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Universality of Transition to Chaos in All Kinds of Nonlinear Differential Equations

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1. Introduction

The basically finished universal theory of dynamical chaos in all kinds of nonlinear differential equations including dissipative and conservative, nonautonomous and autonomous nonlinear systems of ordinary and partial differential equations and differential equations with delay arguments is shortly presented in the paper. Consequence of the theory is an existence of the uniform universal mechanism of self-organizing in the huge class of the mathematical models having the applications in many areas of science and techniques and describing the numerous physical, chemical, biological, economic and social both natural and public phenomena and processes. All theoretical positions and results are received within last several years by extremely author and his pupils and confirmed with numerous examples, illustrations and numerical calculations.

The basis of this theory consists of the Feigenbaum theory of period doubling bifurcations in one-dimensional mappings (Feigenbaum, 1978), the Sharkovskii theory of subharmonic bifurcations of stable cycles of an arbitrary period up to the cycle of period three in one-dimensional mappings (Sharkovskii, 1964), the Magnitskii theory of homoclinic and heteroclinic bifurcations of stable cycles and tori in systems of differential equations and the Magnitskii theory of rotor type singular points of two-dimensional nonautonomous systems of differential equations with periodic coefficients of leading linear parts as a bridge between one-dimensional mappings and differential equations (Magnitskii & Sidorov, 2006; Magnitskii, 2007; Magnitskii, 2008; Magnitskii, 2008b; Magnitskii, 2010).

It is shown that this universal Feigenbaum-Sharkovskii-Magnitskii (FSM) bifurcation theory of transition to dynamical chaos takes place in all classical three-dimensional chaotic dissipative systems of ordinary differential equations including Lorenz hydrodynamic system, Ressler chemical system, Chua electro technical system, Magnitskii macroeconomic system and many others. It takes place also in well-known two-dimensional non-

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autonomous and many-dimensional autonomous nonlinear dissipative systems of ordinary differential equations, such as Duffing-Holmes, Mathieu, Croquette and Rikitaki equations. It takes place also in nonlinear partial differential equations and differential equations with delay arguments, such as Brusselyator, Ginzburg-Landau, Navier-Stokes and Mackey-Glass equations, reaction-diffusion systems and systems of differential equations describing excitable and autooscillating mediums. Moreover, the same scenario of transition to chaos takes place also in conservative and, in particularly, Hamiltonian systems such as Henon-Heiles and Yang-Mills systems, conservative Duffing-Holmes, Mathieu and Croquette equation and many others.

Thus, the question is about discovery and description of the uniform universal mechanism of the arranging of surrounding us infinitely complex and infinitely various nonlinear world. And this nonlinear world is arranged under uniform laws, and these laws are laws of nonlinear dynamics, qualitative theory of nonlinear systems of differential equations and theory of bifurcations in such systems.

2. Dynamical chaos in nonlinear dissipative systems of ordinary differential equations

2.1. Two-dimensional systems with periodic coefficients

Consider a smooth family of two-dimensional real nonlinear non-autonomous systems of ordinary differential equations

$$\dot{u} = D(t,\mu)u(t) + H(u,t,\mu),\ H(0,t,\mu) \equiv 0, \tag{1}$$

with a $T(\mu)$ -periodic matrix $D(t,\mu)$ of the leading linear part depending on a scalar system parameter μ. Expansion of a function $H(u,t,\mu)$ on components of vector u begins with members of the second order. The Floquet theory states that the fundamental matrix solution $U(t,\mu)$ of the linear part of system of Eqs. (1) can be represented in the form $U(t,\mu) = P(t,\mu)e^{B(\mu)t}$, where $P(t,\mu)$ is some T -periodic complex matrix and $B(\mu)$ is some constant complex matrix whose eigenvalues are named as Floquet exponents. It is important that the real linear system can have various complex but not complex-conjugate Floquet exponents $\alpha_1(\mu)$ and $\alpha_2(\mu)$. Real parts $\mathrm{Re}\,\alpha_1(\mu) = \beta_1(\mu),\ \mathrm{Re}\,\alpha_2(\mu) = \beta_2(\mu),$ can be different but imaginary parts $\mathrm{Im}\,\alpha_2(\mu) = \mathrm{Im}\,\alpha_1(\mu) + 2\pi k$ can be equal or differ from each other on $2\pi k$. Singular point $O(0,0)$ of a two-dimensional non-autonomous real system of Eqs. (1) with periodic coefficients of its leading linear part is a **rotor** if corresponding linear system has complex Floquet exponents with equal imaginary and different real parts (Magnitskii, 2008, 2011; Magnitskii & Sidorov, 2006). Canonical form of a rotor is a linear system

$$\begin{aligned}\dot{u}_1 &= \frac{\beta_1 + \beta_2 + (\beta_1 - \beta_2)\cos\omega t}{2}u_1 + \frac{(\beta_1 - \beta_2)\sin\omega t - \omega}{2}u_2 \\ \dot{u}_2 &= \frac{(\beta_1 - \beta_2)\sin\omega t + \omega}{2}u_1 + \frac{\beta_1 + \beta_2 - (\beta_1 - \beta_2)\cos\omega t}{2}u_2\end{aligned} \tag{2}$$

with $2\pi / \omega$ - periodic coefficients. In Eqs. (2) β_1 and β_2 are arbitrary real constants.

2.1.1. FSM – scenario of transition to chaos

If real parts of Floquet exponents depend on parameter μ which is changing, then the Sharkovskii subharmonic cascade of bifurcations of stable limit cycles is realizing in system of Eqs. (1) in accordance with the Sharkovskii order (Magnitskii, 2008; Magnitskii & Sidorov, 2006):

$$\begin{aligned} &1 \lhd 2 \lhd 2^2 \lhd 2^3 \lhd \ldots \lhd 2^2 \cdot 7 \lhd 2^2 \cdot 5 \lhd 2^2 \cdot 3 \lhd \ldots \\ &\qquad \ldots \lhd 2 \cdot 7 \lhd 2 \cdot 5 \lhd 2 \cdot 3 \lhd \ldots \lhd 7 \lhd 5 \lhd 3. \end{aligned} \tag{3}$$

The ordering $n \lhd k$ in (3) means that the existence of a cycle of period k implies the existence of a cycle of period n. So, if a system of Eqs. (1) has a stable limit cycle of period three then it has also all unstable cycles of all periods in accordance with the Sharkovskii order. So, the family of systems of Eqs. (1) can have irregular attractors only at infinitely many accumulation points of bifurcation values of the system parameter. Every such value is a limit of a sequence of values of some Feigenbaum subcascade of period doubling bifurcations in Sharkovskii cascade. Thus, any irregular attractor of the family of systems of Eqs. (1) with rotor type singular point is a **singular** attractor, as it is defined in (Magnitskii & Sidorov, 2006; Magnitskii, 2011). Simple singular attractor is almost stable non-periodic trajectory which is the limit of a sequence of periodic orbits of some Feigenbaum subcascade of period doubling bifurcations. Complex singular attractor exists only in bifurcation values corresponding to homoclinic or heteroclinic separatrix loops. For other values of the parameter μ the family of systems of Eqs. (1) has only regular attractors - asymptotically orbitally stable periodic trajectories, even of a very large period.

