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Chapter 3

Linear Wave Motions in Continua with Nano-Pores

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Additional information is available at the end of the chapter

http://dx.doi.org/10.5772/50768

1. Introduction

The first attempts to describe the behaviour of porous materials with the use of an additional degree of kinematical freedom, in order to refine the Cauchy's theory, are due principally to Nunziato and Cowin [1, 2] and co-workers. Nevertheless, their voids theory can be considered as a particular case of a general theory of continua with microstructure [3] and so, when we have to consider more complex media with nano-pores, we need to use suggestions of this last theory [4]. In fact, a nano-pore in a thermoelastic solid is roughly ellipsoidal, unlike small lacunae finely dispersed in the solid matrix that can be supposed all spherical and for which the volume fraction suffices to describe the microdeformation (see, also, [5, 6]). Cowin itself remarked the importance of the shape of the holes in the description of lacunae containing osteocytes or of bone canaliculi [7, 8]: in the human bone, *e.g.*, the lacunae are almost ellipsoidal with mean values along the axes of about 4 μm, 9 μm and 22 μm. And, as a matter of fact, the voids theory does not predict size effects in torsion of bars in an isotropic material, while they occur both in torsion and in bending, as observed for bones and polymer foam materials in [9]. Even if some problem of physical concreteness or of mathematical hardness could arise [10, 11], a better improvement of the voids theory, within a microstructured scheme, is necessary in order to characterize the more complex structure.

A direct way to proceed is to consider the thermoelastic solid with nano-pores as a continuum with an ellipsoidal microstructure (see [4, 12]) which describes media whose each material element contains a large cavity, that does not diffuse through the skeleton, filled by an elastic inclusion, or an inviscid fluid, both of negligible mass (*e.g.*, composite materials reinforced with chopped elastic fibers, porous media with elastic granular inclusions, real ceramics, *etc.*): this cavity is able to have a microstretch different from, and independent of, the local affine deformation deriving from the macromotion and so can allow distinct microstrains along the principal axes of microdeformation, in absence of microrotations . The "tortuosity" matrix, a macroscopic geometrical symmetric tensor that expresses the effects of the geometry of the microscopic pores' surface, was previously presented in [13], but in [14] the model of a microstretched medium has been firstly used to study materials with distributions of aligned ellipsoidal vacuous pores and explicit computations have been carried

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out for matrix materials subjected to axisymmetric and plane strain loading conditions; then the model has been used to formulate more general multiphase theories of microstructured media [15]. Furthermore, some tension tests and numerical results have been presented in [16] for a similar model of a microstretched medium. The quoted model [4] is surely complementary to the use of the Cosserat theory [9], when microrotations are of interest in the analysis, but no the microdeformations: merely, we wish to observe that our theory contains naturally the voids theory by constraining the microstrain to be spherical. In particular, in [12] it has been observed that, during quasi-static homogeneous motion, the porous solid behaves like an isotropic simple material with fading memory in the linear range and it reduces to a viscoelastic medium when the microstructural variable remains spherical as in [2]. More generally, for complete microdeformations, the framework of media with affine structure better depicts macro- and micro-motion (see [3, 17]). Finally, our model [4] was used to analyze nonlinear wave propagation in constrained porous media [18] and to examine adsorption and diffusion of polluttants in soils, viewed as an immiscible mixture of materials with, and without, microstructure [19, 20].

In this chapter, we extend the linear theory [12] of elastic solids with nano-pores to the thermoelastic case and include a rate effect in the holes' response, which results in internal dissipation from experimental evidence [21]; after we make a complete study of the propagation of linear waves. In particular, in §2 we apply the general theory of continua with microstructure to the ellipsoidal case and furnish balance equations and jump conditions; in §3 we present constitutive equations for kinetic energy and co-energy density and for dependent constitutive fields and, after, we use thermodinamic restrictions; in §4 we define small thermoelastic deformations from the reference placement and obtain the linear field balance equations; in §5 we study linear micro-vibrations for which we obtain three admissible modes; in §6 we analyze the propagation of harmonic plane waves and comment the secular equations governing the eight solutions: two shear optical micro-elastic modes, two coupled transverse elastic waves and four coupled longitudinal thermo-elastic waves; in §7 we get the propagation conditions of the macro-acceleration waves for either a heat-conducting or non-conducting isotropic thermoelastic material with nano-pores (corresponding, respectively, to *homothermal* and *homentropic* waves), as well as for *generalized transverse* waves; in §8 we gain the growth equations which govern the propagation of the macro-acceleration waves and discuss the couplings between the higher order discontinuities.

2. Balance laws and jump conditions

We identify the continuous material body with nano-pores $\mathcal{B}$ with a fixed homogeneous and free of residual stresses region of the three dimensional Euclidean space $\mathcal{G}$, called the "natural" reference placement $\mathcal{B}_*$ (see, *e.g.*, §83 of [22]). We suppose that each material element of the continuum contains a nano-pore which is capable to have a microstretch different from, and independent of, the local affine deformation ensuing from the macromotion. Therefore, if we denote the generic material element of $\mathcal{B}_*$ by $\mathbf{x}_*$, the thermomechanical behaviour of $\mathcal{B}$ is described by three smooth mappings on $\mathcal{B}_* \times \Re$ ($\Re$ is the set of real numbers): the spatial position $\mathbf{x} \in \mathcal{G}$, at time τ, of the material point which occupied the position $\mathbf{x}_*$ in the reference placement $\mathcal{B}_*$, the left Cauchy–Green tensor of the micro-deformation $\mathbf{U} \in \mathrm{Sym}^+$, at time τ, of the associated nano-pore (Sym^+ being the collection of second-order symmetric and positive definite tensor fields) and the absolute positive temperature $\theta > 0$.

The spatial position $\mathbf{x}(\mathbf{x}_*, \tau)$ is a one-to-one correspondence, for each τ, between the reference placement $\mathcal{B}_*$ and the current placement $\mathcal{B}_\tau = \mathbf{x}(\mathcal{B}_*, \tau)$ of the body $\mathcal{B}$ and, so, the deformation gradient $\mathbf{F} := \nabla\mathbf{x}(\mathbf{x}_*, \tau) \ \ (= \partial\mathbf{x}/\partial\mathbf{x}_*)$ is a second order tensor with positive determinant. Through the inverse mapping $\mathbf{x}_*(\mathbf{x}, \tau)$ of $\mathbf{x}$, we can consider all the relevant fields in the theory as defined over the current placement $\mathcal{B}_\tau = \mathbf{x}(\mathcal{B}_*, \tau)$ as well as over the reference placement $\mathcal{B}_*$ of the body $\mathcal{B}$.

Hence, a body with nano-pores is like a medium with ellipsoidal microstructure [4] and a rotation $\mathbf{Q} = \mathrm{e}^{-\mathcal{E}\mathbf{s}}$ of the observer of characteristic vector $\mathbf{s}$, where $\mathcal{E}$ is Ricci's permutation tensor and e the basis of natural logarithms, causes the symmetric tensor $\mathbf{U}$ to change into $\mathbf{U}_s = \mathbf{Q}\mathbf{U}\mathbf{Q}^T$; moreover, the infinitesimal generator $\mathcal{A}$ of the group of rotations on the microstructure in Sym^+, *i.e.*, the operator describing the effect of a rotation of the observer on the value $\mathbf{U}_s$ of the microstructure to the first order in $\mathbf{s}$ (see §3 of [3]), is given by

$$\mathcal{A}(\mathbf{U}) := \left.\frac{d\mathbf{U}_s}{d\mathbf{s}}\right|_{\mathbf{s}=\mathbf{0}}, \tag{1}$$

that is, in components:

$$\mathcal{A}_{ijk} = \mathrm{U}_{il}\mathcal{E}_{ljk} + \mathcal{E}_{ikl}\mathrm{U}_{lj}. \tag{2}$$

$\mathcal{A}$ is a third-order tensor, symmetric and positive definite in the first two indices, that is $\mathcal{A}\mathbf{c} \in \mathrm{Sym}^+$ for all vectors $\mathbf{c}$.

The expression of the kinetic energy density per unit mass of microstructured bodies is the sum of two terms, the classical one $\frac{1}{2}\dot{\mathbf{x}}^2$ due to the translational inertia and the microstructured one $\kappa(\mathbf{U}, \dot{\mathbf{U}})$ due to the inertia related to the admissible expansional micromotions of the pores' boundaries (the superposed dot denotes material time derivative). This additional term is a non-negative scalar function, homogeneous in $\mathbf{U}$, such that $\kappa(\mathbf{U}, \mathbf{0}) = 0$ and $\frac{\partial^2\kappa}{\partial\dot{\mathbf{U}}^2} \neq \mathbf{0}$, and it is related to the kinetic co-energy density $\chi(\mathbf{U}, \dot{\mathbf{U}})$ by the Legendre transform

$$\frac{\partial\chi}{\partial\dot{\mathbf{U}}} \cdot \dot{\mathbf{U}} - \chi = \kappa \tag{3}$$

(see, also, [23]). The kinetic co-energy χ, as κ, must have the same value for all observers at rest, *i.e.*, it must be invariant under the Galilean group and hence satisfy the condition

$$\dot{\mathcal{A}}^* \frac{\partial\chi}{\partial\dot{\mathbf{U}}} = -\mathcal{A}^* \frac{\partial\chi}{\partial\mathbf{U}}, \tag{4}$$

where the third-order tensor $\mathcal{A}^*$ is defined through the relation $(\mathcal{A}^*\mathbf{C}) \cdot \mathbf{c} := \mathbf{C} \cdot (\mathcal{A}\mathbf{c})$, for all second-order tensors $\mathbf{C}$ and all vectors $\mathbf{c}$. The use of Eq. (2) into Eq. (4) and the multiplication of both sides by the Ricci's tensor $\mathcal{E}$ gives the following kinematic compatibility relation

$$\mathrm{skw}\left[\dot{\mathbf{U}}\frac{\partial\chi}{\partial\dot{\mathbf{U}}} + \mathbf{U}\frac{\partial\chi}{\partial\mathbf{U}}\right] = \mathbf{0}, \tag{5}$$

where 'skw' denotes the skew part of a second–order tensor: $\mathrm{skw}\,(\cdot) := \frac{1}{2}\left[(\cdot) - (\cdot)^T\right]$ $\Big($the symmetric one being $\mathrm{sym}\,(\cdot) := \frac{1}{2}\left[(\cdot) + (\cdot)^T\right]\Big)$.

All the admissible thermo-kinetic processes for porous solids with large irregular voids are governed by the following general system of balance equations proposed in [4]; they are the mass conservation, the Cauchy equation, the micromomentum and moment of momentum balances, the Neumann energy equation and the entropy inequality in the Lagrangian description, respectively:

$$\rho_* = \rho \, \det \mathbf{F}, \tag{6}$$

$$\rho_* \ddot{\mathbf{x}} = \rho_* \mathbf{f} + \mathrm{Div}\, \mathbf{P}, \tag{7}$$

$$\rho_* \left[\frac{d}{d\tau} \left(\frac{\partial \chi}{\partial \dot{\mathbf{U}}} \right) - \frac{\partial \chi}{\partial \mathbf{U}} \right] = \rho_* \mathbf{B} - \mathbf{Y} + \mathrm{Div}\, \mathbf{\Lambda}, \tag{8}$$

$$\mathcal{E}\left(\mathbf{P}\mathbf{F}^T\right) = \mathcal{A}^* \mathbf{Y} + \left(\nabla \mathcal{A}^*\right) \mathbf{\Lambda}, \tag{9}$$

$$\rho_* \, \dot{\varepsilon} = \mathbf{P} \cdot \dot{\mathbf{F}} + \mathbf{Y} \cdot \dot{\mathbf{U}} + \mathbf{\Lambda} \cdot \nabla \dot{\mathbf{U}} + \rho_* \lambda - \mathrm{Div}\, \mathbf{h}, \tag{10}$$

$$\rho_* \theta \dot{\eta} \geq \rho_* \lambda - \mathrm{Div}\, \mathbf{h} + \theta^{-1} \mathbf{h} \cdot \nabla \theta, \tag{11}$$

where ρ is the mass density and ρ_* its referential value; Div means the trace of the nabla: $\mathrm{Div}\,(\cdot) := \mathrm{tr}\,(\nabla\,(\cdot))$; $\mathbf{f}$ is the vector body force, $\mathbf{P}$ the Piola-Kirchhoff stress tensor, ε the specific internal energy density per unit mass, λ the rate of heat generation due to irradiation or heating supply, $\mathbf{h}$ the referential heating flux, η the density of entropy.

Moreover, on the left hand side of the balance of micro-momentum (8) the Lagrangian derivative of the kinetic co-energy χ appears, while, on the right hand side, $\rho_* \mathbf{B}$ and $-\mathbf{Y}$ are the resultant second-order symmetric tensor densities of external and internal microactions, respectively: the first one is interpreted as a controlled pore pressure and the other one includes interactive forces between the gross and fine structures as well as internal dissipative contributions due to the stir of the pores' surface. Finally, $\mathbf{\Lambda}$ is the referential microstress third-order tensor, symmetric in the first two indices, which is related to the capability of recognizing boundary microtractions, even if, in some cases, it expresses weakly non-local internal effects due to the impossibility of defining a physically significant connection on the manifold of the microstructural kinetic parameter $\mathbf{U}$ (see [24]).

