

# We are IntechOpen, the world's leading publisher of Open Access books Built by scientists, for scientists

6,900

Open access books available

186,000

International authors and editors

200M

Downloads

Our authors are among the

154

Countries delivered to

TOP 1%

most cited scientists

12.2%

Contributors from top 500 universities



WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index  
in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?  
Contact [book.department@intechopen.com](mailto:book.department@intechopen.com)

Numbers displayed above are based on latest data collected.  
For more information visit [www.intechopen.com](http://www.intechopen.com)



---

# New Approaches to Modeling Elastic Media

---

G. J. Martin, J. Hussan and M. McKubre-Jordens

Additional information is available at the end of the chapter

<http://dx.doi.org/10.5772/54771>

---

## 1. Introduction

In this chapter we suggest new models for the study of deformations of elastic media through the minimization of distortion functionals. This provides a “holistic” approach to this problem and a potential mathematical underpinning of a number of phenomena which are actually observed. The functionals we study are conformally invariant and give measures of the local anisotropic deformation of the material which can be tuned by varying  $p$ -norms and weights. That these functions are conformally invariant offers the opportunity to address multiscale problems in an integrative manner - there is no natural scale. When applied to the problem of deforming elastic bodies through stretching we see phenomena such as tearing naturally arising through bad delocalisation of energy. More formally, we find that deformations either exist within certain ranges or fail to exist outside these ranges – a material can only be stretched so far. We dub this the Nitsche phenomenon after a similar phenomenon was observed in connection with minimal surfaces. We see that for these deformation problems the ranges where deformations can be proven to exist depend on invariants picked up from the profile of the material or its density depending on how one interprets the weight. These invariants are akin to conformal invariants such as moduli for complex analysis. Outside the range where we can prove existence we further show how to prove minimisers do not exist. Thus this gives a complete dichotomy depending on certain structural invariants. Further, when properly parameterised by these invariants, existence of otherwise in the borderline cases depends on more subtle invariants such as structural profiles and their degree of regularity. Our results primarily pertain to two dimensional elasticity, but apply in three dimensions in certain circumstances for particular models.

## 2. Basic notions

Consider an elastic body  $\Omega \subset \mathbb{R}^n$ . We study deformations  $f : \Omega \rightarrow \Omega' \subset \mathbb{R}^n$  of  $\Omega$  of Sobolev class  $f \in W_{loc}^{1,1}(\Omega)$  consisting of functions with first derivatives in  $L_{loc}^1(\Omega)$ . The principle

of interpenetrability of matter allows us to focus on the case that  $f$  is a homeomorphism. We will also assume that the mapping  $f$  has finite distortion. The basics of the theory of mappings of finite distortion were laid down in [3, 16], but see [15, Chapter 6] for a more general discussion. Finite distortion means that the distortion function  $\mathbb{K}(x, f)$ , defined by the formula

$$\mathbb{K}(x, f) = \frac{\|Df(x)\|^n}{J(x, f)}, \quad x \in \Omega \quad (1)$$

is finite almost everywhere. Here  $Df(x)$  is the Jacobian matrix and  $J(x, f)$  its Jacobian determinant. In order to get a viable theory some additional regularity must be placed on  $\mathbb{K}(z, f)$ . The theory of quasiconformal mappings pertains to the case  $\mathbb{K}(x, f) \in L^\infty$ . Most recent developments assume  $\mathbb{K}(x, f)$  is exponentially integrable (or close to it) and hence a close connection with BMO functions. The theory then exploits the  $H^1$ -BMO duality (since we know a Jacobian is in  $H^1$ ) when considering the distortion inequality

$$\|Df(x)\|^n \leq \mathbb{K}(x, f)J(x, f)$$

so the right hand side here might be given meaning as an integrable function.

Note that the distortion functional is conformally invariant. If  $\lambda \in \mathbb{R}_+$ , then

$$\mathbb{K}(z, \lambda f) = \mathbb{K}(z, f) \quad (2)$$

Moreover, for sufficiently regular mappings ( $W_{loc}^{1,n}$  and  $\mathbb{K} \in L_{loc}^\infty$  suffices) we have the multiplicative formula

$$\mathbb{K}(z, f \circ g) \leq \mathbb{K}(z, g) \mathbb{K}(g(z), f) \quad (3)$$

for  $g : \Omega \rightarrow g(\Omega)$  and  $f : g(\Omega) \rightarrow \mathbb{R}^n$ , and in fact if  $\mathbb{K}(z, g) = 1$ , then  $\mathbb{K}(z, f \circ g) = \mathbb{K}(g(z), f)$ .

In two dimensions, the equation  $\mathbb{K}(z, g) = 1$  linearises to the Cauchy-Riemann equations and  $W_{loc}^{1,1}$  solutions will be conformal mappings. In higher dimensions there is the remarkable rigidity theorem of Liouville from 1850 (much improved over the years) that a  $W_{loc}^{1,n}(\Omega)$  solution of the equation  $\mathbb{K}(z, f) = 1$  is the restriction to  $\Omega$  of a  $C^\infty$  Möbius transformation of  $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ . A Möbius transformation is a finite composition of reflections, translations and dilations.

## 2.1. Deformations

The general theory posits that deformations rearrange an object (our domain  $\Omega$ ) so as to minimise a *stored energy functional*. Such functionals are motivated by various physical principals (for instance the principle of least action) and the general theory seeks to

understand the existence, uniqueness and regularity as well as the topological structures of minimisers.

A typical functional might look like

$$f \mapsto \mathcal{E}(f) = \iint_{\Omega} \|Df(x)\|^p + \frac{1}{J^q(x, f)} dx \quad (4)$$

For instance one obtains the Dirichlet energy functional with the choice  $p = 2$ ,  $q = 0$  and the minimisers are well known to be the harmonic mappings. When  $q > 0$  we are penalising the compression of the body as the Jacobian determinant  $J(z, f)$  measures the local change in volume. These (and many other) functionals are often set up so that the direct method has a chance of working and some *a priori* modulus of continuity on a minimising sequence can be found. In this way the direct method (examining the limit of a minimising sequence) gives you something to hold in your hand and the problem becomes one of regularity. The associated Euler-Lagrange equations are highly nonlinear in all but the most elementary cases and can seldom be solved. Indeed the solution is usually obtained through the variational process and regularity.

Here we wish to study functionals of the form

$$f \mapsto \mathcal{E}(f) = \iint_{\Omega} \left( \frac{\|\nabla f(x)\|^n}{J(x, f)} \right)^p dx \quad \left( = \iint_{\Omega} \mathbb{K}^p(x, f) dx \right)$$

or more generally

$$\mathcal{E}_{\Phi}(f) = \iint_{\Omega} \Phi(\mathbb{K}(x, f)) \lambda(x) dx \quad (5)$$

where here  $\Phi(t)$  is convex (eg.  $t \mapsto t^n$ ,  $n \geq 1$ ) and  $\lambda(x)$  is a positive weight describing properties (density or profile) of the material body  $\Omega$ . We call the value of the functional  $\mathcal{E}_{\Phi}(f)$  the  $\Phi$  - conformal energy. It is a measure of the *average local anisotropic stretching* of the material and is clearly conformally invariant. In studying such functionals we find that the direct method in the calculus of variations has little chance of working. For instance

- there are no obvious *a priori* estimates,
- (provably) no modulus of continuity estimates and
- (provably) no compactness theorems for the classes of mappings one might obtain in a minimising sequence.

