jj}=L_{\sigma s}+L_{hs}$ for stator and $L_{JJ}=L_{\Sigma R}+L_{HR}$ for rotor. The mutual inductances of two phases placed on the same part (stator or rotor) have negative values, which are equal to half of the maximum mutual inductances and with the main self-phase component: $M_{jk}=L_{jk}=L_{hj}=L_{hk}$. The expressions in matrix form are:

$$\left[L_{ss}\right]=\begin{bmatrix} L_{\sigma s}+L_{hs} & -(1/2)L_{hs} & -(1/2)L_{hs} \\ -(1/2)L_{hs} & L_{\sigma s}+L_{hs} & -(1/2)L_{hs} \\ -(1/2)L_{hs} & -(1/2)L_{hs} & L_{\sigma s}+L_{hs} \end{bmatrix} \tag{5-1}$$

$$\left[L_{RR}\right]=\begin{bmatrix} L_{\Sigma R}+L_{HR} & -(1/2)L_{HR} & -(1/2)L_{HR} \\ -(1/2)L_{HR} & L_{\Sigma R}+L_{HR} & -(1/2)L_{HR} \\ -(1/2)L_{HR} & -(1/2)L_{HR} & L_{\Sigma R}+L_{HR} \end{bmatrix} \tag{5-2}$$

$$\left[L_{sR}\right]=\left[L_{Rs}\right]_t=L_{sR}\cdot\begin{bmatrix} \cos\theta_R & \cos\left(\theta_R+u\right) & \cos\left(\theta_R+2u\right) \\ \cos\left(\theta_R+2u\right) & \cos\theta_R & \cos\left(\theta_R+u\right) \\ \cos\left(\theta_R+u\right) & \cos\left(\theta_R+2u\right) & \cos\theta_R \end{bmatrix} \tag{5-3}$$

where u denotes the angle of 120^0 (or $2\pi/3$ rad).

The analysis of the induction machine usually reduces the rotor circuit to the stator one. This operation requires the alteration of the rotor quantities with the coefficient $k=W_s/W_R$ by complying with the conservation rules. The new values are:

$$u_{abcr}=k\cdot u_{ABCR};\quad \psi_{abcr}=k\cdot\psi_{ABCR};\quad i_{abcr}=(1/k)\cdot i_{ABCR};$$
$$R_r=k^2\cdot R_R;\quad L_{hr}=k^2\cdot L_{HR}=\left(\frac{W_s}{W_R}\right)^2\cdot\frac{W_R^2}{\Re_h}=\frac{W_s^2}{\Re_h}=L_{hs}; \tag{6}$$

$$L_{\sigma r}=k^2L_{\Sigma R}=\left(\frac{W_s}{W_R}\right)^2\frac{W_R^2}{\Re_{\sigma R}}=\frac{W_s^2}{\Re_{\sigma r}}\approx L_{\sigma s};\ L_{sr}=kL_{sR}=\left(\frac{W_s}{W_R}\right)\frac{W_sW_R}{\Re_h}=L_{hs}$$

where the reluctances of the flux paths have been used. The new matrices, with rotor quantities denoted with lowercase letters are:

$$\left[L_{rr}\right]=k^2\left[L_{RR}\right]=\begin{bmatrix}L_{\sigma r}+L_{hs} & -(1/2)L_{hs} & -(1/2)L_{hs}\\ -(1/2)L_{hs} & L_{\sigma r}+L_{hs} & -(1/2)L_{hs}\\ -(1/2)L_{hs} & -(1/2)L_{hs} & L_{\sigma r}+L_{hs}\end{bmatrix} \tag{7-1}$$

$$\left[L_{sr}\right]=k\left[L_{sR}\right]=\left[L_{rs}\right]_t=L_{hs}\cdot\begin{bmatrix}\cos\theta_R & \cos(\theta_R+u) & \cos(\theta_R+2u)\\ \cos(\theta_R+2u) & \cos\theta_R & \cos(\theta_R+u)\\ \cos(\theta_R+u) & \cos(\theta_R+2u) & \cos\theta_R\end{bmatrix} \tag{7-2}$$

By virtue of these transformations, the voltage equations become:

$$\begin{cases}\left[u_{abcs}\right]=\left[R_s\right]\left[i_{abcs}\right]+\dfrac{d\left[\psi_{abcs}\right]}{dt}=\left[R_s\right]\left[i_{abcs}\right]+\left[L_{ss}\right]\dfrac{d\left[i_{abcs}\right]}{dt}+\dfrac{d\left\{\left[L_{sr}\right]\left[i_{abcr}\right]\right\}}{dt}\\ \left[u_{abcr}\right]=\left[R_r\right]\left[i_{abcr}\right]+\dfrac{d\left[\psi_{abcr}\right]}{dt}=\left[R_r\right]\left[i_{abcr}\right]+\left[L_{rr}\right]\dfrac{d\left[i_{abcr}\right]}{dt}+\dfrac{d\left\{\left[L_{sr}\right]_t\left[i_{abcs}\right]\right\}}{dt}\end{cases} \tag{8}$$

By using the notations:

$$\begin{aligned}(\Sigma L_\Pi)&=L_{\sigma r}(3L_{hs}+L_{\sigma s})+L_{\sigma s}(3L_{hs}+L_{\sigma r})\\ (\Pi L_s)&=L_{\sigma r}(L_{hs}+L_{\sigma s})+L_{\sigma s}(3L_{hs}+L_{\sigma r})\\ (\Pi L_r)&=L_{\sigma s}(L_{hs}+L_{\sigma r})+L_{\sigma r}(3L_{hs}+L_{\sigma s})\end{aligned} \tag{9}$$

and after the separation of the currents derivatives, (8) can be written under operational form as follows:

$$\left(\bar{s}+\frac{R_s\left(\Pi L_s\right)}{L_{\sigma s}\left(\Sigma L_\Pi\right)}\right)\bar{i}_{as}=-\frac{R_s L_{hs} L_{\sigma r}}{L_{\sigma s}\left(\Sigma L_\Pi\right)}\left(\bar{i}_{bs}+\bar{i}_{cs}\right)+\frac{2R_r L_{hs}}{\left(\Sigma L_\Pi\right)}\left[\bar{i}_{ar}\cos\theta_R+\bar{i}_{br}\cos\left(\theta_R+u\right)+\bar{i}_{cr}\cos\left(\theta_R+2u\right)\right]+$$
$$+\dot{\theta}_R\frac{L_{hs}\left(3L_{hs}+2L_{\sigma r}\right)}{\left(\Sigma L_\Pi\right)}\left[\bar{i}_{ar}\sin\theta_R+\bar{i}_{br}\sin\left(\theta_R+u\right)+\bar{i}_{cr}\sin\left(\theta_R+2u\right)\right]+$$
$$+2{,}6\dot{\theta}_R\frac{L_{hs}^2}{\left(\Sigma L_\Pi\right)}\left(\bar{i}_{bs}-\bar{i}_{cs}\right)+\frac{L_{\sigma r}L_{hs}}{L_{\sigma s}\left(\Sigma L_\Pi\right)}\left(\bar{u}_{as}+\bar{u}_{bs}+\bar{u}_{cs}\right)+\frac{3L_{hs}+2L_{\sigma r}}{\left(\Sigma L_\Pi\right)}\bar{u}_{as}-$$
$$-\frac{2L_{hs}}{\left(\Sigma L_\Pi\right)}\left[\bar{u}_{ar}\cos\theta_R+\bar{u}_{br}\cos\left(\theta_R+u\right)+\bar{u}_{cr}\cos\left(\theta_R+2u\right)\right],$$

$$\left(\bar{s}+\frac{R_s\left(\Pi L_s\right)}{L_{\sigma s}\left(\Sigma L_\Pi\right)}\right)\bar{i}_{bs}=-\frac{R_s L_{hs} L_{\sigma r}}{L_{\sigma s}\left(\Sigma L_\Pi\right)}\left(\bar{i}_{cs}+\bar{i}_{as}\right)+\frac{2R_r L_{hs}}{\left(\Sigma L_\Pi\right)}\left[\bar{i}_{br}\cos\theta_R+\bar{i}_{cr}\cos\left(\theta_R+u\right)+\bar{i}_{ar}\cos\left(\theta_R+2u\right)\right]+$$
$$+\dot{\theta}_R\frac{L_{hs}\left(3L_{hs}+2L_{\sigma r}\right)}{\left(\Sigma L_\Pi\right)}\left[\bar{i}_{br}\sin\theta_R+\bar{i}_{cr}\sin\left(\theta_R+u\right)+\bar{i}_{ar}\sin\left(\theta_R+2u\right)\right]+$$
$$+2{,}6\dot{\theta}_R\frac{L_{hs}^2}{\left(\Sigma L_\Pi\right)}\left(\bar{i}_{cs}-\bar{i}_{as}\right)+\frac{L_{\sigma r}L_{hs}}{L_{\sigma s}\left(\Sigma L_\Pi\right)}\left(\bar{u}_{as}+\bar{u}_{bs}+\bar{u}_{cs}\right)+\frac{3L_{hs}+2L_{\sigma r}}{\left(\Sigma L_\Pi\right)}\bar{u}_{bs}-$$
$$-\frac{2L_{hs}}{\left(\Sigma L_\Pi\right)}\left[\bar{u}_{br}\cos\theta_R+\bar{u}_{cr}\cos\left(\theta_R+u\right)+\bar{u}_{ar}\cos\left(\theta_R+2u\right)\right],$$

$$\left(\bar{s}+\frac{R_s\left(\Pi L_s\right)}{L_{\sigma s}\left(\Sigma L_\Pi\right)}\right)\bar{i}_{cs}=-\frac{R_s L_{hs} L_{\sigma r}}{L_{\sigma s}\left(\Sigma L_\Pi\right)}\left(\bar{i}_{as}+\bar{i}_{bs}\right)+\frac{2R_r L_{hs}}{\left(\Sigma L_\Pi\right)}\left[\bar{i}_{cr}\cos\theta_R+\bar{i}_{ar}\cos\left(\theta_R+u\right)+\bar{i}_{br}\cos\left(\theta_R+2u\right)\right]+$$
$$+\dot{\theta}_R\frac{L_{hs}\left(3L_{hs}+2L_{\sigma r}\right)}{\left(\Sigma L_\Pi\right)}\left[\bar{i}_{cr}\sin\theta_R+\bar{i}_{ar}\sin\left(\theta_R+u\right)+\bar{i}_{br}\sin\left(\theta_R+2u\right)\right]+$$
$$+2{,}6\dot{\theta}_R\frac{L_{hs}^2}{\left(\Sigma L_\Pi\right)}\left(\bar{i}_{as}-\bar{i}_{bs}\right)+\frac{L_{\sigma r}L_{hs}}{L_{\sigma s}\left(\Sigma L_\Pi\right)}\left(\bar{u}_{as}+\bar{u}_{bs}+\bar{u}_{cs}\right)+\frac{3L_{hs}+2L_{\sigma r}}{\left(\Sigma L_\Pi\right)}\bar{u}_{cs}-$$
$$-\frac{2L_{hs}}{\left(\Sigma L_\Pi\right)}\left[\bar{u}_{cr}\cos\theta_R+\bar{u}_{ar}\cos\left(\theta_R+u\right)+\bar{u}_{br}\cos\left(\theta_R+2u\right)\right],$$

$$\left(\bar{s}+\frac{R_r\left(\Pi L_r\right)}{L_{\sigma r}\left(\Sigma L_\Pi\right)}\right)\bar{i}_{ar}=\frac{2L_{hs}R_s}{\left(\Sigma L_\Pi\right)}\left[\bar{i}_{as}\cos\theta_R+\bar{i}_{bs}\cos\left(\theta_R+2u\right)+\bar{i}_{cs}\cos\left(\theta_R+u\right)\right]+2{,}6\dot{\theta}_R\frac{L_{hs}^2}{\left(\Sigma L_\Pi\right)}\left(-\bar{i}_{br}+\bar{i}_{cr}\right)-$$
$$-\frac{2L_{hs}}{\left(\Sigma L_\Pi\right)}\left[\bar{u}_{as}\cos\theta_R+\bar{u}_{bs}\cos\left(\theta_R+2u\right)+\bar{u}_{cs}\cos\left(\theta_R+u\right)\right]+\frac{L_{\sigma s}L_{hs}}{L_{\sigma r}\left(\Sigma L_\Pi\right)}\left(\bar{u}_{ar}+\bar{u}_{br}+\bar{u}_{cr}\right)+$$
$$+\frac{2L_{\sigma s}+3L_{hs}}{\left(\Sigma L_\Pi\right)}\bar{u}_{ar}+\dot{\theta}_R L_{hs}\frac{2L_{\sigma s}+3L_{hs}}{\left(\Sigma L_\Pi\right)}\left[\bar{i}_{as}\sin\theta_R+\bar{i}_{bs}\sin\left(\theta_R+2u\right)+\bar{i}_{cs}\sin\left(\theta_R+u\right)\right]-$$
$$-\frac{L_{\sigma s}L_{hs}R_r}{L_{\sigma r}\left(\Sigma L_\Pi\right)}\left(\bar{i}_{br}+\bar{i}_{cr}\right),$$

$$\left(\bar{s}+\frac{R_r\left(\Pi L_r\right)}{L_{\sigma r}\left(\Sigma L_\Pi\right)}\right)\bar{i}_{br}=\frac{2L_{hs}R_s}{\left(\Sigma L_\Pi\right)}\left[\bar{i}_{bs}\cos\theta_R+\bar{i}_{cs}\cos\left(\theta_R+2u\right)+\bar{i}_{as}\cos\left(\theta_R+u\right)\right]+2{,}6\dot{\theta}_R\frac{L_{hs}^2}{\left(\Sigma L_\Pi\right)}\left(-\bar{i}_{cr}+\bar{i}_{ar}\right)-$$
$$-\frac{2L_{hs}}{\left(\Sigma L_\Pi\right)}\left[\bar{u}_{bs}\cos\theta_R+\bar{u}_{cs}\cos\left(\theta_R+2u\right)+\bar{u}_{as}\cos\left(\theta_R+u\right)\right]+\frac{L_{\sigma s}L_{hs}}{L_{\sigma r}\left(\Sigma L_\Pi\right)}\left(\bar{u}_{ar}+\bar{u}_{br}+\bar{u}_{cr}\right)+$$
$$+\frac{2L_{\sigma s}+3L_{hs}}{\left(\Sigma L_\Pi\right)}\bar{u}_{br}+\dot{\theta}_R L_{hs}\frac{2L_{\sigma s}+3L_{hs}}{\left(\Sigma L_\Pi\right)}\left[\bar{i}_{bs}\sin\theta_R+\bar{i}_{cs}\sin\left(\theta_R+2u\right)+\bar{i}_{as}\sin\left(\theta_R+u\right)\right]-$$
$$-\frac{L_{\sigma s}L_{hs}R_r}{L_{\sigma r}\left(\Sigma L_\Pi\right)}\left(\bar{i}_{cr}+\bar{i}_{ar}\right),$$

$$\left(\bar{s}+\frac{R_r\left(\Pi L_r\right)}{L_{\sigma r}\left(\Sigma L_\Pi\right)}\right)\bar{i}_{cr}=\frac{2L_{hs}R_s}{\left(\Sigma L_\Pi\right)}\left[\bar{i}_{cs}\cos\theta_R+\bar{i}_{as}\cos\left(\theta_R+2u\right)+\bar{i}_{bs}\cos\left(\theta_R+u\right)\right]+2{,}6\dot{\theta}_R\frac{L_{hs}^2}{\left(\Sigma L_\Pi\right)}\left(-\bar{i}_{ar}+\bar{i}_{br}\right)-$$
$$-\frac{2L_{hs}}{\left(\Sigma L_\Pi\right)}\left[\bar{u}_{cs}\cos\theta_R+\bar{u}_{as}\cos\left(\theta_R+2u\right)+\bar{u}_{bs}\cos\left(\theta_R+u\right)\right]+\frac{L_{\sigma s}L_{hs}}{L_{\sigma r}\left(\Sigma L_\Pi\right)}\left(\bar{u}_{ar}+\bar{u}_{br}+\bar{u}_{cr}\right)+ \quad (10)$$
$$+\frac{2L_{\sigma s}+3L_{hs}}{\left(\Sigma L_\Pi\right)}\bar{u}_{cr}+\dot{\theta}_R L_{hs}\frac{2L_{\sigma s}+3L_{hs}}{\left(\Sigma L_\Pi\right)}\left[\bar{i}_{cs}\sin\theta_R+\bar{i}_{as}\sin\left(\theta_R+2u\right)+\bar{i}_{bs}\sin\left(\theta_R+u\right)\right]-$$
$$-\frac{L_{\sigma s}L_{hs}R_r}{L_{\sigma r}\left(\Sigma L_\Pi\right)}\left(\bar{i}_{ar}+\bar{i}_{br}\right),$$

Besides (10), the equations concerning mechanical quantities must be added. To this end, the electromagnetic torque has to be calculated. To this effect, we start from the coenergy expression, W'_m, of the 6 circuits (3 are placed on stator and the other 3 on rotor) and we take into consideration that the leakage fluxes, which are independent of rotation angle of the rotor, do not generate electromagnetic torque, that is:

$$W'_m=\frac{1}{2}\left[i_{abcs}\right]_t\left(\left[L_{ss}\right]-L_{\sigma s}\left[1\right]\right)\left[i_{abcs}\right]+\frac{1}{2}\left[i_{abcr}\right]_t\left(\left[L_{rr}\right]-L_{\sigma r}\left[1\right]\right)\left[i_{abcr}\right]+\left[i_{abcs}\right]_t\left[L_{sr}\left(\theta_R\right)\right]\left[i_{abcr}\right] \quad (11)$$

The magnetic energy of the stator and the rotor does not depend on the rotation angle and consequently, for the electromagnetic torque calculus nothing but the last term of (11) is used. One obtains:

$$T_e=\frac{1}{2}p\left[i_{abcs}\right]_t\frac{d\left[L_{sr}\left(\theta_R\right)\right]}{d\theta_R}\left[i_{abcr}\right]=$$
$$=\frac{1}{2}pL_{hs}\sin\theta_R\left[i_{as}\left(-2i_{ar}+i_{br}+i_{cr}\right)+i_{bs}\left(+i_{ar}-2i_{br}+i_{cr}\right)+i_{cs}\left(+i_{ar}+i_{br}-2i_{cr}\right)\right]+ \quad (12)$$
$$+\frac{\sqrt{3}}{2}pL_{hs}\cos\theta_R\left[i_{as}\left(i_{cr}-i_{br}\right)+i_{bs}\left(i_{ar}-i_{cr}\right)+i_{cs}\left(i_{br}-i_{ar}\right)\right]$$

The equation of torque equilibrium can now be written under operational form as:

$$\bar{\omega}_R\left(\frac{J\bar{s}+k_z}{p}\right)=\frac{1}{2}pL_{hs}\left\{\sin\theta_R\cdot\left[i_{as}\left(-2i_{ar}+i_{br}+i_{cr}\right)+i_{bs}\left(i_{ar}-2i_{br}+i_{cr}\right)+\right.\right.$$
$$\left.\left.+i_{cs}\left(i_{ar}+i_{br}-2i_{cr}\right)\right]+\sqrt{3}\cos\theta_R\cdot\left[i_{as}\left(i_{cr}-i_{br}\right)+i_{bs}\left(i_{ar}-i_{cr}\right)+i_{cs}\left(i_{br}-i_{ar}\right)\right]\right\}-T_{st}$$
$$\bar{s}\bar{\theta}_R=\omega_R=\dot{\theta}_R \quad (13)$$

where ω_R represents the *rotational pulsatance* (or *rotational pulsation*).