The balance of moment of momentum (9) assumes a more significant expression when we use the representation (2) for $\mathcal{A}$; in fact we have

$$\mathrm{skw}\left(\mathbf{P}\mathbf{F}^T\right) = 2\,\mathrm{skw}\left[\mathbf{U}\mathbf{Y} + \nabla \mathbf{U} \odot \mathbf{\Lambda}\right], \tag{12}$$

where the tensor product $\odot$ between third-order tensors is so defined:

$$\left(\nabla \mathbf{U} \odot \Lambda\right)_{ij} := \mathrm{U}_{ih,L} \Lambda_{jhL}. \tag{13}$$

Remark. We observe that the voids theories [1, 5] are immediately recovered when the microstretch $\mathbf{U}$ is constrained to be spherical (see, also, §5 of [4]).

In addition to balances (6-8) and (10-12), we need the balance equations at a surface of discontinuity, namely a propagating wave Σ. As it is customary, we assume that the smooth movable surface Σ, that traverses the body $\mathcal{B}$, is oriented and we denote by $\mathbf{n}$ the unit normal vector to Σ in the reference placement $\mathcal{B}_*$ and by v_n the corresponding non-zero normal speed of displacement of Σ at point $(\mathbf{x}_*, \tau)$ in the reference placement. We further assume that some

field related to the motion of $\mathcal{B}$ (excepting $\mathbf{x}$, $\mathbf{U}$ and θ) suffers jump discontinuity across Σ and so we employ the usual notation $[\![\,\cdot\,]\!]$ for jumps, so that

$$[\![f]\!] = f^+ - f^-, \tag{14}$$

where f^+ or f^- refers to the limit of f as the wave is approached from the right or left, respectively.

Therefore, we can write classical Kotchine's equations, as modified in order to take into account microstructural effects, and a relation that restricts the jump of micromomentum (see [25]-[27]) as it follows:

$$[\![\, \rho_* (v_n - \dot{\mathbf{x}} \cdot \mathbf{n})]\!] = 0, \quad [\![\, \rho_* v_n \dot{\mathbf{x}} + \mathbf{P}\,\mathbf{n}]\!] = \mathbf{0}, \quad \left[\!\!\left[\rho_* v_n \frac{\partial \kappa}{\partial \dot{\mathbf{U}}} (\mathbf{U}, \dot{\mathbf{U}}) + \mathbf{\Lambda}\,\mathbf{n} \right]\!\!\right] = \mathbf{0}, \tag{15}$$

$$\left[\!\!\left[\rho_* v_n \left(\epsilon + \frac{1}{2}\dot{\mathbf{x}}^2 + \kappa(\mathbf{U}, \dot{\mathbf{U}}) \right) \right]\!\!\right] = \left[\!\!\left[\left(\mathbf{h} - \mathbf{P}^T \dot{\mathbf{x}} - \mathbf{\Lambda}^* \dot{\mathbf{U}} \right) \cdot \mathbf{n} \right]\!\!\right], \; [\![\, \rho_* v_n \theta \eta]\!] \geq [\![\, \mathbf{h} \cdot \mathbf{n}]\!]. \tag{16}$$

The form of the jumps across a propagating wave of higher order time derivatives of principal fields can be obtained from the balance equations (6-8) and (10).

3. Thermodynamic restrictions on the constitutive assumptions

The first proposal is of kinematic character and leads to an expression for the microstructural kinetic co-energy χ for $\mathcal{B}$. We are interested in the linear theory from the next section, so the simplest assumption is to assume the function χ to be quadratic in $\dot{\mathbf{U}}$

$$\chi = \frac{1}{2} (\dot{\mathbf{U}} \mathbf{J}_*) \cdot \dot{\mathbf{U}}, \tag{17}$$

where the second-order referential microinertia tensor field $\mathbf{J}_* \in \mathrm{Sym}^+$ is constant (see [3, 12, 28]). As a consequence of this definition, the left-hand side of Eq. (8) reduces to $(\rho_* \ddot{\mathbf{U}} \mathbf{J}_*)$ and we meet also the requirements of positivity and quadratic form for κ: in fact from (3) we have

$$\kappa = \chi = \frac{1}{2} (\dot{\mathbf{U}} \mathbf{J}_*) \cdot \dot{\mathbf{U}}. \tag{18}$$

In order to study thermodynamic restrictions for thermoelastic materials with nano-pores we must define the Helmholtz free energy density per unit mass $\psi := \epsilon - \theta\eta$ and insert it in the axiom of dissipation (11); after, by using Eq. (10), we obtain the reduced version of the entropy inequality:

$$\rho_* \left(\dot{\psi} + \dot{\theta}\eta \right) \leq \mathbf{P} \cdot \dot{\mathbf{F}} + \mathbf{Y} \cdot \dot{\mathbf{U}} + \mathbf{\Lambda} \cdot \nabla \dot{\mathbf{U}} - \theta^{-1} \mathbf{h} \cdot \nabla \theta. \tag{19}$$

Here we generalize constitutive prescriptions presented in [12, 26] by considering both conduction of heat and inelastic surface effects associated to changes in the deformation of nano-pores in the vicinity of the hole boundaries. Therefore, let us call the array $\mathcal{S} := \{\mathbf{F}, \mathbf{U}, \nabla\mathbf{U}, \theta\}$ of independent variables the elastic state of the material with nano–pores and, assuming the equipresence principle, let us postulate the following constitutive relations of thermoelastic kind:

$$\{\psi, \eta, \mathbf{P}, \mathbf{Y}, \mathbf{\Lambda}, \mathbf{h}\} = \{\tilde{\psi}, \tilde{\eta}, \tilde{\mathbf{P}}, \tilde{\mathbf{Y}}, \tilde{\mathbf{\Lambda}}, \tilde{\mathbf{h}}\} \, (\mathcal{S}, \dot{\mathbf{U}}, \nabla\theta). \tag{20}$$

Now we have to check the compatibility of these prescriptions with the Clausius-Duhem inequality in its reduced version (19) that must be valid for any choice of the fields in the set $(\mathcal{S}, \dot{\mathbf{U}}, \nabla\theta)$. Consequently, by using the chain rule of differentiation, when the terms are appropriately ordered, the inequality reads:

$$\left(\rho_* \frac{\partial\psi}{\partial\mathbf{F}} - \mathbf{P}\right) \cdot \dot{\mathbf{F}} + \left(\rho_* \frac{\partial\psi}{\partial\mathbf{U}} - \mathbf{Y}\right) \cdot \dot{\mathbf{U}} + \left(\rho_* \frac{\partial\psi}{\partial(\nabla\mathbf{U})} - \mathbf{\Lambda}\right) \cdot \nabla\dot{\mathbf{U}} +$$
$$+\rho_* \left(\eta + \frac{\partial\psi}{\partial\theta}\right) \dot{\theta} + \rho_* \frac{\partial\psi}{\partial\dot{\mathbf{U}}} \cdot \ddot{\mathbf{U}} + \rho_* \frac{\partial\psi}{\partial(\nabla\theta)} \cdot \nabla\dot{\theta} + \frac{1}{\theta}\,\mathbf{h} \cdot \nabla\theta \leq 0. \tag{21}$$

The left-hand member of 21 is linear with respect to $\dot{\mathbf{F}}, \nabla\dot{\mathbf{U}}, \dot{\theta}, \ddot{\mathbf{U}}$ and $\nabla\dot{\theta}$, quantities that take up arbitrary values; thus the respective coefficients in the linear expression must vanish, and hence:

$$\psi = \tilde{\psi}(\mathcal{S}), \quad \mathbf{P} = \rho_* \frac{\partial\psi}{\partial\mathbf{F}}, \quad \mathbf{\Lambda} = \rho_* \frac{\partial\psi}{\partial(\nabla\mathbf{U})}, \quad \eta = -\frac{\partial\psi}{\partial\theta}. \tag{22}$$

These relations mean that the Helmholtz free energy ψ, the Piola-Kirchhoff stress tensor $\mathbf{P}$, the microstress $\mathbf{\Lambda}$ and the entropy η depend upon the elastic state of the material only; moreover, $\mathbf{P}$, $\mathbf{\Lambda}$ and η are determined as soon as the constitutive equation for ψ is known.

The residual inequality defines the dissipation $\mathcal{D}$ of the thermo-kinetic process

$$\mathcal{D} := \mathbf{H} \cdot \dot{\mathbf{U}} + \frac{1}{\theta}\,\mathbf{h} \cdot \nabla\theta \leq 0, \tag{23}$$

where $\mathbf{H} := \rho_* \frac{\partial\tilde{\psi}}{\partial\mathbf{U}}(\mathcal{S}) - \tilde{\mathbf{Y}}(\mathcal{S}, \dot{\mathbf{U}}, \nabla\theta)$ is the dissipative part of internal microactions, a symmetric second-order tensor.

For a thermally isotropic porous material which is a *definite heat conductor* [29], Fourier's law gives

$$\mathbf{h} = -\xi(\mathcal{S})\,\nabla\theta, \quad \text{with} \quad \xi \geq 0; \tag{24}$$

thus, by using in the energy Eq. (10) relations (22), (24) and $\epsilon = \psi + \theta\eta$, we have that, for a thermoelastic medium with nano–pores, it becomes

$$\rho_*\theta\frac{d\eta}{d\tau} + \mathbf{H} \cdot \dot{\mathbf{U}} = \rho_*\lambda + \mathrm{Div}\,(\xi\,\nabla\theta). \tag{25}$$

4. Linear field equations

Now we need to introduce the displacement field $\mathbf{u}$ of a material element of the body: it is

$$\mathbf{u}(\mathbf{x}_*, \tau) := \mathbf{x}(\mathbf{x}_*, \tau) - \mathbf{x}_*; \tag{26}$$

therefore, the deformation gradient $\mathbf{F}$ is expressed by:

$$\mathbf{F}(\mathbf{x}_*, \tau) = \mathbf{I} + \nabla\mathbf{u} \tag{27}$$

The natural reference placement $\mathcal{B}_*$ is homogeneous, free of residual macro- and micro-stresses, so $\mathcal{S}_* = \{\mathbf{I}, \mathbf{I}, \mathbf{O}, \theta*\}$ and $\mathbf{P}_*$, $\mathbf{Y}_*$, $\mathbf{\Lambda}_*$, ψ_* all vanish in $\mathcal{B}_*$ ($\mathbf{I} := (\delta_{ik})$ is the identity tensor); moreover, the reference microinertia tensor field $\mathbf{J}_*$ has spherical value:

$\mathbf{J}_* = \kappa_* \mathbf{I}$, where $\kappa_* \geq 0$ is the non-negative microinertia coefficient depending on the reference geometric features of the pores.

Besides, we may take the displacement field $\mathbf{u}$, the infinitesimal strain tensor $\mathbf{E}$, the microstrain tensor $\mathbf{V}$, with its reference gradient $\nabla\mathbf{V}$, and the temperature ϑ and mass ϱ variations from the reference placement as measures of "small" thermoelastic deformations from the reference placement $\mathcal{B}_*$; they are defined by Eq. (26) and as it follows:

$$\mathbf{E} := \operatorname{sym}(\nabla \mathbf{u}), \quad \mathbf{V} := \mathbf{U} - \mathbf{I}, \quad \nabla \mathbf{V} = \nabla \mathbf{U}, \quad \vartheta := \theta - \theta_* \quad \text{and} \quad \varrho := \rho - \rho_*. \tag{28}$$

Hence, in the linear theory we can change the choice of variables of the elastic state $\mathcal{S}$, the new ones being the following $\tilde{\mathcal{S}} := \{\mathbf{E}, \mathbf{V}, \nabla\mathbf{V}, \vartheta\}$; then, in the natural reference placement $\mathcal{B}_*$, it is $\tilde{\mathcal{S}}_* = \{\mathbf{0}, \mathbf{0}, \mathbf{O}, 0\}$.

As observed in the previous section, the free energy ψ determines much of the behaviour of the nano-porous material, thus we suppose that the reference placement $\mathcal{B}_*$ of the body is also a placement of minimum for the free energy and, therefore, we choose the most general homogeneous, quadratic and positive definite form for the free energy ψ valid for a centrosymmetric isotropic linear thermoelastic solids with nano-pores (see [12, 30, 31]):

$$\begin{aligned}
\psi(\tilde{\mathcal{S}}) = {} & \frac{(v_l^2 - 2v_t^2)}{2} (\operatorname{tr}\mathbf{E})^2 + v_t^2 \, \mathbf{E}\cdot\mathbf{E} + \frac{\kappa_* \lambda_3}{2} (\operatorname{tr}\mathbf{V})^2 + \kappa_* \lambda_4 \, \mathbf{V}\cdot\mathbf{V} + \kappa_* \lambda_5 (\operatorname{tr}\mathbf{E})(\operatorname{tr}\mathbf{V}) + \\
& + \kappa_* \lambda_6 \, \mathbf{E}\cdot\mathbf{V} + \kappa_* (v_{tm}^2 - v_{sm}^2) \operatorname{Div}\mathbf{V} \cdot \operatorname{Div}\mathbf{V} + \frac{\kappa_*}{2} v_{sm}^2 \, \nabla\mathbf{V}\cdot\nabla\mathbf{V} + \\
& + \kappa_* \lambda_1 \nabla(\operatorname{tr}\mathbf{V}) \cdot \operatorname{Div}\mathbf{V} + \frac{\kappa_* \lambda_2}{2} \nabla(\operatorname{tr}\mathbf{V}) \cdot \nabla(\operatorname{tr}\mathbf{V}) + \frac{\gamma_1}{2} \vartheta^2 + \gamma_2 \, \vartheta \operatorname{tr}\mathbf{E} + \kappa_* \gamma_3 \, \vartheta \operatorname{tr}\mathbf{V},
\end{aligned} \tag{29}$$

where $\operatorname{tr}(\cdot)$ denotes the trace of a tensor, *i.e.*, $\operatorname{tr}\mathbf{E} := \mathbf{E}\cdot\mathbf{I}$.