However our study is aided by the chance that geometric methods and conformal invariants interplay and these invariants can be used to identify a minima and establish existence or otherwise of a minimum.

## 2.2. Non-convex case

We take a moment to remark on the hypothesis that  $\Phi$  is convex. In fact if  $\Phi$  is a function of sublinear growth minimisers almost never exist. From [5, Theorem 5.3], we note

**Lemma 1.** *Let  $\phi(t)$  be a positive strictly increasing function of sublinear growth:*

$$\lim_{t \rightarrow \infty} \frac{\phi(t)}{t} = 0$$

Let  $B = D(z_0, r)$  be a round disk and suppose that  $f_0 : B \rightarrow \mathbb{C}$  is a homeomorphism of finite distortion with  $\iint_B \phi(\mathbb{K}(z, f_0)) < \infty$ . Then there is a sequence of mappings of finite distortion  $f_n : B \rightarrow f_0(B)$  with  $f_n(\zeta) = f_0(\zeta)$  near  $\partial B$  and with

- $\mathbb{K}(z, f_n) \rightarrow 1$  uniformly on compact subsets of  $B$ , and
- as  $n \rightarrow \infty$ ,

$$\iint_B \phi(\mathbb{K}(z, f_n)) \rightarrow \iint_B \phi(1)$$

Hence:

**Corollary 1.** *Let  $\phi(t)$  be a positive strictly increasing function of sublinear growth, let  $\Omega$  be a domain and let  $\lambda(z) \in L^\infty(\Omega)$  be a positive weight. Suppose*

$$\iint_\Omega \phi(\mathbb{K}(z, f_0)) dz < \infty.$$

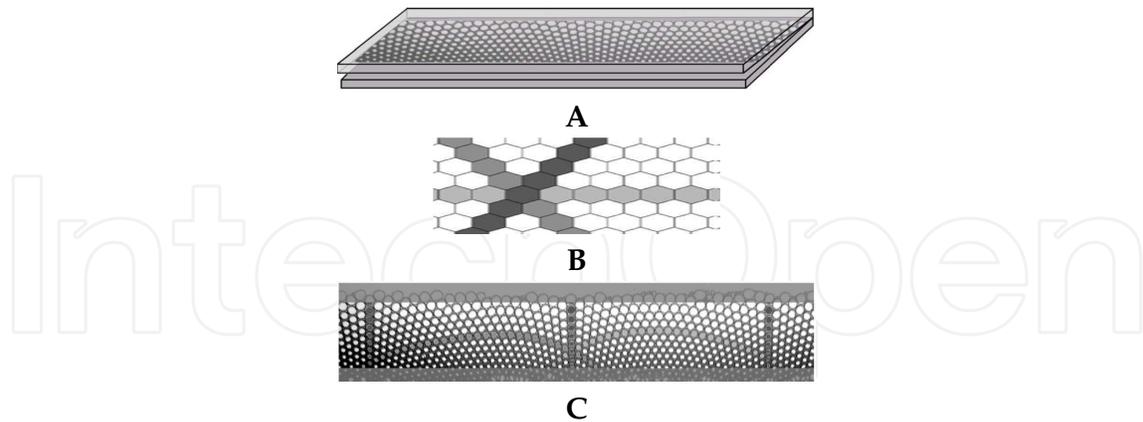
Then

$$\min_{\mathcal{F}} \iint_\Omega \phi(\mathbb{K}(z, f_0)) \lambda(z) dz = \phi(1) \iint_\Omega \lambda(z) dz$$

with equality achieved by a mapping of finite distortion if and only if the boundary values of  $f_0$  are shared by a conformal mapping. Here  $\mathcal{F}$  consists of homeomorphisms of finite distortion  $f$  with  $f|_{\partial\Omega} = f_0$ .

## 2.3. Applications and the problem

As we have noted, our functionals measure the local anisotropic stretching of an object, and it is to the physical situations where this appears to be the dominant mechanism of deformation of a body that our models should have most applicability. In particular the modeling of films (eg soap films) and the elastic stretching of tissues and muscles are being currently considered [11]. For instance one achieves a deformation of a 2D-cellular structure of soap films in the following manner: consider the cellular structure of soap bubbles - finite in extent - trapped between two close parallel panes of glass. The steady state with bubbles of uniform size is closely approximated by a regular 2D hexagonal tessellation. Now vary



**Figure 1.** Bubbles trapped between two glass planes. Reproduced from Figure 1 and 3 from [6].

the angle between the panes. The new configuration is expected to be achieved by a “nearly” conformal deformation (that is  $\mathbb{K} \approx 1$  - we cannot have  $\mathbb{K} = 1$  because of rigidity unless we are in quite exceptional circumstances). This is because the equations for minimising the interfacial (film) energy give, locally, a trivalent equal angle (of  $2\pi/3$ ) hexagonal tiling. The initial configuration also exhibits  $2\pi/3$  symmetry. Scale invariance and pressure/volume considerations suggest the free boundary “free energy minimisation” problem may be solved by an angle preserving deformation - that is, a conformal mapping.

Next, in another more constricted regime, one can use microfabrication techniques to create accordion-like honeycomb microstructure to yield porous, elastomeric three-dimensional scaffolds with controllable stiffness and anisotropy. It has been demonstrated that the neonatal rat cardiac cell based tissue, cultured using these honeycomb scaffolds, closely matched the mechanical properties of rat ventricular myocardium. In this case there is a relatively stiff interface which is being deformed *at the boundary* and quite different effects are modeled.

Our aim would be to identify a first attempt at a “holistic” mathematical formalism which might encompass theoretical analysis of structures which appear at both ends of this spectrum. Of course, the energy functional characterising tissues is rather more complex and we should only expect a fairly coarse analogy – however, near a steady state most symmetries and integral invariants are conserved and we will be able to assert existence and regularity of minimisers of our models in reasonable physical situations.