Obviously, the simplest singular attractor is the Feigenbaum attractor, i.e. the first non-periodic attractor existing in the family of systems of Eqs. (1) for $\mu = \mu_\infty$, where the value μ_∞ is the first limit of the sequence of bifurcation values μ for which period doubling bifurcations of the original cycle take place. Note that the Feigenbaum cascade of period doubling bifurcations is the beginning of the Sharkovskii subharmonic cascade. Note also, that the subharmonic cascade of bifurcations in accordance with the Sharkovskii order (3) does not exhaust the entire complexity of transition to chaos in two-dimensional nonlinear nonautonomous dissipative systems of ordinary differential equations with rotors. It can be continued at least by the Magnitskii homoclinic cascade of bifurcations of stable cycles converging to a homoclinic loop of the rotor type singular point.

As an example, we consider a simplest two-dimensional nonlinear non-autonomous system of Eqs. (1) with leading linear part (2) in which $\beta_1(\mu) = 2\mu,\ \beta_2(\mu) = 2\mu - 4$:

$$\begin{aligned} \dot{u}_1 &= 2(\mu - 1 + \cos(\omega t))u_1 + (2\sin(\omega t) - \omega / 2)u_2 - u_2^2, \\ \dot{u}_2 &= (2\sin(\omega t) + \omega / 2)u_1 + 2(\mu - 1 - \cos(\omega t))u_2. \end{aligned} \tag{4}$$

For $\omega = 4$ and for growth of the parameter μ stable cycles are generated in the system of Eqs. (4) in accordance with the Sharkovskii order (3) and then in accordance with the Magnitskii homoclinic order. These cycles of period two and three, one of the singular attractors and homoclinic cycles of periods four and five are presented in Fig.1 (Magnitskii & Sidorov, 2006). Thus, in this system full bifurcation FSM (Feigenbaum-Sharkovskii-Magnitskii) scenario of transition to dynamical chaos is realized.

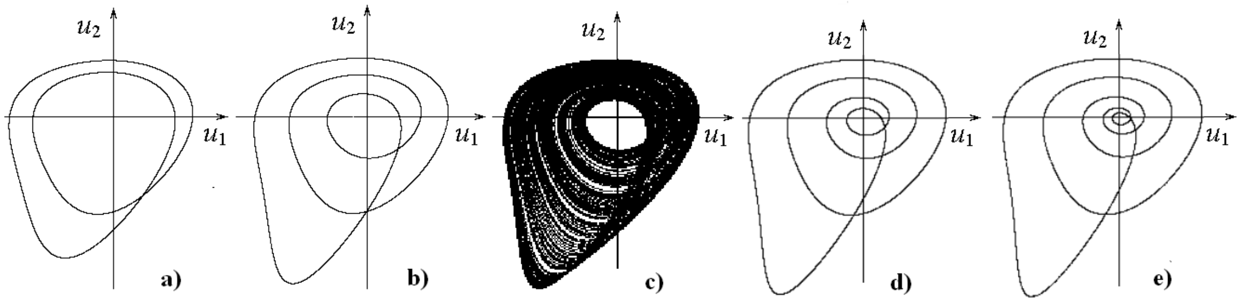

Figure 1. Stable cycles of period two (a) and period three (b), singular attractor (c) and homoclinic cycles of periods four (d) and five (e) in the system of Eqs. (4).

2.1.2. Topological structure of singular attractors

The problem which can be named as a main problem of chaotic dynamics of nonlinear systems of differential equations, is to find out how the boundary of the separatrix surface of the original singular cycle becomes more complex as the bifurcation parameter increases and how the onset of infinitely many regular and singular attractors of the system settle down on this separatrix manifold in accordance with a certain order (Sharkovskii order, homoclinic or heteroclinic order).

Note, that the simplest performance of a two-dimensional manifold in three-dimensional space on which all cycles in the Sharkovskii order and singular attractors can be placed without self-intersections was found in (Gilmore & Lefranc, 2002) in the form of branching manifold with the use of the Birman-Williams theorem and the principles of symbolic dynamics. However, such manifold must have a gluing, so that one can use it to explain the chaotic structure of semiflows but cannot generalize these results to flows, because this contradicts with the uniqueness theorem for solutions of differential equations. Hence, the representation given in (Gilmore & Lefranc, 2002) cannot be considered satisfactory.

We obtained a representation of the boundary of the separatrix surface of an original singular cycle of an arbitrary nonlinear dissipative system in a form of an infinitely folded two-dimensional heteroclinic separatrix manifold which Poincare section is named as **heteroclinic separatrix zigzag** (Magnitskii, 2010). It spanned by Moebius bands joining various cycles from the Feigenbaum period doubling cascade of bifurcations. From this consideration it becomes clear how and why cycles are arranged on this manifold in subharmonic and homoclinic order in the case of sufficiently strong dissipation, and why this order can be violated in systems with small dissipation and in conservative systems.

Rewrite the system of Eqs. (4) in the form of autonomous 4d-system

$$\begin{aligned} \dot{u} &= (2(\mu - b) + 2bp)u + (2bq - \omega / 2)v - v^2, \quad \dot{p} = -\omega q \\ \dot{v} &= (2bq + \omega / 2)u + (2(\mu - b) - 2bp)v, \quad \dot{q} = \omega p \end{aligned} \tag{5}$$

with $b = 1, u_2 = v$ and with the cycle $p^2 + q^2 = 1$. The parameter μ in system of Eqs. (5) is a bifurcation parameter, and the parameter b is responsible for dissipation. For small b and small μ the system is weakly dissipative, for large b and small μ it is strongly dissipative. Besides at $\mu = b$ the system of Eqs. (5) is conservative.

As a rule, all known dissipative systems of nonlinear differential equations are strongly dissipative, which has for many decades prevented one from studying the structure of their irregular attractors even with the use of most advanced computers. Last circumstance stimulated the development of numerous definitions of irregular attractors, ostensibly distinguished in their topological structure (strange, chaotic, stochastic, etc.). We illustrate this circumstance by the example of system of Eqs. (5) with strong dissipation for $b = 1, \omega = 4$, that is for the system of Eqs. (4). In this case, as the parameter $\mu > 0$ increases, system of Eqs. (5) has not only a complete subharmonic cascade of bifurcations in accordance with the Sharkovskii order, but also it has complete homoclinic cascade of bifurcations of cycles converging to the rotor homoclinic loop. The cause is clarified in Fig. 2a in which the Poincare section $(q = 0, p > 0)$ of the singular attractor of system of Eqs. (5) for $\mu = 0.12$ lying between cycles of period 5 and 3 in the Sharkovskii order is shown. The graph of the section almost coincides with the graph of one-dimensional unimodal mapping of a segment into itself, which has the above-listed cascades of bifurcations (Feigenbaum, 1978; Sharkovskii, 1964; Magnitskii & Sidorov, 2006). The projection of the manifold of the singular attractor onto the plane (p, u) corresponding to the section is shown in Fig. 2b.