The positiveness of the expression in (29) assures us that the thirteen constant thermoelastic coefficients v_l, v_t, λ_i $(i = 1, \ldots, 6)$, γ_j $(j = 1, 2, 3)$, v_{tm} and v_{sm} must resolve the following system of inequalities:

$$\begin{aligned}
& 3v_l^2 > 4v_t^2 > 0, \quad (3v_l^2 - 4v_t^2)(3\lambda_3 + 2\lambda_4) > \kappa_* (3\lambda_5 + \lambda_6)^2, \quad 4\lambda_4 v_t^2 > \kappa_* \lambda_6^2, \\
& v_{tm}^2 > v_{sm}^2 > 0, \quad v_{sm}^2 (6\lambda_1 + 9\lambda_2 + 2v_{tm}^2 + v_{sm}^2) > 4(v_{tm}^2 - v_{sm}^2)^2, \quad \gamma_1 > 0, \\
& \gamma_1 \left[\rho_* \kappa_*^{-1} (3v_l^2 - 4v_t^2)(3\lambda_3 + 2\lambda_4) - (3\lambda_5 + \lambda_6)^2\right] + 6\gamma_2\gamma_3(3\lambda_5 + \lambda_6) > \\
& \quad > 3\gamma_2^2 {\kappa_*}^{-1} (3\lambda_3 + 2\lambda_4) + 3\gamma_3^2 (3v_l^2 - 4v_t^2), \quad \gamma_1(3\lambda_3 + 2\lambda_4) > 3\kappa_* \gamma_3^2;
\end{aligned} \tag{30}$$

alternatively to the last inequality of relation (30), the following one holds:

$$\gamma_1 (3v_l^2 - 4v_t^2) > 3\gamma_2^2. \tag{31}$$

Finally, we need to express the dissipative part $\mathbf{H}$ of internal microactions $\mathbf{Y}$ and the referential heating flux $\mathbf{h}$ within the same linear approximation as the other constitutive terms in the balance equations, thus, since $\mathbf{H}$ must vanish whenever $\dot{\mathbf{V}} = \mathbf{0}$, we take

$$\mathbf{H} = -\rho_* \kappa_* \left[\omega(\operatorname{tr}\dot{\mathbf{V}})\mathbf{I} + 2\sigma\dot{\mathbf{V}}\right] \quad \text{and} \quad \mathbf{h} = -\xi_* \nabla\theta, \tag{32}$$

with ω and σ inelastic constants and $\xi_* := \xi(\tilde{\mathcal{S}}_*)$. By inserting relations (32) in the dissipation imbalance (23), we obtain in the linear approximation

$$\rho_*\kappa_* \left[\omega(\mathrm{tr}\,\dot{\mathbf{V}})^2 + 2\,\sigma\,\dot{\mathbf{V}}^2\right] + \frac{\xi_*}{\theta}\,(\nabla\,\theta)^2 \geq 0, \tag{33}$$

which is verified if and only if

$$3\omega + 2\sigma \geq 0, \quad \sigma \geq 0, \quad \text{and} \quad \xi_* \geq 0. \tag{34}$$

Therefore, we derive from constitutive equations (22) and (32) and the definition of $\mathbf{H}$ (see, also, [12]) the following linear constitutive expressions for the dependent fields:

$$\begin{aligned}
\mathbf{P} &= \rho_* \left\{ \left[(v_l^2 - 2v_t^2)\,\mathrm{tr}\,\mathbf{E} + \kappa_*\lambda_5\,\mathrm{tr}\,\mathbf{V} + \gamma_2\vartheta \right] \mathbf{I} + 2\,v_t^2\,\mathbf{E} + \kappa_*\lambda_6\mathbf{V} \right\}, \\
\mathbf{Y} &= \rho_*\kappa_* \left\{ \left[\mathrm{tr}\,\left(\lambda_3\mathbf{V} + \lambda_5\mathbf{E} + \omega\dot{\mathbf{V}}\right) + \gamma_3\,\vartheta \right] \mathbf{I} + 2\,\lambda_4\,\mathbf{V} + \lambda_6\,\mathbf{E} + 2\,\sigma\,\dot{\mathbf{V}} \right\}, \\
\mathbf{\Lambda} &= \rho_*\kappa_* \{ 2\,(v_{tm}^2 - v_{sm}^2)\,\mathrm{syml}\,(\mathrm{Div}\,\mathbf{V} \otimes \mathbf{I}) + v_{sm}^2 \nabla\,\mathbf{V} + \\
&\qquad + \lambda_1\,[\mathbf{I} \otimes \mathrm{Div}\,\mathbf{V} + \mathrm{syml}\,(\nabla\,(\mathrm{tr}\,\mathbf{V}) \otimes \mathbf{I})] + \lambda_2\,\mathbf{I} \otimes \nabla\,(\mathrm{tr}\,\mathbf{V}) \}, \\
\eta &= -\gamma_1\vartheta - \gamma_2\,\mathrm{tr}\,\mathbf{E} - \kappa_*\gamma_3\,\mathrm{tr}\,\mathbf{V},
\end{aligned} \tag{35}$$

where the left-symmetric part "syml" of a third-order tensor $\mathbf{\Omega}$ means:

$$(\mathrm{syml}\,\mathbf{\Omega})_{ijl} := \frac{1}{2}\left(\Omega_{ijl} + \Omega_{jil}\right), \quad \forall\, i,j,k = 1,2,3. \tag{36}$$

In addition we need the expression of the determinant of the deformation gradient $\mathbf{F}$ in the linear theory:

$$\det\mathbf{F} = \det(\mathbf{I} + \nabla\mathbf{u}) \simeq 1 + \mathrm{Div}\,\mathbf{u} \quad \left[\Rightarrow (\det\mathbf{F})^{-1} \simeq 1 - \mathrm{Div}\,\mathbf{u}\right]. \tag{37}$$

Remark. When the pores are absent, v_l and v_t reduce to be the usual propagation speeds of dilatational and distortional waves in the linear isothermal elasticity, respectively.

The balance equations for a thermoelastic solid with nano-pores, governing the mass density ρ, the displacement field $\mathbf{u}$, the microstrain tensor $\mathbf{V}$ and the temperature change ϑ, are obtained by substituting constitutive relations (17) and (35) and Eq. $(37)_2$ into the Eqs. (6)-(8) and after by using $(28)_1$ and the fact that $\mathbf{J}_* = \kappa_*\mathbf{I}$; besides, last equation is get by Eq. (25) when linearized relations $(28)_4$, (32) and $(35)_4$ are applied:

$$\varrho = -\rho_*\mathrm{Div}\,\mathbf{u}, \tag{38}$$

$$\ddot{\mathbf{u}} = v_t^2\Delta\mathbf{u} + \nabla\left[(v_l^2 - v_t^2)\mathrm{Div}\,\mathbf{u} + \kappa_*\lambda_5\,\mathrm{tr}\,\mathbf{V} + \gamma_2\vartheta\right] + \kappa_*\lambda_6\,\mathrm{Div}\,\mathbf{V} + \mathbf{f}, \tag{39}$$

$$\begin{aligned}
\ddot{\mathbf{V}} &= v_{sm}^2\,\Delta\mathbf{V} + 2(v_{tm}^2 - v_{sm}^2)\,\mathrm{sym}\,[\nabla\,(\mathrm{Div}\,\mathbf{V})] + \lambda_1\nabla^2(\mathrm{tr}\,\mathbf{V}) + \\
&\quad + \left[\lambda_1\mathrm{Div}\,(\mathrm{Div}\,\mathbf{V}) + \lambda_2\Delta(\mathrm{tr}\,\mathbf{V}) - \mathrm{tr}\,(\lambda_3\mathbf{V} + \omega\dot{\mathbf{V}}) - \lambda_5\mathrm{Div}\,\mathbf{u} - \gamma_3\vartheta\right]\mathbf{I} - \\
&\qquad - \lambda_6\,\mathrm{sym}\,(\nabla\,\mathbf{u}) - 2\lambda_4\mathbf{V} - 2\sigma\dot{\mathbf{V}} + \kappa_*^{-1}\mathbf{B} \qquad \text{and}
\end{aligned} \tag{40}$$

$$0 = \gamma\Delta\vartheta + \gamma_1\dot{\vartheta} + \gamma_2\mathrm{Div}\,\dot{\mathbf{u}} + \kappa_*\gamma_3\mathrm{tr}\,\dot{\mathbf{V}} + \theta_*^{-1}\lambda, \tag{41}$$

with $\gamma := \xi_*(\rho_*\theta_*)^{-1} \geq 0$.

Moreover, we observe that the balance of moment of momentum (12) in the linear approximation assumes the following more simple expression: $\mathrm{skw}\,(\mathbf{P}-2\mathbf{Y})=\mathbf{0}$, which is satisfied by the symmetric tensors $(35)_{1,2}$ identically.

At the end we can write the linear equations of jumps by inserting constitutive relations (35) in Eqs.(15) and (16) and by ignoring terms of higher order: therefore they are the following ones:

$$[\![\,\rho_*(v_n-\dot{\mathbf{x}}\cdot\mathbf{n})]\!]=0, \tag{42}$$

$$\Big[\!\Big[v_n\dot{\mathbf{u}}+\Big[(v_l^2-2v_t^2)\,\mathrm{Div}\,\mathbf{u}+\kappa_*\lambda_5\,\mathrm{tr}\,\mathbf{V}+\gamma_2\vartheta\Big]\,\mathbf{n}\Big]\!\Big]+$$
$$+\Big[\!\Big[v_t^2\,[\mathrm{Div}\,(\mathbf{u}\otimes\mathbf{n})+\nabla\,(\mathbf{u}\cdot\mathbf{n})]+\kappa_*\lambda_6\mathbf{V}\mathbf{n}\Big]\!\Big]=\mathbf{0}, \tag{43}$$

$$\Big[\!\Big[v_n\dot{\mathbf{V}}+2\,(v_{tm}^2-v_{sm}^2)\,\mathrm{sym}\;(\mathrm{Div}\,\mathbf{V}\otimes\mathbf{n})+v_{sm}^2\,(\nabla\,\mathbf{V})\,\mathbf{n}\Big]\!\Big]+$$
$$+[\![\lambda_1\,\{(\mathbf{n}\cdot\mathrm{Div}\,\mathbf{V})\mathbf{I}+\mathrm{sym}\;[\nabla\,(\mathrm{tr}\,\mathbf{V})\otimes\mathbf{n}]\}+\lambda_2\,[\mathbf{n}\cdot\nabla\,(\mathrm{tr}\,\mathbf{V})]\,\mathbf{I}]\!]=\mathbf{O}, \tag{44}$$

$$[\![v_n\,(\gamma_1\vartheta+\gamma_2\,\mathrm{Div}\,\mathbf{u}+\kappa_*\gamma_3\,\mathrm{tr}\,\mathbf{V})-\gamma\,\nabla\,\vartheta\cdot\mathbf{n}]\!]=0, \tag{45}$$

last imbalance being satisfied identically by Eq. (45).

Now we can uncouple the spherical and deviatoric components of the linear balance of micromomentum (39) to obtain, respectively:

$$\ddot{\nu}=\left(\frac{1}{3}v_{sm}^2+\frac{2}{3}v_{tm}^2+2\lambda_1+3\lambda_2\right)\Delta\nu+\Big[2(v_{tm}^2-v_{sm}^2)+3\lambda_1\Big]\,\mathrm{Div}\,(\mathrm{Div}\,\mathbf{V}^D)-$$
$$-(3\lambda_3+2\lambda_4)\,\nu-(3\omega+2\sigma)\dot{\nu}-(3\lambda_5+\lambda_6)\mathrm{Div}\,\mathbf{u}-3\gamma_3\vartheta+\iota, \tag{46}$$

$$\ddot{\mathbf{V}}^D=v_{sm}^2\,\Delta\mathbf{V}^D+2(v_{tm}^2-v_{sm}^2)\,\Big\{\mathrm{sym}\,\Big[\nabla\,(\mathrm{Div}\,\mathbf{V}^D)\Big]\Big\}^D-2\lambda_4\mathbf{V}^D-2\sigma\dot{\mathbf{V}}^D+$$
$$+\left[\frac{2}{3}(v_{tm}^2-v_{sm}^2)+\lambda_1\right](\nabla^2\nu)^D-\lambda_6\,[\mathrm{sym}\,(\nabla\,\mathbf{u})]^D+\kappa_*^{-1}\mathbf{B}^D, \tag{47}$$

where ν and ι are the traces of $\mathbf{V}$ and $\kappa_*^{-1}\mathbf{B}$, respectively, while the deviatoric part is defined by: $\mathbf{A}^D:=\mathbf{A}-3^{-1}(\mathrm{tr}\,\mathbf{A})\mathbf{I}$, for each symmetric second-order tensor $\mathbf{A}$.