We seek to minimize the energy

$$\inf_{f \in \mathcal{F}} \mathcal{E}_\Phi(f)$$

where our class of mappings  $\mathcal{F}$  consists of all mappings  $f : \Omega \rightarrow \tilde{\Omega}$  of finite distortion, perhaps with prescribed the boundary values,

$$f|_{\partial\Omega} = f_0 : \partial\Omega \rightarrow \partial\tilde{\Omega}$$



**Figure 2.** Sequence of catenoids degenerating - mathematical models of soap films

### 3. Modeling elastic media : Examples

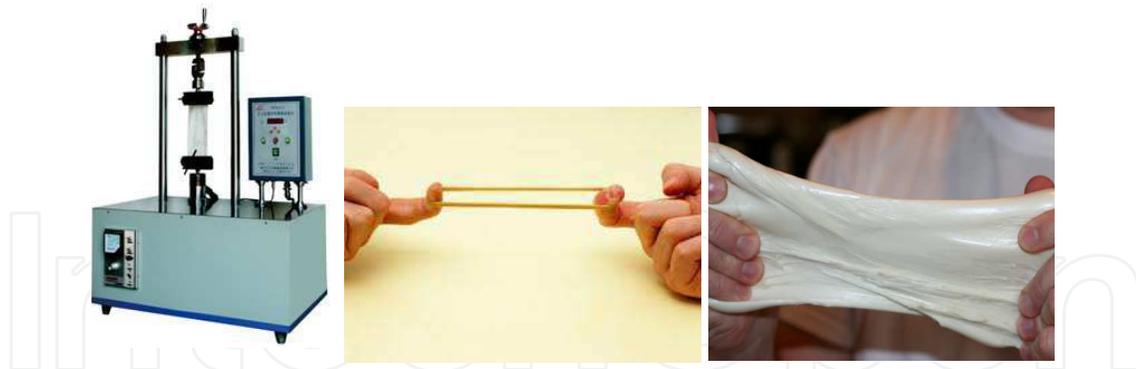
We want to model the degeneration of elastic media under deformations. For instance, consider deforming the soap film between two parallel rings by moving the rings further apart. Classically the solution (shape of the *minimal* surface) is described by the catenoid  $\{r = \cosh(z)\}$ . Thus we should see a family of constant negative curvature rotationally symmetric surfaces as illustrated Figure 2.

Of course what actually happens is that when the rings are sufficiently far apart other forces come into play and material properties of the film can no longer support a solution - the film “pops”. In fact just before the film pops, the surface looks as indicated.

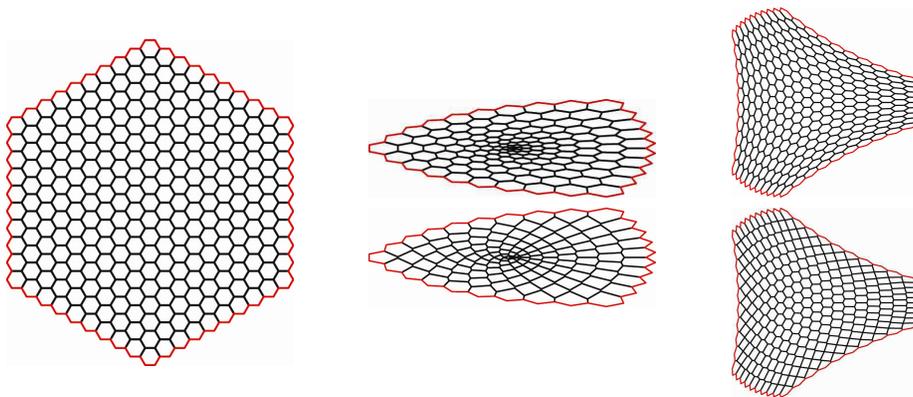


There are a couple of things we want to note from this example. First, we can actually arbitrarily compress the catenoid - the problem comes when we try to stretch it too far. Second, as we see from this illustration, it appears negative curvature is lost and that the surface has become very flat (curvature = 0) just before the solution ceases to exist. We will see both these phenomena in our model.

As another example to model we would like to consider what happens when we stretch a material with defects. Here is an experiment you can try at home. Take two rubber bands of varying widths  $T_1 > T_2$ . Depending on the thickness these can be stretched a certain distance  $d_i$  before breaking. However if one introduces a slit cut in  $T_1$  so that the width at the end of the slit is  $T_2$  when the first band is stretched it will fail before stretching to  $d_2$  - there is a bad localization of “energy” at the cut. If one makes a V shaped cut, then the band stretches further before failing, but still not as far as the band of with  $T_2$ . Yet if one makes a very smooth cut, the band will stretch as far. The moral here is that it is not the thickness of the



**Figure 3.** Stretching of highly elastic media.



**Figure 4.** Deformations of cellular structures. Bottom figures minimise the mean distortion of the boundary deformation which is then relaxed to minimise interfacial energy (thus preserving angles) giving top figures.

material but the profile which limits how far it can be deformed. This is what we will see in our models as well.

Images from our computational investigations in Figure 4 show the deformations of elastic cellular structures close to the  $L^1$ -case. Here we minimize the conformal energy of all possible extensions of the boundary values (to give the lower configurations). We then minimise the interfacial tension along the cellular walls, leading to the  $2\pi/3$  symmetry at vertices.

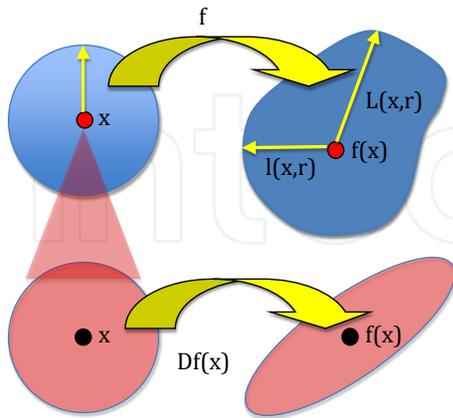
#### 4. Geometric interpretations of distortion

Our functionals measure average distortion and the distortion of deformation at a point has a couple of important geometric interpretation which we now discuss. We recall  $\mathbb{K}(x, f) = \frac{\|Df(x)\|^2}{J(x, f)}$ . If we put  $\mu(z) = f_{\bar{z}}(z)/f_z(z)$  (known as the Beltrami coefficient of the mapping  $f$ ), then we compute

$$\mathbb{K}(z, f) = \frac{|f_z|^2 + |f_{\bar{z}}|^2}{|f_z|^2 - |f_{\bar{z}}|^2} = \frac{1 + |\mu|^2}{1 - |\mu|^2}$$

### 4.1. Linear distortion

If we define



$$L(x, r) = \max_{|x-y|=r} |f(x) - f(y)|$$

$$\ell(x, r) = \min_{|x-y|=r} |f(x) - f(y)|$$

Then

$$\mathbb{K}(x, f) = \limsup_{r \rightarrow 0} \frac{1}{2} \left( \frac{L(x, r)}{\ell(x, r)} + \frac{\ell(x, r)}{L(x, r)} \right)$$

A point needs to be made here about the use of  $L/\ell + \ell/L$  instead of just  $L/\ell$  say. The point is simply that the quantities  $L$  and  $\ell$  are clearly related to the singular values of  $Df$  and while these might vary smoothly, as they cross the maximum will not necessarily vary smoothly and the differentiability of the distortion is lost. Thus techniques available through the calculus of variations (looking for Euler Lagrange equations and so forth) can no longer be used.