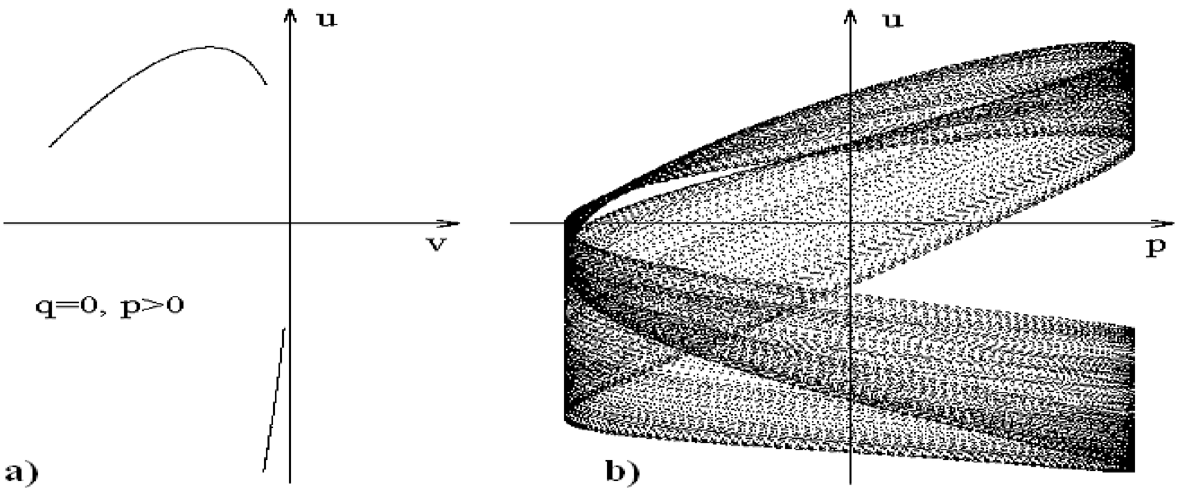

Figure 2. The projection of the Poincare section $(q = 0, p > 0)$ of solution of system of Eqs. (5) for $b = 1, \omega = 4, \mu = 0.12$ (a) and the projection of the manifold of the singular attractor onto the plane (p, u) corresponding to the section (b).

It seems that this is a two-dimensional strip whose lower part rotates around the original cycle, goes into its upper part in a revolution around it without twisting, and, in turn, the upper part goes into the lower part with twisting by 180 degrees in the next revolution. But in this case, to avoid contradiction with uniqueness theorem for solutions of systems of differential equations, two branches of the upper part should go into two branches of the lower part, which can be detected even under tenfold magnification (Fig.2a). Therefore, the upper part of the graph of the section in Fig. 2a should also consist of two branches, which makes its lower part to consist of four branches and so on. Consequently, the invariant manifold of the singular attractor shown in Fig. 2 should be a two-dimensional infinitely-sheeted folded surface. However, strong dissipation of the system in this case prevents correct understanding a topological structure of separatrix manifold of original singular cycle.

So, let us analyze the behavior of attractors of the system of Eqs. (5) with weak dissipation for $b = 0.05, \omega = 0.8$. A stable cycle of the double period, which is the boundary of the unstable Moebius band (an unstable two-dimensional manifold) of the original unit singular cycle $p^2 + q^2 = 1$, is generated in system of Eqs. (5) for small $\mu > 0$. It is an ordinary simple cycle of the period $4/\pi$ in the projection onto the two-dimensional subspace (u,v) . Initially this cycle has two multipliers lying on the positive part of the real axis inside the unit circle and moving towards each other as the parameter μ grows. Then multipliers meet, become complex conjugated and continue to move on positive and negative half-circles inside an unit circle towards the negative part of the real axis. In this case, the unstable Moebius band of the original singular cycle becomes a complex roll around the stable cycle of the double period. The frequency of rotation of a trajectory on the roll around the stable cycle of the double period is specified by the frequency ω and also by imaginary parts of complex conjugated multipliers. Therefore, the approach of the multipliers to the negative part of the real axis leads to the flattening of the roll in one direction and to its degeneration into a stable Moebius band around the stable cycle of the double period.

Further multipliers of the cycle begin to move along the negative part of the real axis in opposite directions, which leads to appearance of two stable two-dimensional manifolds in the form of two transversal Moebius bands for the cycle of double period. Therefore, the cycle of double period becomes a singular stable cycle. Next, at the moment of intersection of the unit circle by one of the multiplies at the point -1, the cycle of double period becomes an unstable singular cycle, whose stable and unstable manifolds are two transversal Moebius bands. The boundary of its unstable manifold is a stable cycle of quadruple period. Thus, we came to an original situation, but for a singular cycle of double period.

The cascade of Feigenbaum period doubling bifurcations continues, up to infinity, the process of construction of a two-dimensional heteroclinic separatrix manifold, which consists of Moebius bands, joining the unstable singular cycles of the cascade. The self-similar separatrix figure obtained in the Poincare section is referred to as **Feigenbaum separatrix tree**. A nonperiodic stable trajectory passes through the top of the Feigenbaum tree and through the endpoints of all of its branches, and each neighborhood of that trajectory contains singular unstable cycles from the Feigenbaum cascade. This nonperiodic almost stable trajectory is the Feigenbaum attractor, which is the first and the simplest attractor in the infinite family of singular attractors.

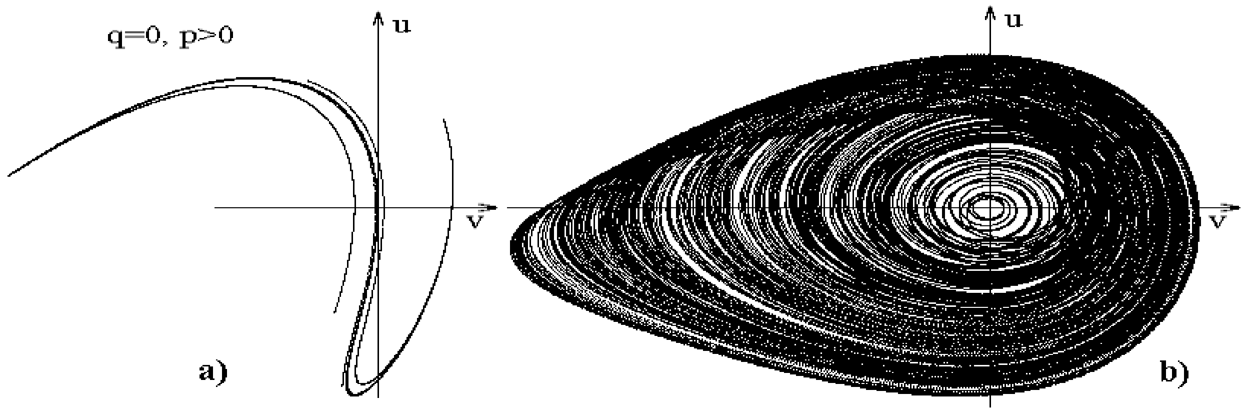

Figure 3. Projection of the Poincare section $(q = 0, p > 0)$ of solution of system of Eqs. (5) for $b = 0.05, \omega = 0.8, \mu = 0.02$ (a) and the projection of the manifold of the singular attractor onto the plane (u, v) corresponding to the section (b).