With the same procedure, the respective jump (44) is splitted in the following spherical and deviatoric relations, respectively:

$$\Big[\!\Big[v_n\dot{\nu}+\Big[\tfrac{1}{3}(v_{sm}^2+2v_{tm}^2)+2\,\lambda_1+3\,\lambda_2\Big]\,\nabla\,\nu\cdot\mathbf{n}+[2\,(v_{tm}^2-v_{sm}^2)+3\,\lambda_1]\mathrm{Div}\,\mathbf{V}^D\cdot\mathbf{n}\Big]\!\Big]=0,$$

$$[\![v_n\dot{\mathbf{V}}^D+v_{sm}^2\,(\nabla\,\mathbf{V}^D)\,\mathbf{n}+ \tag{48}$$
$$+\Big\{\mathrm{sym}\,\Big\{\Big[\lambda_1\nabla\,\nu+2\,(v_{tm}^2-v_{sm}^2)\,\Big(\mathrm{Div}\,\mathbf{V}^D+\tfrac{1}{3}\nabla\,\nu\Big)\Big]\otimes\mathbf{n}\Big\}\Big\}^D]\!]=\mathbf{O}.$$

5. Micro-vibrations in solids with nano-pores

Micro-vibrations, produced during various operations from railway and/or roads to foot traffic and propagated from one medium to another, are one of the main factor for fatigue

in structures; moreover, they could also cause serious damages in producing micro and nano scale equipments, other than errors during experiments in high-precision laboratories equipped with lasers, sensors or microscopes.

To study their propagation, let us assume that external volume contributions are null, *i.e.* $\mathbf{f} = \mathbf{0},\ \mathbf{B} = \mathbf{O}$ and $\lambda = 0$; moreover, there are no macro-displacements in the system, *i.e.* $\mathbf{u} = \mathbf{0}$ and $\varrho = 0$, then let us consider solutions of Eqs. (39), (46), (47) and (41) of the form of thermal micro-vibrations in absence of dissipation as the following ones:

$$\nu = \hat{\nu}\, e^{ib\tau},\ \mathbf{V}^D = \hat{\mathbf{V}}\, e^{ib\tau},\ \vartheta = \hat{\vartheta}\, e^{ib\tau},\ \omega = \sigma = 0, \tag{49}$$

where $\hat{\nu},\ \hat{\mathbf{V}}$ and $\hat{\vartheta}$ are constant amplitudes, b is the frequency and i is the imaginary unit. Then Eq. (39) is satisfied identically, while the other equations become

$$\left(b^2 - 3\lambda_3 - 2\lambda_4\right)\hat{\nu} = 3\gamma_3\hat{\vartheta}, \quad \left(b^2 - \lambda_4\right)\hat{\mathbf{V}}^D = \mathbf{O}, \quad \gamma_1\hat{\vartheta} + \kappa_*\gamma_3\hat{\nu} = 0. \tag{50}$$

Therefore we have the following admissible results:

○) *dilatational mode:*

$$\begin{gathered} b_d = \sqrt{3\lambda_3 + 2\lambda_4 - 3\kappa_*\gamma_1^{-1}\gamma_3^2}, \\ \hat{\mathbf{V}}_{11} = \hat{\mathbf{V}}_{22} = \hat{\mathbf{V}}_{33}\ (\Rightarrow \hat{\nu} = 3\,\hat{\mathbf{V}}_{11}), \quad \hat{\mathbf{V}}_{ij} = 0,\ \forall\ i \neq j, \quad \hat{\vartheta} = -\kappa_*\gamma_1^{-1}\gamma_3\hat{\nu}; \end{gathered} \tag{51}$$

we observe that the frequency b_d of this spatio-thermal oscillation is real for the restriction $(30)_8$ of the free energy density ψ to be a positive definite form;

○) *extensional modes with a constant volume:*

$$b_e = \sqrt{\lambda_4}, \quad \hat{\nu} = \hat{\vartheta} = 0\ (\Rightarrow \hat{\mathbf{V}}_{33} = -\hat{\mathbf{V}}_{11} - \hat{\mathbf{V}}_{22}), \quad \hat{\mathbf{V}}_{ij} = 0,\ \forall\ i \neq j; \tag{52}$$

also in these modes the frequency b_e of the micro-oscillations is real for the restriction $(30)_3$, while no thermal vibrations are present;

○) *shear modes:*

$$b_s = \sqrt{\lambda_4}, \quad \hat{\mathbf{V}}_{ij} \neq 0,\ \forall\ i \neq j, \quad \hat{\mathbf{V}}_{ii} = 0,\ \forall\ i,\ \Rightarrow \hat{\nu} = \hat{\vartheta} = 0; \tag{53}$$

their frequency b_s coincides with the real frequency b_e of the extensional modes and neither here there are thermal vibrations.

Remark. When we neglect thermic phenomena, our oscillating solutions recover three of the mechanical micro-vibrations obtained for general microstructure in [17].

6. Dispersion relations for plane waves

Now we draw here some results on the propagation of plane wave motion in a linear thermoelastic solids with big pores. We seek solutions of the system of linear balance Eqs. (38), (39), (46), (47) and (41) in the form of traveling harmonic waves (see, also, [32]):

$$\mathbf{u} = \phi(\mathbf{x}_*, \tau)\,\mathbf{w},\ \nu = \mu\,\phi(\mathbf{x}_*, \tau),\ \mathbf{V}^D = \phi(\mathbf{x}_*, \tau)\,\mathbf{S},\ \vartheta = \bar{\vartheta}\,\phi(\mathbf{x}_*, \tau),\ \varrho = \bar{\varrho}\,\phi(\mathbf{x}_*, \tau) \tag{54}$$

where $\mathbf{w}$, μ, $\mathbf{S}$ $\bar{\vartheta}$ and $\bar{\varrho}$ are constants which represent the wave amplitude of, respectively, the macro-displacement vector, the micro-strain trace, the deviatoric part of the micro-strain tensor and the temperature and mass fluctuation; besides, the wave function ϕ can be generally represented by the real (or the imaginary) part of a complex function

$$\phi(\mathbf{x}_*, \tau) = \exp(ib\tau - \delta\, \mathbf{n}\cdot\mathbf{x}_*) \quad \text{with} \quad \delta := a + ib/c, \tag{55}$$

where δ is the wave number; $b > 0$ is the frequency; $a(b) > 0$ and $c(b)$ are the wave attenuation and the wave speed, respectively; $\mathbf{n}$ is the unit vector representing the direction of wave propagation, while the unit vector $\hat{\mathbf{w}}$ defines the direction of motion. The specific loss l is defined by $l := 4\pi\left|\frac{ac}{b}\right|$.

Again we suppose that all external sources are zero, *i.e.* $\mathbf{f} = \mathbf{0}$, $\mathbf{B} = \mathbf{O}$ and $\lambda = 0$, hence, by substituting Eqs. (54) and (55) in the linear system (38), (39), (46), (47) and (41), we obtain the following relations:

$$(b^2 + v_t^2\delta^2)\mathbf{w} + (v_l^2 - v_t^2)\delta^2(\mathbf{w}\cdot\mathbf{n})\mathbf{n} - \kappa_*\delta\left[\lambda_6\mathbf{S}\mathbf{n} + \left(\frac{\lambda_6}{3} + \lambda_5\right)\mu\,\mathbf{n}\right] - \gamma_2\delta\bar{\vartheta}\mathbf{n} = \mathbf{0}, \tag{56}$$

$$\left[b^2 + \left(\frac{1}{3}v_{sm}^2 + \frac{2}{3}v_{tm}^2 + 2\lambda_1 + 3\lambda_2\right)\delta^2 - 2\lambda_4 - 3\lambda_3 - ib(2\sigma + 3\omega)\right]\mu + \\ + \left[2(v_{tm}^2 - v_{sm}^2) + 3\lambda_1\right]\delta^2(\mathbf{S}\mathbf{n})\cdot\,\mathbf{n} + (\lambda_6 + 3\lambda_5)\,\delta(\mathbf{w}\cdot\mathbf{n}) - 3\gamma_3\bar{\vartheta} = 0, \tag{57}$$

$$\left(b^2 + v_{sm}^2\delta^2 - 2\lambda_4 - 2i\sigma b\right)\mathbf{S} + 2(v_{tm}^2 - v_{sm}^2)\delta^2\left[\text{sym}\ (\mathbf{S}\mathbf{n}\otimes\mathbf{n})\right]^D + \\ + \lambda_6\delta\left[\text{sym}\,(\mathbf{w}\otimes\mathbf{n})\right]^D + \left[\frac{2}{3}(v_{tm}^2 - v_{sm}^2) + \lambda_1\right]\delta^2\mu(\mathbf{n}\otimes\mathbf{n})^D = \mathbf{O}, \tag{58}$$

$$(i\gamma\delta^2 - b\gamma_1)\bar{\vartheta} + \gamma_2 b\delta\mathbf{w}\cdot\mathbf{n} - \kappa_*\gamma_3 b\nu = 0 \quad \text{and} \quad \bar{\varrho} = \rho_*\psi h_n. \tag{59}$$

This algebraic system of eleven equations may be combined into five independent systems through linear combinations of those equations: two uncoupled relations, two coupled systems of two equations each and one coupled system of five equations. The study of all five systems needs the introduction of two unit vectors, $\mathbf{e}$ and $\mathbf{f}$, in the plane orthogonal to the direction of propagation $\mathbf{n}$ and such that $\mathbf{e}\cdot\mathbf{f} = 0$. Therefore, we have the following particular occurrences.

6.1. Shear optical waves

From the deviatoric Eq. (58), we obtain two independent dispersion relations relating frequencies b and wave numbers δ; they are two different shear optical micro-waves :

$$\left(b^2 + v_{sm}^2\delta^2 - 2\lambda_4 - 2i\sigma b\right)S_{ef} = 0 \quad \text{and} \tag{60}$$

$$\left(b^2 + v_{sm}^2\delta^2 - 2\lambda_4 - 2i\sigma b\right)(S_{ee} - S_{ff}) = 0, \tag{61}$$

where the subscripts indicates tensor components.

These shear optical micro-modes propagate with attenuation $a_s = \frac{\sigma c}{v_{sm}^2}$ and velocity given by $c_s^2 = \frac{v_{sm}^2}{2\sigma^2}\left[2\lambda_4 - b^2 + \sqrt{(2\lambda_4 - b^2)^2 + 4\,\sigma^2 b^2}\right]$ without modifying the thermo-elastic

features of the matrix material of the porous medium; then the specific loss is $l_s = 4\pi \left| \frac{2\lambda_4 - b^2}{2b\sigma} + \sqrt{1 + \left(\frac{2\lambda_4 - b^2}{2b\sigma} \right)^2} \right|$.

For high frequencies all quantities grow with b, while for low frequencies, the speed and the attenuation approach $v_{sm}\sqrt{2\lambda_4}/\sigma$ and $\sqrt{2\lambda_4}/v_{sm}$, respectively, while l_s is big. Moreover, it is also possible a static solution with attenuation $a_s = \frac{\sqrt{2\lambda_4}}{v_{sm}}$.

6.2. Transverse waves

We get also two different systems of transverse waves, for $j = e, f$, from Eqs. (56) and (58):

$$(b^2 + v_t^2\delta^2)\, w_j - \kappa_* \lambda_6 \delta\, S_{nj} = 0, \tag{62}$$

$$\lambda_6 \delta\, w_j + 2\left(b^2 + v_{tm}^2\delta^2 - 2\lambda_4 - 2i\sigma b\right) S_{nj} = 0. \tag{63}$$

This homogeneous system has a nontrivial solution for the amplitudes w_j and S_{nj} if and only if the following dispersion relation is satisfied by δ:

$$2\left(v_t^2\delta^2 + b^2\right)\left(v_{tm}^2\delta^2 + b^2 - 2\lambda_4 - 2i\sigma b\right) + \kappa_* \lambda_6^2 \delta^2 = 0. \tag{64}$$

Eq. (64) is similar to the dispersion relation for plane thermoelastic waves studied in [33] and our analysis will use results there obtained. The first transverse solution of (64) is associated predominantly with the elastic properties of the material (v_t) and denoted by δ_t; the second one, δ_{tm}, with the properties governing elastic and dissipative changes in porosity (v_{tm}, λ_4, λ_6 and σ).

The analytical solutions of the dispersion relation (64) are quite cryptic and they are summarized in Table 1 of [34] (*modulo* some innocuous identification in notations); in this work we only report their physical interpretation without big difficulties. The coupling of motion Eqs. (56) and (58) of linear macro- and micro-momentum does the wave of dispersive kind, while the presence of big voids adds a dissipative mechanism associated with nano-pores which yields both waves to attenuate. If the dissipation coefficient σ is null, then we recover the presence of the resonance.

When frequencies are low, the elastic wave propagates with speed $v_t\sqrt{1-\zeta}$, where $0 \leq \zeta \left(:= \frac{\kappa_* \lambda_6^2}{4\lambda_4 v_t^2}\right) < 1$ for the inequality $(30)_3$, while the attenuation coefficient and the specific loss remain very small and approach zero with the frequency itself. The predominantly micro-transverse wave propagates with constant speed $\frac{v_{tm}}{\sigma}\sqrt{2\lambda_4(1-\zeta)}$ and with constant attenuation $\frac{\sqrt{2\lambda_4(1-\zeta)}}{v_{tm}}$; nevertheless, its specific loss $l_{tm} = \frac{2\lambda_4(1-\zeta)}{\sigma b}$ is large and in inverse proportion to the frequency b.

At high frequencies, the predominantly elastic transverse wave propagates with the classical speed v_t and, as the frequency approaches infinity, the attenuation coefficient a_{te} and the specific loss l_{te} are very small and approach zero, as for low frequencies. Instead the predominantly micro-transverse wave propagates with attenuation σ/v_{tm} and with constant speed v_{tm}, but with a small specific loss which approach zero when the frequency approaches infinity.