### 4.2. Angular distortion

The distortion function  $\mathbb{K}(z, f)$  also controls the infinitesimal distortion of angles as explained in [11]. Indeed one may define the maximum and minimum angular distortion for a linear map by consideration of the distortion of the unit circle  $\{e^{i\theta} : 0 \leq \theta < 2\pi\}$ . If the mapping is

$$z \mapsto \alpha(z + \mu\bar{z}), \quad \alpha \in \mathbb{C} \setminus \{0\}, \quad |\mu| < 1$$

then the distortions are

$$\Theta_{max} = \max_{\theta \in [0, 2\pi]} \left| \frac{d}{d\theta} (e^{i\theta} + \mu e^{-i\theta}) \right|, \quad \Theta_{min} = \min_{\theta \in [0, 2\pi]} \left| \frac{d}{d\theta} (e^{i\theta} + \mu e^{-i\theta}) \right|. \tag{6}$$

We calculate that

$$\left| \frac{d}{d\theta} (e^{i\theta} + \mu e^{-i\theta}) \right|^2 = 1 + |\mu|^2 - 2\Re(\mu e^{-2i\theta}). \tag{7}$$

Accordingly, we may apply these observations to the differential of a Sobolev mapping via the approximation  $f(z) \approx f(z_0) + f_z(z_0)(z - z_0) + f_{\bar{z}}(z_0)(\bar{z} - \bar{z}_0)$  to get

$$\Theta_{max} = 1 + |\mu|, \quad \Theta_{min} = 1 - |\mu|.$$

So the angular distortion of a mapping  $f$  is

$$\frac{\Theta_{max}(z)}{\Theta_{min}(z)} + \frac{\Theta_{min}(z)}{\Theta_{max}(z)} = 2 \frac{1 + |\mu|^2}{1 - |\mu|^2} = 2\mathbb{K}(z, f) \quad (8)$$

From both of these calculations one can immediately see that the distortion measures the anisotropic properties of the deformation. Further the connections between preserving angles of interfaces (and hence conformal mappings) can be brought out by minimising the deformations of angles at interfaces.

## 5. Connections with harmonic mappings

The first surprising observation that we want to point out is a remarkable connection with harmonic mappings that motivated some of the computational investigation above. After a change of variables  $f = g^{-1}$

$$\begin{aligned} \int_{\mathbb{A}} \mathbb{K}(z, g) \lambda &= \int_{\mathbb{A}'} \frac{\|Dg(f)\|^2}{J_g(f)} \lambda(f) J_f = \int_{\mathbb{A}'} \frac{\|Dg(f)\|^2}{J_g(f) J_f} J_f^2 \lambda(f) \\ &= \int_{\mathbb{A}'} \|(Df)^{-1} J_f\|^2 \lambda(f) = \int_{\mathbb{A}'} \|Df\|^2 \lambda(f) \end{aligned}$$

We must note that there are considerable technical difficulties in ensuring enough regularity here for these mappings and their inverses so that we can make this change of variables, this is discussed in [9].

Stationary points of the functional

$$f \mapsto \int_{\mathbb{A}'} \|Df\|^2 \lambda(f)$$

are the maps which are harmonic with respect to the metric  $ds = \lambda(z)|dz|$  and satisfy the nonlinear harmonic map equation

$$\Delta g + (\log \lambda)_{\bar{w}}(g) g_z g_{\bar{z}} = 0$$

widely studied in geometry, topology and analysis - but unfortunately non-linear in a bad way since the solution is tangled up with the coefficients (the term  $\lambda(f)$ ). We expect that the approach through minimisers of conformal energy functionals might yield a new approach to these problems.

## 6. Boundary value problems in two-dimensions

Here we give an account of the problem of minimising distortion functions with  $\mathbb{K} \in L^1$  for the boundary value problem on the unit disk in the complex plane (see [5] and [18]).

### 6.1. $L^1$ -Boundary value problem for the disk

Minimise among homeomorphisms of finite distortion  $f : \mathbb{D} \rightarrow \mathbb{D}$ , subject to the boundary values  $f|_{\partial\mathbb{D}} = f_0$ , the quantity

$$\iint_{\mathbb{D}} \mathbb{K}(z, f) dz$$

It was then shown that there exists a minimiser if and only if

$$E = \int_{\mathbb{S}} \int_{\mathbb{S}} \log |f_0(\zeta) - f_0(\eta)| d\zeta d\bar{\eta} < \infty$$

In fact the minimum value attained for the problem is  $E$ ! When it exists, the minimiser is a smooth diffeomorphism which is quasiconformal if and only if the boundary values  $f_0$  are bilipschitz. What this means is that the minimum has bounded distortion  $\mathbb{K}(z, f) \in L^\infty(\mathbb{D})$  if and only if we have, for some  $1 \leq L < \infty$ , the bilipschitz estimate on the boundary values

$$\frac{1}{L} |\eta - \zeta| \leq |f_0(\eta) - f_0(\zeta)| \leq L |\eta - \zeta| \quad (9)$$

This is surprising in that the minimiser will most often have unbounded distortion, a quite unexpected phenomenon. This occurs for instance if there is no uniform lower bound on  $|f_0'(\zeta)|$ .

Even more remarkable is that there is a moduli structure to the minimisers. If  $f$  is a solution to the minimisation problem, then the Beltrami coefficient

$$\mu = f_{\bar{z}}/f_z \quad (10)$$

itself satisfies the (nonlinear) Beltrami equation

$$\mu_z = \bar{\mu}\mu_{\bar{z}}$$

This implies a number of interesting consequences. For instance the distortion satisfies a maximum principle - the maximal distortion occurs on the boundary [4].

## 6.2. Teichmüller problems

There is another novel phenomena to observe here. Suppose we try and deform the interior of an object so as to minimise mean distortion, yet try and keep the boundary values fixed (say the identity). In the literature this is known as a “Teichmüller problem”, see [18]. Let us discuss two borderline cases. The classical  $L^\infty$  problem, where there is always a solution, and the  $L^1$  problem, where there is never a solution ( $r \neq 0$ ).

**Problem.** For  $0 \leq r < 1$ , let

$$\mathcal{M}_T^\infty(r) = \|\mathbb{K}(z, f)\|_{L^\infty(\mathbb{D})}, \quad \text{and} \quad \mathcal{M}_T^1(r) = \inf \left\{ \frac{1}{\pi} \iint_{\mathbb{D}} \mathbb{K}(z, f) \, dz \right\}, \quad (11)$$

where the infimum is taken over all mappings  $f : \mathbb{D} \rightarrow \mathbb{D}$  of finite distortion such that  $f$  has a homeomorphic extension to  $\overline{\mathbb{D}}$  and

- $f|_{\partial\mathbb{D}} \rightarrow \partial\mathbb{D}$  is the identity mapping,
- $f(0) = r$ .