Along with separatrix branches connecting unstable singular cycles of the Feigenbaum cascade, the Feigenbaum separatrix tree contains stable one-sided separatrix branches only entering cycles. These branches begin to close each other and form heteroclinic separatrix folds passing through various stable and unstable cycles from the Sharkovskii subharmonic cascade of bifurcations, which are generated during saddle-node bifurcations. New Feigenbaum separatrix trees are generated on the separatrices of the newly generated singular cycles, and the tops of these trees contain more complicated singular attractors. An infinitely folded separatrix two-dimensional manifold, which Poincare section is referred to as a **heteroclinic separatrix zigzag** is thereby generated. It is shown in Fig. 3a. In the case of weak dissipation Poincare section of solutions of the system (a heteroclinic separatrix zigzag) is already not close to the graph of the one-dimensional unimodal mapping, which leads to the violation of the Sharkovskii order in its right-hand side, i.e. cycles of periods 7, 5 and 3 may not exist in the system but may also be stable either simultaneously with cascades of bifurcations of some other cycle or without them. For example, system of Eqs. (5) for $b = 0.05, \omega = 1.5$ has simultaneously two stable cycles of periods one and three (for $\mu = 0.0355$).

Thus, any unstable cycle of the system is unstable singular cycle joining neiboring separatrices of a heteroclinic zigzag. Any simple singular attractor is almost stable nonperiodic trajectory passing through vertices of some infinite Feigenbaum tree. Any trajectory of system from the attraction domain of the separatrix zigzag is first attracted to it along the nearest stable Moebius band, then approaches unstable sheets, goes along them, and tends either to a stable cycle or to a singular attractor depending on value of bifurcation parameter.

2.1.3. Some examples of classical two-dimensional nonautonomous systems

Consider three classical nonlinear ordinary differential equations of the second order with periodic coefficients such as Duffing-Holmes equation

$$\ddot{x} + k\dot{x} + \omega^2 x + \mu x^3 = f_0 \cos\Omega t, \tag{6}$$

modified dissipative Mathieu equation

$$\ddot{x} + \mu\dot{x} + (\delta + \varepsilon\cos\omega t)x + \alpha x^3 = 0, \tag{7}$$

and Croquette dissipative equation

$$\ddot{x} + \mu\dot{x} + \alpha\sin x + \beta\sin(x - \omega t) = 0. \tag{8}$$

All these equations are equivalent to two-dimensional nonlinear dissipative systems of ordinary differential equations with periodic coefficients and all of them have the same universal FSM scenario of transition to dynamical chaos (Magnitskii & Sidorov, 2006). For these equations, some important stable cycles and singular subharmonic attractors in accordance with the Sharkovskii order are presented in Fig. 4 - Fig. 6.

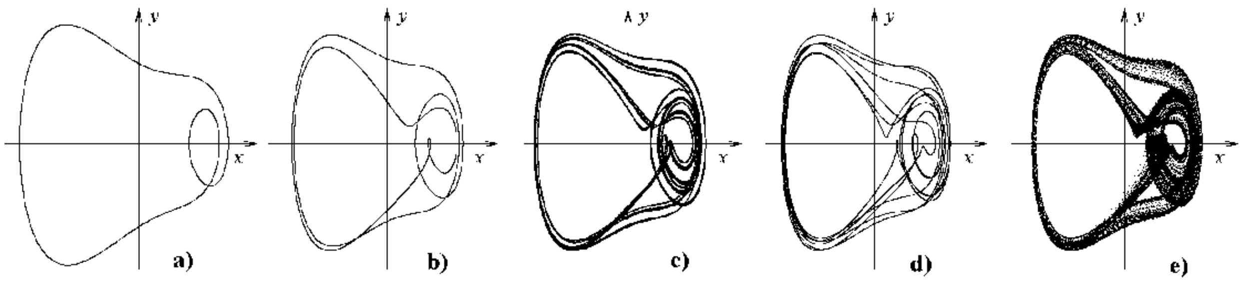

Figure 4. Original cycle (a), cycle of period two (b), Feigenbaum attractor (c), cycle of period six (d) from subharmonic cascade and more complex singular attractor (e) in the Duffing-Holmes equation (6).

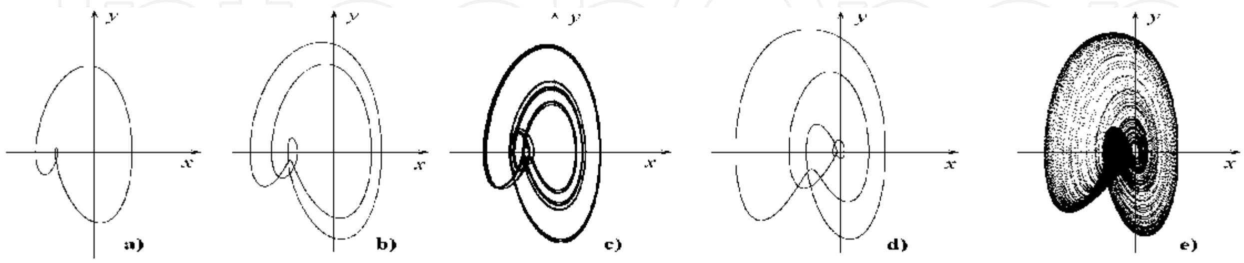

Figure 5. Original cycle (a), cycle of period two (b), Feigenbaum attractor (c), cycle of period three (d) from subharmonic cascade and more complex singular attractor (e) in the Mathieu equation (7).

Note that double period bifurcations were found also in (Awrejcewicz, 1989; Awrejcewicz 1991) for some other nonlinear ordinary differential equations of the second order with periodic coefficients.