The simulation of the induction machine operation in Matlab-Simulink environment on the basis of the above equations system is rather complicated. Moreover, since all equations depend on the angular speed than the precision of the results could be questionable mainly for the study of rapid transients. Consequently, the use of other variables is understandable. Further, we shall use the *total fluxes* of the windings (3 motionless windings on stator and other rotating 3 windings on rotor).

It is well known that the total fluxes have a self-component and a mutual one. Taking into consideration the rules of reducing the rotor circuit to the stator one, the matrix of inductances can be written as follows:

$$\left[L_{abcabc}\right]=L_{hs}\cdot\begin{bmatrix} 1+l_{\sigma s} & -(1/2) & -(1/2) & \cos\theta_R & \cos(\theta_R+u) & \cos(\theta_R+2u) \\ -(1/2) & 1+l_{\sigma s} & -(1/2) & \cos(\theta_R+2u) & \cos\theta_R & \cos(\theta_R+u) \\ -(1/2) & -(1/2) & 1+l_{\sigma s} & \cos(\theta_R+u) & \cos(\theta_R+2u) & \cos\theta_R \\ \cos\theta_R & \cos(\theta_R+2u) & \cos(\theta_R+u) & 1+l_{\sigma r} & -(1/2) & -(1/2) \\ \cos(\theta_R+u) & L_{hs}\cos\theta_R & \cos(\theta_R+2u) & -(1/2) & 1+l_{\sigma r} & -(1/2) \\ \cos(\theta_R+2u) & \cos(\theta_R+u) & \cos\theta_R & -(1/2) & -(1/2) & 1+l_{\sigma r} \end{bmatrix} \quad (14)$$

Now, the equation system (8) can be written shortly as:

$$\left[u_{abcabc}\right]=\left[R_{s,r}\right]\left[i_{abcabc}\right]+\frac{d\left[\psi_{abcabc}\right]}{dt},\quad where:\ \left[\psi_{abcabc}\right]=\left[L_{abcabc}\right]\left[i_{abcabc}\right] \quad (15)$$

By using the multiplication with the reciprocal matrix:

$$\left[L_{abcabc}\right]^{-1}\left[\psi_{abcabc}\right]=\left[L_{abcabc}\right]^{-1}\left[L_{abcabc}\right]\left[i_{abcabc}\right],\quad or\ \left[i_{abcabc}\right]=\left[L_{abcabc}\right]^{-1}\left[\psi_{abcabc}\right] \quad (16)$$

than (15) becomes:

$$\left[u_{abcabc}\right]=\left[R_{s,r}\right]\left[L_{abcabc}\right]^{-1}\left[\psi_{abcabc}\right]+\frac{d\left[\psi_{abcabc}\right]}{dt} \quad (17)$$

This is an expression that connects the voltages to the total fluxes with no currents involvement. Now, practically the reciprocal matrix must be found. To this effect, we suppose that the reciprocal matrix has a similar form with the direct matrix. If we use the condition: $\left[L_{abcabc}\right]^{-1}\left[L_{abcabc}\right]=\left[1\right]$, than through term by term identification is obtained:

$$\left[L_{abcabc}\right]^{-1}=\frac{1}{(\Pi LD)}\cdot\begin{bmatrix} \Pi L_{s\sigma} & L_{hs}L_{\sigma r}^2 & L_{hs}L_{\sigma r}^2 & \Gamma\cos\theta_R & \Gamma\cos(\theta_R+u) & \Gamma\cos(\theta_R+2u) \\ L_{hs}L_{\sigma r}^2 & \Pi L_{s\sigma} & L_{hs}L_{\sigma r}^2 & \Gamma\cos(\theta_R+2u) & \Gamma\cos\theta_R & \Gamma\cos(\theta_R+u) \\ L_{hs}L_{\sigma r}^2 & L_{hs}L_{\sigma r}^2 & \Pi L_{s\sigma} & \Gamma\cos(\theta_R+u) & \Gamma\cos(\theta_R+2u) & \Gamma\cos\theta_R \\ \Gamma\cos\theta_R & \Gamma\cos(\theta_R+2u) & \Gamma\cos(\theta_R+u) & \Pi L_{r\sigma} & L_{hs}L_{\sigma s}^2 & L_{hs}L_{\sigma s}^2 \\ \Gamma\cos(\theta_R+u) & \Gamma\cos\theta_R & \Gamma\cos(\theta_R+2u) & L_{hs}L_{\sigma s}^2 & \Pi L_{r\sigma} & L_{hs}L_{\sigma s}^2 \\ \Gamma\cos(\theta_R+2u) & \Gamma\cos(\theta_R+u) & \Gamma\cos\theta_R & L_{hs}L_{\sigma s}^2 & L_{hs}L_{\sigma s}^2 & \Pi L_{r\sigma} \end{bmatrix} \quad (18)$$

where the following notations have been used:

$$\begin{gathered}(\Pi LD)=\left(3L_{hs}L_{\sigma s}+3L_{hs}L_{\sigma r}+2L_{\sigma r}L_{\sigma s}\right)L_{\sigma r}L_{\sigma s};\qquad \Gamma=-2L_{hs}L_{\sigma s}L_{\sigma r};\\ \Pi L_{s\sigma}=\left(L_{hs}L_{\sigma r}+3L_{hs}L_{\sigma s}+2L_{\sigma r}L_{\sigma s}\right)L_{\sigma r};\quad \Pi L_{r\sigma}=\left(L_{hs}L_{\sigma s}+3L_{hs}L_{\sigma r}+2L_{\sigma r}L_{\sigma s}\right)L_{\sigma s}\end{gathered} \quad (19)$$

Further, the matrix product is calculated: $\left[R_{s,r}\right]\left[L_{abcabc}\right]^{-1}\left[\psi_{abcabc}\right]$, which is used in (17). After a convenient grouping, the system becomes:

$$\frac{d\psi_{as}}{dt}+\frac{\Pi L_{s\sigma}R_s}{(\Pi LD)}\psi_{as}=u_{as}-\frac{L_{hs}L_{\sigma r}^2R_s}{(\Pi LD)}(\psi_{bs}+\psi_{cs})+\frac{L_{hs}L_{\sigma s}L_{\sigma r}R_s}{(\Pi LD)}\times \\ \times\left[(2\psi_{ar}-\psi_{br}-\psi_{cr})\cos\theta_R+\sqrt{3}(\psi_{cr}-\psi_{br})\sin\theta_R\right] \tag{20-1}$$

$$\frac{d\psi_{bs}}{dt}+\frac{\Pi L_{s\sigma}R_s}{(\Pi LD)}\psi_{bs}=u_{bs}-\frac{L_{hs}L_{\sigma r}^2R_s}{(\Pi LD)}(\psi_{cs}+\psi_{as})+\frac{L_{hs}L_{\sigma s}L_{\sigma r}R_s}{(\Pi LD)}\times \\ \times\left[(-\psi_{ar}+2\psi_{br}-\psi_{cr})\cos\theta_R+\sqrt{3}(\psi_{ar}-\psi_{cr})\sin\theta_R\right] \tag{20-2}$$

$$\frac{d\psi_{cs}}{dt}+\frac{\Pi L_{s\sigma}R_s}{(\Pi LD)}\psi_{cs}=u_{cs}-\frac{L_{hs}L_{\sigma r}^2R_s}{(\Pi LD)}(\psi_{as}+\psi_{bs})+\frac{L_{hs}L_{\sigma s}L_{\sigma r}R_s}{(\Pi LD)}\times \\ \times\left[(-\psi_{ar}-\psi_{br}+2\psi_{cr})\cos\theta_R+\sqrt{3}(\psi_{br}-\psi_{ar})\sin\theta_R\right] \tag{20-3}$$

$$\frac{d\psi_{ar}}{dt}+\frac{\Pi L_{r\sigma}R_r}{(\Pi LD)}\psi_{ar}=u_{ar}-\frac{L_{hs}L_{\sigma s}^2R_r}{(\Pi LD)}(\psi_{br}+\psi_{cr})+\frac{L_{hs}L_{\sigma s}L_{\sigma r}R_r}{(\Pi LD)}\times \\ \times\left[(2\psi_{as}-\psi_{bs}-\psi_{cs})\cos\theta_R+\sqrt{3}(\psi_{bs}-\psi_{cs})\sin\theta_R\right] \tag{20-4}$$

$$\frac{d\psi_{br}}{dt}+\frac{\Pi L_{r\sigma}R_r}{(\Pi LD)}\psi_{br}=u_{br}-\frac{L_{hs}L_{\sigma s}^2R_r}{(\Pi LD)}(\psi_{cr}+\psi_{ar})+\frac{L_{hs}L_{\sigma s}L_{\sigma r}R_r}{(\Pi LD)}\times \\ \times\left[(-\psi_{as}+2\psi_{bs}-\psi_{cs})\cos\theta_R+\sqrt{3}(\psi_{cs}-\psi_{as})\sin\theta_R\right] \tag{20-5}$$

$$\frac{d\psi_{cr}}{dt}+\frac{\Pi L_{r\sigma}R_r}{(\Pi LD)}\psi_{cr}=u_{cr}-\frac{L_{hs}L_{\sigma s}^2R_r}{(\Pi LD)}(\psi_{ar}+\psi_{br})+\frac{L_{hs}L_{\sigma s}L_{\sigma r}R_r}{(\Pi LD)}\times \\ \times\left[(-\psi_{as}-\psi_{bs}+2\psi_{cs})\cos\theta_R+\sqrt{3}(\psi_{as}-\psi_{bs})\sin\theta_R\right] \tag{20-6}$$

For the calculation of the *electromagnetic torque* we can use the principle of energy conservation or the expression of stored magnetic energy. The expression of the electromagnetic torque corresponding to a multipolar machine (p is the number of pole pairs) can be written in a matrix form as follows:

$$T_e=-\frac{p}{2}\cdot\left\{\left[\psi_{abcabc}\right]_t\cdot\frac{d\left[L_{abcabc}\right]^{-1}}{d\theta_R}\cdot\left[\psi_{abcabc}\right]\right\} \tag{21}$$

To demonstrate the validity of (21), one uses the expression of the matrix $\left[L_{abcabc}\right]^{-1}$, (18), in order to calculate its derivative:

$$\frac{d}{d\theta_R}\left[L_{abcabc}\right]^{-1}=\Lambda_3\cdot$$
$$\cdot\begin{bmatrix} 0 & 0 & 0 & \sin\theta_R & \sin(\theta_R+u) & \sin(\theta_R+2u) \\ 0 & 0 & 0 & \sin(\theta_R+2u) & \sin\theta_R & \sin(\theta_R+u) \\ 0 & 0 & 0 & \sin(\theta_R+u) & \sin(\theta_R+2u) & \sin\theta_R \\ \sin\theta_R & \sin(\theta_R+2u) & \sin(\theta_R+u) & 0 & 0 & 0 \\ \sin(\theta_R+u) & \sin\theta_R & \sin(\theta_R+2u) & 0 & 0 & 0 \\ \sin(\theta_R+2u) & \sin(\theta_R+u) & \sin\theta_R & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

where the following notation has been used:

$$\Lambda_3=\frac{1}{(3/2)(L_{\sigma s}+L_{\sigma r})+L_{\sigma r}L_{\sigma s}/L_{hs}} \tag{23}$$

This expression defines the permeance of a three-phase machine for the mathematical model in total fluxes.

Observation: One can use the general expression of the electromagnetic torque where the direct and reciprocal matrices of the inductances (which link the currents with the fluxes) should be replaced, that is:

$$T_e=\frac{1}{2}p\left[i_{abcabc}\right]_t\frac{d\left[L_{abcabc}\right]}{d\theta_R}\left[i_{abcabc}\right]=\frac{1}{2}p\cdot\left[\psi_{abcabc}\right]_t\left[L\right]_t^{-1}\cdot\frac{d\left[L_{abcabc}\right]}{d\theta_R}\cdot\left[L\right]^{-1}\left[\psi_{abcabc}\right]$$
$$T_e=-\frac{1}{2}p\left[\psi_{abcabc}\right]_t\frac{d\left[L_{abcabc}\right]^{-1}}{d\theta_R}\left[\psi_{abcabc}\right] \tag{24}$$

A more convenient expression that depends on $\sin\theta_R$ and $\cos\theta_R$, leads to the electromagnetic torque equation *in fluxes* alone:

$$T_e=-(1/2)p\Lambda_3\{\left[\psi_{as}(2\psi_{ar}-\psi_{br}-\psi_{cr})+\psi_{bs}(2\psi_{br}-\psi_{cr}-\psi_{ar})+\psi_{cs}(2\psi_{cr}-\psi_{ar}-\psi_{br})\right]\sin\theta_R+$$
$$+\sqrt{3}\left[\psi_{as}(\psi_{br}-\psi_{cr})+\psi_{bs}(\psi_{cr}-\psi_{ar})+\psi_{cs}(\psi_{ar}-\psi_{br})\right]\cos\theta_R\} \tag{25}$$

Ultimately, by getting together the equations of the 6 electric circuits and the movement equations we obtain an 8 equation system, which can be written under operational form:

$$\bar{\psi}_{as}\left(\bar{s}+\frac{\Pi L_{s\sigma}R_s}{(\Pi LD)}\right)=\bar{u}_{as}-\frac{L_{hs}L_{\sigma r}^2R_s}{(\Pi LD)}\left(\bar{\psi}_{bs}+\bar{\psi}_{cs}\right)+\frac{L_{hs}L_{\sigma s}L_{\sigma r}R_s}{(\Pi LD)}\times$$
$$\times\left[\left(2\bar{\psi}_{ar}-\bar{\psi}_{br}-\bar{\psi}_{cr}\right)\cos\theta_R+\sqrt{3}\left(\bar{\psi}_{cr}-\bar{\psi}_{br}\right)\sin\theta_R\right] \tag{26-1}$$

$$\bar{\psi}_{bs}\left(\bar{s}+\frac{\Pi L_{s\sigma}R_s}{(\Pi LD)}\right)=\bar{u}_{bs}-\frac{L_{hs}L_{\sigma r}^2R_s}{(\Pi LD)}\left(\bar{\psi}_{cs}+\bar{\psi}_{as}\right)+\frac{L_{hs}L_{\sigma s}L_{\sigma r}R_s}{(\Pi LD)}\times$$
$$\times\left[\left(-\bar{\psi}_{ar}+2\bar{\psi}_{br}-\bar{\psi}_{cr}\right)\cos\theta_R+\sqrt{3}\left(\bar{\psi}_{ar}-\bar{\psi}_{cr}\right)\sin\theta_R\right] \tag{26-2}$$

$$\bar{\psi}_{cs}\left(\bar{s}+\frac{\Pi L_{s\sigma}R_s}{(\Pi LD)}\right)=\bar{u}_{cs}-\frac{L_{hs}L_{\sigma r}^2R_s}{(\Pi LD)}\left(\bar{\psi}_{as}+\bar{\psi}_{bs}\right)+\frac{L_{hs}L_{\sigma s}L_{\sigma r}R_s}{(\Pi LD)}\times \\ \times\left[\left(-\bar{\psi}_{ar}-\bar{\psi}_{br}+2\bar{\psi}_{cr}\right)\cos\theta_R+\sqrt{3}\left(\bar{\psi}_{br}-\bar{\psi}_{ar}\right)\sin\theta_R\right] \tag{26-3}$$

$$\bar{\psi}_{ar}\left(\bar{s}+\frac{\Pi L_{r\sigma}R_r}{(\Pi LD)}\right)=\bar{u}_{ar}-\frac{L_{hs}L_{\sigma s}^2R_r}{(\Pi LD)}\left(\bar{\psi}_{br}+\bar{\psi}_{cr}\right)+\frac{L_{hs}L_{\sigma s}L_{\sigma r}R_r}{(\Pi LD)}\times \\ \times\left[\left(2\bar{\psi}_{as}-\bar{\psi}_{bs}-\bar{\psi}_{cs}\right)\cos\theta_R+\sqrt{3}\left(\bar{\psi}_{bs}-\bar{\psi}_{cs}\right)\sin\theta_R\right] \tag{26-4}$$