### 6.2.1. The $L^\infty$ result

Let  $\mathcal{K}$  be the complete elliptic integral of the first kind,

$$\mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}.$$

Set

$$\mu(r) = \frac{\pi}{2} \frac{\mathcal{K}(\sqrt{1-r^2})}{\mathcal{K}(r)} \quad (12)$$

We note the estimate

$$\mu(x) = \log \left( \frac{4}{x} - x - \delta(x) \right) \quad (13)$$

where  $x^3/4 < \delta(x) < 2x^3$  given in [17, pp 62].

Teichmüller’s theorem states

**Theorem 1.**

$$\mathcal{M}_T^\infty(r) = \frac{1}{2} \left( \coth^2 \left( \frac{\mu(r)}{2} \right) + \tanh^2 \left( \frac{\mu(r)}{2} \right) \right) \quad (14)$$

and there is a mapping realising this minimum. If we let  $d = \rho_{\mathbb{D}}(0, r) = \log \frac{1+r}{1-r}$  (the hyperbolic distance between 0 and  $r$ ) we have the asymptotics

- $\mathcal{M}_T^\infty(r) \approx \frac{8}{\pi^4} d^2$ , as  $d \rightarrow \infty$ .
- $\mathcal{M}_T^\infty(r) \approx 1 + \frac{d^2}{4}$ , as  $d \rightarrow 0$ .

6.2.2. The  $L^1$  result

**Theorem 2.** The minimal mean distortion function  $\mathcal{M}_T^1(r)$  has the following asymptotics

- As  $r \rightarrow 1$

$$\frac{2}{\pi^2} \log \frac{1+r}{1-r} + \frac{17 \log 2}{4\pi^2} \leq \mathcal{M}_T(r) \leq \frac{2}{\pi^2} \log \frac{1+r}{1-r} + \frac{4}{3} - \frac{7+8 \log 2}{2\pi^2}$$

up to an  $O(1-r)$  term.

- As  $r \rightarrow 0$

$$\mathcal{M}_T(r) \leq 1 + \left( \frac{20 - 8 \log(2)}{\pi^2} - \frac{7}{6} \right) r^2 + O(r^4)$$

The minimum value  $\mathcal{M}_T^1(r)$  is never attained for a homeomorphism of finite distortion.

In terms of the hyperbolic distance  $d = \rho_{\mathbb{D}}(0, r)$  we have

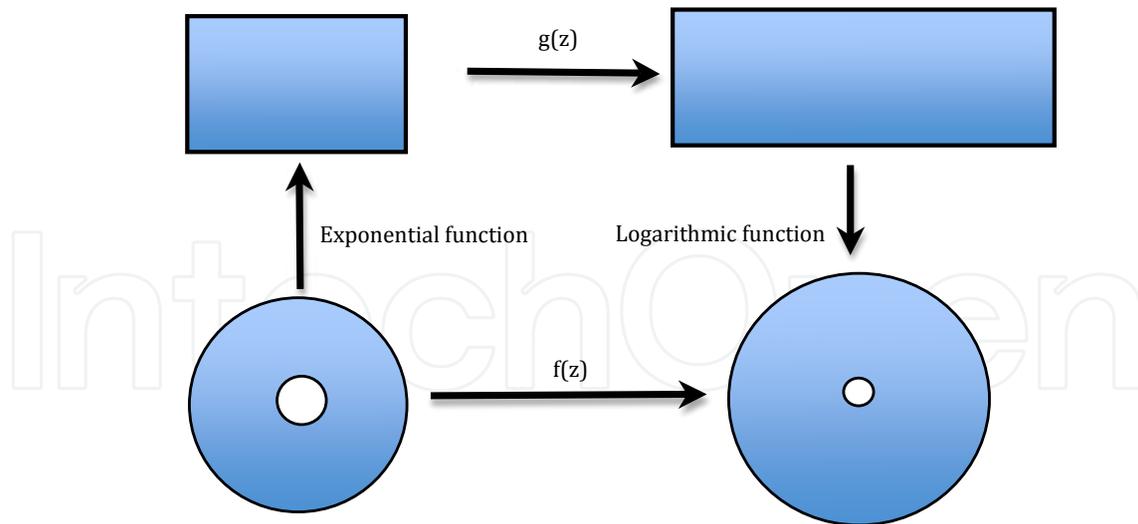
- $\mathcal{M}_T^1(r) \approx \frac{2}{\pi^2} d$ , as  $d \rightarrow \infty$ .
- $\mathcal{M}_T^1(r) \leq 1 + \left( \frac{5-2 \log(2)}{\pi^2} - \frac{7}{24} \right) d^2$ , as  $d \rightarrow 0$ .

Notice the constant here  $\frac{5-2 \log(2)}{\pi^2} - \frac{7}{24} = 0.07447 < \frac{1}{4}$  - the constant for the maximal distortion.

## 7. Nitsche phenomena

We now want to discuss the results of our paper [19]. A harmonic mapping defined on a domain  $\Omega \subset \mathbb{C}$  is a mapping  $h = h_1 + ih_2$  and each  $h_i : \Omega \rightarrow \mathbb{R}$  is harmonic with respect to the Euclidean metric,  $\Delta h_i = 0$ . Thus conformal mappings are certainly harmonic.

In 1962 Nitsche was able to show, in connection with problems concerning minimal surfaces, that there was no harmonic homeomorphism  $f : \mathbb{A}_1 \rightarrow \mathbb{A}_2$  between annuli  $\mathbb{A}_1$  and  $\mathbb{A}_2$  if  $\mathbb{A}_1$  is too thick relative to  $\mathbb{A}_2$ . That is  $\text{Mod}(\mathbb{A}_1) \gg \text{Mod}(\mathbb{A}_2)$ .



**Figure 5.** The equivalence between the Grötzsch problem and the Nitsche type problem is effected by conformal change of coordinates

Here the modulus of an annulus  $\mathbb{A} = \{z : r < |z| < R\}$  is defined to be  $Mod(A) = \log \frac{R}{r}$ .

Nitsche’s argument was basically by compactness (using a Harnack inequality), but he did conjecture the sharp relationship between moduli necessary for existence.

We call this interesting phenomena, the nonexistence of expected minimisers outside a range of moduli, the **Nitsche phenomenon** and we now explore this in considerably more generality and seek applications in material science.

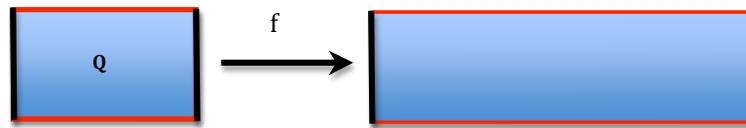
### 8. Distortion functionals & Grötzsch problem

In two dimensions the problems of stretching the catenoid or a strip of material have a common formulation as the exp and log functions are conformal.