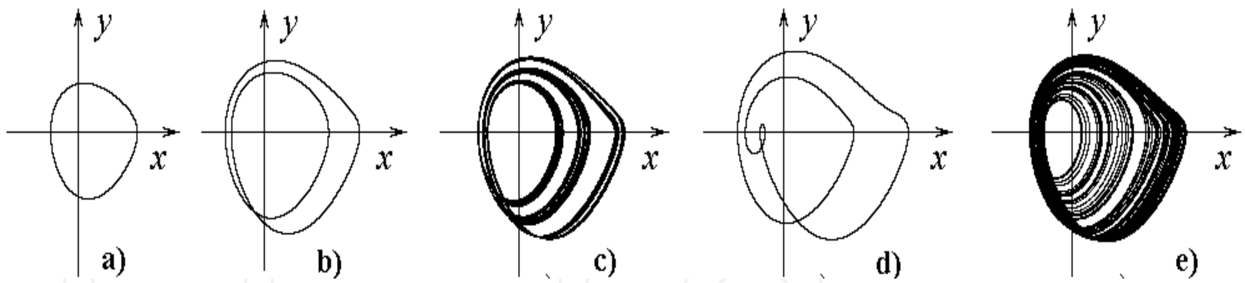

Figure 6. Original cycle (a), cycle of period two (b), Feigenbaum attractor (c), cycle of period three (d) from subharmonic cascade and more complex singular attractor (e) in the Croquette equation (8).

2.2. Three-dimensional autonomous systems

Consider a smooth family of three-dimensional nonlinear dissipative autonomous systems of ordinary differential equations

$$\dot{x} = F(x,\mu), \qquad x \in M \subset R^3, \quad \mu \in I \subset R, \quad F \in C^\infty, \tag{9}$$

depending on a scalar system parameter μ.

It is shown by the author in (Magnitskii & Sidorov, 2006; Magnitskii, 2008) that if a three-dimensional system of Eqs. (9) has a singular cycle of period T defined by complex Floquet exponents with equal imaginary parts (i.e. Moebius bands are its stable and unstable invariant manifolds), then by passing to a coordinate system rotating around the cycle, one can reduce such system to a two-dimensional nonautonomous system in coordinates, transversal to the singular cycle with zero rotor-type singular point corresponding to the cycle. So, all arguments listed in the previous section hold completely for autonomous three-dimensional systems with singular cycles.

2.2.1. FSM – scenario of transition to chaos

Therefore, three-dimensional autonomous system with singular cycle should have the same FSM scenario of transition to chaos as two-dimensional nonautonomous system with periodic coefficients and zero rotor-type singular point. As an example, consider the autonomous three-dimensional system

$$\begin{aligned}
\dot{x}_1 &= -\omega x_2 + x_1[((\mu-1)\sqrt{x_1^2+x_2^2}+x_1)(x_1^2+x_2^2-1)+(2x_2-\omega/2)x_3],\\
\dot{x}_2 &= \omega x_1 + x_2[((\mu-1)\sqrt{x_1^2+x_2^2}+x_1)(x_1^2+x_2^2-1)+(2x_2-\omega/2)x_3],\\
\dot{x}_3 &= (x_2+\omega/4)(x_1^2+x_2^2-1)+2(\mu-1-x_1)x_3.
\end{aligned} \tag{10}$$

For $\mu<1$ system of Eqs.(10) has the singular point $(0,0,\omega/8(\mu-1))$ and limit cycle $x_0(t,\mu)=(\cos\omega t,\sin\omega t,0)^T$ with period $T=2\pi/\omega$ in the plane of variables (x_1,x_2). By changing the variables $x(t,\mu)=x_0(t,\mu)+Q(t,\mu)(0,u_1(t),u_2(t))^T$ with $2\pi/\omega$ -periodic matrix $Q(t)=\left(\dot{x}_0(t),x_0(t),(0,0,1)^T\right)$, one can reduce the system of Eqs. (10) to two-dimensional nonautonomous system with $2\pi/\omega$ -periodic coefficients and zero rotor-type singular point

$$\begin{aligned}\dot{u}_1 &= 2(\mu - 1 + \cos\omega t)u_1 + (2\sin\omega t - \omega/2)u_2 + h_1(u_1, u_2, t, \omega, \mu),\\ \dot{u}_2 &= (2\sin\omega t + \omega/2)u_1 + 2(\mu - 1 - \cos\omega t)u_2 + h_2(u_1, u_2, t, \omega, \mu),\end{aligned} \tag{11}$$

where

$$\begin{aligned}h_1 &= (\mu - 1 + \cos\omega t)((2u_1 + u_1^2)^2 + u_1^2) + ((2u_1 + 4)\sin\omega t - \omega/2)u_1 u_2,\\ h_2 &= (2\sin\omega t + (u_1 + 1)\sin\omega t + \omega/4)u_1^2 - +2u_1 u_2 \cos\omega t.\end{aligned}$$

Leading linear part of system of Eqs. (11) coincides with the linear part of system of Eqs. (4) with rotor. So, for $\mu < 0$ zero solution of system of Eqs. (11) and singular cycle $x_0(t, \mu)$ of system of Eqs. (10) are stable. For $\mu > 0$ all cascades of bifurcations in accordance with the theory FSM take place in both systems. Some cycles and singular attractors from these cascades are presented in Fig. 7, rotor and singular cycle separatrix loops are presented in Fig. 8. Thus, if parameter μ is changing, then the Sharkovskii subharmonic and Magnitskii

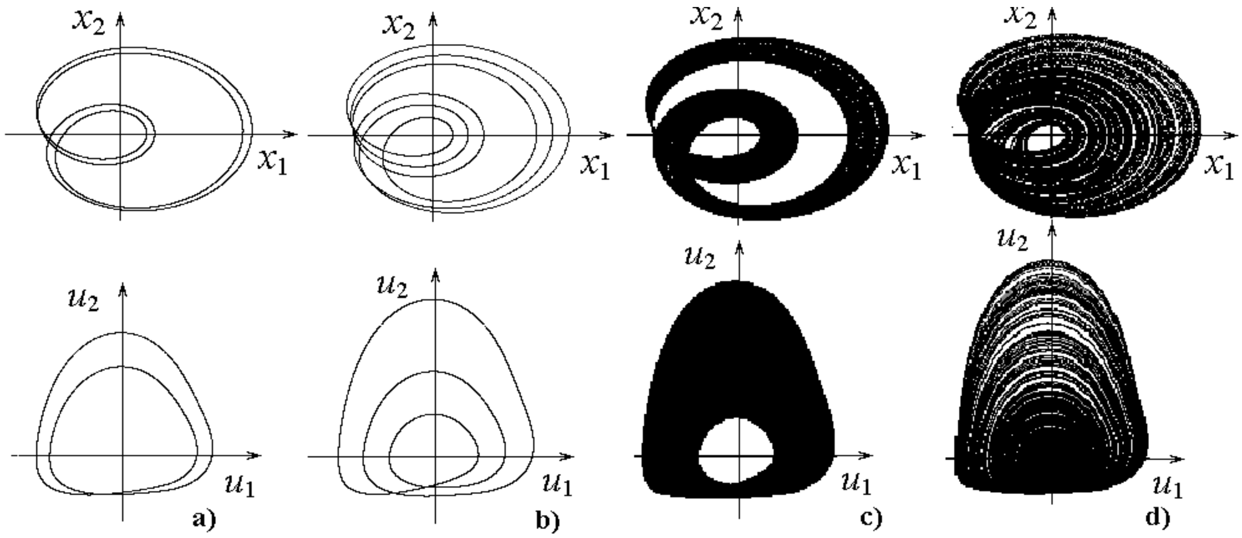

Figure 7. Projections of period four and six cycles and singular attractors of system of Eqs. (10) (above) and corresponding to them period two and three cycles and singular attractors of system of Eqs. (11) (below).