The amplitude ratio R of the micro-wave to the macro-wave is obtained from (62):

$$R = \frac{S_{nj}}{w_j} = \frac{(b^2 + v_t^2\delta^2)}{\kappa_* \lambda_6 \delta}. \tag{65}$$

For the solution predominantly elastic δ_t at large frequencies, the ratio R_t is a constant; at small frequencies it approaches zero with the frequency itself. Instead, the micro-mode δ_{tm} at large frequencies gives a ratio R_{tm} very big, while at low frequencies it is constant.

At the end, we can also have a static solution with attenuation $a_{tm} = \frac{\sqrt{2\lambda_4(1-\zeta)}}{v_{tm}}$ in which the amplitudes are related by $w_j = \frac{\kappa_* \lambda_6}{v_t^2\, a} S_{nj}$, for $j = e, f$.

6.3. Longitudinal waves

The remaining equations of the system (56)-(59) furnish the solutions for the longitudinal waves, the only ones which present thermal effects:

$$(b^2 + v_l^2\delta^2) w_n - \kappa_* \delta \left[\lambda_6 S_{nn} + \left(\frac{\lambda_6}{3} + \lambda_5 \right) \mu \right] - \gamma_2 \delta \bar{\vartheta} = 0, \tag{66}$$

$$\left[b^2 + \left(\frac{1}{3} v_{sm}^2 + \frac{2}{3} v_{tm}^2 + 2\lambda_1 + 3\lambda_2 \right) \delta^2 - 2\lambda_4 - 3\lambda_3 - ib(2\sigma + 3\omega) \right] \mu +$$
$$+ (\lambda_6 + 3\lambda_5)\, \delta w_n + \left[\frac{2}{3}(v_{tm}^2 - v_{sm}^2) + \lambda_1 \right] \delta^2 S_{nn} - 3\gamma_3 \bar{\vartheta} = 0, \tag{67}$$

$$\frac{2}{3}\lambda_6 \delta\, w_n + \left[b^2 + \frac{1}{3} \left(4v_{tm}^2 - v_{sm}^2 \right) \delta^2 - 2\lambda_4 - 2i\sigma b \right] S_{nn} + \tag{68}$$
$$+ \frac{2}{3} \left[\frac{2}{3}(v_{tm}^2 - v_{sm}^2) + \lambda_1 \right] \delta^2 \mu = 0,$$

$$\gamma_2 b \delta\, w_n - \kappa_* \gamma_3 b \mu + (i\gamma\delta^2 - b\gamma_1)\bar{\vartheta} = 0 \quad \text{and} \quad \bar{\varrho} = \rho_* \psi w_n. \tag{69}$$

Last equation rules the propagation of mass wave and it relates the mass amplitude $\bar{\varrho}$ directly with that of normal displacement w_n, so when this is calculated, the first one is get by Eq. $(69)_2$.

For the residual amplitudes w_n, μ, S_n and $\bar{\vartheta}$, we must pose the determinant of their coefficients equal to zero in order to have a nontrivial solution of the system (66)-$(69)_1$; therefore, we get a 4^{th}-order equation in δ^2, which can be resolved with the Ferrari-Cardano derivation of the quartic formula, after the application of the Tchirnhaus transformation by mean of numerical techniques (see [32] and [35]). By the way, an exact analytical solution of the dispersion relation for longitudinal waves is very complicated and without interest to be reported here explicitly: instead we are concerned to summarize the behaviour of all wave numbers δ_e^2, δ_d^2, δ_v^2 and δ_{th}^2, which are dominated by displacement, deviatoric and spherical parts of microstrain and thermal fields, respectively. Hence, there exist four coupled longitudinal waves: the first one δ_e^2 is predominantly an elastic wave of dilatation, the second one δ_d^2 is associated with an equivoluminal microelastic wave, the third one δ_v^2 is predominantly a volume fraction wave

of pure dilatation, the last one corresponding to δ_{th}^2 is similar in character to a thermal wave; from numerical outcomes, the two micro-waves result to be slower than elastic and thermal waves, the elastic one being the fastest: this is in accordance with experimental evidence.

By disregarding micro-rotation and thermal effects, we are able to observe that the micro-wave solution above can be acknowledged in some developments of §8 of [17] for elastic plates and, peculiarly, the velocity of the elastic wave is less than that which would be calculated for classical elasticity v_l due to the phenomenon of the compliance of pores. In addition, if we neglect non-spherical contributions to the microstrain in constitutive equations (35), we find solutions of voids theory [36].

The longitudinal waves are all dispersive in character, because of the coupling of Eqs. (66)-$(69)_1$, and suffer attenuation (the thermal mode with a large coefficient) due to the thermal coupling and to the presence of voids. Furthermore, if the two dissipation coefficient σ and ω are zero, we can observe the phenomenon of the resonance.

For low frequencies, there is no damping effect in either of the four modes. The only significant wave is the δ_e-one, because the other three modes almost do not exist and their attenuation coefficients remain very small and approach zero with the frequency itself; instead the velocity v_l of the δ_e-wave is increased by a small amount due to the thermomechanical coupling, but decreased significantly because of nano-porosity effects; the attenuation is a quite small constant.

If frequencies are high, the δ_d micro-wave, of speed $v_d^2 = 2v_{tm}^2 - v_{sm}^2 + \lambda_1 > 0$ by inspection, and the δ_v one, which travels with velocity $v_v^2 = \frac{5}{3}v_{sm}^2 - \frac{2}{3}v_{tm}^2 + 3\lambda_2 > 0$ for $(30)_{4,5}$, are not accompanied by elastic or thermal modes, which are instead coupled, and vice versa; attenuation coefficients for micro-modes remains small but constant. Elastic mode propagates with the classical speed v_l and attenuation coefficient which approaches zero slowly with the frequency itself; instead the propagation velocity and the attenuation coefficient of the thermal wave δ_{th} sharply increase with the frequency itself, being diffusive in nature. We notice that the high frequency limits of the two micro-elastic waves correspond to the velocities of acceleration waves in the same material, undeformed and at rest, obtained in [26].

Finally, it can be observed numerically that the specific loss l is significantly large when the wave velocity has quite small value in some regions of frequency. The loss due to energy dissipation is comparatively high in case of δ_e and δ_{th}-modes and moderate for predominantly dilatation micro-elastic modes.

7. Macro-acceleration waves

We are now in a position to study acceleration wave propagation. We shall consider the surfaces Σ that are *weak singularities*, defined as those carrying only jumps of the derivatives of order 2 of the macro- and micro-displacement vectors and of order 1 of the thermal variables; these singular surfaces of order 2 are called *macro-acceleration waves* and all external forces and supplies, *i.e.* $\mathbf{f}$, $\mathbf{B}$ and λ, are supposed continuous across them with all the derivatives (see, also, [1, 37]).

Therefore, these peculiar discontinuity surfaces suffer 2^{nd}-order derivative jumps of the displacement $\mathbf{u}$ and 1^{st}-order derivative jumps of the temperature variation ϑ and of the

microstrain tensor $\mathbf{V}$, which is, in our context, directly related to the left Cauchy-Green tensor $\mathbf{U}$ of the micro-deformation.

Remark: It is noteworthy that our definition of macro-acceleration wave differs from usual definitions of acceleration waves in, *e.g.*, [27, 38, 39] because they examine singularities of order 2 for both macro- and micro-structural kinematic variables $\mathbf{u}$ and $\mathbf{V}$. Moreover, our study of standard acceleration waves in [26] shows that, in the linear theory, these jumps are in general uncoupled unless the instantaneous acoustic macro- and micro-tensor have some eigenvalue coincident (see, also, the comments about the non-linear theory in §8 of [39]).

The normal velocity of displacement v_n of the macro-acceleration wave is continuous everywhere in the body and, hence, the following Hugoniot-Hadamard compatibility condition for the jump across Σ of the derivatives of an arbitrary field $\mathbf{\Psi}$ in $\mathcal{B}_*$ holds (see, *e.g.*, $(2.7)_2$ of [40]):

$$[\![\nabla \mathbf{\Psi}]\!] = \nabla [\![\mathbf{\Psi}]\!] - v_n^{-1} [\![\dot{\mathbf{\Psi}}]\!] \otimes \mathbf{n}. \tag{70}$$

In particular, if the field $\mathbf{\Psi}$ is continuous in $\mathcal{B}_*$, Eq. (70) reduces to

$$v_n [\![\nabla \mathbf{\Psi}]\!] = -[\![\dot{\mathbf{\Psi}}]\!] \otimes \mathbf{n}. \tag{71}$$

7.1. Homothermal case

In the linear approximation, jump Eqs. (42) and (43) along a macro-acceleration wave Σ are identically satisfied; instead the jump balance of energy (45) reduces to the Fourier condition

$$-\gamma [\![\nabla \vartheta]\!] \cdot \mathbf{n} = \gamma v_n^{-1} [\![\dot{\vartheta}]\!] = 0, \tag{72}$$

where Eq. (71) were used. Thus, *in linear porous thermoelasticity*, the first derivatives of ϑ are continuous and *every macro-acceleration wave is homothermal*.

The last jump condition (44) gives the following relation

$$\begin{aligned} &v_n [\![\dot{\mathbf{V}}]\!] + v_{sm}^2 [\![\nabla \mathbf{V}]\!] \mathbf{n} + 2(v_{tm}^2 - v_{sm}^2) \operatorname{sym} ([\![\operatorname{Div} \mathbf{V}]\!] \otimes \mathbf{n}) + \\ &+\lambda_1 [(\mathbf{n} \cdot [\![\operatorname{div} \mathbf{V}]\!]) \mathbf{I} + \operatorname{sym} ([\![\nabla \nu]\!] \otimes \mathbf{n})] + \lambda_2 (\mathbf{n} \cdot [\![\nabla \nu]\!]) \mathbf{I} = \mathbf{O}, \end{aligned} \tag{73}$$

while the jumps of the balance laws (39) and (41) and of the derivative of Eq. (38) furnish these other ones, when Eq. (72) is also used:

$$[\![\ddot{\mathbf{u}}]\!] = v_t^2 [\![\Delta \mathbf{u}]\!] + (v_l^2 - v_t^2) [\![\nabla (\operatorname{Div} \mathbf{u})]\!] + \kappa_* (\lambda_5 [\![\nabla \nu]\!] + \lambda_6 [\![\operatorname{Div} \mathbf{V}]\!]), \tag{74}$$

$$\gamma [\![\Delta \vartheta]\!] + \gamma_2 [\![\operatorname{Div} \dot{\mathbf{u}}]\!] + \kappa_* \gamma_3 [\![\dot{\nu}]\!] = 0, \quad [\![\dot{\rho}]\!] + \rho_* [\![\operatorname{Div} \dot{\mathbf{u}}]\!] = 0. \tag{75}$$

Now we can use the Hugoniot-Hadamard condition (71) to get a system of algebraic equations where the amplitudes of the discontinuities $[\![\dot{\rho}]\!]$, $[\![\ddot{\mathbf{u}}]\!]$, $[\![\dot{\mathbf{V}}]\!]$ and $[\![\ddot{\vartheta}]\!]$ are the unknown quantities. With this end in view, we employ the following definitions of instantaneous *homothermal acoustic macro-tensor* $\mathcal{U}(\mathbf{n} \otimes \mathbf{n})$ and *micro-tensor* $\mathcal{C}(\mathbf{n} \otimes \mathbf{n})$ (see, for analogy, [26, 39]):

$$\mathcal{U}(\mathbf{n}\otimes\mathbf{n}) := \left[v_t^2\,\mathbf{I} + (v_l^2 - v_t^2)\mathbf{n}\otimes\mathbf{n}\right], \tag{76}$$

$$\mathcal{C}(\mathbf{n}\otimes\mathbf{n}) := [v_{sm}^2\,\mathcal{I} + (v_{tm}^2 - v_{sm}^2)(\mathbf{\Phi}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{I}\otimes\mathbf{n}) + \\ +\lambda_2\,\mathbf{I}\otimes\mathbf{I} + \lambda_1\,(\mathbf{I}\otimes\mathbf{n}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{n}\otimes\mathbf{I})], \tag{77}$$

where we introduced the fourth-order tensor $\mathcal{I}$ and third-order tensor $\mathbf{\Phi}$ of components, respectively: $\mathcal{I}_{ijkl} := \delta_{ik}\delta_{jl}$ and $\Phi_{ijk} := \delta_{ik}n_j$.

Therefore, from Eqs. (73)-(75), we are able to write the system of equations for the unknown amplitudes in the following manner (see, also, [41]):

$$v_n[\![\,\dot\rho\,]\!] = \rho_*\,[\![\,\ddot{\mathbf{u}}\,]\!]\cdot\mathbf{n}, \tag{78}$$

$$\left[v_n^2\mathbf{I} - \mathcal{U}(\mathbf{n}\otimes\mathbf{n})\right][\![\,\ddot{\mathbf{u}}\,]\!] = -\kappa_* v_n(\lambda_5\,\mathbf{n}\otimes\mathbf{I} + \lambda_6\,\mathbf{I}\otimes\mathbf{n})[\![\,\dot{\mathbf{V}}\,]\!], \tag{79}$$

$$\left[v_n^2\,\mathcal{I} - \mathcal{C}(\mathbf{n}\otimes\mathbf{n})\right][\![\,\dot{\mathbf{V}}\,]\!] = \mathbf{0}, \tag{80}$$

$$\gamma\,[\![\,\ddot\vartheta\,]\!] = \kappa_*\gamma_3 v_n^2\,\mathbf{I}\cdot[\![\,\dot{\mathbf{V}}\,]\!] - \gamma_2 v_n[\![\,\ddot{\mathbf{u}}\,]\!]\cdot\mathbf{n}. \tag{81}$$

In conclusion, the jump of macro-acceleration $[\![\,\ddot{\mathbf{u}}\,]\!]$, ruled by Eq. (79), is in general coupled to the discontinuities in the microstructural variable $[\![\,\dot{\mathbf{V}}\,]\!]$, unlike the purely acceleration waves studied in [26], as observed in the previous remark.