We can therefore make the following calculation:

$$\begin{aligned} \mathbb{K}(z, g) &= \mathbb{K}\left(z, \frac{1}{2\pi} \log (f(e^{2\pi z}))\right) = \mathbb{K}(z, f(e^{2\pi z})), \\ \iint_{\mathbb{Q}} \Phi(\mathbb{K}(z, g)) \lambda(z) |dz|^2 &= \iint_{\mathbb{Q}} \Phi(\mathbb{K}(z, f(e^{2\pi z})) \lambda(z) |dz|^2 \\ &= 4\pi^2 \iint_{\mathbb{A}} \Phi(\mathbb{K}(w, f)) \lambda(z) e^{-4\pi \Re(z)} |dw|^2. \end{aligned}$$

Thus, basically all that has changed is the weight function (in a well defined way). However, this equivalence is only precise should the minimiser have enough symmetry that it can be lifted from the problem of mapping rectangles to that of annuli. This leads us to the Grötzsch Problem.



**Figure 6.** The Grötzsch problem: we seek a minimiser of the conformal energy at (15) and which is a homeomorphism and which respects the boundary as indicated

### 8.1. The Grötzsch problem

Our conformally invariant functional is

$$\mathcal{E}(f) = \iint_Q \Phi(\mathbb{K}(z, f)) \eta(z) dz \tag{15}$$

and our problem is to minimize this functional subject to \$f\$ being a (sufficiently regular) map \$Q \to Q'\$ preserving the edges.

In order to study this problem we need some basic facts:

- Convexity:

$$(X, Y, J) \rightarrow \frac{|X|^2 + |Y|^2}{J}$$

is convex on \$\mathbb{C} \times \mathbb{C} \times \mathbb{R}\$; the graph of the function lies above its tangent plane.

- Natural extrema : The mapping between rectangles of the form

$$f_0(z) = u(x) + iy,$$

is a natural candidate for an extremum if the weight depends only on \$x\$.

- For such a map we have \$(f\_0)\_x = u\_x\$ and \$(f\_0)\_y = i\$. Therefore, if we set \$\omega(x) = 1/u\_x(x)\$, a real valued function, we have the identity

$$|\omega(x)f_x + if_y|^2 \geq 0 \tag{16}$$

with equality holding only for \$f\_0\$.

We now expand (16) to see

$$\begin{aligned} 0 &\leq |\omega(x)f_x + if_y|^2 = (\omega(x)f_x + if_y)(\omega(x)\overline{f_x} - i\overline{f_y}) \\ &= \omega^2(x)|f_x|^2 + |f_y|^2 - 2\Im(\omega(x)f_y\overline{f_x}) \\ \omega^2(x)|f_x|^2 + |f_y|^2 &\geq 2\omega(x)\Im(f_y\overline{f_x}) = 2\omega(x)J(z, f). \end{aligned}$$

So

$$\omega^2(x)|f_x|^2 + |f_y|^2 \geq 2\omega(x)J(z, f)$$

with equality if and only if  $f = f_0$ . This gives us estimates on the distortion function (writing  $J = J(z, f)$ ),

$$\mathbb{K}(z, f) \geq (1 - \omega^{-2}(x)) \frac{|f_y|^2}{J} + 2\omega^{-1}(x)$$

Thus, for a general mapping  $f$ :

$$\mathbb{K}(z, f) - \mathbb{K}(z, f_0) \geq (1 - \omega^{-2}(x)) \left[ \frac{|f_y|^2}{J} - \frac{|(f_0)_y|^2}{J_0} \right]$$

We now want to apply the convexity inequality. If  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  is convex, then its graph lies above any tangent line:

$$\Phi(\mathbb{K}) - \Phi(\mathbb{K}_0) \geq \Phi'(\mathbb{K}_0)(\mathbb{K} - \mathbb{K}_0).$$

Hence

$$\begin{aligned} & \Phi(\mathbb{K}(z, f)) - \Phi(\mathbb{K}(z, f_0)) \\ & \geq (1 - \omega^{-2}(x)) \Phi'(\mathbb{K}_0) \times \left[ 2 \Re e \left( \frac{(f_0)_y}{J_0} (f_y - (f_0)_y) \right) - \frac{|(f_0)_y|^2}{J_0^2} (J - J_0) \right] \end{aligned}$$

Now  $(f_0)_y = i$  and  $(f_0)_x = 1/\omega(x) = J_0$  so

$$\begin{aligned} & \Phi(\mathbb{K}(z, f)) - \Phi(\mathbb{K}(z, f_0)) \\ & \geq (1 - \omega^2(x)) \Phi'(\mathbb{K}_0) [2 \Re e(f_x - (f_0)_x) - (J - J_0)] \end{aligned}$$

Now we multiply by  $\lambda(x)$  and integrate. Choose  $f_0$  so that

$$\lambda(x)(1 - \omega^2(x)) \Phi'(\mathbb{K}_0) = \alpha \neq 0 \tag{17}$$

for a real constant  $\alpha$ .

This equation is

$$\lambda(x) \left(1 - \frac{1}{u_x^2}\right) \Phi' \left(u_x + \frac{1}{u_x}\right) = \alpha$$

Finally, consideration of the boundary values gives

$$\iint_{Q_1} \Re(f_x - (f_0)_x) |dz|^2 = \int_0^1 \left[ \int_0^\ell \Re(f_x - (f_0)_x) dx \right] dy = 0.$$

and for an arbitrary Sobolev mapping

$$\iint_Q J |dz|^2 \leq |Q'| = \iint_Q J_0 |dz|^2$$

Putting the discussion above together gives us the following theorem.

**Theorem 3.** Let  $\lambda(x) > 0$  be a positive weight,  $\Phi$  convex and  $u : [0, \ell] \rightarrow [0, L]$ ,  $u(0) = 0$ ,  $u(\ell) = L$  a solution to the ordinary differential equation

$$\lambda(x) \left(1 - \frac{1}{u_x^2(x)}\right) \Phi' \left(u_x(x) + \frac{1}{u_x(x)}\right) = \alpha \in \mathbb{R} \tag{18}$$

Set  $f_0(z) = u(x) + iy$ . If  $f$  is a mapping of finite distortion mapping  $Q$  to  $Q'$  and satisfying our edge condition, then

$$\iint_Q \Phi(\mathbb{K}(z, f)) \lambda(x) |dz|^2 \geq \iint_Q \Phi(\mathbb{K}(z, f_0)) \lambda(x) |dz|^2.$$

Equality holds if and only if  $f = f_0$ .

Actually following the construction above, outside the range providing existence by the theorem, one can identify an obvious degenerate minimiser  $f_0$  for which strict inequality still holds. Once we can find a minimising sequence converging to this one can prove more. Namely

**Theorem 4.** If (18) can not be solved for  $f_0$  (that is there is no good choice of  $\alpha$ ), then there cannot be a minimiser in the class of homeomorphisms of finite distortion.