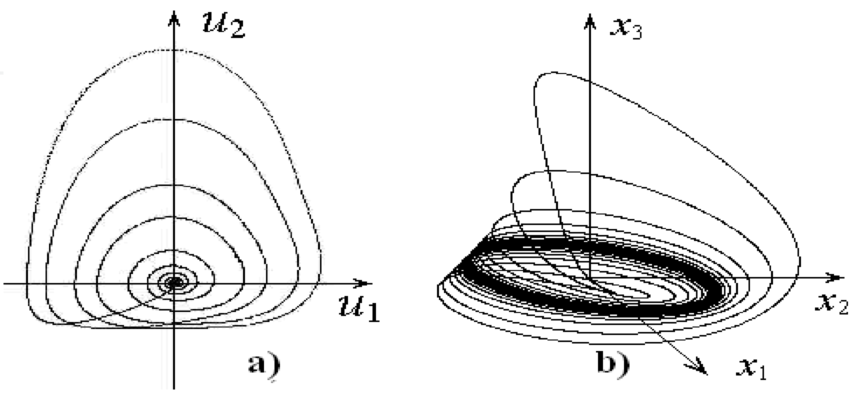

Figure 8. Rotor separatrix loop of system of Eqs. (11) (a) and corresponding to it separatrix loop of singular cycle of system of Eqs. (10) (b).

homoclinic cascades of bifurcations of stable limit cycles are realizing in any system of Eqs. (9) in accordance with the Sharkovskii and homoclinic orders. Cycle of period three is the last cycle in the Sharkovskii order. Therefore, to verify an existence of subharmonic cascade of bifurcations in any system one should to find a stable cycle of period three in this system or any stable homoclinic cycle.

2.2.2. Topological structure of singular attractors

The three-dimensional phase space of three-dimensional autonomous system containing the original singular cycle of period T is diffeomorphic to three-dimensional manifold of an autonomous four-dimensional system of the form of Eqs. (5), the first two equations of which have linear part of the form of Eqs. (2), and the remaining two equations with some condition define a motion on a plane along a simple cycle of period T. Therefore, the separatrix heteroclinic manifold constructed in previous Section for system of Eqs. (5) in the section $(q = 0, p > 0)$ (in the section of the singular cycle corresponding to the rotor) should be completely similar to the separatrix heteroclinic manifold of a three-dimensional autonomous system in the section of the original singular cycle.

As an example, consider the autonomous three-dimensional system:

$$\begin{aligned}
\dot{x} &= -\omega y - \omega xz/2 - ((\mu - b)x + b)(1 - x^2 - y^2), \\
\dot{y} &= \omega x + 2(b - \omega y/4)z - (\mu - b)y(1 - x^2 - y^2) \\
\dot{z} &= 2(\mu - b - bx)z - (by + \omega/4)(1 - x^2 - y^2).
\end{aligned} \tag{12}$$

System of Eqs. (12) has the periodic solution (the cycle) $x_0(t,\mu) = (\cos\omega t, \sin\omega t, 0)^T$, which lies in the plane of the variables (x, y) and has the period $2\pi/\omega$. By linearizing system of Eqs. (12) on the cycle with respect to deviations y_1, y_2, y_3 from the cycle and by performing the change of variables $y(t) = Q(t)z(t)$ with $2\pi/\omega$- periodic matrix $Q(t) = (\dot{x}_0(t), x_0(t), (0,0,1)^T)$, one can obtain the following system of equations in the rotating variables transversal to the cycle:

$$\begin{aligned}
\dot{z}_2 &= (2(\mu - b) + 2b\cos\omega t)z_2 + (2b\sin\omega t - \omega/2)z_3, \\
\dot{z}_3 &= (2b\sin\omega t + \omega/2)z_2 + (2(\mu - b) - 2b\cos\omega t)z_3.
\end{aligned} \tag{13}$$

System of Eqs. (13) coincides with the linear part of system of Eqs. (4) considered in previous Section, and in addition, the coordinate tangent to the cycle has the form $\dot{z}_1 = ((-2b/\omega)\sin\omega t)z_2 + ((2b/\omega)\cos\omega t)z_3$, and does not influence the generation of the dynamics of solutions in a neighborhood of the cycle. Consequently, the heteroclinic separatrix manifold generated around the cycle of system of Eqs. (12) as the bifurcation parameter μ grows has the same structure as that of the heteroclinic separatrix manifold of a rotor-type singular point and should be similar to a heteroclinic separatrix zigzag in the Poincare section for small dissipation parameter b. The projection of Poincare section $(y = -0.1, x < 0)$ of solution of system of Eqs. (12) is presented in Fig. 9a.

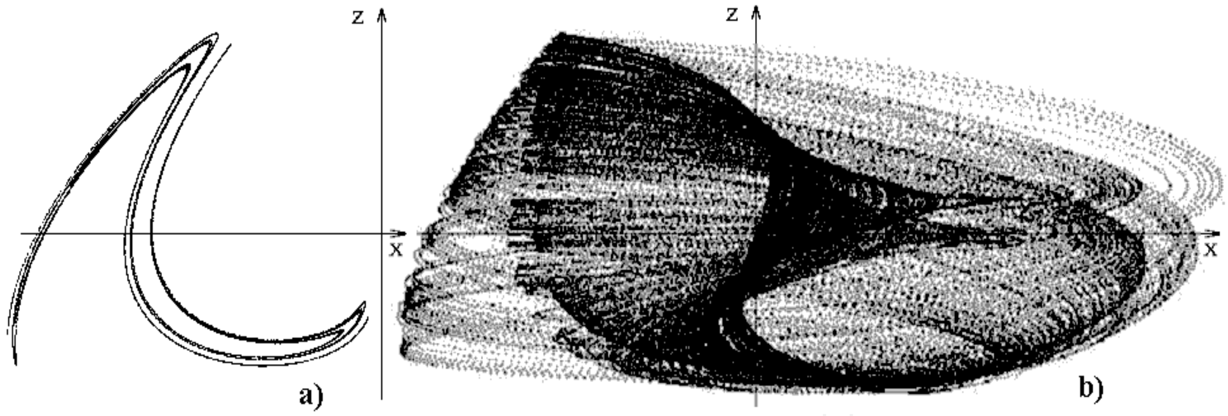

Figure 9. Projection of the Poincare section $(y=-0.1, x<0)$ of solution of system of Eqs. (12) for $b=0.08, \omega=1, \mu=0.051$ (a) and the projection of the manifold of the singular attractor onto the plane (x,z) corresponding to the section (b).