Hence to classify possible macro-acceleration waves Σ we must analyze in advance Eq. (80) that gives three possible speed of displacement v_n for the surface Σ related to:

○) *Two shear optical micro-waves,* whose speed propagation is $v_n = v_{sm}$; the amplitude of the discontinuity is $[\![\,\dot{\mathbf{V}}\,]\!]_{sm} = \alpha(\mathbf{e}\otimes\mathbf{e} - \mathbf{f}\otimes\mathbf{f}) + \beta(\mathbf{e}\otimes\mathbf{f} + \mathbf{f}\otimes\mathbf{e})$, where α and β are the scalar components of the wave amplitude and, as before in the plane waves study, $\mathbf{e}$ and $\mathbf{f}$ are the unit vectors in the plane orthogonal to $\mathbf{n}$ such that $\mathbf{e}\cdot\mathbf{f} = 0$.

By inserting this solution in the other three Eqs. (78), (79) and (81), we obtain, in general, that $[\![\,\ddot{\mathbf{u}}\,]\!]_{sm} = \mathbf{0}$, $[\![\,\dot\rho\,]\!]_{sm} = 0$ and $[\![\,\ddot\vartheta\,]\!]_{sm} = 0$, that is, in essence, the waves carry predominantly a change in the nano-pore structure without altering the thermoelastic features of the matrix material.

Observation. Instead, in the particular case in which one eigenvalue of $\mathcal{U}$ coincides with that of $\mathcal{C}$, *i.e.*, $v_t = v_{sm}$, (or $v_l = v_{sm}$), there could be also an associated transverse (or longitudinal, respectively) macro-wave of free amplitude which not alters (or which alters, respectively) mass and temperature fields; in this last case we have $[\![\,\dot\rho\,]\!]_{sm} = \rho_* v_{sm}^{-1}[\![\,\ddot u_n\,]\!]_{sm}$ and $[\![\,\ddot\vartheta\,]\!]_{sm} = -\gamma^{-1}\gamma_2 v_{sm}[\![\,\ddot u_n\,]\!]_{sm}$.

○) *Two transverse micro-waves,* with propagation velocity $v_n = v_{tm}$. Their amplitude is of the form $[\![\,\dot{\mathbf{V}}\,]\!]_{tm} = \chi_e(\mathbf{n}\otimes\mathbf{e} + \mathbf{e}\otimes\mathbf{n}) + \chi_f(\mathbf{n}\otimes\mathbf{f} + \mathbf{f}\otimes\mathbf{n})$, with χ_e and χ_f the components of the amplitude itself. Now, there is a coupled *transverse macro-wave*, obtained by the study of equation (79), of amplitude:

$$[\![\,\ddot{\mathbf{u}}\,]\!]_{tm} = \frac{\kappa_* v_{tm}\lambda_6}{v_t^2 - v_{tm}^2}(\chi_e\,\mathbf{e} + \chi_f\,\mathbf{f}), \tag{82}$$

but no discontinuities along Σ in the mass and temperature fields, for equations (78) and (81), as in the classical case, in fact $[\![\,\dot{\rho}\,]\!]_{tm} = 0$ and $[\![\,\ddot{\vartheta}\,]\!]_{tm} = 0$.

○) *One extensional micro-wave,* whose vector amplitude is

$$[\![\,\dot{\mathbf{V}}\,]\!]_{em} = \delta(\zeta\,\mathbf{n}\otimes\mathbf{n} + \mathbf{e}\otimes\mathbf{e} + \mathbf{f}\otimes\mathbf{f});$$

here δ is its scalar amplitude, ζ the constant so defined: $\zeta := \frac{2(v_{tm}^2 - v_{sm}^2 + \lambda_1) - \lambda_2 + \sqrt{D}}{2(\lambda_1+\lambda_2)}$, D the discriminant $D := 12\lambda_1(\lambda_1+\lambda_2) + 9\lambda_8^2 + 4(v_{tm}^2 - v_{sm}^2)(v_{tm}^2 - v_{sm}^2 + 2\lambda_1 - \lambda_2) > 0$ by inspection for inequalities $(30)_{4,5}$. This wave propagates at a constant speed $v_n = v_{em}$, with $v_{em}^2 := v_{tm}^2 + \lambda_1 + \frac{3}{2}\lambda_2 + \frac{1}{2}\sqrt{D}$: $v_{tm}^2 + \lambda_1 + \frac{3}{2}\lambda_2 > 0$ for the same inequalities and so $v_{em} > v_{tm}$; obviously this result holds only if $(\lambda_1+\lambda_2) \neq 0$.
The coupled *macro-wave* is now *longitudinal*, that is,

$$[\![\,\ddot{\mathbf{u}}\,]\!]_{em} = \delta\,\varpi\,v_{em}\,\mathbf{n}, \quad \text{with the constant } \varpi := \frac{\kappa_*}{v_l^2 - v_{em}^2}[2\lambda_5 + (\lambda_5+\lambda_6)\zeta], \tag{83}$$

and the discontinuity amplitudes in the mass and temperature variations are, respectively:

$$[\![\,\dot{\rho}\,]\!]_{em} = \rho_*\delta\,\varpi \quad \text{and} \quad [\![\,\ddot{\vartheta}\,]\!]_{em} = \delta\,\check{\vartheta}, \tag{84}$$

with the constant $\check{\vartheta} := \gamma^{-1}v_{em}^2[\kappa_*\gamma_3(2+\zeta) - \gamma_2\varpi]$.

Remark. The *longitudinal micro-waves* of compaction or distention, usually predicted in the voids theory [1, 38], is here excluded, in general, unless we impose the additional condition of the subsequent point ii) on the constitutive thermoelastic constants.

If $(\lambda_1+\lambda_2) = 0$, we have other two significant subcases:

i) *one purely transverse micro-wave,* of vector amplitude $[\![\,\dot{\mathbf{V}}\,]\!]_{pm} = \varrho(\mathbf{I} - \mathbf{n}\otimes\mathbf{n})$ and speed $v_{pm}^2 := v_{sm}^2 - 2\lambda_1$, which corresponds to another *longitudinal macro-, mass, temperature wave* of amplitudes

$$[\![\,\ddot{\mathbf{u}}\,]\!]_{pm} = \varrho\,\varsigma\,v_{pm}\,\mathbf{n}, \quad \text{with} \quad \varsigma = \frac{2\kappa_*\,\lambda_5}{v_l^2 - v_{pm}^2}, \tag{85}$$

$$[\![\,\dot{\rho}\,]\!]_{pm} = \rho_*\varrho\,\varsigma, \quad [\![\,\ddot{\vartheta}\,]\!]_{pm} = \tilde{\vartheta}\varrho, \quad \text{with} \quad \tilde{\vartheta} := \frac{v_{pm}^2}{\gamma}(2\kappa_*\gamma_3 - \gamma_2\varsigma); \tag{86}$$

ii) *a purely longitudinal micro-wave,* which recovers the quoted prediction of voids theories and which propagates at constant speed $v_{lm}^2 := 2v_{tm}^2 - v_{sm}^2 + \lambda_1$; the amplitude of the discontinuity is $[\![\,\dot{\mathbf{V}}\,]\!]_{lm} = \hat{\delta}\,\mathbf{n}\otimes\mathbf{n}$: thus it is a wave of compaction, if the scalar amplitude $\hat{\delta} < 0$, and of distention, if $\hat{\delta} > 0$. In this last case the coupled wave is again *longitudinal* and the connected amplitudes are

$$[\![\,\ddot{\mathbf{u}}\,]\!]_{lm} = \hat{\delta}\,\hat{\varpi}\,v_{lm}\,\mathbf{n}, \quad \text{with} \quad \hat{\varpi} = \frac{\kappa_*(\lambda_5+\lambda_6)}{v_l^2 - v_{lm}^2}, \tag{87}$$

$$[\![\,\dot{\rho}\,]\!]_{lm} = \rho_*\hat{\delta}\,\hat{\varpi}, \quad [\![\,\ddot{\vartheta}\,]\!]_{lm} = \hat{\vartheta}\hat{\delta}, \quad \text{with} \quad \hat{\vartheta} := \frac{v_{lm}^2}{\gamma}(\kappa_*\gamma_3 - \gamma_2\hat{\varpi}). \tag{88}$$

7.2. Homentropic modes

In the particular case when the solid with nano-pores does not conduct heat, *i.e.*, $\mathbf{h} \equiv 0$ whatever $\nabla\vartheta$ we choose, then the energy balance (25) may be written, in the linear theory, in the following form $\theta_* \dot{\eta} = \lambda$ and the jump across a macro-acceleration wave Σ shows that

$$[\![\, \dot{\eta} \,]\!] = 0. \tag{89}$$

Thus, we have established the following result: *In a non-conducting thermoelastic body with nano-pores, every macro-acceleration wave is homentropic*.

The jump Eqs. (78) and (80) remain unchanged, while, since $[\![\, \dot{\vartheta} \,]\!] \neq 0$, Eqs. (79) and (81) are replaced by the following ones:

$$\left[v_n^2 \mathbf{I} - \mathcal{U}(\mathbf{n} \otimes \mathbf{n}) \right] [\![\, \ddot{\mathbf{u}} \,]\!] = -\kappa_* v_n (\lambda_5\, \mathbf{n} \otimes \mathbf{I} + \lambda_6\, \mathbf{I} \otimes \mathbf{n}) [\![\, \dot{\mathbf{V}} \,]\!] - \gamma_2 v_n [\![\, \dot{\vartheta} \,]\!] \mathbf{n}, \tag{90}$$

$$\gamma_1\, v_n [\![\, \dot{\vartheta} \,]\!] = \gamma_2 [\![\, \ddot{\mathbf{u}} \,]\!] \cdot \mathbf{n} - \kappa_* \gamma_3 v_n\, \mathbf{I} \cdot [\![\, \dot{\mathbf{V}} \,]\!], \tag{91}$$

where the relation $(35)_4$ and the condition (71) were used in the jump of Eq. (39) and in Eq. (89); by substituting Eq. (91) into Eq. (90) we obtain the following jump, similar to (79),

$$\left[v_n^2 \mathbf{I} - \tilde{\mathcal{U}}(\mathbf{n} \otimes \mathbf{n}) \right] [\![\, \ddot{\mathbf{u}} \,]\!] = -\kappa_* v_n (\tilde{\lambda}_5\, \mathbf{n} \otimes \mathbf{I} + \lambda_6\, \mathbf{I} \otimes \mathbf{n}) [\![\, \dot{\mathbf{V}} \,]\!], \tag{92}$$

but where we introduced the instantaneous *homentropic acoustic macro-tensor*

$$\tilde{\mathcal{U}}(\mathbf{n} \otimes \mathbf{n}) := \left[v_t^2\, \mathbf{I} + \left(\tilde{v}_l^2 - v_t^2 \right) \mathbf{n} \otimes \mathbf{n} \right], \tag{93}$$

with $\tilde{v}_l^2 := v_l^2 - \gamma_2^2 \gamma_1^{-1}$, and the constant $\tilde{\lambda}_5 := \lambda_5 - \gamma_2 \gamma_3 \gamma_1^{-1}$.

The linear algebraic system of Eqs. (78), (92) and (80) for the amplitudes of the macro-acceleration waves Σ in the homentropic case has the same five solutions with the same propagation speeds, as the homothermal one: two shear optical micro-waves, two transverse micro-waves and one extensional micro-wave. The only variation is in the extensional one and consists in the change of constants v_l and λ_5 with $\tilde{v}_l$ and $\tilde{\lambda}_5$, respectively.

Finally, the temperature jump (91) is absent in all shear optical and transverse micro-waves, while in the extensional one we have:

$$[\![\, \dot{\vartheta} \,]\!]_{em} = \gamma_1^{-1} [\gamma_2 \varsigma - \kappa_* \gamma_3 (2 + \zeta)]\, \delta, \tag{94}$$

with the constant $\tilde{\varpi} := \frac{\kappa_*}{\tilde{v}_l^2 - v_{em}^2} [2\tilde{\lambda}_5 + (\tilde{\lambda}_5 + \lambda_6)\zeta]$.

7.3. Generalized transverse case

This last subsection concerns the behaviour of macro-acceleration waves that are both homothermal and homentropic and which are usually called *generalized transverse waves*. In physical terms these waves are uninfluenced by thermo-mechanical coupling effects in the transmitting matrix material of the porous solid.

Both conditions $[\![\,\dot{\vartheta}\,]\!] = 0$ and $\gamma = 0$ apply; thus, from Eq. (91) (or from (81)), we obtain the following compatibility condition for the generalized transverse wave:

$$[\![\,\ddot{\mathbf{u}}\,]\!] \cdot \mathbf{n} = \kappa_* \gamma_3 \gamma_2^{-1} v_n \, \mathbf{I} \cdot [\![\,\dot{\mathbf{V}}\,]\!]. \tag{95}$$

Also now we have shear optical and transverse micro-waves as in the previous instances; instead, in general, extensional macro-acceleration waves do not occur, unless the condition (95) is satisfied, that is: $\gamma_3(2+\zeta)(v_l^2 - v_{em}^2) = \gamma_2[2\lambda_5 + (\lambda_5 + \lambda_6)\zeta]$.