### 9. Applications

These theorems motivate us to study the ordinary differential equation

$$\lambda(x) \left(1 - \frac{1}{u_x^2(x)}\right) \Phi' \left(u_x(x) + \frac{1}{u_x(x)}\right) = \alpha$$

Actually the transformation from the Nitsche type problem to the Grötzsch problem yields a significantly simpler equation to study – in fact it’s not really an ODE at all since the only term here is  $u_x$ . This is one of the reasons why we transformed to the Grötzsch problem.

### 9.1. Sharp bounds for Nitsche conjecture

In the situation of Nitsche’s original conjecture we have  $\lambda(x) = e^{4\pi x}$  as  $\eta(w) = 1$  and  $\Phi(t) = t$ .

$$1 - \frac{1}{u_x^2(x)} = \alpha e^{-4\pi x}, \quad u_x(x) = \frac{1}{\sqrt{1 - \alpha e^{-4\pi x}}}$$

$$u(x) = \int \frac{e^{2\pi x} dx}{\sqrt{e^{4\pi x} - \alpha}} = \frac{1}{2\pi} \int \frac{dt}{\sqrt{t^2 - \alpha}}, \quad t = e^{2\pi x}.$$

So

$$u(x) = \frac{1}{2\pi} \log \left( \frac{e^{2\pi x} + \sqrt{e^{4\pi x} - \alpha}}{1 + \sqrt{1 - \alpha}} \right), \quad \alpha \neq 0$$

We must choose  $\alpha$  to solve  $u(\ell) = L$

$$L = \frac{1}{2\pi} \log \left( \frac{e^{2\pi \ell} + \sqrt{e^{4\pi \ell} - \alpha}}{1 + \sqrt{1 - \alpha}} \right) \tag{19}$$

As  $\alpha \rightarrow -\infty$  we can make the RHS of (19) arbitrarily small. Thus there is always a minimiser if  $L \leq \ell$ . Next, if  $\alpha > 0$ .

$$u_x(x) = \frac{1}{\sqrt{1 - \alpha e^{-4\pi x}}}$$

requires  $\alpha < 1$  so that

$$L < \frac{1}{2\pi} \log \left( e^{2\pi \ell} + \sqrt{e^{4\pi \ell} - 1} \right)$$

and when unwound, this is precisely Nitsche’s conjecture.

Most recently Iwaniec, Kovalev and Onninen have completed Nitsche’s problem by showing that within this range there are not even local minima - that is, there are no harmonic mappings whatsoever [12].

## 9.2. More general weights $\lambda(x)$

For more general weights  $\lambda(x)$ , we put  $\lambda_0 = \min_{x \in [0, \ell]} \lambda(x)$ . The solution is dominated by the one with the choice  $\alpha = \lambda_0$  and the issue is to decide whether

$$\int_0^\ell \frac{1}{\sqrt{\lambda(x) - \lambda_0}} dx < \infty.$$

If this integral is finite, then we will observe Nitsche type phenomena; non-existence of minima outside a range of moduli.

## 9.3. Observing Nitsche phenomena

Without going into excessively fine details, convergence of

$$\int \frac{1}{\sqrt{\lambda(x) - \alpha}}$$

will require that

$$\lambda(x) \approx \lambda_0 + x^{2s}, \quad s < 1$$

near the minimum  $\lambda_0$  of  $\lambda(x)$ . A few simple calculations will reveal

- **Existence I.** if  $\lambda$  is a smooth positive weight and  $\lambda'(x) = 0$  at its minimum (which may well occur at the endpoints), then we can always solve the deformation problem.
- **Existence II.** if  $\lambda(x)$  is constant,  $u_x(x)$  is constant and therefore  $f$  will be a linear mapping.
- **Non Existence I.** if  $\lambda'(x) \neq 0$  at a minimum (ie minimum at the boundary), then the Nitsche phenomenon occurs.
- **Non Existence II.** if  $\lambda'(x) = 0$  at a minimum, but  $\lambda$  is not twice differentiable, then we see the Nitsche phenomenon, (actually  $C^{1,1}$  is the cutoff).

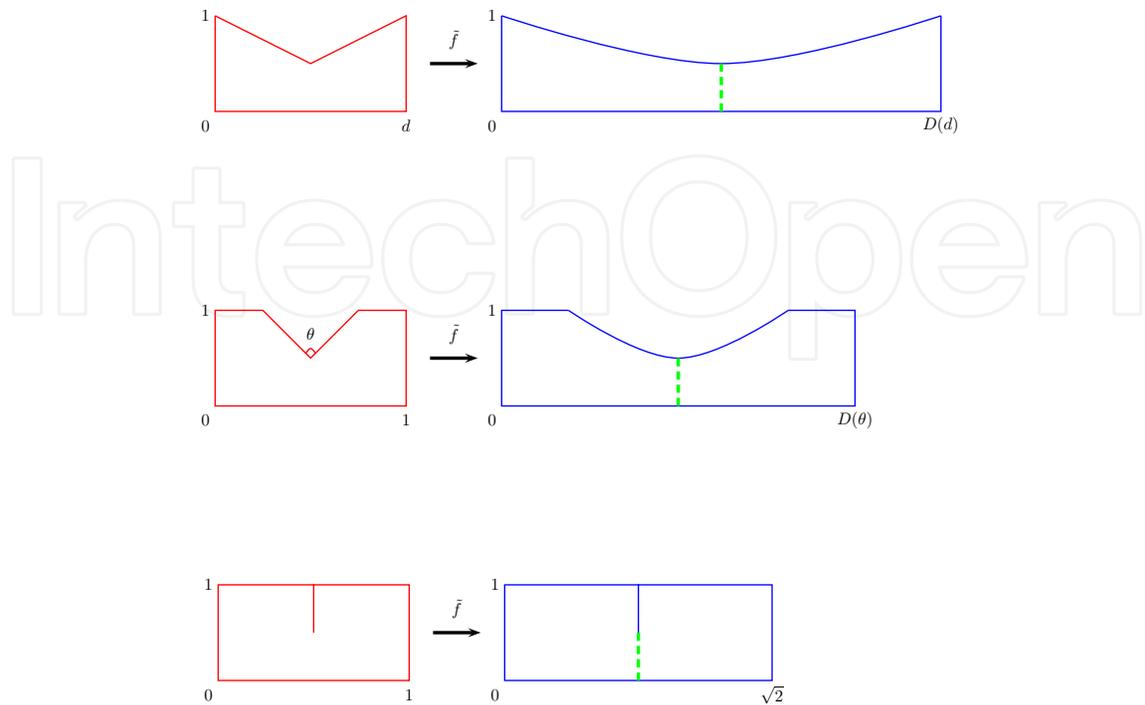
Suppose the weight function  $\lambda(x)$  is viewed as a thickness profile of the material. Then “cuts” gives a little more insight to Nitsche-type phenomenon. Find  $\alpha$  to determine how far the final stretch can be;

This illustrates the sort of thing we were talking about before when we were discussing the stretching of rubber bands. Some precise calculations are indicated in Figure 7.