$$\bar{\psi}_{br}\left(\bar{s}+\frac{\Pi L_{r\sigma}R_r}{(\Pi LD)}\right)=\bar{u}_{br}-\frac{L_{hs}L_{\sigma s}^2R_r}{(\Pi LD)}\left(\bar{\psi}_{cr}+\bar{\psi}_{ar}\right)+\frac{L_{hs}L_{\sigma s}L_{\sigma r}R_r}{(\Pi LD)}\times \\ \times\left[\left(-\bar{\psi}_{as}+2\bar{\psi}_{bs}-\bar{\psi}_{cs}\right)\cos\theta_R+\sqrt{3}\left(\bar{\psi}_{cs}-\bar{\psi}_{as}\right)\sin\theta_R\right] \tag{26-5}$$

$$\bar{\psi}_{cr}\left(\bar{s}+\frac{\Pi L_{r\sigma}R_r}{(\Pi LD)}\right)=\bar{u}_{cr}-\frac{L_{hs}L_{\sigma s}^2R_r}{(\Pi LD)}\left(\bar{\psi}_{ar}+\bar{\psi}_{br}\right)+\frac{L_{hs}L_{\sigma s}L_{\sigma r}R_r}{(\Pi LD)}\times \\ \times\left[\left(-\bar{\psi}_{as}-\bar{\psi}_{bs}+2\bar{\psi}_{cs}\right)\cos\theta_R+\sqrt{3}\left(\bar{\psi}_{as}-\bar{\psi}_{bs}\right)\sin\theta_R\right] \tag{26-6}$$

$$\dot{\theta}_R\left(\bar{s}+k_z/J\right)=(p/J)\left\langle-(1/2)p\Lambda_3\left\{\sin\theta_R\left[\bar{\psi}_{as}\left(2\bar{\psi}_{ar}-\bar{\psi}_{br}-\bar{\psi}_{cr}\right)+\right.\right.\right. \\ +\bar{\psi}_{bs}\left(2\bar{\psi}_{br}-\bar{\psi}_{cr}-\bar{\psi}_{ar}\right)+\bar{\psi}_{cs}\left(2\bar{\psi}_{cr}-\bar{\psi}_{ar}-\bar{\psi}_{br}\right)\Big]+\sqrt{3}\cos\theta_R \\ \left.\left.\left.\cdot\left[\bar{\psi}_{as}\left(\bar{\psi}_{br}-\bar{\psi}_{cr}\right)+\bar{\psi}_{bs}\left(\bar{\psi}_{cr}-\bar{\psi}_{ar}\right)+\bar{\psi}_{cs}\left(\bar{\psi}_{ar}-\bar{\psi}_{br}\right)\right]\right\}-T_{st}\right\rangle \tag{26-7}$$

$$\frac{d\theta_R}{dt}=\dot{\theta}_R=\omega_R \tag{26-8}$$

This equation system, (26-1)-(26-8) allows the study of any operation duty of the three-phase induction machine: steady state or transients under balanced or unbalanced condition, with simple or double feeding.

3. Mathematical models used for the study of steady-state under balanced and unbalanced conditions

Generally, the symmetrical three-phase squirrel cage induction machine has the stator windings connected to a supply system, which provides variable voltages according to certain laws but have the same pulsation. Practically, this is the case with 4 wires connection, 3 phases and the neutral. The sum of the phase currents gives the current along neutral and the homopolar component can be immediately defined. The analysis of such a machine can use the symmetric components theory. This is the case of the machine with *two unbalances* as concerns the supply. The study can be done either using

the equation system (26-1...8) or on the basis of symmetric components theory with three distinct mathematical models for each component (positive sequence, negative sequence and homopolar).

The vast majority of electric drives uses however the 3 wires connection (no neutral). Consequently, there is no homopolar current component, the homopolar fluxes are zero as well and the sum of the 3 phase total fluxes is null. This is an asymmetric condition with *single unbalance,* which can be studied by using the direct and inverse sequence components when the transformation from 3 to 2 axes is mandatory. This approach practically replaces the three-phase machine with unbalanced supply with two symmetric three-phase machines. One of them produces the positive torque and the other provides the negative torque. The resultant torque comes out through superposition of the effects.

3.1. The abc-$\alpha\beta$0 model in total fluxes

The operation of the machine with 2 unbalances can be analyzed by considering certain expressions for the instantaneous values of the stator and rotor quantities (voltages, total fluxes and currents eventually, which can be transformed from (a, b, c) to (α, β, 0) reference frames in accordance with the following procedure :

$$\begin{bmatrix} \psi_{\alpha s} \\ \psi_{\beta s} \\ \psi_{0s} \end{bmatrix} = \sqrt{\frac{2}{3}} \cdot \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \cdot \begin{bmatrix} \psi_{as} \\ \psi_{bs} \\ \psi_{cs} \end{bmatrix} \tag{27}$$

We define the following notations:

$$\begin{aligned} &\frac{\Pi L_{s\sigma} R_s}{(\Pi LD)} = \frac{\left(L_{hs}L_{\sigma r} + 3L_{hs}L_{\sigma s} + 2L_{\sigma r}L_{\sigma s}\right)}{\left(3L_{hs}L_{\sigma r} + 3L_{hs}L_{\sigma s} + 2L_{\sigma r}L_{\sigma s}\right)}\left(\frac{R_s}{L_{\sigma s}}\right) \cong \frac{2}{3}\left(\frac{R_s}{L_{\sigma s}}\right) = \nu_{st}; \\ &\frac{L_{hs}L_{\sigma r}^2 R_s}{(\Pi LD)} = \frac{L_{hs}L_{\sigma r}R_s}{\left(3L_{hs}L_{\sigma s} + 3L_{hs}L_{\sigma r} + 2L_{\sigma r}L_{\sigma s}\right)L_{\sigma s}} \cong \frac{1}{6}\left(\frac{R_s}{L_{\sigma s}}\right) = \nu_{sr}; \\ &\nu_{st} - \nu_{sr} = \frac{3 + 2\left(L_{\sigma r}/L_{hs}\right)}{3\left(L_{\sigma r}/L_{\sigma s}\right) + 3 + 2\left(L_{\sigma r}/L_{hs}\right)}\left(\frac{R_s}{L_{\sigma s}}\right) \cong \frac{1}{2}\left(\frac{R_s}{L_{\sigma s}}\right) = \nu_s \end{aligned} \tag{28-1}$$

$$\begin{aligned} &\frac{\Pi L_{r\sigma} R_r}{(\Pi LD)} = \frac{\left(L_{hs}L_{\sigma s} + 3L_{hs}L_{\sigma r} + 2L_{\sigma r}L_{\sigma s}\right)}{\left(3L_{hs}L_{\sigma r} + 3L_{hs}L_{\sigma s} + 2L_{\sigma r}L_{\sigma s}\right)}\left(\frac{R_r}{L_{\sigma r}}\right) \cong \frac{2}{3}\left(\frac{R_r}{L_{\sigma r}}\right) = \nu_{rt}; \\ &\frac{L_{hs}L_{\sigma s}^2 R_r}{(\Pi LD)} = \frac{L_{hs}L_{\sigma s}R_r}{\left(3L_{hs}L_{\sigma s} + 3L_{hs}L_{\sigma r} + 2L_{\sigma r}L_{\sigma s}\right)L_{\sigma r}} \cong \frac{1}{6}\left(\frac{R_r}{L_{\sigma r}}\right) = \nu_{rs}; \\ &\nu_{rt} - \nu_{rs} = \frac{3 + 2\left(L_{\sigma s}/L_{hs}\right)}{3\left(L_{\sigma s}/L_{\sigma r}\right) + 3 + 2\left(L_{\sigma s}/L_{hs}\right)}\left(\frac{R_r}{L_{\sigma r}}\right) \cong \frac{1}{2}\left(\frac{R_r}{L_{\sigma r}}\right) = \nu_r \end{aligned} \tag{28-2}$$

$$\frac{3L_{hs}L_{\sigma s}L_{\sigma r}R_s}{(\Pi LD)}=\frac{3L_{\sigma s}}{\left(3L_{\sigma s}+3L_{\sigma r}+2L_{\sigma r}L_{\sigma s}/L_{hs}\right)}\left(\frac{R_s}{L_{\sigma s}}\right)\cong\frac{1}{2}\left(\frac{R_s}{L_{\sigma s}}\right)=\nu_{\sigma s};\tag{28-3}$$

$$\frac{3L_{hs}L_{\sigma s}L_{\sigma r}R_r}{(\Pi LD)}=\frac{3L_{\sigma r}}{\left(3L_{\sigma s}+3L_{\sigma r}+2L_{\sigma r}L_{\sigma s}/L_{hs}\right)}\left(\frac{R_r}{L_{\sigma r}}\right)\cong\frac{1}{2}\left(\frac{R_r}{L_{\sigma r}}\right)=\nu_{\sigma r};\tag{28-4}$$

By using these notations in (17) and after convenient groupings we obtain:

$$\frac{d\psi_{as}}{dt}+\nu_{st}\psi_{as}=u_{as}-\nu_{sr}\left(\psi_{bs}+\psi_{cs}\right)+\frac{1}{3}\nu_{\sigma s}\times\\\times\left[\left(2\psi_{ar}-\psi_{br}-\psi_{cr}\right)\cos\theta_R+\sqrt{3}\left(\psi_{cr}-\psi_{br}\right)\sin\theta_R\right]\tag{29-1}$$

$$\frac{d\psi_{bs}}{dt}+\nu_{st}\psi_{bs}=u_{bs}-\nu_{sr}\left(\psi_{cs}+\psi_{as}\right)+\frac{1}{3}\nu_{\sigma s}\times\\\times\left[\left(-\psi_{ar}+2\psi_{br}-\psi_{cr}\right)\cos\theta_R+\sqrt{3}\left(\psi_{ar}-\psi_{cr}\right)\sin\theta_R\right]\tag{29-2}$$

$$\frac{d\psi_{cs}}{dt}+\nu_{st}\psi_{cs}=u_{cs}-\nu_{sr}\left(\psi_{as}+\psi_{bs}\right)+\frac{1}{3}\nu_{\sigma s}\times\\\times\left[\left(-\psi_{ar}-\psi_{br}+2\psi_{cr}\right)\cos\theta_R+\sqrt{3}\left(\psi_{br}-\psi_{ar}\right)\sin\theta_R\right]\tag{29-3}$$

$$\frac{d\psi_{ar}}{dt}+\nu_{rt}\psi_{ar}=u_{ar}-\nu_{rs}\left(\psi_{br}+\psi_{cr}\right)+\frac{1}{3}\nu_{\sigma r}\times\\\times\left[\left(2\psi_{as}-\psi_{bs}-\psi_{cs}\right)\cos\theta_R+\sqrt{3}\left(\psi_{bs}-\psi_{cs}\right)\sin\theta_R\right]\tag{29-4}$$

$$\frac{d\psi_{br}}{dt}+\nu_{rt}\psi_{br}=u_{br}-\nu_{rs}\left(\psi_{cr}+\psi_{ar}\right)+\frac{1}{3}\nu_{\sigma r}\times\\\times\left[\left(-\psi_{as}+2\psi_{bs}-\psi_{cs}\right)\cos\theta_R+\sqrt{3}\left(\psi_{cs}-\psi_{as}\right)\sin\theta_R\right]\tag{29-5}$$

$$\frac{d\psi_{cr}}{dt}+\nu_{rt}\psi_{cr}=u_{cr}-\nu_{rs}\left(\psi_{ar}+\psi_{br}\right)+\frac{1}{3}\nu_{\sigma r}\times\\\times\left[\left(-\psi_{bs}+2\psi_{cs}-\psi_{as}\right)\cos\theta_R+\sqrt{3}\left(\psi_{as}-\psi_{bs}\right)\sin\theta_R\right]\tag{29-6}$$

Typical for the cage machine or even for the wound rotor after the starting rheostat is short-circuited is the fact that the *rotor voltages become zero*. The equations of the six circuits get different as a result of certain convenient math operations. (29-2) and (29-3) are multiplied by (-1/2) and afterwards added to (29-1); (29-3) is subtracted from (29-2); (29-1), (29-2) and (29-3) are added together. We obtain three equations that describe the stator. Similarly, (29-4), (29-5) and (29-6) are used for the rotor equations. The new equation system is:

$$\begin{cases} \dfrac{d\psi_{\alpha s}}{dt} + \nu_s \psi_{\alpha s} = u_{\alpha s} + \nu_{\sigma s}\left(\psi_{\alpha r}\cos\theta_R - \psi_{\beta r}\sin\theta_R\right) \\ \dfrac{d\psi_{\beta s}}{dt} + \nu_s \psi_{\beta s} = u_{\beta s} + \nu_{\sigma s}\left(\psi_{\alpha r}\sin\theta_R + \psi_{\beta r}\cos\theta_R\right) \\ \dfrac{d\psi_{0s}}{dt} + \left(\nu_{st} + 2\nu_{sr}\right)\psi_{0s} = u_{0s} \end{cases} \quad (30\text{-}1, 2, 3)$$

$$\begin{cases} \dfrac{d\psi_{\alpha r}}{dt} + \nu_r \psi_{\alpha r} = u_{\alpha r} + \nu_{\sigma r}\left(\psi_{\alpha s}\cos\theta_R + \psi_{\beta s}\sin\theta_R\right) \\ \dfrac{d\psi_{\beta r}}{dt} + \nu_r \psi_{\beta r} = u_{\beta r} + \nu_{\sigma r}\left(-\psi_{\alpha s}\sin\theta_R + \psi_{\beta s}\cos\theta_R\right) \\ \dfrac{d\psi_{0r}}{dt} + \left(\nu_{rt} + 2\nu_{rs}\right)\psi_{0r} = u_{0r} \end{cases} \quad (30\text{-}4, 5, 6)$$

Further, the movement equation has to be attached. It is necessary to establish the detailed expression of the electromagnetic torque in *fluxes* alone starting with (25) and using convenient transformations:

$$T_e = -\left(3/2\right)p\Lambda_3\left[\left(\psi_{\alpha s}\psi_{\alpha r} + \psi_{\beta s}\psi_{\beta r}\right)\sin\theta_R + \left(\psi_{\alpha s}\psi_{\beta r} - \psi_{\beta s}\psi_{\alpha r}\right)\cos\theta_R\right] \quad (31)$$

Ultimately, the 8 equation system under operational form is:

$$\overline{\psi}_{\alpha s}\left(\bar{s} + \nu_s\right) = \overline{u}_{\alpha s} + \nu_{\sigma s}\left(\overline{\psi}_{\alpha r}\cos\theta_R - \overline{\psi}_{\beta r}\sin\theta_R\right) \quad (32\text{-}1)$$

$$\overline{\psi}_{\beta s}\left(\bar{s} + \nu_s\right) = \overline{u}_{\beta s} + \nu_{\sigma s}\left(\overline{\psi}_{\alpha r}\sin\theta_R + \overline{\psi}_{\beta r}\cos\theta_R\right) \quad (32\text{-}2)$$

$$\overline{\psi}_{0s}\left(\bar{s} + \nu_{st} + 2\nu_{sr}\right) = \overline{u}_{0s} \quad (32\text{-}3)$$

$$\overline{\psi}_{\alpha r}\left(\bar{s} + \nu_r\right) = \overline{u}_{\alpha r} + \nu_{\sigma r}\left(\overline{\psi}_{\alpha s}\cos\theta_R + \overline{\psi}_{\beta s}\sin\theta_R\right) \quad (32\text{-}4)$$

$$\overline{\psi}_{\beta r}\left(\bar{s} + \nu_r\right) = \overline{u}_{\beta r} + \nu_{\sigma r}\left(-\overline{\psi}_{\alpha s}\sin\theta_R + \overline{\psi}_{\beta s}\cos\theta_R\right) \quad (32\text{-}5)$$

$$\overline{\psi}_{0r}\left(\bar{s} + \nu_{rt} + 2\nu_{rs}\right) = \overline{u}_{0r} \quad (32\text{-}6)$$

$$\dot{\theta}_R\left(\bar{s} + k_z / J\right) = \left(p/J\right)\cdot\left\{-\left(3/2\right)p\Lambda_3\left[\left(\overline{\psi}_{\alpha s}\overline{\psi}_{\alpha r} + \overline{\psi}_{\beta s}\overline{\psi}_{\beta r}\right)\sin\theta_R + \left(\overline{\psi}_{\alpha s}\overline{\psi}_{\beta r} - \overline{\psi}_{\beta s}\overline{\psi}_{\alpha r}\right)\cos\theta_R\right] - T_{st}\right\} \quad (32\text{-}7)$$

$$\frac{d\theta_R}{dt} = \dot{\theta}_R = \omega_R \quad (32\text{-}8)$$

These equations allow the study of three-phase induction machine for any duty. It has to be mentioned that the electromagnetic torque expression has no homopolar components of the total fluxes.