8. Evolution equations for wave amplitudes

Let us study now the growth or the decay of the macro-acceleration waves Σ which travel through the thermo-elastic material with nano-pores, thus we restrict ourselves to plane waves which are of uniform scalar amplitude with assigned initial value, uniform in the sense that the scalar amplitude does not vary with position on Σ.

For this purpose, we differentiate twice with respect to time each term of Eq. (38) and once those of Eqs. (39) and (41), take into account the balance of micromomentum (40) and form jumps of all equations across the wave Σ to have:

$$\begin{aligned}
&[\![\,\ddot{\rho}\,]\!] = -\rho_* [\![\,\mathrm{Div}\,\ddot{\mathbf{u}}\,]\!], \\
&\mathbf{d} = v_t^2 [\![\,\Delta\dot{\mathbf{u}}\,]\!] + (v_l^2 - v_t^2)[\![\,\nabla\,(\mathrm{Div}\,\dot{\mathbf{u}})\,]\!] + \kappa_* \left(\lambda_5 [\![\,\nabla\,(\mathrm{tr}\,\dot{\mathbf{V}})\,]\!] + \lambda_6 [\![\,\mathrm{Div}\,\dot{\mathbf{V}}\,]\!]\right) + \gamma_2 [\![\,\nabla\,\dot{\vartheta}\,]\!], \\
&[\![\,\ddot{\mathbf{V}}\,]\!] = v_{sm}^2 [\![\,\Delta\mathbf{V}\,]\!] + 2(v_{tm}^2 - v_{sm}^2)\,\mathrm{sym}\,[\![\,\nabla\,(\mathrm{Div}\,\mathbf{V})\,]\!] + \lambda_1 [\![\,\nabla^2(\mathrm{tr}\,\mathbf{V})\,]\!] + \\
&\qquad\qquad + \left\{\lambda_1 [\![\,\mathrm{Div}\,(\mathrm{Div}\,\mathbf{V})\,]\!] + \lambda_2 [\![\,\Delta(\mathrm{tr}\,\mathbf{V})\,]\!] - \omega\,\mathbf{I} \cdot [\![\,\dot{\mathbf{V}}\,]\!]\right\}\mathbf{I} - 2\sigma [\![\,\dot{\mathbf{V}}\,]\!], \\
&\gamma [\![\,\Delta\dot{\vartheta}\,]\!] + \gamma_1 [\![\,\ddot{\vartheta}\,]\!] + \gamma_2 [\![\,\mathrm{Div}\,\ddot{\mathbf{u}}\,]\!] + \gamma_3\,\mathbf{I} \cdot [\![\,\ddot{\mathbf{V}}\,]\!] = 0,
\end{aligned} \tag{96}$$

where $\mathbf{d}\,(\equiv\ d_n\mathbf{n} + d_e\mathbf{e} + d_f\mathbf{f})$ represents the jump in the third time-derivative of the displacement field $\mathbf{u}$.

Algebraic computations, very similar to those carried out in [26, 42], with the use of the Hugoniot-Hadamard compatibility condition (70) and of definitions (76) and (77) of the homothermal acoustic tensors, we obtain the following evolution equations for the propagating wave Σ:

$$[\![\,\ddot{\rho}\,]\!] = \rho_* (v_n^{-1} d_n - \mathrm{Div}\,[\![\,\ddot{\mathbf{u}}\,]\!]), \tag{97}$$

$$\begin{aligned}
&\left[v_n^2\mathbf{I} - \mathcal{U}(\mathbf{n}\otimes\mathbf{n})\right]\mathbf{d} = \kappa_* v_n \left\{v_n \left[\lambda_5 \nabla\,(\mathrm{tr}\,[\![\,\dot{\mathbf{V}}\,]\!]) + \lambda_6 \mathrm{Div}\,[\![\,\dot{\mathbf{V}}\,]\!]\right] - \left[\lambda_5 (\mathrm{tr}\,[\![\,\ddot{\mathbf{V}}\,]\!])\mathbf{n} + \lambda_6 [\![\,\ddot{\mathbf{V}}\,]\!]\mathbf{n}\right]\right\} + \\
&\quad + v_n \left\{(v_t^2 - v_l^2)\left[(\mathrm{Div}\,[\![\,\ddot{\mathbf{u}}\,]\!])\mathbf{n} + \nabla\,([\![\,\ddot{\mathbf{u}}\,]\!] \cdot \mathbf{n})\right] - 2v_t^2 (\nabla\,[\![\,\ddot{\mathbf{u}}\,]\!])\mathbf{n} - \gamma_2 [\![\,\ddot{\vartheta}\,]\!]\mathbf{n}\right\} + \gamma_2 v_n^2 \nabla\,[\![\,\dot{\vartheta}\,]\!],
\end{aligned} \tag{98}$$

$$\begin{aligned}
&\left[v_n^2\,\mathcal{I} - \mathcal{C}(\mathbf{n}\otimes\mathbf{n})\right][\![\,\ddot{\mathbf{V}}\,]\!] = 2v_n(v_{sm}^2 - v_{tm}^2)\,\mathrm{sym}\,\left[(\mathrm{Div}\,[\![\,\dot{\mathbf{V}}\,]\!]) \otimes \mathbf{n} + \nabla\,([\![\,\dot{\mathbf{V}}\,]\!]\mathbf{n})\right] - \\
&\quad - 2\,v_n \left\{v_{sm}^2 (\nabla\,[\![\,\dot{\mathbf{V}}\,]\!])\mathbf{n} + \lambda_1 \left\{\mathrm{sym}\,[\nabla\,(\mathrm{tr}\,[\![\,\dot{\mathbf{V}}\,]\!]) \otimes \mathbf{n}] + (\mathbf{n} \cdot \mathrm{Div}\,[\![\,\dot{\mathbf{V}}\,]\!])\mathbf{I}\right\}\right\} - \\
&\qquad\qquad - 2\,v_n\,\lambda_2 [\mathbf{n} \cdot \nabla\,(\mathrm{tr}\,[\![\,\dot{\mathbf{V}}\,]\!])]\mathbf{I} - v_n^2 [\omega\,\mathrm{tr}\,[\![\,\dot{\mathbf{V}}\,]\!]\,\mathbf{I} + 2\,\sigma\,[\![\,\dot{\mathbf{V}}\,]\!]],
\end{aligned} \tag{99}$$

$$\begin{aligned}
&\gamma\Theta = v_n \left[2\gamma(\mathbf{n} \cdot \nabla\,[\![\,\ddot{\vartheta}\,]\!]) + \gamma_2 d_n\right] - \\
&\qquad\qquad - v_n^2 \left(\gamma\Delta[\![\,\dot{\vartheta}\,]\!] + \gamma_1 [\![\,\ddot{\vartheta}\,]\!] + \gamma_2 \mathrm{Div}\,[\![\,\ddot{\mathbf{u}}\,]\!] + \kappa_* \gamma_3 \mathrm{tr}\,[\![\,\ddot{\mathbf{V}}\,]\!]\right),
\end{aligned} \tag{100}$$

with Θ that indicates the jump in the third time-derivative of the temperature change field ϑ.

Therefore, the transport equations in the linearized case will give standard evolution laws of the type $f' = -\mu f$ and hence $f = f_0\,\mathrm{e}^{-\mu\phi}$, where μ is a constant, f_0 is the strength of the wave at $\tau = \tau_0$ and ϕ is the increasing distance, measured along the normal to the wave, from the wave front at the same time.

8.1. Evolution of homothermal waves

Now, since for a plane homothermal macro-acceleration wave entering the natural reference placement $\mathcal{B}_*$ the jump of $\dot{\vartheta}$ vanishes for the Fourier condition (72), by developing the analysis of equations (97)-(100) we get the following consequences:

8.1.1. Shear optical case

By inserting the micro-wave solution of §7.1 of amplitude $[\![\,\dot{\mathbf{V}}\,]\!]_{sm}$ and speed v_{sm}, which have fields $\dot{u}$, $\dot{\rho}$ and $\ddot{\vartheta}$ continuous through the wave, we obtain that:

$$[\![\,\ddot{\rho}\,]\!] = \rho_*\, d_n v_{sm}^{-1}, \quad \left[v_{sm}^2\mathbf{I} - \mathcal{U}(\mathbf{n}\otimes\mathbf{n})\right]\mathbf{d} = -\kappa_* v_{sm}\left[\lambda_5(\mathrm{tr}\,[\![\,\ddot{\mathbf{V}}\,]\!])\mathbf{n} + \lambda_6[\![\,\ddot{\mathbf{V}}\,]\!]\mathbf{n}\right], \tag{101}$$

$$\left[v_{sm}^2\,\mathcal{I} - \mathcal{C}(\mathbf{n}\otimes\mathbf{n})\right][\![\,\ddot{\mathbf{V}}\,]\!] = \\ = 2v_{sm}^3\left\{\left[\frac{d\alpha}{dn} + \frac{\sigma\alpha}{v_{sm}}\right](\mathbf{f}\otimes\mathbf{f} - \mathbf{e}\otimes\mathbf{e}) - \left[\frac{d\beta}{dn} + \frac{\sigma\beta}{v_{sm}}\right](\mathbf{e}\otimes\mathbf{f} + \mathbf{f}\otimes\mathbf{e})\right\}, \tag{102}$$

$$\gamma\Theta = \gamma_2 v_{sm} d_n - \kappa_*\gamma_3\, v_{sm}^2\,\mathrm{tr}\,[\![\,\ddot{\mathbf{V}}\,]\!]; \tag{103}$$

hence, from Eq. (102), it must be $[\![\,\ddot{V}\,]\!]_{11} = [\![\,\ddot{V}\,]\!]_{12} = [\![\,\ddot{V}\,]\!]_{13} = 0$ and $[\![\,\ddot{V}\,]\!]_{33} = -[\![\,\ddot{V}\,]\!]_{22}$, while $[\![\,\ddot{V}\,]\!]_{22}$ and $[\![\,\ddot{V}\,]\!]_{23}$ remain undefined,

$$\alpha(\tau) = \alpha_0\exp\left(-\frac{\sigma}{v_{sm}}\phi\right) \text{ and } \beta(\tau) = \beta_0\exp\left(-\frac{\sigma}{v_{sm}}\phi\right), \tag{104}$$

with α_0 and β_0 the values at $\tau = \tau_0$. Instead, by analysing Eq. $(101)_2$, we have that $\mathbf{d}_{sm} = \mathbf{0}$ and thus $[\![\,\ddot{\rho}\,]\!]_{sm} = 0$ and $\Theta_{sm} = 0$.

This kind of micro-wave does not cause any disturbance in the mechanical and thermal fields and the scalar amplitudes α and β decay to zero as the time interval $(\tau - \tau_0)$ increases indefinitely (because ϕ behaves so).

8.1.2. Transverse micro-wave

For the transverse solutions of §7.1, whose amplitude is $[\![\,\dot{\mathbf{V}}\,]\!]_{tm}$, speed v_{tm} and jump $[\![\,\dot{\mathbf{u}}\,]\!]_{tm}$ given by equation (82) (while $\dot{\rho}$ and $\ddot{\vartheta}$ are continuous), the algebraic system of evolution Eqs. (97)-(100) reduces to the following one:

$$[\![\,\ddot{\rho}\,]\!] = \rho_* v_{tm}^{-1} d_n, \quad \left[v_{tm}^2\mathbf{I} - \mathcal{U}(\mathbf{n}\otimes\mathbf{n})\right]\mathbf{d} = \tag{105}$$

$$= \kappa_* v_{tm}^2\lambda_6\frac{v_{tm}^2 + v_t^2}{v_{tm}^2 - v_t^2}\left(\frac{d\chi_e}{dn}\mathbf{e} + \frac{d\chi_f}{dn}\mathbf{f}\right) - \kappa_* v_{tm}\left[\lambda_5(\mathrm{tr}\,[\![\,\ddot{\mathbf{V}}\,]\!])\mathbf{n} + \lambda_6[\![\,\ddot{\mathbf{V}}\,]\!]\mathbf{n}\right],$$

$$\left[v_{tm}^2\,\mathcal{I} - \mathcal{C}(\mathbf{n}\otimes\mathbf{n})\right][\![\,\ddot{\mathbf{V}}\,]\!] = -4v_{tm}^3\mathrm{sym}\left[\left(\frac{d\chi_e}{dn} + \frac{\sigma\chi_e}{v_{tm}}\right)\mathbf{e} + \left(\frac{d\chi_f}{dn} + \frac{\sigma\chi_f}{v_{tm}}\right)\mathbf{f}\right]\otimes\mathbf{n}, \tag{106}$$

$$\gamma\Theta = \gamma_2 v_{tm} d_n - \kappa_*\gamma_3\, v_{tm}^2\mathrm{tr}\,[\![\,\ddot{\mathbf{V}}\,]\!]. \tag{107}$$

Therefore, we have from Eq. (106) that all the components of $[\![\,\ddot{\mathbf{V}}\,]\!]_{tm}$ are equal to zero except for the undetermined $[\![\,\ddot{V}\,]\!]_{12}$ and $[\![\,\ddot{V}\,]\!]_{13}$, while the scalar amplitudes are given by

$$\chi_i(\tau) = \chi_{i0} \exp\left(-\frac{\sigma}{v_{tm}}\phi\right), \text{ for } i = e, f, \tag{108}$$

with χ_{i0} the values at $\tau = \tau_0$. Instead, Eqs. (105), (107) and (108) establish that

$$(d_n)_{tm} = [\![\,\ddot{\rho}\,]\!]_{tm} = \Theta_{tm} = 0,\ (d_i)_{tm} = \frac{\kappa_* v_{tm}\,\lambda_6}{v_t^2 - v_{tm}^2}\left([\![\,\ddot{V}\,]\!]_{1i} + \frac{\sigma(v_{tm}^2 + v_t^2)}{v_{tm}^2 - v_t^2}\chi_i\right), \text{ for } i = e, f. \tag{109}$$

Also in this case the scalar amplitudes decay to zero as the time interval $(\tau - \tau_0)$ increases indefinitely, but, unlike shear optical waves, we have here a macro-acceleration jump with a third order discontinuity related to the elastic properties of nano-pores and to a part that decays to zero.