Next, should the convex function  $\Phi$  have *unbounded derivative*, then there is *always a minimiser*.

$$\left(1 - \frac{1}{u_x^2(x)}\right) \Phi' \left(u_x(x) + \frac{1}{u_x(x)}\right) = \frac{\alpha}{\lambda(x)}$$

This is simply an application of the maximum principle. If  $\alpha \searrow 0$ , then  $u_x \searrow 1$  and  $\alpha \nearrow \infty$  gives  $u_x \nearrow \infty$ , so the intermediate value theorem provides us with a solution.



**Figure 7.** Calculations indicating the maximal stretching deformations of an elastic three dimensional material with given profile. Notice smoothness of the profile at the critical stretching. Beyond this maximal value no homeomorphic solution of finite conformal energy exists

In particular we do not see the Nitsche phenomenon for the  $L^p$ -norms of mean distortion: we can always minimise

$$\iint_Q \mathbb{K}(z, f)^p dz, \quad p > 1$$

While for  $p < 1$  there is never a minimiser (unless it is the identity).

## 10. Conclusions

Here we have suggested new models for the study of deformations of elastic media through the minimisation of distortion functionals which seem to provide a “holistic” approach to this problem and a potential mathematical underpinning of a number of phenomena which are actually observed.

We have developed the theory from a mathematical perspective using conformally invariant functionals measuring the local anisotropic deformation of the material. We expect that there are good physical reasons for the efficacy and validation of these models but we have not considered that in this article. However close reading of related models [6, 8, 21] suggest that these are relatively good models and the relationship between these minimisation problems and conformal mappings evidenced in soap film models and elsewhere is borne out in mathematical calculations in explained in [11]. Basically minimising conformal invariant functionals is “equivalent” to minimising the angular distortion at interfaces when applied to

cellular structures (so length scales are implicit). These interfacial angles are often prescribed by the physics involved and therefore themselves are invariant - hence the effectiveness of conformal mappings in free boundary problems as they preserve angles.

The use of conformally invariant functionals in materials science offers the opportunity to address multiscale problems in an integrative manner - particularly when the effects they model dominate. Our models provide a possible mathematical explanation (and perhaps a predictive model when fully understood) for phenomena such as tearing and an analysis of how the process occurs. More formally, we discussed the general Nitsche phenomenon - deformations either exist within certain ranges or fail to exist outside these ranges - a material can only be stretched so far.

Our results primarily pertain to two dimensional elasticity, but in identifying the function  $\lambda(z)$  as a profile and not a weight we obtain three dimensions in certain circumstances. There remains the obvious problem of developing this research into higher dimensions. Some related work in higher dimensions can be found in [13, 14] but not for the sorts of conformally invariant functionals we propose to consider. Presently we have only partial results and these will be communicated elsewhere.

## Author details

G. J. Martin<sup>1,\*</sup>, J. Hussan<sup>2</sup> and M. McKubre-Jordens<sup>3</sup>

\* Address all correspondence to: g.j.martin@massey.ac.nz

1 Massey University, Auckland, New Zealand

2 The University of Auckland, Auckland, New Zealand

3 Canterbury University, Christchurch, New Zealand

## References

- [1] K. Astala, T. Iwaniec and G.J. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, Princeton Mathematical Series, 48. Princeton University Press, Princeton, NJ, 2009
- [2] K. Astala, T. Iwaniec and G.J. Martin, *Deformations of annuli with smallest mean distortion*, Arch. Rational Mech. Anal., 195, (2010), 899–921.
- [3] K. Astala, T. Iwaniec, P. Koskela and G.J. Martin, *Mappings of BMO-bounded distortion II*, Math. Annalen, 317, (2000), 703–726.
- [4] K. Astala, T. Iwaniec and G.J. Martin, *Bi-Lipschitz homeomorphisms of the circle and non-linear Beltrami equations*, On the notions of solution to nonlinear elliptic problems: results and developments, 105D115, Quad. Mat., 23, Dept. Math., Seconda Univ. Napoli, Caserta, 2008
- [5] K. Astala, T. Iwaniec, G.J. Martin and J. Onninen, *Extremal mappings of finite distortion*, Proc. London Math. Soc., 91, (2005), 655D702.

- [6] W. Drenckhan, D. Weaire, and S.J. Cox, *The demonstration of conformal maps with two-dimensional foams*, Eur. J. Phys., 2004, 25, 429-438
- [7] P. L. Duren, *Harmonic mappings in a plane*, Cambridge University Press, 2004.
- [8] G. C. Engelmayr Jr., M. Cheng, C. J. Bettinger, J. T. Borenstein, R. Langer, and L. E. Freed, *Accordion-like honeycombs for tissue engineering of cardiac anisotropy*, Nature Mat. 7 (2008), no. 12, 1003–1010.
- [9] S. Hencl and P. Koskela, *Regularity of the inverse of a planar Sobolev homeomorphism*, Arch. Ration. Mech. Anal., 180, (2006), 75D95.
- [10] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1993, Oxford Science Publications.
- [11] J. Hussan and G.J. Martin, *Non-linear models for the deformation of cellular structures*, to appear.
- [12] T. Iwaniec , L. Kovalev and J. Onninen, *The Nitsche conjecture*, J. Amer. Math. Soc., 24, (2011), 345–373.
- [13] T. Iwaniec and J. Onninen, *Hyperelastic deformations of smallest total energy*, Arch. Ration. Mech. Anal., 194, (2009), 927–986.
- [14] T. Iwaniec and J. Onninen, *Deformations of finite conformal energy: existence and removability of singularities* Proc. Lond. Math. Soc., 100, (2010), 1–23.
- [15] T. Iwaniec and G.J. Martin, *Geometric Function Theory and Nonlinear Analysis*, Oxford University Press, 2001.
- [16] T. Iwaniec, P. Koskela and G.J. Martin, *Mappings of BMO-bounded distortion and Beltrami type operators*, Dedicated to the memory of Tom Wolff., J. Anal. Math. 88, (2002), 337–381.
- [17] O. Lehto and K. Virtanen, *Quasiconformal mappings in the plane*, Springer-Verlag, 1973.
- [18] G. Martin, *The Teichmüller problem for mean distortion*, Ann. Acad. Sci. Fenn. Math., 34, (2009), 233–247.
- [19] G. Martin and M. McKubre-Jordens, *Deformations with smallest weighted  $L^p$  average distortion and Nitsche-type phenomena*, J. Lond. Math. Soc., 85, (2012), 282–300
- [20] R Schoen and S-T. Yau, *Lectures on harmonic maps*, Conference Proceedings and Lecture Notes in Geometry and Topology, II, International Press, Cambridge, MA, 1997.
- [21] D. Weaire and N. Rivier, *Soap, cells and statistics - random patterns in two dimensions*, Contemp. Phys. 50 (2009), no. 1, 199–239.