3.2. The abc-dq model in total fluxes

For the study of the single unbalance condition is necessary to consider expressions of the instantaneous values of the stator and rotor quantities (voltages, total fluxes and eventually currents in *a,b,c* reference frame) whose sum is null. The real quantities can be transformed to (d,q) reference frame (Simion et al., 2011). By using the notations (28-1), (28-2), (28-3) and (28-4) then after convenient grouping we obtain (Simion, 2010):

$$\begin{aligned}\frac{d\psi_{\alpha s}}{dt}+\nu_s\psi_{\alpha s}&=u_{\alpha s}+\nu_{\sigma s}\left(\psi_{\alpha r}\cos\theta_R-\psi_{\beta r}\sin\theta_R\right)\\ \frac{d\psi_{\beta s}}{dt}+\nu_s\psi_{\beta s}&=u_{\beta s}+\nu_{\sigma s}\left(\psi_{\alpha r}\sin\theta_R+\psi_{\beta r}\cos\theta_R\right)\end{aligned} \tag{33-1, 2}$$

$$\begin{aligned}\frac{d\psi_{\alpha r}}{dt}+\nu_r\psi_{\alpha r}&=\nu_{\sigma r}\left(\psi_{\alpha s}\cos\theta_R+\psi_{\beta s}\sin\theta_R\right)\\ \frac{d\psi_{\beta r}}{dt}+\nu_r\psi_{\beta r}&=\nu_{\sigma r}\left(-\psi_{\alpha s}\sin\theta_R+\psi_{\beta s}\cos\theta_R\right)\end{aligned} \tag{33-3, 4}$$

Further, the movement equation (31) must be attached. The operational form of the equation system (4 electric circuits and 2 movement equations) is:

$$\overline{\psi}_{\alpha s}\left(\bar{s}+\nu_s\right)=\overline{u}_{\alpha s}+\nu_{\sigma s}\left(\overline{\psi}_{\alpha r}\cos\theta_R-\overline{\psi}_{\beta r}\sin\theta_R\right) \tag{34-1}$$

$$\overline{\psi}_{\beta s}\left(\bar{s}+\nu_s\right)=\overline{u}_{\beta s}+\nu_{\sigma s}\left(\overline{\psi}_{\alpha r}\sin\theta_R+\overline{\psi}_{\beta r}\cos\theta_R\right) \tag{34-2}$$

$$\overline{\psi}_{\alpha r}\left(\bar{s}+\nu_r\right)=\nu_{\sigma r}\left(\overline{\psi}_{\alpha s}\cos\theta_R+\overline{\psi}_{\beta s}\sin\theta_R\right) \tag{34-3}$$

$$\overline{\psi}_{\beta r}\left(\bar{s}+\nu_r\right)=\nu_{\sigma r}\left(-\overline{\psi}_{\alpha s}\sin\theta_R+\overline{\psi}_{\beta s}\cos\theta_R\right) \tag{34-4}$$

$$\dot{\theta}_R\left(\bar{s}+k_z/J\right)=\left(p/J\right)\cdot\left\{-\left(3/2\right)p\Lambda_3\left[\left(\overline{\psi}_{\alpha s}\overline{\psi}_{\alpha r}+\overline{\psi}_{\beta s}\overline{\psi}_{\beta r}\right)\sin\theta_R+\left(\overline{\psi}_{\alpha s}\overline{\psi}_{\beta r}-\overline{\psi}_{\beta s}\overline{\psi}_{\alpha r}\right)\cos\theta_R\right]-T_{st}\right\} \tag{34-5}$$

$$\frac{d\theta_R}{dt}=\dot{\theta}_R=\omega_R \tag{34-6}$$

The equation sets (33-1...4) and (34-1...6) prove that a three-phase induction machine connected to the supply system by 3 wires can be studied similarly to a two-phase machine

(*two-phase* mathematical model). Its parameters can be deduced by linear transformations of the original parameters including the supply voltages (Fig. 2a).

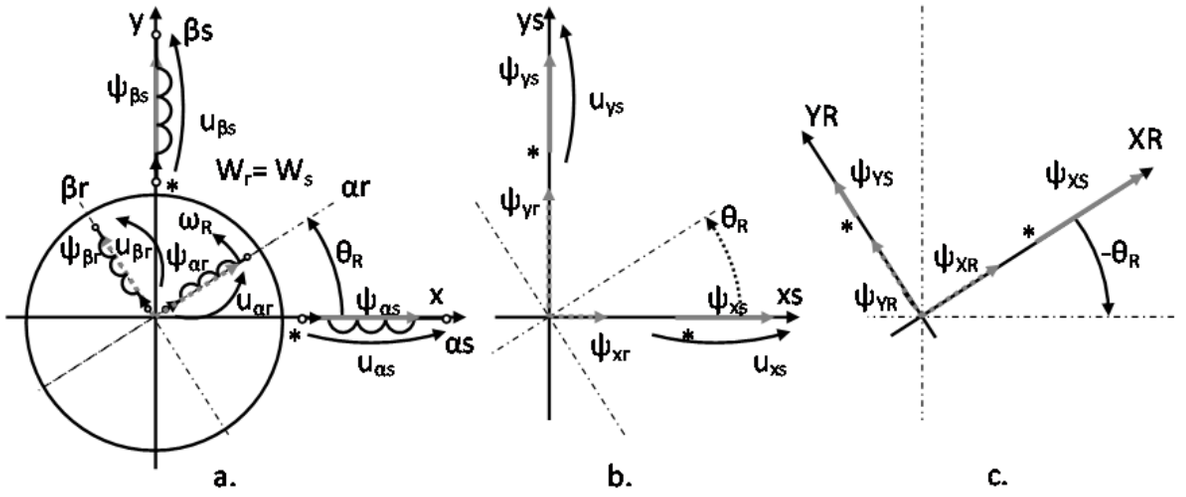

Figure 2. Induction machine schematic view: a.Two-phase model; b. Simplified view of the total fluxes in stator reference frame; c. Idem, but in rotor reference frame

The windings of two-phase model are denoted with (*αs, βs*) and (*αr, βr*) in order to trace a correspondence with the real two-phase machine, whose subscripts are (*as, bs*) and (*ar, br*) respectively. We shall use the subscripts *xs* and *ys* for the quantities that corresponds to the three-phase machine but transformed in its two-phase model. This is a rightful assumption since (*αs, βs*) axes are collinear with (*x, y*) axes, which are commonly used in analytic geometry. Further, new notations (35) for the flux linkages of the right member of the equations (33-1...4) will be defined by following the next rules:

- projection sums corresponding to rotor flux linkages from (*αr, βr*) axes along the two stator axes (denoted with *x* and *y* that is ψ_{xr}, ψ_{yr}) when they refer to the flux linkages from the right member of the first two equations, Fig. 2b.
- projection sums corresponding to stator flux linkages from (*αs, βs*) axes along the two rotor axes (denoted with *X* and *Y* that is ψ_{XS}, ψ_{YS}) when they refer to the flux linkages from the last two equations, Fig. 2c.

$$\begin{cases} \psi_{xr} = \psi_{\alpha r}\cos\theta_R - \psi_{\beta r}\sin\theta_R, & \psi_{yr} = \psi_{\alpha r}\sin\theta_R + \psi_{\beta r}\cos\theta_R \\ \psi_{XS} = \psi_{\alpha s}\cos\theta_R + \psi_{\beta s}\sin\theta_R, & \psi_{YS} = -\psi_{\alpha s}\sin\theta_R + \psi_{\beta s}\cos\theta_R \end{cases} \tag{35}$$

Some aspects have to be pointed out. When the machine operates under motoring duty, the pulsation of the stator flux linkages from (*αs, βs*) axes is equal to ω_s. Since the rotational pulsation is ω_R then the pulsation of the rotor quantities from (*αr, βr*) axes is equal to $\omega_r = s\omega_s = \omega_s - \omega_R$. The pulsation of the rotor quantities projected along the stator axes with the subscripts *xr* and *yr* is equal to ω_s. The pulsation of the stator quantities projected along the rotor axes with the subscripts *XS* and *YS* is equal to ω_r. The equations (33-1...4) become:

$$\overline{\psi}_{\alpha s}(\bar{s}+\nu_s)=\bar{u}_{\alpha s}+\nu_{\sigma s}\overline{\psi}_{xr} \tag{36-1}$$

$$\overline{\psi}_{\beta s}(\bar{s}+\nu_s)=\bar{u}_{\beta s}+\nu_{\sigma s}\overline{\psi}_{yr} \tag{36-2}$$

$$\overline{\psi}_{\alpha r}(\bar{s}+\nu_r)=\nu_{\sigma r}\overline{\psi}_{XS} \tag{36-3}$$

$$\overline{\psi}_{\beta r}(\bar{s}+\nu_r)=\nu_{\sigma r}\overline{\psi}_{YS} \tag{36-4}$$

The first two equations join the quantities with the pulsation ω_s and the other two, the quantities with the pulsation $\omega_r = s\omega_s$. The expression of the magnetic torque, in *total fluxes* and *rotor position angle* becomes:

$$T_e=-(3/2)p\Lambda_3\left(\overline{\psi}_{\alpha s}\overline{\psi}_{yr}-\overline{\psi}_{\beta s}\overline{\psi}_{xr}\right) \tag{37}$$

or a second equivalent expression:

$$T_e=(3/2)p\Lambda_3\left(\overline{\psi}_{\alpha r}\overline{\psi}_{YS}-\overline{\psi}_{\beta r}\overline{\psi}_{XS}\right) \tag{38}$$

which shows the "total symmetry" of the two-phase model of the three-phase machine regarding both stator and rotor. The equations of the four circuits together with the movement equation (37) under operational form give:

$$\overline{\psi}_{\alpha s}\left(\bar{s}+\nu_s\right)=\bar{u}_{\alpha s}+\nu_{\sigma s}\overline{\psi}_{xr} \tag{39-1}$$

$$\overline{\psi}_{\beta s}\left(\bar{s}+\nu_s\right)=\bar{u}_{\beta s}+\nu_{\sigma s}\overline{\psi}_{yr} \tag{39-2}$$

$$\overline{\psi}_{\alpha r}\left(\bar{s}+\nu_r\right)=\nu_{\sigma r}\overline{\psi}_{XS} \tag{39-3}$$

$$\overline{\psi}_{\beta r}\left(\bar{s}+\nu_r\right)=\nu_{\sigma r}\overline{\psi}_{YS} \tag{39-4}$$

$$\dot{\theta}_R\left(\bar{s}+k_z/J\right)=(p/J)\cdot\left\{\ (3/2)p\Lambda_3\left(\overline{\psi}_{\beta s}\overline{\psi}_{xr}-\overline{\psi}_{\alpha s}\overline{\psi}_{yr}\right)-T_{st}\right\} \tag{39-5}$$

$$\frac{d\theta_R}{dt}=\dot{\theta}_R=\omega_R \tag{39-6}$$

This last equation system allows the study of transients under single unbalance condition. It is similar with the frequently used equations (Park) but contains as variables only total fluxes and the rotation angle. There are no currents or angular speed in the voltage equations.

4. Expressions of electromagnetic torque

For the steady state analysis of the symmetric three-phase induction machine, one can define the simplified space phasor of the stator flux, which is collinear to the total flux of the

(αs) axis and has a $\sqrt{3}$ times higher modulus. In a similar way can be obtained the space phasors of the stator voltages and rotor fluxes and the system equation (39-1...6) that describe the steady state becomes:

$$\begin{aligned} \underline{U}_{sR3} &= \left(\nu_s + j\omega_s\right)\underline{\Psi}_{sR3} - \nu_{\sigma s}\underline{\Psi}_{rR3} = \left(\omega_s - j\nu_s\right)\Psi_{sR3}e^{j\alpha_s} + j\nu_{\sigma s}\Psi_{rR3}e^{j\alpha_r} \\ 0 &= \nu_{\sigma r}\underline{\Psi}_{sR3} - \left(\nu_r + js\omega_s\right)\underline{\Psi}_{rR3} = -j\nu_{\sigma r}\Psi_{sR3}e^{j\alpha_s} + \left(j\nu_r - s\omega_s\right)\Psi_{rR3}e^{j\alpha_r} \end{aligned} \tag{40}$$

$$T_e = \left(3/2\right)p\Lambda_3\Psi_{sR3}\Psi_{rR3}\sin\left(\alpha_s - \alpha_r\right)$$

When the speed regulation of the cage induction machine is employed by means of voltage and/or frequency variation then the simultaneous control of the two total flux space vectors is difficult. As consequence, new strategies more convenient can be chosen. To this effect, we shall deduce expressions of the electromagnetic torque that include only one of the total flux space vectors either from stator or rotor.

4.1. Variation of the torque with the stator total flux space vector

One of the methods used for the control of induction machine consists in the operation with *constant stator total flux space vector*. From (40), the rotor total flux space vector is:

$$\begin{aligned} \underline{\Psi}_{rR3} &= \frac{\nu_{\sigma r}}{\nu_r + js\omega_s}\underline{\Psi}_{sR3} = \frac{\nu_{\sigma r}\underline{\Psi}_{sR3}}{\sqrt{\omega_s^2 s^2 + \nu_r^2}}\left(\frac{\nu_r}{\sqrt{\omega_s^2 s^2 + \nu_r^2}} - j\frac{s\omega_s}{\sqrt{\omega_s^2 s^2 + \nu_r^2}}\right) = \\ &= \frac{\nu_{\sigma r}\underline{\Psi}_{sR3}}{\sqrt{\omega_s^2 s^2 + \nu_r^2}}e^{-j\theta};(\theta = \alpha_s - \alpha_r);\sin\theta = \frac{s\omega_s}{\sqrt{\omega_s^2 s^2 + \nu_r^2}};\cos\theta = \frac{\nu_r}{\sqrt{\omega_s^2 s^2 + \nu_r^2}} \end{aligned} \tag{41}$$

where θ is the angle between stator and rotor total flux space vectors. This angle has the meaning of an *internal angle of the machine*.

The expression of the magnetic torque that depends with the stator total flux space vector becomes:

$$\begin{aligned} T_e &= -\left(\frac{3}{2}\right)p\Lambda_3\,\mathrm{Re}\left(j\underline{\Psi}_{sR3}\underline{\Psi}^*_{rR3}\right) = -\left(\frac{3}{2}\right)p\Lambda_3\,\mathrm{Re}\left\{j\underline{\Psi}_{sR3}\frac{\nu_{\sigma r}}{\sqrt{\omega_s^2 s^2 + \nu_r^2}}\cdot\underline{\Psi}^*_{sR3}\left(\cos\theta + j\sin\theta\right)\right\} = \\ &= \frac{3}{2}\frac{\nu_{\sigma r}}{\nu_r}p\Lambda_3\Psi^2_{sR3}\frac{s\omega_s\nu_r}{\omega_s^2 s^2 + \nu_r^2} = \frac{3}{4}\frac{\nu_{\sigma r}}{\nu_r}p\Lambda_3\Psi^2_{sR3}\sin 2\theta. \end{aligned} \tag{42}$$

Assuming the ideal hypothesis of maintaining constant the stator flux, for example equal to the no-load value, then the pull-out torque, T_{emax}, corresponds to $\sin 2\theta = 1$ that is:

$$\begin{aligned} &2\sin\theta\cos\theta = 1 \leftrightarrow s_{cr}\omega_s = \nu_r, and\ T_{e\max} = \frac{3}{4}\frac{\nu_{\sigma r}}{\nu_r}p\Lambda_3\Psi^2_{sR3}; \\ &or\ T_{e\max} = \frac{3}{2}\nu_{\sigma r}p\Lambda_3\left(\frac{U_{sR3}}{\omega_s}\right)^2\frac{\nu_r}{\nu_s^2\left(\nu_r/\omega_s\right)^2 + 2\nu_{\sigma s}\nu_{\sigma r}\nu_r/\omega_s + \nu_{tt}^2 + 2\nu_r^2} \end{aligned} \tag{43}$$

Now an observation can be formulated. Let us suppose an ideal static converter that operates with a U/*f*=*constant*=k_1 strategy. For low supply frequencies, the pull-out torque decreases in value since the denominator increases with the pulsatance decrease, ω_s (Fig. 3). Within certain limits at low frequencies, an increase of the supply voltage is necessary in order to maintain the pull-out torque value. In other words, $U/f = k_2$, and $k_2 > k_1$.

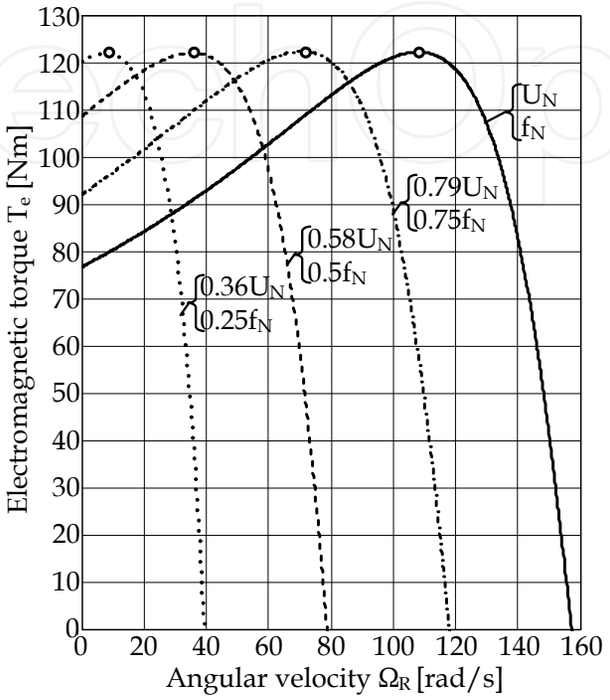

Figure 3. Mechanical characteristics, $M_e=f(\Omega_R)$ at $\Psi_{sR3}=const.$

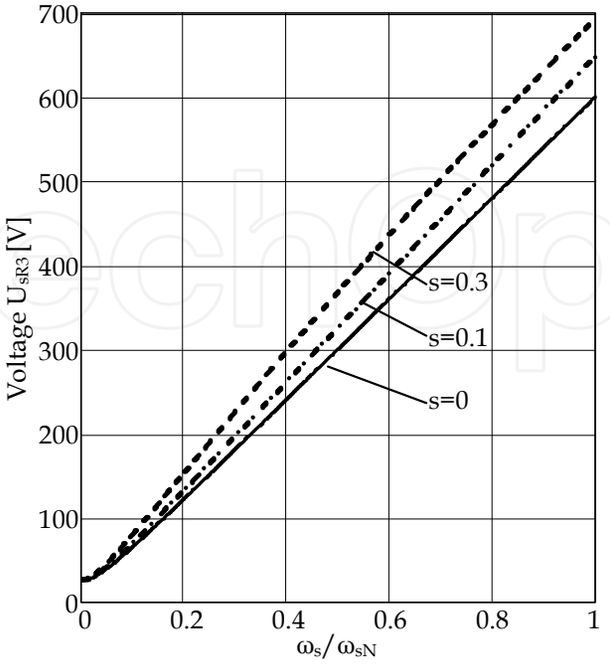

Figure 4. Resultant stator voltage vs. pulsatance $U_{sR3}=f(\omega_s)$ *at* $\Psi_{sR3}=const.$ *(1,91Wb)*

A proper control of the induction machine requires a strategy based on *U/f = variable*. More precisely, for low frequency values it is necessary to increase the supply voltage with respect to the values that result from *U/f = const.* strategy. At a pinch, when the frequency becomes zero, the supply voltage must have a value capable to compensate the voltage drops upon the equivalent resistance of the windings. Lately, the modern static converters can be parameterized on the basis of the catalog parameters of the induction machine or on the basis of some laboratory tests results.