8.1.3. Extensional mode

In this case the solutions of §7.1 for the amplitude and the speed are $[\![\,\dot{\mathbf{V}}\,]\!]_{em}$ and v_{em}, respectively, and thus, by applying Eqs (83) and (84), we have:

$$[\![\,\ddot{\rho}\,]\!] = \rho_* d_n v_{em}^{-1} - \rho_* v_{em}\varpi\frac{d\delta}{dn}, \qquad \left[v_{em}^2\mathbf{I} - \mathcal{U}(\mathbf{n}\otimes\mathbf{n})\right]\mathbf{d} = \tag{110}$$

$$= \kappa_* v_{em}\left\{v_{em}\left[2\lambda_5 + (\lambda_5 + \lambda_6)\zeta\right]\frac{v_{em}^2 + v_l^2}{v_{em}^2 - v_l^2}\,\frac{d\delta}{dn}\,\mathbf{n} - \left[\lambda_5(\mathrm{tr}\,[\![\,\ddot{\mathbf{V}}\,]\!])\mathbf{n} + \lambda_6[\![\,\ddot{\mathbf{V}}\,]\!]\mathbf{n}\right]\right\},$$

$$\left[v_{em}^2\,\mathbf{I} - \mathcal{C}(\mathbf{n}\otimes\mathbf{n})\right][\![\,\ddot{\mathbf{V}}\,]\!] = -v_{em}\left\{2\frac{d\delta}{dn}\left[\lambda_1\zeta + \lambda_2(2+\zeta)\right] + v_{em}\omega(2+\zeta)\delta\right\}\mathbf{I} +$$

$$+2v_{em}\left\{\frac{d\delta}{dn}\left[(v_{sm}^2 - 2v_{tm}^2)\zeta - \lambda_1(2+\zeta)\right] - v_{em}\sigma\zeta\delta\right\}(\mathbf{n}\otimes\mathbf{n}) \tag{111}$$

$$-2v_{em}\left(v_{sm}^2\frac{d\delta}{dn} + v_{em}\sigma\delta\right)(\mathbf{e}\otimes\mathbf{e} + \mathbf{f}\otimes\mathbf{f}),$$

$$\gamma\Theta = v_{em}\left(2\,\gamma\,\breve{\vartheta}\frac{d\delta}{dn} + \gamma_2 d_n\right) - v_{em}^2\left(\gamma_1\breve{\vartheta}\delta + \gamma_2 v_{em}\varpi\frac{d\delta}{dn} + \kappa_*\gamma_3\,\mathrm{tr}\,[\![\,\ddot{\mathbf{V}}\,]\!]\right). \tag{112}$$

The solutions of the system (111) are $[\![\,\ddot{V}\,]\!]_{ij} = 0$, if $i \neq j$, and $[\![\,\ddot{V}\,]\!]_{33} = [\![\,\ddot{V}\,]\!]_{22}$, while, to determine δ, it is necessary to know either of $[\![\,\ddot{V}\,]\!]_{11}$ or $[\![\,\ddot{V}\,]\!]_{22}$ previously, otherwise it remains undetermined and we cannot say anything about the growth or the decay of this wave; vice versa, if we are able to assign the behaviour of the amplitude δ, we can resolve the remaining two jumps. Moreover, from Eqs. (110) and (112), we obtain that $(d_e)_{em} = (d_f)_{em} = 0$, while also $[\![\,\ddot{\rho}\,]\!]_{em}$, $(d_n)_{em}$ and Θ_{em} suffer of the same undeterminacies already spoken about. The micro-wave is then accompanied by second and third order discontinuities in macro-mechanical, mass and thermal fields.

8.1.4. $\lambda_1 + \lambda_2 = 0$ case.

In this peculiar subcase we observed in §7.1:

i) *A purely transverse micro-wave* of amplitude $[\![\dot{\mathbf{V}}]\!]_{pm}$ and speed v_{pm}, the other jumps being given by Eqs. (85) and (86). By performing same developments of previous solutions, we obtain that $[\![\ddot{V}]\!]_{ij} = 0$, if $i \neq j$, and $[\![\ddot{V}]\!]_{33} = [\![\ddot{V}]\!]_{22}$ (which remain undetermined); moreover, $[\![\ddot{V}]\!]_{11} = \frac{2v_{pm}^2\omega}{v_{tm}^2-2v_{pm}^2-\lambda_1}\varrho$, with the scalar amplitudes ϱ given by $\varrho(\tau) = \varrho_0 \exp\left(-\frac{\omega+\sigma}{v_{pm}}\phi\right)$ (ϱ_0 being the value at $\tau = \tau_0$). Thus, it results that $(d_e)_{pm} = (d_f)_{pm} = 0$, while $(d_n)_{pm} = \Gamma\varrho - \frac{2\kappa_*\lambda_5 v_{pm}}{v_{pm}^2-v_l^2}[\![\ddot{V}]\!]_{22}$, $[\![\ddot{\rho}]\!]_{pm} = \rho_* v_{pm}^{-1} d_n$ and $\gamma\Theta_{pm} = \gamma_2 v_{pm} d_n - 2\kappa_*\gamma_3 v_{pm}^2 [\![\ddot{V}]\!]_{22} - \Pi\,\varrho$, where Γ and Π are constants related to previous defined constitutive constants.

Hence the third order discontinuity of the longitudinal macro-wave, induced by the purely transverse micro-wave, has a first part that decays to zero as the time interval $(\tau - \tau_0)$ go to infinity and a second one related to the elastic properties of nano-pores, as well as discontinuities in the mass and temperature derivatives.

ii) *A purely longitudinal micro-wave* of amplitude $[\![\dot{\mathbf{V}}]\!]_{lm}$ and speed of propagation v_{lm} for which we have that all $[\![\ddot{V}]\!]_{ij} = 0$, if $i \neq j$, $[\![\ddot{V}]\!]_{11}$ remains undefined and $[\![\ddot{V}]\!]_{33} = [\![\ddot{V}]\!]_{22} = \frac{\omega}{v_{lm}^2-v_{sm}^2+2\lambda_1}\hat{\delta}$, with $\hat{\delta}(\tau) = \hat{\delta}_0 \exp\left(-\frac{\omega+\sigma}{2v_{lm}}\phi\right)$ and $\hat{\delta}_0$ its value at $\tau = \tau_0$; in addition, also now $(d_e)_{lm} = (d_f)_{lm} = 0$, while $(d_n)_{lm} = \Xi\,\hat{\delta} - \kappa_* v_{lm}(\lambda_5+\lambda_6)[\![\ddot{V}]\!]_{11}$, $[\![\ddot{\rho}]\!]_{lm} = \rho_* v_{lm}^{-1} d_n$ and $\gamma\Theta_{lm} = \gamma_2 v_{lm} d_n - \kappa_*\gamma_3 v_{lm}^2 [\![\ddot{V}]\!]_{11} - \mathrm{Y}\,\hat{\delta}$, with Ξ and Y constants related to constitutive constants.

Therefore, also the pure micro-wave of compaction or distention is accompanied by a third order discontinuities in the mechanical field with a first part that decays to zero with the increasing of the time interval $(\tau - \tau_0)$ and a second one related to nano-pores properties.

8.2. Homentropic and generalized transverse evolution instances

○) *Homentropic macro-acceleration waves:* When the solid with nano-pores does not conduct heat (see §7.2), we have to substitute evolution Eq. (100) with the following one:

$$\gamma_1 [\![\ddot{\vartheta}]\!] = \gamma_2 v_n^{-1} d_n - \gamma_2 \mathrm{Div}\,[\![\ddot{\mathbf{u}}]\!] - \kappa_*\gamma_3 \mathrm{tr}\,[\![\dot{\mathbf{V}}]\!], \tag{113}$$

which is obtained by deriving with respect to the time τ the energy balance $\dot{\eta} = \theta_*^{-1}\lambda$, by using relation $(35)_4$ and by taking its jump.

As we observed in §7.2, discussions about this subcase follow closely those carried out for the homothermal one with respect to shear optical and transverse macro-acceleration waves; instead, for the extensional, purely transverse and purely longitudinal ones the only change consists in the choose of constants $\tilde{v}_l$ and $\tilde{\lambda}_5$ in place of v_l and λ_5: for example, Eq. $(110)_2$ must be substituted by

$$\left[v_{em}^2\mathbf{I} - \mathcal{U}(\mathbf{n}\otimes\mathbf{n})\right]\mathbf{d}_{em} = \tag{114}$$
$$= \kappa_* v_{em}\left\{v_{em}\left[2\tilde{\lambda}_5 + (\tilde{\lambda}_5+\lambda_6)\zeta\right]\frac{v_{em}^2+v_l^2}{v_{em}^2-\tilde{v}_l^2}\frac{d\delta}{dn}\mathbf{n} - \left[\lambda_5(\mathrm{tr}\,[\![\ddot{\mathbf{V}}]\!]_{em})\mathbf{n} + \lambda_6[\![\ddot{\mathbf{V}}]\!]_{em}\mathbf{n}\right]\right\},$$

while Eq. (113) gives $\gamma_1[\![\ddot{\vartheta}]\!]_{em} = \gamma_2 v_{em}^{-1}(d_n)_{em} - \gamma_2\tilde{\omega}v_{em}\frac{d\delta}{dn} - \kappa_*\gamma_3([\![\ddot{V}]\!]_{11} + 2[\![\ddot{V}]\!]_{22})$.

Hence, with this simple change of constants, the conclusions about the evolution of amplitudes of the macro-acceleration waves remain the same as in the corresponding homothermal case.

◦) *Generalized Transverse Case:* This last peculiar wave, both homothermal and homentropic, has the same shear optical and transverse solutions as the homothermal case (see §7.2) and so the solutions of the evolution equations have the same behaviour.

The extensional micro-wave, and so comments about its amplitude evolution, occurrs only if the condition (95) is satisfied, *i.e.*, $\gamma_3(2+\zeta)(v_l^2 - v_{em}^2) = \gamma_2[2\lambda_5 + (\lambda_5 + \lambda_6)\zeta]$.

9. Conclusions

The results showed in this chapter can be outlined as it follows:

a) *Linear thermo-dynamic theory.* We derived the linear theory of a thermoelastic solid with nano-pores which includes inelastic surface effects associated with changes in the deformation of the holes in the vicinity of void boundaries and which generalizes classical voids theories. In order to get the fields equations we used the principles of objectivity and equipresence, besides the compatibility with the Clausius-Duhem inequality.

b) *Micro-vibrations.* The first application to micro-vibrations in absence of dissipation gives origin to three admissible results: a dilatational micro-thermal oscillation and two solutions, both with no thermal vibrations, with the same frequency and with null trace: a shear mode and an extensional mode with constant volume.

c) *Plane waves.* Here we presented the solutions of secular equations governing the propagation of harmonic plane waves in the porous thermoelastic medium: there can exist two shear optical micro-elastic waves, two coupled transverse elastic waves and four coupled longitudinal thermo-elastic waves. The exact or approximate values of the phase speeds, specific losses, attenuation factors and amplitude ratios are discussed for large and small frequencies.

d) *Macro-acceleration waves.* Last investigation regarded the propagation conditions and the growth equations which govern the motion of particular weak singularities, called macro-acceleration waves, for which only jumps of the derivatives of the macro- and micro-displacement of order 2 and of the temperature of order 1 are of interest in the theory.

We observed that, for a linear conducting homogeneous centrosymmetric isotropic material with nano-pores, every macro-acceleration wave is homothermal and only three speeds of propagation are possible: i) one related to two shear-optical micro-modes completely decoupled from the mass and thermoelastic macro-properties of the matrix material and which decay to zero when the time interval increases; ii) the second velocity associated to two transverse micro-modes coupled with a transverse macro-acceleration wave, spreading without perturbing mass and thermal fields: the micro-modes decay still to zero, while the associated macro-ones have a constant part and an added contribution that decays still to zero; iii) the third one linked to one extensional micro-wave coupled with a longitudinal macro-wave and with discontinuities in the second and third order of derivatives of mass and thermal fields.

Instead in the non-conducting case every wave is homentropic, but we obtained the same number of propagation velocities and of macro-acceleration waves as in the previous homothermal instance.

At the end, for generalized transverse macro-acceleration waves, only the extensional micro-mode does not occurr, in general.

Acknowledgements

The support of the Italian "Gruppo Nazionale per la Fisica Matematica" of the "Istituto Nazionale di Alta Matematica" and of the Department of Mechanics and Materials of the "Mediterranean" University of Reggio Calabria (Italy) is gratefully acknowledged.

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