2.2.3. Some examples of classical tree-dimensional autonomous nonlinear systems

For instance let consider four classical tree-dimensional chaotic systems of nonlinear ordinary differential equations describing different natural and social processes:

the Lorenz hydrodynamic system

$$\dot{x}=\sigma(y-x), \quad \dot{y}=x(r-z)-y, \quad \dot{z}=xy-bz, \tag{14}$$

the Ressler chemical system

$$\dot{x}=-(y+z), \dot{y}=x+ay, \dot{z}=b+z(x-\mu), \tag{15}$$

the Chua electro technical system

$$\dot{x}=\mu[y-h(x)], \ \ \dot{y}=x-y+z, \ \ \dot{z}=-\beta y, \tag{16}$$

where $h\,(x)$ is a piecewise linear function; and the Magnitskii macroeconomic system

$$\dot{x}=bx((1-\sigma)z-\delta y), \dot{y}=x(1-(1-\delta)y+\sigma z), \dot{z}=a(y-dx). \tag{17}$$

To demonstrate that the transition to chaos under variation of a system parameter in all these classical chaotic systems occurs in accordance with the described above unique FSM scenario, let show that all these systems have period three stable cycles in accordance with the Sharkovskii order (3). This stable period three cycles are presented in Fig. 10.

In Fig. 11 it is presented homoclinic cascade of bifurcations of stable cycles in the Lorenz system and the most complex separatrix contour in this system named as **heteroclinic butterfly** which is the limit of the heneroclinic cascade of bifurcations of stable heteroclinic cycles (Magnitskii & Sidorov, 2006; Magnitskii , 2008; Magnitskii , 2011).

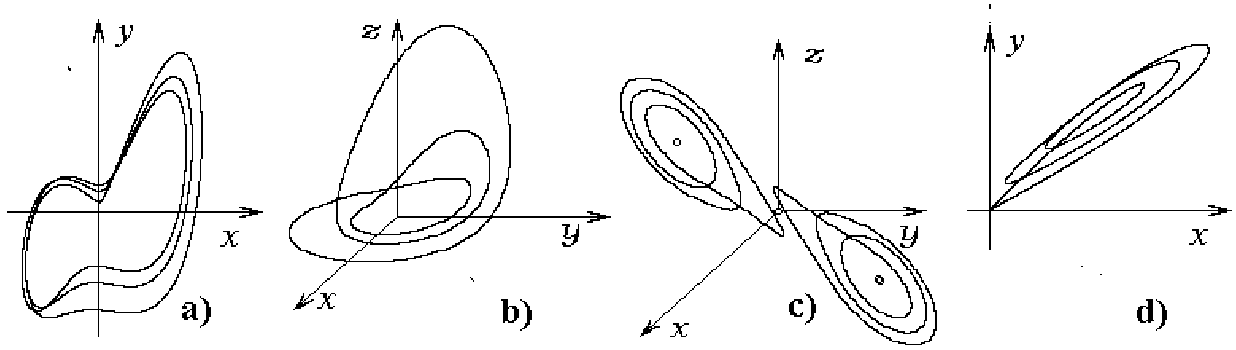

Figure 10. Cycles of period three in Lorenz (14) (a), Ressler (15) (b), Chua (16) (c) and Magnitskii (17) (d) systems.

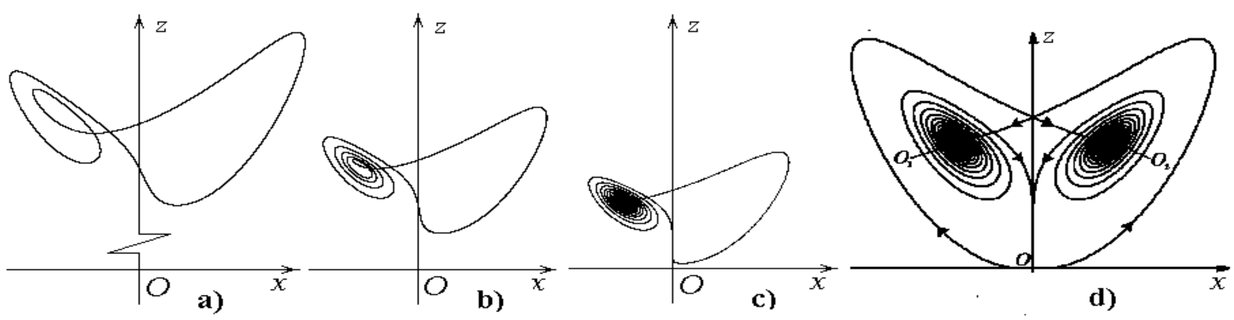

Figure 11. Homoclinic cascade of bifurcations of stable cycles (a)-(c) and heteroclinic butterfly separatrix contour (d) in the Lorenz system (14).

All these classical systems have also arbitrary stable cycles from the Sharkovskii subharmonic cascade of bifurcations and all singular attractors from this cascade. Moreover, these systems have also more complex complete or incomplete homoclinic or heteroclinic cascades of bifurcations which take place after Sharkovskii cascade and infinitely many homoclinic or heteroclinic singular attractors (Magnitskii & Sidorov, 2006; Magnitskii , 2008; Magnitskii , 2011).

In conclusion of this Section note that also very many other nonlinear three-dimensional autonomous systems of ordinary differential equations considered in the scientific literature have the same universal scenario of transition to dynamical chaos in accordance with the Feigenbaum-Sharkovskii-Magnitskii (FSM) theory. Among them there are systems of: Vallis, Anishchenko-Astakhov, Rabinovich-Fabricant, Pikovskii-Rabinovich-Trakhtengertz, Sviregev, Volterra-Gause, Sprott, Chen, Rucklidge, Genezio-Tesi, Wiedlich-Trubetskov and many others (Magnitskii , 2011; Magnitsky , 2007).

2.3. Many- and infinitely- dimensional autonomous systems

2.3.1. Transition to chaos through bifurcation cascades of stable cycles

At the beginning let us show that the scenario of transition to chaos through the Sharkovskii subharmonic and homoclinic cascades of bifurcations of stable cycles takes place also in many-dimensional dissipative nonlinear systems of ordinary differential equations. For example consider Rikitaki system

$$\dot{x} = -\mu x + yz,\ \dot{y} = -\mu y + xu,\ \dot{z} = 1 - xy - bz,\ \dot{u} = 1 - xy - cu, \tag{18}$$

modelling a change in dynamics of magnetic poles of the Earth. Some main cycles of subharmonic cascade of bifurcations in the system of Eqs. (18) and some singular attractors are presented in Fig. 12.