From (40) we can deduce:

$$\underline{\Psi}_{sR3} = -j\frac{\underline{U}_{sR3}}{\omega_s}\frac{(s\omega_s - j\nu_r)}{(s\omega_s - \nu_{tt}) - j(\nu_r + s\nu_s)} \leftrightarrow U_{sR3}^2 = \frac{\Psi_{sR3}^2\omega_s^2\left(As^2 + 2Bs + C\right)}{\omega_s^2 s^2 + \nu_r^2} \tag{44}$$

and further:

$$\frac{U_{sR3}^2}{\omega_s^2} = \Psi_{sR3}^2\left[1 + \frac{\nu_s^2 s^2 + 2\nu_{\sigma s}\nu_{\sigma r}s + \nu_{tt}^2}{\omega_s^2 s^2 + \nu_r^2}\right] \leftrightarrow \Psi_{sR3} = \frac{U_{sR3}}{\omega_s}\sqrt{\frac{F(s)}{F(s) + sG(s)}} \tag{45}$$

$$where: \quad F(s) = \omega_s^2 s^2 + \nu_r^2; \quad G(s) \approx \nu_s^2 s + 2\nu_{\sigma s}\nu_{\sigma r}$$

if the term ν_{tt} was neglected. By inspecting the square root term, which is variable with the slip (and load as well), we can point out the following observations.

- Constant maintaining of the stator flux for *low pulsations* (that is low angular velocity values including start-up) can be obtained with a significant increase of the supply voltage. The "additional" increasing of the voltage depends proportionally on the load value. Analytically, this fact is caused by the predominance of the term G against F, (45). From the viewpoint of physical phenomena, a higher voltage in case of severe start-up or low frequency operation is necessary for the compensation of the leakage fluxes after which the stator flux must keep its prescribed value.
- Constant maintaining of the stator flux for *high pulsations* (that is angular speeds close or even over the rated value) requires an insignificant rise of the supply voltage. The *U/f* ratio is close to its rated value (rated values of *U* and *f*) especially for low load torque values. However, a certain increase of the voltage is required proportionally with the load degree. Analytically, this fact is now caused by the predominance of the term F against G, (45).
- In conclusion, the resultant stator flux remain constant for *U/f =constant=k1* strategy if the load torque is small. For high loads (especially if the operation is close to the pull-out point), the maintaining of the stator flux requires an increase of the *U/f* ratio, which means a significant rise of the voltage and current.

If the machine parameters are established, then a variation rule of the supply voltage can be settled in order to have a constant stator flux (equal, for example, to its no-load value) both for frequency and load variation.

Fig. 4 presents (for a machine with predetermined parameters: supply voltage with the amplitude of 490 V (U_{as}=346.5V); R_s=R_r=2; L_{hs}=0,09; $L_{\sigma s}$= $L_{\sigma r}$=0,01; J=0,05; p=2; k_z=0,02;

ω_1=314,1 (SI units)) the variation of the resultant stator voltage with the pulsatance (in per unit description) for three constant slip values. The variation is a straight line for reduced loads and has a certain inflection for low frequency values (a few Hz). For under-load operation, a significant increase of the voltage with the frequency is necessary. This fact is more visible at high slip values, close to pull-out value (in our example the pull-out slip is of 0,33).

The variation rule based on $U_{sR}=f(\omega_s)$ strategy (applied to the upper curve from Fig. 4) provide an operation of the motor within a large range of angular speeds (from start-up to rated point) under a developed torque, whose value is close to the pull-out one. Obviously, the input current is rather high (4-5 I_{1N}) and has to be reduced. Practically, the operation points must be placed within the upper and the lower curves, Fig. 4. It is also easy to notice that the operation with higher frequency values than the rated one does not generally require an increase of the supply voltage but the developed torque is lower and lower. In this case, the output power keeps the rated value.

4.2. Variation of the torque with the rotor total flux space vector

Usually, the electric drives that demand high value starting torque use *constant rotor total flux space vector* strategy. The stator total flux space vector can be written from (41) as:

$$\begin{aligned}&\underline{\Psi}_{sR3}=\frac{\nu_r+js\omega_s}{\nu_{\sigma r}}\underline{\Psi}_{rR3}\Leftrightarrow\Psi_{sR3}=\Psi_{rR3}\frac{\sqrt{\omega_s^2s^2+\nu_r^2}}{\nu_{\sigma r}};\underline{\Psi}_{sR3}=\frac{\sqrt{\omega_s^2s^2+\nu_r^2}}{\nu_{\sigma r}}\underline{\Psi}_{rR3}e^{j\theta};\\&(\theta=\alpha_s-\alpha_r);\sin\theta=\frac{s\omega_s}{\sqrt{\omega_s^2s^2+\nu_r^2}};\cos\theta=\frac{\nu_r}{\sqrt{\omega_s^2s^2+\nu_r^2}}\end{aligned}\tag{46}$$

and the expression of the electromagnetic torque on the basis of rotor flux alone becomes:

$$T_e=-\left(\frac{3}{2}\right)p\Lambda_3\,\mathrm{Re}\left(j\underline{\Psi}_{sR3}\underline{\Psi}^*_{rR3}\right)=\frac{3}{2}\frac{p\Lambda_3}{\nu_{\sigma r}}\Psi^2_{rR3}s\omega_s\tag{47}$$

Assuming the ideal hypothesis of maintaining constant the rotor flux, for example equal to the no-load value, then the electromagnetic torque expression is:

$$T_e=\frac{3}{2}\frac{p\Lambda_3}{\nu_{\sigma r}}\Psi^2_{rR30}s\omega_s\approx\frac{3}{2}\frac{p\Lambda_3}{\nu_{\sigma r}}\left(\frac{\nu_{\sigma r}U_{sR3}}{\nu_r\omega_s}\right)^2s\omega_s=\frac{3}{2}\frac{p\Lambda_3\nu_{\sigma r}}{\nu_r^2}\left(\frac{U^2_{sR3}}{\omega_s^2}\right)\left(\omega_s-p\Omega_R\right)\tag{48}$$

where the voltage and pulsation is supposed to have rated values. Taking into discussion a machine with predetermined parameters (supply voltage with the amplitude of 490 V (U_{as}=346.5V); $R_s=R_r$=2; L_{hs}=0,09; $L_{\sigma s}=L_{\sigma r}$=0,01; J=0,05; p=2; k_z=0,02; ω_1=314,1 (SI units)) then the expression of the mechanical characteristic is:

$$T_e=\frac{3}{2}\frac{2\cdot32,14\cdot96,43}{103,57^2}\left(\frac{U^2_{sR3N}}{\omega^2_{sN}}\right)\left(\omega_s-2\Omega_R\right)=3,17\left(\omega_s-2\Omega_R\right)\tag{49}$$

which is a straight line, A1 in Fig. 5. The two intersection points with the axes correspond to synchronism (T_e=0, $\Omega_R=\omega_s/2$=157) and start-up (T_e=995 Nm, Ω_R=0) respectively.

The pull-out torque is extremely high and acts at start-up. This behavior is caused by the hypothesis of maintaing constant the rotor flux at a value that corresponds to no-load operation (when the rotor reaction is null) no matter the load is. The compensation of the magnetic reaction of the rotor under load is *hypothetical* possible through an unreasonable increase of the supply voltage. Practically, the pull-out torque is much lower.

Another unreasonable possibility is the maintaining of the rotor flux to a value that corresponds to start-up (*s = 1*) and the supply voltage has its rated value. In this case the expression of the mechanical characteristic is (50) and the intersection points with the axes (line A2, Fig. 5) correspond to synchronism (T_e=0, $\Omega_R=\omega_s/2$=157) and start-up (T_e=78 Nm, Ω_R=0) respectively.

$$T_e = \frac{3}{2}\frac{2\cdot 32,14\cdot}{96,43}\Psi_{rRk}^2\left(\omega_s - 2\Omega_R\right) = 0,25\left(\omega_s - 2\Omega_R\right) \tag{50}$$

The supply of the stator winding with constant voltage and rated pulsation determines a variation of the resultant rotor flux within the short-circuit value (Ψ_{rRk}=0,5Wb) and the synchronism value (Ψ_{rR0}=1,78Wb). The operation points lie between the two lines, A1 and A2, on a position that depends on the load torque. When the supply pulsation is two times smaller (and the voltage itself is two times smaller as well) and the resultant rotor flux is maintained constant to the value Ψ_{rR0}=1,78Wb, then the mechanical characteristic is described by the straight line B1, which is parallel to the line A1. Similarly, for Ψ_{rRk}=0,5Wb, the mechanical characteristic become the line B2, which is parallel to A2.

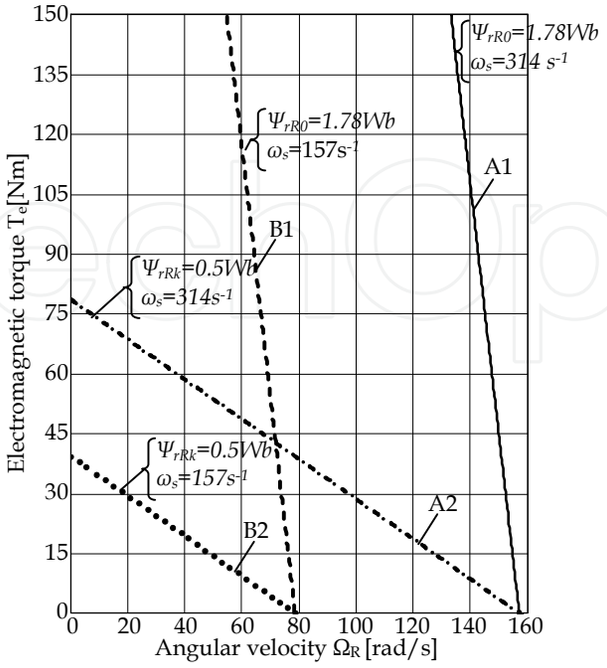

Figure 5. Mechanical characteristics $T_e=f(\Omega_R)$, $\Psi_{rR}=const.$

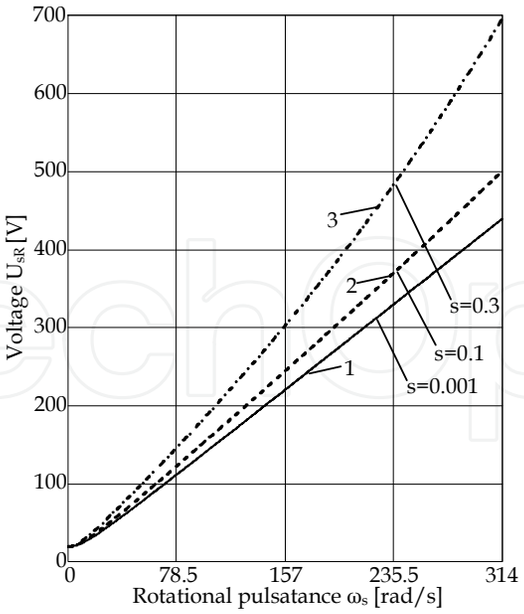

Figure 6. Resultant stator voltage vs. pulsatance, $U_{sR}=f(\omega_s)$ at $\Psi_{rR}=const.$ *(1.3Wb)*

When the applied voltage and pulsation are two times smaller regarding the rated values then the operation points lie between B1 and B2 since the rotor flux varies within Ψ_{rRk}=0,5Wb (short-circuit) and Ψ_{rR0}=1,78Wb (synchronism).

The control based on constant rotor flux strategy ensures parallel mechanical characteristics. This is an important advantage since the induction machine behaves like shunt D.C. motor. A second aspect is also favorable in the behavior under this strategy. The mechanical characteristic has no sector of unstable operation as the usual induction machine has.

The modification of the flux value (generally with decrease) leads to a different slope of the characteristics, which means a significant decrease of the torque for a certain angular speed.

The question is "what variation rule of U_{sR}/ω_s must be used in order to have constant rotor flux"? The expression of the modulus of the resultant rotor flux can be written as:

$$\frac{U_{sR3}^2}{\omega_s^2}=\Psi_{rR3}^2\frac{As^2+2Bs+C}{\nu_{\sigma r}^2}\leftrightarrow\frac{U_{sR3}}{\omega_s}=\frac{\Psi_{rR3}}{\nu_{\sigma r}}\sqrt{As^2+2Bs+C} \tag{51}$$

$$with:\quad A=\omega_s^2+\nu_s^2;B=\nu_{\sigma s}\nu_{\sigma r};C=\nu_r^2+\nu_{tt}^2;\nu_{tt}^2=(\nu_s\nu_r-\nu_{\sigma s}\nu_{\sigma r})^2/\omega_s^2\ .$$

Fig. 6 presents the variation of the stator voltage with pulsatance at constant resultant rotor flux (1,3 Wb), which are called the *control characteristics* of the static converter connected to the induction machine. The presented characteristics correspond to three constant slip values, s=0,001 (no-load)-curve 1, s=0,1 (rated duty)-curve 2 and s=0,3 (close to pull-out point)-curve 3. It can be seen that the operation with high slip values (high loads) require an increased stator voltage for a certain pulsation. As a matter of fact, the ratio U_{sR3}/ω_s must be increased with the load when the pulsatance (pulsation) and the angular speed rise as well. Such a strategy is indicated for fans, pumps or load machines with speed-dependent torque.

When the pulsation of the stator voltage is low (small angular velocities) then the torque that has to be overcame is small too, but it will rise with the speed and the frequency along a parabolic variation. Since the upper limit of the torque is given by the limited power of the machine (thermal considerations) then this strategy requires additional precautions as concern the safety devices that protect both the static converter and the supply source itself.

The analysis of the square root term from (51) generates similar remarks as in the above discussed control strategy.

Finally is important to say that a control characteristic must be prescribed for the static converter. This characteristic should be simplified and generally reduced to a straight line placed between the curves 1 and 2 from Fig. 6.

5. Study of the unbalanced duties

The unbalanced duties (generated by supply asymmetries) are generally analyzed by using the theory of symmetric components, according to which any asymmetric three-phase system with *single unbalance* (the sum of the applied instantaneous voltages is always zero) can be equated with two symmetric systems of opposite sequences: positive (+) (or direct) and negative (-) (or inverse) respectively. There are two possible ways for the analysis of this problem.

a. When the amplitudes of the phase voltages are different and/or the angles of phase difference are not equal to 2π/3 then the *unbalanced three-phase* system can be replaced with an equivalent *unbalanced two-phase* system, which further is taken apart in two systems, one of *direct sequence* with higher two-phase amplitude voltages and the other of *inverse sequence* with lower two-phase amplitude voltages. Usually, this equivalence process is obtained by using an orthogonal transformation. Not only voltages but also the total fluxes and eventually the currents must be established for the two resulted systems. The quantities of the unbalanced two-phase system can be written as follows:

$$\begin{bmatrix} \underline{U}_{\alpha s} \\ \underline{U}_{\beta s} \\ \underline{U}_{0s} \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \underline{U}_{as} \\ \underline{U}_{bs} \\ \underline{U}_{cs} \end{bmatrix} \leftrightarrow \begin{cases} \underline{U}_{\alpha s} = \sqrt{\frac{3}{2}} \underline{U}_{as}; \underline{U}_{\beta s} = \sqrt{\frac{3}{2}} \frac{\underline{U}_{bs} - \underline{U}_{cs}}{\sqrt{3}} \\ \underline{U}_{0s} = 0; \quad \underline{U}_{as} + \underline{U}_{bs} + \underline{U}_{cs} = 0 \end{cases} \tag{52}$$

Further, the unbalanced quantities are transformed to balanced quantities and we obtain:

$$\begin{bmatrix} \underline{U}_{\alpha s(+)} \\ \underline{U}_{\alpha s(-)} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & j \\ 1 & -j \end{bmatrix} \begin{bmatrix} \underline{U}_{\alpha s} \\ \underline{U}_{\beta s} \end{bmatrix}, \text{ or: } \begin{cases} \underline{U}_{\alpha s(+)} = \left(\underline{U}_{as} e^{j\pi/6} + j\underline{U}_{bs} \right) / \sqrt{2}; \\ \underline{U}_{\alpha s(-)} = \left(\underline{U}_{as} e^{-j\pi/6} - j\underline{U}_{bs} \right) / \sqrt{2} \end{cases} \tag{53}$$

The quantities of the three-phase system with *single unbalance* can be written as follows:

$$u_{as} = U\sqrt{2}\cos\omega t \Leftrightarrow \underline{U}_{as} = Ue^{j0}; \underline{U}_{bs} = kUe^{-j\beta}; \underline{U}_{cs} = -U(1 + ke^{-j\beta}) \tag{54}$$

and further:

$$\underline{U}_{\alpha s(+)} = U(e^{j\pi/6} + ke^{j(\pi/2-\beta)})/\sqrt{2}; \underline{U}_{\alpha s(-)} = U(e^{-j\pi/6} - ke^{j(\pi/2-\beta)})/\sqrt{2} \tag{55}$$

Modulus of these components can be determined at once with:

$$U_{\alpha s(+)} = U\sqrt{1+k^2+2k\sin(\beta+\pi/6)}/\sqrt{2}; U_{\alpha s(-)} = U\sqrt{1+k^2-2k\sin(\beta-\pi/6)}/\sqrt{2} \tag{56}$$

For the transformation of the unbalanced two-phase quantities in balanced two-phase components (53) must be used:

$$\begin{cases} \underline{U}_{\alpha s} = \underline{U}_{\alpha s(+)} + \underline{U}_{\alpha s(-)} \\ \underline{U}_{\beta s} = -j\underline{U}_{\alpha s(+)} + j\underline{U}_{\alpha s(-)} \end{cases} \tag{57}$$

The matrix equation of the two-phase model is written in a convenient way hereinafter:

$$\begin{bmatrix} \underline{U}_{\alpha s} \\ \underline{U}_{\beta s} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \nu_s + j\omega_s & 0 & 0 & -\nu_{\sigma s} \\ 0 & \nu_s + j\omega_s & -\nu_{\sigma s} & 0 \\ 0 & \nu_{\sigma r} & -(\nu_r + j\omega_s) & \omega_R \\ \nu_{\sigma r} & 0 & -\omega_R & -(\nu_r + j\omega_s) \end{bmatrix} \times \begin{bmatrix} \underline{\Psi}_{\alpha s} \\ \underline{\Psi}_{\beta s} \\ \underline{\Psi}_{yr} \\ \underline{\Psi}_{xr} \end{bmatrix} \tag{58}$$

Using elementary math (multiplications with constants, addition and subtraction of different equations) we can obtain the equations of the two-phase *direct* (M2D) and *inverse* (M2I) models:

$$\text{(M2D)} \qquad \begin{bmatrix} \underline{U}_{\alpha s(+)} \\ 0 \end{bmatrix} = \begin{bmatrix} \nu_s + j\omega_s & -\nu_{\sigma s} \\ \nu_{\sigma r} & -\left(\nu_r + js_d\omega_s\right) \end{bmatrix} \times \begin{bmatrix} \underline{\Psi}_{\alpha s(+)} \\ \underline{\Psi}_{xr(+)} \end{bmatrix} \tag{59}$$

$$\text{(M2I)} \qquad \begin{bmatrix} \underline{U}_{\alpha s(-)} \\ 0 \end{bmatrix} = \begin{bmatrix} \nu_s + j\omega_s & -\nu_{\sigma s} \\ \nu_{\sigma r} & -\left(\nu_r + js_i\omega_s\right) \end{bmatrix} \times \begin{bmatrix} \underline{\Psi}_{\alpha s(-)} \\ \underline{\Psi}_{xr(-)} \end{bmatrix} \tag{60}$$

We have defined the slip values for the direct (+) and respectively inverse (-) machines: $s_d = s = \frac{\omega_s - \omega_R}{\omega_s}$; $s_i = \frac{\omega_s + \omega_R}{\omega_s}$ with the interrelation expression: $s_i = 2 - s$.

The two machine-models create self-contained torques, which act simultaneously upon rotor. The resultant torque emerges from superposition effects procedure (Simion et al., 2009; Simion & Livadaru, 2010). The equation set (59), for M2D, gives two equations:

$$\underline{U}_{\alpha s(+)} = (\nu_s + j\omega_s)\underline{\Psi}_{\alpha s(+)} - \nu_{\sigma s}\underline{\Psi}_{xr(+)}; \quad 0 = \nu_{\sigma r}\underline{\Psi}_{\alpha s(+)} - \left(\nu_r + js\omega_s\right)\underline{\Psi}_{xr(+)} \tag{61}$$

which give further

$$\underline{\Psi}_{\alpha s(+)}=\frac{\left(\nu_r+js\omega_s\right)\underline{U}_{\alpha s(+)}}{\underline{\Delta}_{(+)}};\underline{\Psi}_{xr(+)}=\frac{\nu_{\sigma r}\underline{U}_{\alpha s(+)}}{\underline{\Delta}_{(+)}};\underline{\Delta}_{(+)}=\left(\nu_s+j\omega_s\right)\left(\nu_r+js\omega_s\right)-\nu_{\sigma s}\nu_{\sigma r} \tag{62}$$

Similarly, for M2I we obtain:

$$\underline{U}_{\alpha s(-)}=(\nu_s+j\omega_s)\underline{\Psi}_{\alpha s(-)}-\nu_{\sigma s}\underline{\Psi}_{xr(-)};\quad 0=\nu_{\sigma r}\underline{\Psi}_{\alpha s(-)}-\left[\nu_r+j(2-s)\omega_s\right]\underline{\Psi}_{xr(-)} \tag{63}$$

$$\underline{\Psi}_{\alpha s(-)}=\frac{\left[\nu_r+j\left(2-s\right)\omega_s\right]\underline{U}_{\alpha s(-)}}{\underline{\Delta}_{(-)}};\underline{\Psi}_{xr(-)}=\frac{\nu_{\sigma r}\underline{U}_{\alpha s(-)}}{\underline{\Delta}_{(-)}};\underline{\Delta}_{(-)}=\left(\nu_s+j\omega_s\right)\left[\nu_r+j\left(2-s\right)\omega_s\right]-\nu_{\sigma s}\nu_{\sigma r} \tag{64}$$

To determine the electromagnetic torque developed under unbalanced supply condition we use the symmetric components and the superposition effect. The *mean electromagnetic torque M2D* results from (25) but transformed in simplified complex quantities:

$$T_{e(+)}=-\frac{3p}{2}\Lambda_3\cdot 2\mathrm{Re}\left(j\underline{\Psi}_{\alpha s(+)}\cdot\underline{\Psi}'^{*}_{xr(+)}\right)=\frac{3p\nu_{\sigma r}\Lambda_3}{2\omega_s}\cdot\frac{2U^2_{\alpha s(+)}s}{As^2+2Bs+C} \tag{65}$$

Similarly, the expression of the *mean electromagnetic torque M2D* is:

$$T_{e(-)}=-\frac{3p}{2}\Lambda_3\cdot 2\mathrm{Re}\left(j\underline{\Psi}_{\alpha s(-)}\cdot\underline{\Psi}^{*}_{xr(-)}\right)=\frac{3p\nu_{\sigma r}\Lambda_3}{2\omega_s}\cdot\frac{2U^2_{\alpha s(-)}\left(2-s\right)}{A\left(2-s\right)^2+2B\left(2-s\right)+C} \tag{66}$$

The *mean resultant torque*, as a difference of the torques produced by M2D and M2I, can be written by using (65) and (66):

$$T_{erez}=\frac{3p\nu_{\sigma r}\Lambda_3}{2\omega_s}\left[\frac{s\cdot 2U^2_{\alpha s(+)}}{As^2+2Bs+C}-\frac{\left(2-s\right)\cdot 2U^2_{\alpha s(-)}}{A(2-s)^2+2B(2-s)+C}\right] \tag{67}$$

where we have defined the notations: $\omega_s^2+\nu_s^2=A;\quad \nu_{\sigma s}\nu_{\sigma r}=B;\quad \nu_r^2+\nu_{tt}^2=C;$ and

$$\sqrt{2}U_{\alpha s(+)}=U\sqrt{1+k^2+2k\sin(\beta+\pi/6)};\sqrt{2}U_{\alpha s(-)}=U\sqrt{1+k^2-2k\sin(\beta-\pi/6)} \tag{68}$$

Finally, the expression of the mean resultant torque with the slip is:

$$T_{erez}=\frac{3p\nu_{\sigma r}\Lambda_3U^2}{2\omega_s}\left[\frac{1+k^2+2k\sin(\beta+\pi/6)}{As^2+2Bs+C}s-\frac{1+k^2-2k\sin(\beta-\pi/6)}{A(2-s)^2+2B(2-s)+C}\left(2-s\right)\right] \tag{69}$$

The influence of the supply unbalances upon $T_e=f(s)$ characteristic are presented in Fig. 7. To this effect, let us take again into discussion the machine with the following parameters: supply voltages with the amplitude of 490 V (U_{as}=346.5V) and 2π/3 rad. shifted in phase; $R_s=R_r=2$; $L_{hs}=0{,}09$; $L_{\sigma s}=L_{\sigma r}=0{,}01$; $J=0{,}05$; $p=2$; $k_z=0{,}02$; $\omega_1=314{,}1$ (SI units). The characteristic corresponding to the three-phase symmetric machine is the curve A (the motoring pull-out

torque is equal to 124 Nm and obviously $U_{as(-)} = 0$). If the voltage on phase *b* keeps the same amplitude as the voltage in phase *a*, for example, but the angle of phase difference changes with $\pi/24=7,5$ degrees (from $2\pi/3=16\pi/24$ to $17\pi/24$ rad.) then the new characteristic is the B curve. The pull-out torque value decreases with approx. 12% but the pull-out slip keeps its value. Other two unbalance degrees are presented in Fig. 7 as well.

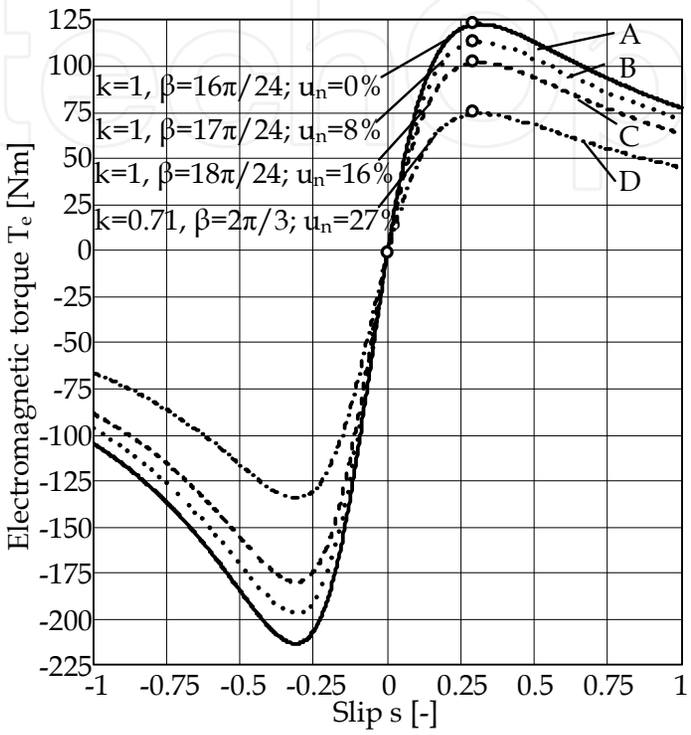

Figure 7. $T_e=f(s)$ characteristic for different unbalance degrees

Usually, the *unbalance degree* of the supply voltage is defined as the ratio of inverse and direct components:

$$u_n = \frac{U_{\alpha s(-)}}{U_{\alpha s(+)}} = \frac{\sqrt{1+k^2-2k\sin(\beta-\pi/6)}}{\sqrt{1+k^2+2k\sin(\beta+\pi/6)}} \cdot 100[\%] \tag{70}$$

The curves A, B, C, and D from Fig. 7 correspond to the following values of the unbalance degree: u_n= 0; 8%; 16% and 27%. The highest unbalance degree (27% - curve D) causes a decrease of the pull-out torque by 40%.

b. The second approach takes into consideration the following reasoning. When the amplitudes of the three-phase supply system and/or the angles of the phase difference are not equal to $2\pi/3$ then the *unbalanced system* can be replaced by two *balanced three-phase* systems that act in opposition. One of them is the *direct sequence* system and has higher voltages and the other is the *inverse sequence* system and has lower voltages. A transformation of the unbalanced voltages and total fluxes into two symmetric systems is again necessary. In other words, there is an unbalanced voltage system ($\underline{U}_{as}$, $\underline{U}_{bs}$, $\underline{U}_{cs}$),

which is replaced by the *direct* and *inverse* symmetric systems. The mean resultant torque is the difference between the torques developed by the two symmetric machine-models. Taking into consideration their slip values ($s_d = s$ and $s_i = 2\text{-}s$) we can deduce the torque expression:

$$T_{erez} = -(3/2)p\Lambda_3 \cdot [3\mathrm{Re}\left(j\underline{\Psi}_{as1}\underline{\Psi}'^{*}_{ar1}\right) - 3\mathrm{Re}\left(j\underline{\Psi}_{as2}\underline{\Psi}'^{*}_{ar2}\right)] \tag{71}$$

$$T_{erez} = \frac{3p\nu_{\sigma r}\Lambda_3}{2\omega_s}\left[\frac{3sU_{as1}^2}{As^2+2Bs+C} - \frac{3(2-s)U_{as2}^2}{A(2-s)^2+2B(2-s)+C}\right] \tag{72}$$

and this is the same with (69) as we expected.

6. Simulation study upon some transient duties of the three-phase induction machine

6.1. Symmetric supply system

The mathematical model described by the equation system (26-1…8) allows a complete simulation study of the operation of the three-phase induction machine, which include start-up, any sudden change of the load and braking to stop eventually. To this end, the machine parameters (resistances, main and leakage phase inductances, moments of inertia corresponding to the rotor and the load, coefficients that characterize the variable speed and torque, etc.) have to be calculated or experimentally deduced. At the same time, the values of the load torque and the expressions of the instantaneous voltages applied to each stator phase winding are known, as well. The rotor winding is considered short-circuited. Using the above mentioned equation system, the structural diagram in the Matlab-Simulink environment can be carried out. Additionally, for a complete evaluation, virtual oscillographs for the visualization of the main physical parameters such as voltage, current, magnetic flux, torque, speed, rotation angle and current or specific characteristics (mechanical characteristic, angular characteristic or flux hodographs) fill out the structural diagram.

The study of the *symmetric three-phase* condition in the Matlab-Simulink environment takes into consideration the following parameter values: three identical supply voltages with the amplitude of 490 V (U_{as}=346.5V) and $2\pi/3$ rad. shifted in phase; $u_{ar}=u_{br}=u_{cr}=0$ since the rotor winding is short-circuited; $R_s=R_r=2$; L_{hs}=0,09; $L_{\sigma s}= L_{\sigma r}$=0,01; J=0,05; p=2; k_z=0,02; ω_1=314,1 (SI units). The equation system becomes:

$$\left(\bar{s}+135{,}71\right)\overline{\psi}_{as} = \overline{u}_{as} - 32{,}14\left(\overline{\psi}_{bs}+\overline{\psi}_{cs}\right) + 32{,}14\left(2\overline{\psi}_{ar}-\overline{\psi}_{br}-\overline{\psi}_{cr}\right)\cos\theta_R + 55{,}67(\overline{\psi}_{cr}-\overline{\psi}_{br})\sin\theta_R$$

$$\left(\bar{s}+135{,}71\right)\overline{\psi}_{bs} = \overline{u}_{bs} - 32{,}14\left(\overline{\psi}_{cs}+\overline{\psi}_{as}\right) + 32{,}14\left(2\overline{\psi}_{br}-\overline{\psi}_{cr}-\overline{\psi}_{ar}\right)\cos\theta_R + 55{,}67(\overline{\psi}_{ar}-\overline{\psi}_{cr})\sin\theta_R$$

$$\left(\bar{s}+135,71\right)\overline{\psi}_{cs}=\bar{u}_{cs}-32,14\left(\overline{\psi}_{as}+\overline{\psi}_{bs}\right)+32,14\left(2\overline{\psi}_{cr}-\overline{\psi}_{ar}-\overline{\psi}_{br}\right)\cos\theta_R+55,67(\overline{\psi}_{br}-\overline{\psi}_{ar})\sin\theta_R$$

$$\left(\bar{s}+135,71\right)\overline{\psi}_{ar}=0-32,14\left(\overline{\psi}_{br}+\overline{\psi}_{cr}\right)+32,14\left(2\overline{\psi}_{as}-\overline{\psi}_{bs}-\overline{\psi}_{cs}\right)\cos\theta_R+55,67(\overline{\psi}_{bs}-\overline{\psi}_{cs})\sin\theta_R$$

$$\left(\bar{s}+135,71\right)\overline{\psi}_{br}=0-32,14\left(\overline{\psi}_{cr}+\overline{\psi}_{ar}\right)+32,14\left(2\overline{\psi}_{bs}-\overline{\psi}_{cs}-\overline{\psi}_{as}\right)\cos\theta_R+55,67(\overline{\psi}_{cs}-\overline{\psi}_{as})\sin\theta_R$$

$$\left(\bar{s}+135,71\right)\overline{\psi}_{cr}=0-32,14\left(\overline{\psi}_{ar}+\overline{\psi}_{br}\right)+32,14\left(2\overline{\psi}_{cs}-\overline{\psi}_{as}-\overline{\psi}_{bs}\right)\cos\theta_R+55,67(\overline{\psi}_{as}-\overline{\psi}_{bs})\sin\theta_R$$

$$\begin{aligned}\dot{\theta}_R\left(\bar{s}+0,4\right)=\left(40\right)\langle-\left(32,14\right)\{\ &\sin\theta_R\left[\overline{\psi}_{as}\left(2\overline{\psi}_{ar}-\overline{\psi}_{br}-\overline{\psi}_{cr}\right)+\overline{\psi}_{bs}\left(2\overline{\psi}_{br}-\overline{\psi}_{cr}-\overline{\psi}_{ar}\right)+\right.\\&\left.+\overline{\psi}_{cs}\left(2\overline{\psi}_{cr}-\overline{\psi}_{ar}-\overline{\psi}_{br}\right)\right]+\sqrt{3}\cos\theta_R\cdot\left[\overline{\psi}_{as}\left(\overline{\psi}_{br}-\overline{\psi}_{cr}\right)+\overline{\psi}_{bs}\left(\overline{\psi}_{cr}-\overline{\psi}_{ar}\right)+\overline{\psi}_{cs}\left(\overline{\psi}_{ar}-\overline{\psi}_{br}\right)\right]\ \}-T_{st}\rangle\end{aligned}\tag{73-1-7}$$

$$\theta_R=\omega_R\frac{1}{s}\tag{73-8}$$

$$\begin{gathered}\bar{u}_{as}\leftrightarrow\frac{490}{\sqrt{2}}e^{j(314,1t)};\bar{u}_{bs}\leftrightarrow\frac{490}{\sqrt{2}}e^{j(314,1t-2,094)};\bar{u}_{cs}\leftrightarrow\frac{490}{\sqrt{2}}e^{j(314,1t-4,188)};\\U_{as\max}=U_{bs\max}=U_{cs\max}=490\end{gathered}\tag{73-9}$$

It has to be mentioned again that the above equation system allows the analysis of the three-phase induction machine under any condition, that is transients, steady state, symmetric or unbalanced, with one or both windings (from stator and rotor) connected to a supply system. Generally, a supplementary requirement upon the stator supply voltages is not mandatory. The case of short-circuited rotor winding, when the rotor supply voltages are zero, include the wound rotor machine under rated operation since the starting rheostat is short-circuited as well.

The presented simulation takes into discussion a varying duty, which consists in a *no-load* start-up (the load torque derives of frictions and ventilation and is proportional to the angular speed and have a steady state rated value of approx. 3 Nm) followed after 0,25 seconds by a sudden loading with a constant torque of 50 Nm. The simulation results are presented in Fig. 8, 10, 12, 14 and 15 and denoted by the symbol *RS-50.* A second simulation iterates the presented varying duty but with a load torque of 120 Nm, symbol *RS-120*, Fig. 9, 11 and 13. Finally, a third simulation takes into consideration a load torque of 125 Nm, which is a value over the pull-out torque. Consequently, the falling out and the stop of the motor in t≈0,8 seconds mark the varying duty (symbol *RS-125*, Fig. 16, 17, 18 and 19).

The *RS-50* simulation shows an upward variation of the angular speed to the no-load value (in t ≈ 0,1 seconds), which has a weak overshoot at the end, Fig. 8. The 50 Nm torque enforcement determines a decrease of the speed corresponding to a slip value of s ≈ 6,5%. In the case of the *RS-120* simulation, the start-up is obviously similar but the loading torque determines a much more significant decrease of the angular speed and the slip value gets to s ≈ 25%, Fig. 9.

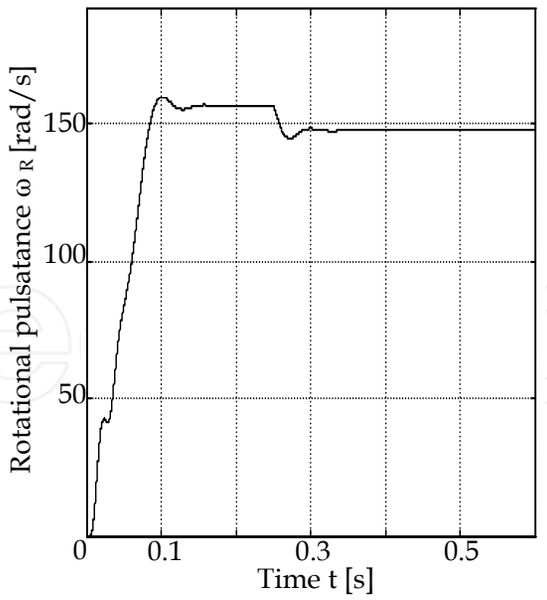

Figure 8. Time variation of rotational pulsatance – *RS-50*

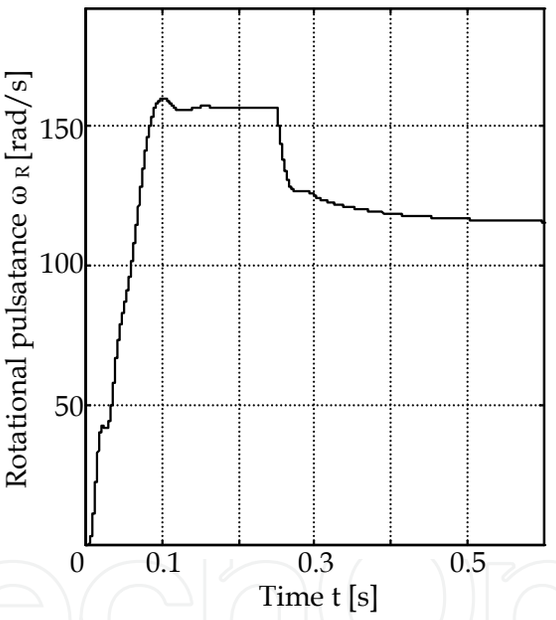

Figure 9. Time variation of rotational pulsatance – *RS-120*

In the first moments of the start-up, the electromagnetic torque oscillates around 100 Nm and after the load torque enforcement, it gets to approx. 53 Nm for *RS-50*, Fig. 10 and to approx. 122 Nm for *RS-120*, Fig. 11. The operation of the motor remains stable for the both duties.

The behavior of the machine is very interesting described by the hodograph of the resultant rotor flux (the locus of the head of the resultant rotor flux phasor), Fig. 12 and 13. With the connecting moment, the rotor fluxes start from 0 (O points on the hodograph) and track a corkscrew to the maximum value that corresponds to synchronism (ideal no-load operation), S points on the hodographs.

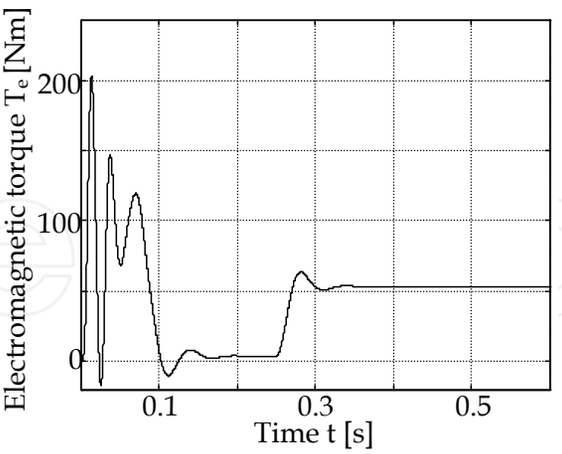

Figure 10. Time variation of electromagnetic torque – *RS-50*

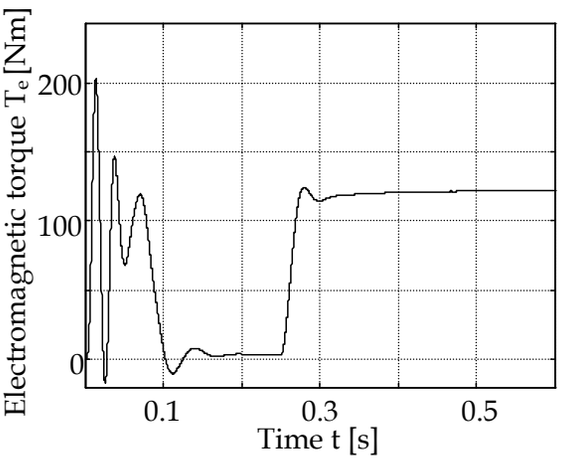

Figure 11. Time variation of electromagnetic torque – *RS-120*

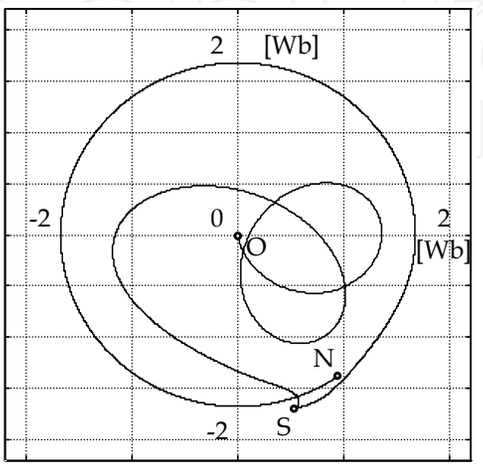

Figure 12. Hodograph of resultant rotor flux – *RS-50*

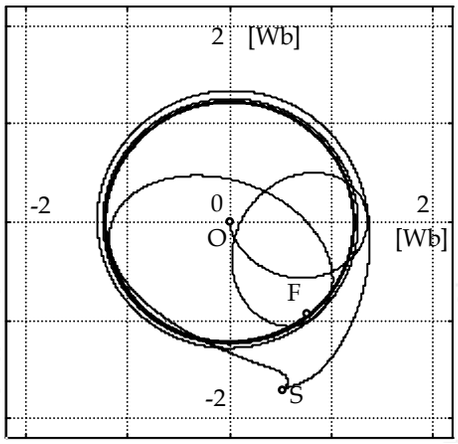

Figure 13. Hodograph of resultant rotor flux – *RS-120*

The enforcement of the load torque determines a decrease of the resultant rotor flux, which is proportional to the load degree, and is due to the rotor reaction. The locus of the head of the phasor becomes a circle whose radius is proportional to the amplitude of the resultant rotor flux. The speed on this circle is given by the rotor frequency that is by the slip value. It is interesting to notice that the load torque of 50 Nm causes a unique rotation of the rotor flux whose amplitude becomes equal to the segment ON (Fig. 12) whereas the 120 Nm torque causes approx. 4 rotations of the rotor flux and the amplitude OF is significantly smaller (Fig. 13).

If the expressions (1) and (2) are also used in the structural diagram then both stator and rotor phase currents can be plotted. The stator current corresponding to *as* phase has the frequency f_1=50 Hz and gets a start-up amplitude of approx. 70 A. This value decreases to approx. 6 A (no-load current) and after the torque enforcement (50 Nm) it rises to a stable value of approx. 14 A, Fig. 14. The rotor current on phase *ar*, which has a frequency value of $f_2 = s \cdot f_1$, gets a similar (approx. 70 A) start-up variation but in opposition to the stator current, i_{as}. Then, its value decrease and the frequency go close to zero. The loading of the machine has as result an increase of the rotor current up to 13 A and a frequency value of $f_2 \approx 3$Hz, Fig. 15. The fact that the current variations are sinusoidal and keep a constant frequency is an argument for a stable operation under symmetric supply conditions.

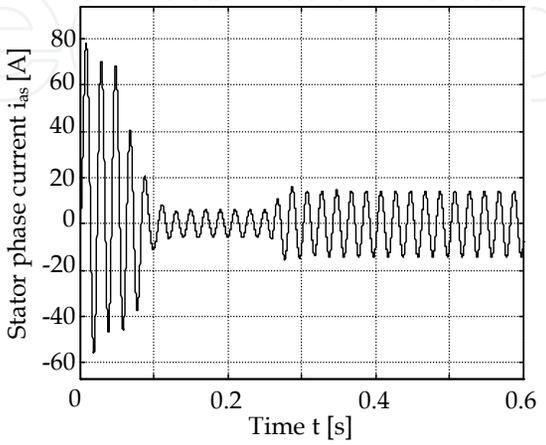

Figure 14. Time variation of stator phase current – *RS-50*

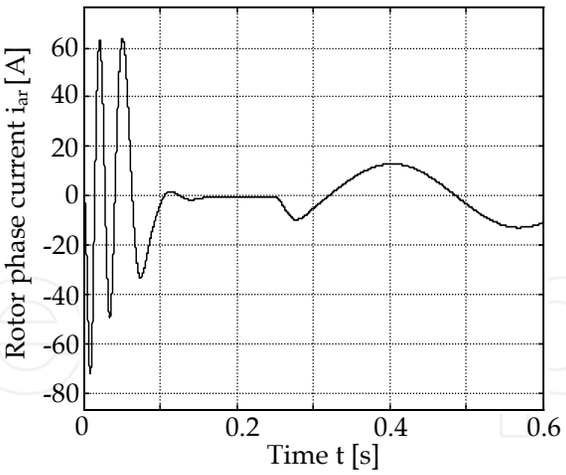

Figure 15. Time variation of rotor phase current – *RS-50*

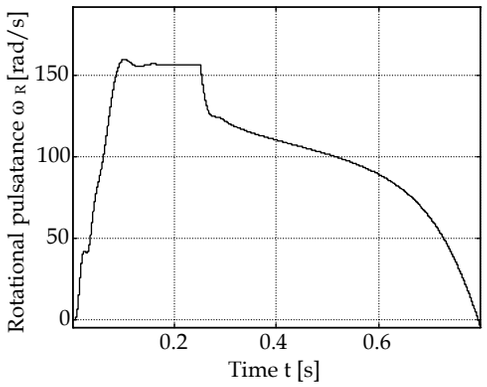

Figure 16. Time variation of rotational pulsatance – *RS-125 (start-up to locked-rotor)*

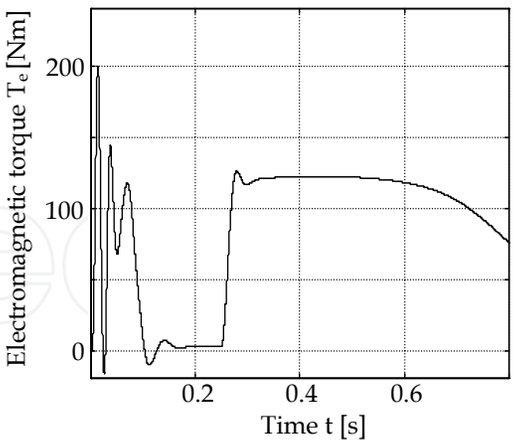

Figure 17. Time variation of electromagnetic torque – *RS-125*

The third simulation, *RS-125*, has a similar start-up but the enforcement of the load torque determines a fast deceleration of the rotor. The pull-out slip (s≈33%) happens in t≈0,5 seconds after which the machine falls out. The angular speed reaches the zero value in t≈0,8 seconds, Fig. 16, and the electromagnetic torque get a value of approx. 78 Nm. This value can be considered the locked-rotor (starting) torque of the machine, Fig. 17.

The described critical duty that involves no-load start-up and operation, overloading, falling out and stop is plotted in terms of resultant rotor flux and angular speed versus electromagnetic torque. The hodograph (Fig. 18) put in view a cuasi corkscrew section, corresponding to the start-up, characterized by its maximum value represented by the segment OS. The falling out tracks the corkscrew SP with a decrease of the amplitude, which is proportional to the deceleration of the rotor. The point P corresponds to the locked-rotor position (s=1). Fig. 19 presents the dynamic mechanical characteristic, which shows the variation of the electromagnetic torque under variable operation condition. During the no-load start-up, the operation point tracks successively the points O, M, L and S, that is from locked-rotor to synchronism with an oscillation of the electromagnetic torque inside certain limits (≈+200Nm to ≈-25Nm). The enforcement of the overload torque leads the operation point along the *downward* curve SK characterized by an *oscillation* section followed by the unstable falling out section, KP. The PKS curve, together with the marked points (Fig. 19) can be considered the *natural mechanical characteristic* under motoring duty.

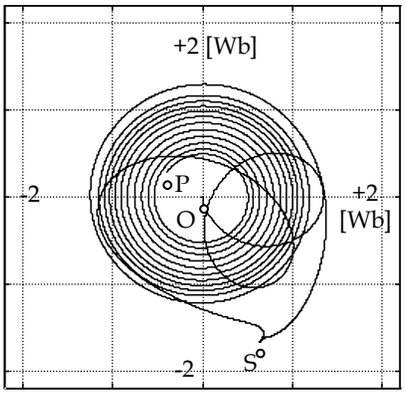

Figure 18. Hodograph of resultant rotor flux – *RS-125 (start-up to locked-rotor)*

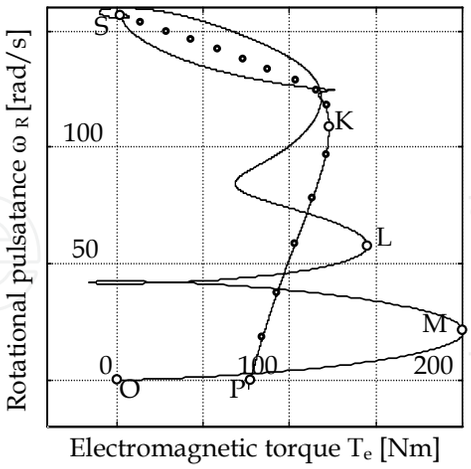

Figure 19. Rotational pulsatance vs. torque – *RS-125 (start-up to locked-rotor)*

6.2. Asymmetric supply system

A simulation study of the three-phase induction machine under unbalanced supply condition and varying duty (start-up, sudden torque enforcement and braking to stop

eventually) is possible by using the same mathematical model described by the equation system (26-1…8). The values of the resistant torques and the expressions of the instantaneous phase voltages have to be stated. Since the rotor winding is short-circuited, the supply rotor voltages are $u_{ar}=u_{br}=u_{cr}=0$. On this basis, the structural diagram has been put into effect in the Matlab-Simulink environment. As regards the unbalanced three-phase supply system, it has to be mentioned that the phase voltages are no more equal in amplitude and the angles of phase difference may have other values than $2\pi/3$ rad. In any event, the sum of the instantaneous values of the applied voltages must be zero, that is $u_{as}+u_{bs}+u_{cs}=0$. As an argument for this seemingly constraint stands the fact that the vast majority of the three-phase induction machines are connected to the industrial system via three supply leads (no neutral).

The simulation presented here takes into discussion an induction machine with the same parameters as above that is: $R_s=R_r=2$; $L_{hs}=0,09$; $L_{\sigma s}= L_{\sigma r}=0,01$; J=0,05; p=2; $k_z=0,02$; $\omega_1=314,1$ (SI units). Consequently, the equations (73-1) - (73-8) keep unchanged. The expressions (73-9) have to be modified in accordance with the asymmetry degree.

Two varying duties under unbalanced condition have been simulated. The first (denoted *RNS-1*) is characterized by an asymmetry degree, u_n = 16,5% and the following supply voltages:

$$\underline{u}_{as} \leftrightarrow \frac{490}{\sqrt{2}} e^{j(314,1t)}; \underline{u}_{bs} \leftrightarrow \frac{375}{\sqrt{2}} e^{j(314,1t-1,96)}; \underline{u}_{cs} \leftrightarrow \frac{490}{\sqrt{2}} e^{j(314,1t-3,927)}; u_n = 16,5\% \qquad (74)$$

The simulation results are presented in Fig. 20, 22, 24, 25 and 28. The second study simulation (denoted *RNS-2*) has an asymmetry degree of u_n = 27% given by the following stator voltages:

$$\underline{u}_{as} \leftrightarrow \frac{490}{\sqrt{2}} e^{j(314,1t)}; \underline{u}_{bs} \leftrightarrow \frac{346,43}{\sqrt{2}} e^{j(314,1t-2,357)}; \underline{u}_{cs} \leftrightarrow \frac{346,43}{\sqrt{2}} e^{j(314,1t-3,295)}; u_n = 27\% \qquad (75)$$

The simulation results are presented in Fig. 21, 23, 26, 27 and 29. The varying duties are similar to those discussed above and consist in a *no-load* start-up (the load torque derives of frictions and ventilation and is proportional to the angular speed and have a steady state rated value of approx. 3 Nm) followed after 0,25 seconds by a sudden loading with a constant torque of 50 Nm.

In comparison to symmetric supply, the unbalanced voltage system causes a longer start-up time with approx. 20% for *RNS-1* (Fig. 20) and with 50% for *RNS-2* (Fig. 21). Moreover, the higher asymmetry degree of *RNS-2* leads to the cancelation of the overshoot at the end of the start-up process. At the same time, significant speed oscillations are noticeable during the operation (no matter the load degree), which are higher with the increase of the asymmetry degree. These oscillations have a constant frequency, which is twice of the supply voltage frequency. They represent the main cause that determines the specific noise of the machines with unbalanced supply system.

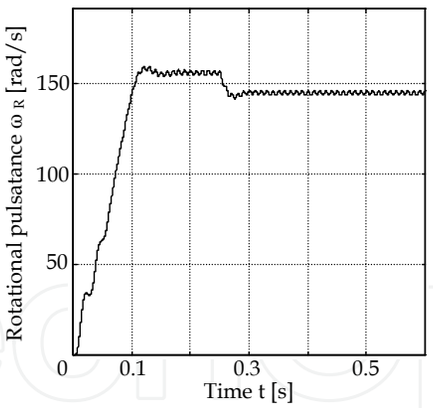

Figure 20. Time variation of rotational pulsatance – *RNS-1 (start-up + sudden load)*

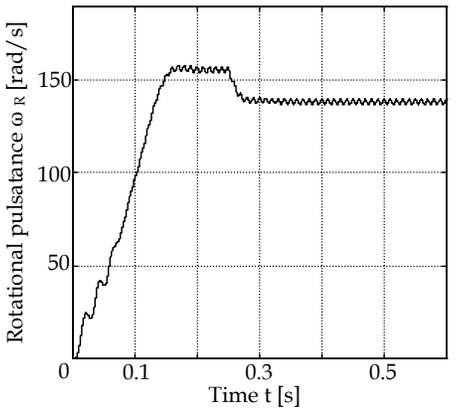

Figure 21. Time variation of rotational pulsatance – *RNS-2 (start-up + sudden load)*

The inspection of the electromagnetic torque variation (Fig. 22 and 23) shows the presence of a variable oscillating torque, whose frequency is twice the supply voltage frequency (in our case 100 Hz) and overlaps the average torque. *This oscillating component is demonstrated by the analytic expression of the instantaneous torque, which is written using nothing but total flux linkages* (25). The symmetric components theory, for example, is not capable to provide information about these oscillating torques. At the most, this theory evaluates the average torque, probably with inherent errors. Coming back to the torque variations, one can see that the amplitude oscillations increase with the asymmetry degree, but their frequency keeps unchanged.

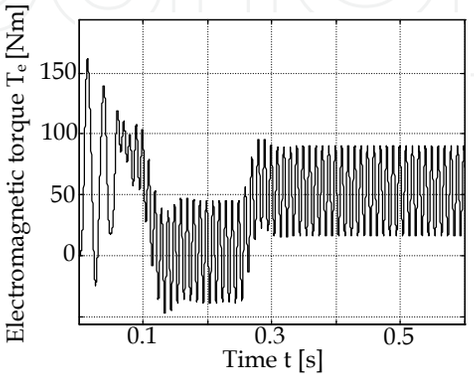

Figure 22. Time variation of electromagnetic torque – *RNS-1*

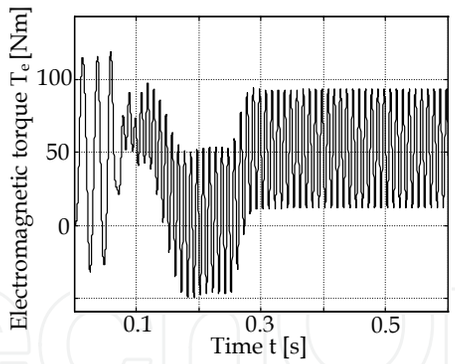

Figure 23. Time variation of electromagnetic torque – *RNS-2*

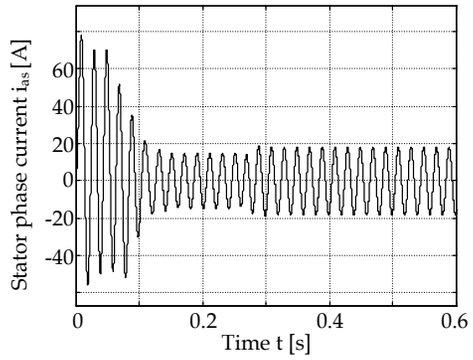

Figure 24. Time variation of stator phase current – *RNS-1*

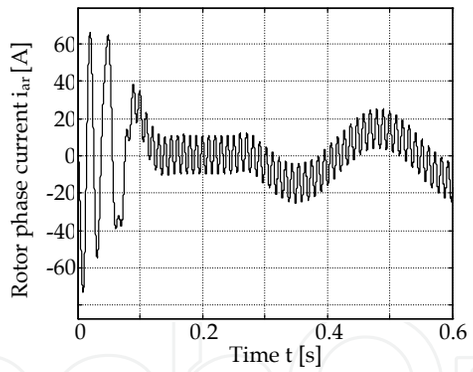

Figure 25. Time variation of rotor phase current – *RNS-1*

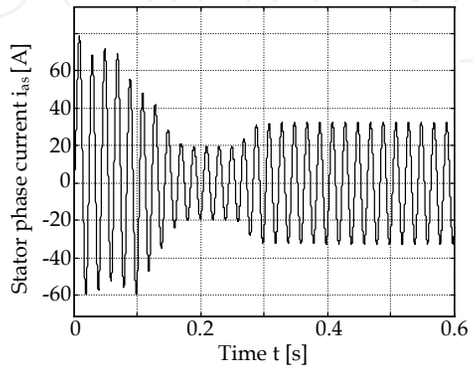

Figure 26. Time variation of stator phase current – *RNS-2*

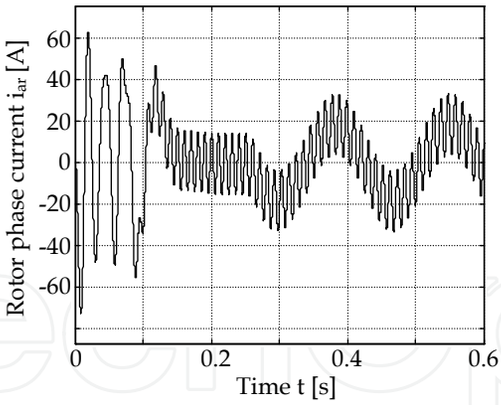

Figure 27. Time variation of rotor phase current – *RNS-2*

The stator currents variation, Fig. 24 and 26, have a sinusoidal shape and an unmodified frequency of 50 Hz. Their amplitude increases however with the asymmetry degree (approx. 18 A for *RNS-1* and approx. 32 A for *RNS-2*). As a consequence of this fact, both power factor and efficiency decrease. The rotor currents (Fig. 25 and 27) include besides the main component of $f_2=s\cdot f_1$ frequency a second oscillating component of high frequency, $f'_2=(2-s)f_1$, which is responsible for parasitic torques and vibrations of the rotor. The amplitude of these oscillating currents increases with the asymmetry degree.

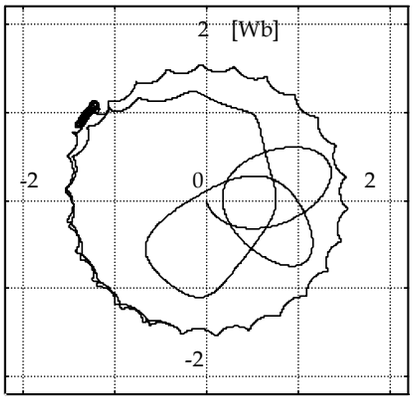

Figure 28. Hodograph of resultant rotor flux – *RNS-1*

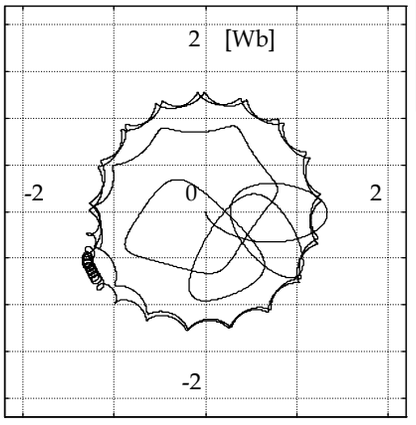

Figure 29. Hodograph of resultant rotor flux – *RNS-2*

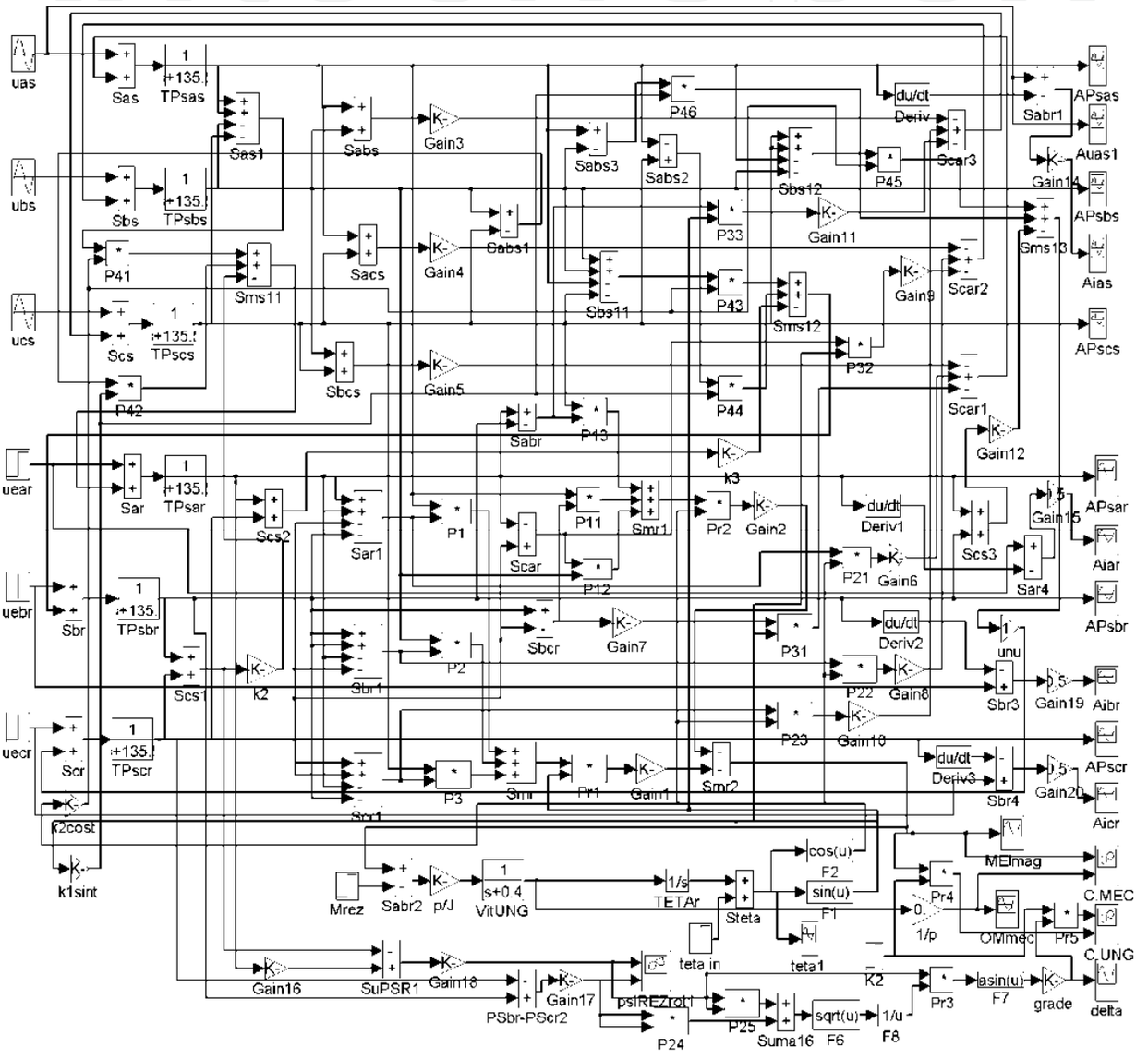

Figure 30. Structural diagram of the three-phase induction machine

The hodographs of the resultant rotor flux show a very interesting behavior of the unbalanced machines, Fig. 28 and 29. In comparison to the symmetric supply cases where the hodograph is a circle under steady state, the asymmetric system distort the curve into a „gear wheel" with a lot of teeth placed on a mean diameter whose magnitude depends inverse proportionally with the asymmetry degree. Generally, these curves do not overlap and prove that during the operation the interaction between stator and rotor fluxes is not constant in time since the rotor speed is not constant. Consequently, the rotor vibrations are usually propagated to the mechanical components and working machine.

In order to point out the superiority of the proposed mathematical model, Fig. 30 shows the structural diagram used in Simulink environment. The diagram is capable to simulate any steady-state and transient duty under balanced or unbalanced state of the induction machine including doubly-fed operation as generator or motor by simple modification of the input data. To prove this statement, a simulation of an unbalanced doubly-fed operation has been performed. The operation cycle involves: I. A no-load start-up (the wound rotor winding is short-circuited); II. Application of a supplementary output torque of (-70) Nm (at the moment t=0.4 sec.) which leads the induction machine to the generating duty (over synchronous speed); III. Supply of two series connected rotor phases with d.c. current (U_{ar}=+40V, U_{br}= −40V, U_{cr}=0V), at the moment time t=0.6 sec., which change the operation of the induction generator into a synchronized induction generator (SIG).

Fig. 31 and 32 show the dynamic mechanical characteristic, $Te=f(\Omega_R)$ and the hodograph of the resultant rotor flux respectively. The start-up corresponds to A-S1 curve, the over synchronous acceleration is modeled by S1-S curve and the operation under SIG duty corresponds to S-S2 curve. A few observations regarding Fig. 32 are necessary as well. The rotor flux hodograph is rotating in a *counterclockwise direction* corresponding to motoring duty, in a *clockwise direction* for generating duty and stands still at synchronism. The "in time" modification and the position of the hodograph corresponding to SIG duty depend on the moment of d.c. supply and the load angle of the machine.

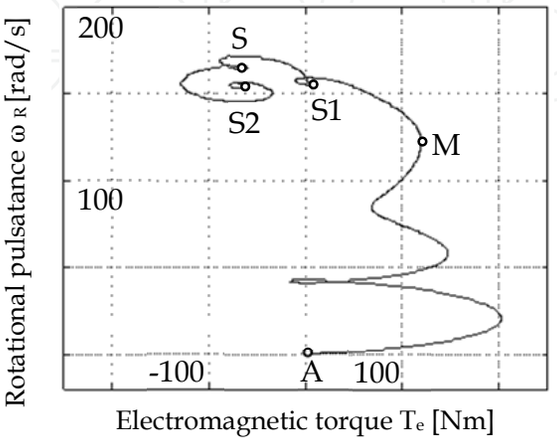

Figure 31. Dynamic mechanical characteristic

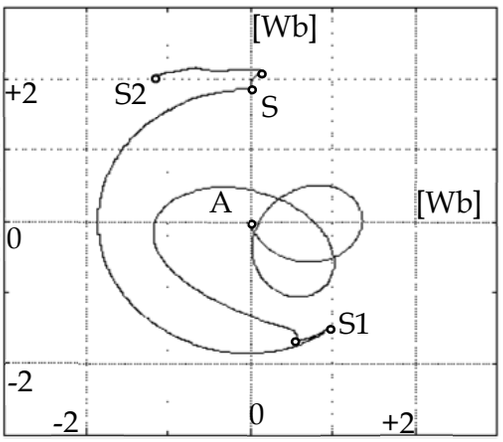

Figure 32. Hodograph of resultant rotor flux

7. Conclusion

The mathematical model presented in this contribution is characterized by the total lack of the winding currents and angular speed in the voltage equations. Since these parameters are differential quantities of other electric parameters, they usually bring supplementary calculus errors mainly for the dynamic duty analysis. Their removal assures a high accuracy of the results. If their variation is however necessary to be known then simple subsequent calculations can be performed.

The use of the mathematical model in total fluxes is appropriate for the study of the electric machines with permanent magnets where the definitive parameter is the magnetic flux and not the electric current.

The coefficients defined by (28.1-4), which depend on resistances and inductances, take into consideration the saturation. Consequently, the study of the induction machine covers more than the linear behavior of the magnetization phenomenon.

The most important advantage of the proposed mathematical model is its generality degree. Any operation duty, such as steady-state or transients, balanced or unbalanced, can be analyzed. In particular, the double feeding duty and the synchronized induction machine operation (feeding with D.C. current of a rotor phase while the other two are short-circuited) can be simulated as well.

The results obtained by simulation are based on the transformation of the equations in structural diagrams under Matlab-Simulink environment. They present the variation of electrical quantities (voltages and currents corresponding to stator and rotor windings), of mechanical quantities (expressed through rotational pulsatance) and of magnetic

parameters (electromagnetic torque, resultant rotor and stator fluxes). They put in view the behavior of the induction machine for different transient duties. In particular, they prove that any unbalance of the supply system generates important variations of the electromagnetic torque and rotor speed. This fact causes vibrations and noise.

Author details

Alecsandru Simion, Leonard Livadaru and Adrian Munteanu
"Gh. Asachi" Technical University of Iaşi, Electrical Engineering Faculty, Romania

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