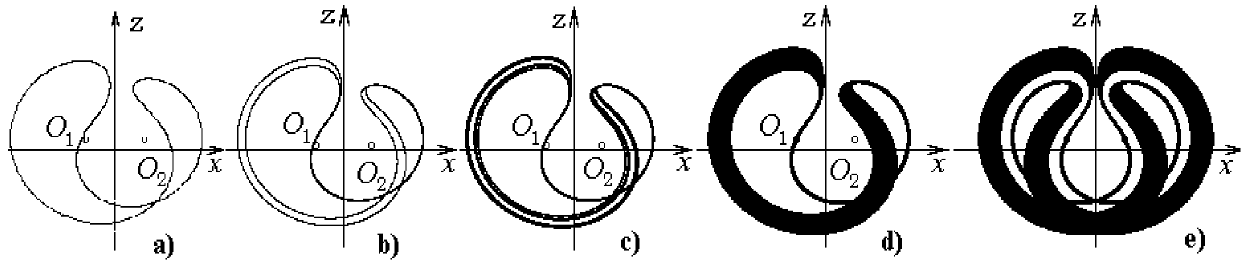

Figure 12. Projections of original singular cycle (a), cycle of period two (b), Feigenbaum attractor (c) and two more complex singular attractors in the Rikitaki system (d)-(e).

2.3.2. Transition to chaos through bifurcation cascades of stable two-dimensional tori

Besides described above mechanism of transition to chaos in accordance with subharmonic and homoclinic cascades of bifurcations of stable cycles, in many-dimensional dissipative nonlinear systems of ordinary differential equations there exists a scenario of transition to chaos through subharmonic and homoclinic cascades of bifurcations of stable two-dimensional or many-dimensional tori along any one or several frequencies simultaneously. The mechanism of this cascade of bifurcations has the same above considered FSM nature, and presently there not discovered really any other scenarios of transition to chaos in many-dimensional nonlinear systems of ordinary differential equations. Such a scenario of transition to chaos takes place in complex five-dimensional Lorenz system

$$\dot{X} = -\sigma X + \sigma Y, \quad \dot{Y} = -XZ + rX - aY, \quad \dot{Z} = -bZ + (X^{*}Y + XY^{*})/2 \tag{19}$$

of two complex variables $X = x_1 + ix_2$ and $Y = y_1 + iy_2$ and one real variable Z. If values of parameters a, b, σ and $\mathrm{Re}\, r$ are fixed and the value of parameter $\mathrm{Im}\, r$ is decreasing, then at first a stable invariant torus is appearing from the stable cycle as a result of Andronov-Hopf bifurcation. After that the period two invariant torus is appearing from this original singular saddle torus as a result of double period bifurcation (Fig. 13). That is the beginning of Feigenbaum cascade of period doubling bifurcations. Then, after further decreasing of bifurcation parameter $\mathrm{Im}\, r$, all subharmonic cascade of bifurcations of stable two-dimensional tori with arbitrary period in accordance with the Sharkovskii order (3) takes place in the complex Lorenz system. Projections of sections of period one, two and three two-dimensional invariant tori and one of the toroidal singular attractor are presented in Fig. 13.

This example shows that the FSM (Feigenbaum-Sharkovskii-Magnitskii) scenario of transition to dynamical chaos in two-dimensional nonautonomous and three-dimensional autonomous systems of ordinary differential equations takes place also in many-dimensional systems. So, appearance of three-dimensional torus is not necessary condition for generation of chaotic dynamics in dissipative many-dimensional systems of differential equations.

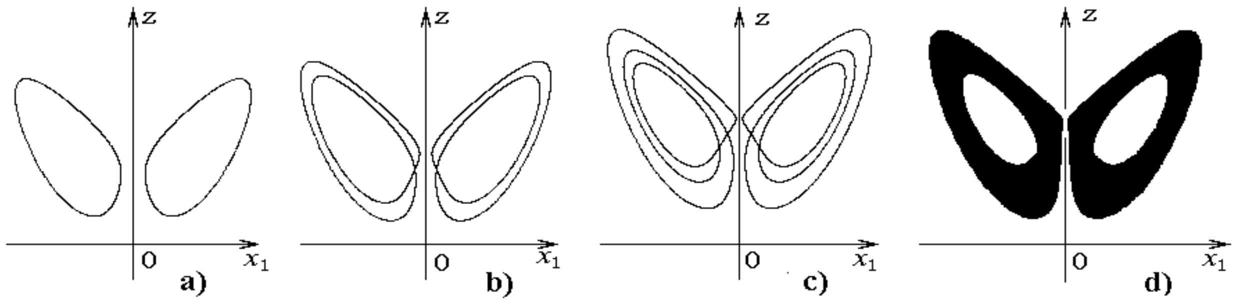

Figure 13. Projections of sections of two-dimensional invariant tori of period one (a), two (b) and three (c) and one of the toroidal singular attractor (d) in complex Lorenz system (19).

2.3.3. Transition to chaos in nonlinear equations with delay argument

Let us show now that the FSM scenario of transition to chaos is realized also in infinitely-dimensional nonlinear autonomous dissipative systems of ordinary differential equations, namely in nonlinear ordinary differential equations with delay arguments. For instance such scenario of transition to chaos through the Sharkovskii subharmonic cascade of bifurcations of stable cycles with arbitrary period in accordance with the Sharkovskii order (3) takes place in well-known Mackey-Glass equation (Mackey & Glass, 1977).

$$\dot{x}(t) = -ax(t) + \beta_0 \frac{\theta^n x(t-\tau)}{\theta^n + x^n(t-\tau)}. \tag{20}$$

In this equation the delay argument τ is a bifurcation parameter. When a value of parameter τ is small, then Mackey-Glass equation has unique stable stationary state. When τ is increasing, then at first a stable cycle is appearing in phase space of the equation from the stable stationary state as a result of Andronov-Hopf bifurcation. After that the period two stable cycle is appearing from this original singular cycle as a result of double period bifurcation. That is the beginning of the Feigenbaum cascade of period doubling bifurcations. Then, after further increasing of bifurcation parameter τ, all subharmonic cascade of bifurcations of stable cycles with arbitrary period in accordance with the Sharkovskii order takes place in the Mackey-Glass equation. Projections of some main stable cycles and singular attractors of the Mackey-Glass equation are presented in Fig. 14.

Thus we can make a conclusion that universal bifurcation Feigenbaum-Sharkovskii-Magnitskii theory describes transition to dynamical chaos in all nonlinear dissipative systems of ordinary differential equations. Scenario of transition to chaos consists of subharmonic and homoclinic (heteroclinic) cascades of bifurcations of stable cycles or stable two- or many-dimensional tori.

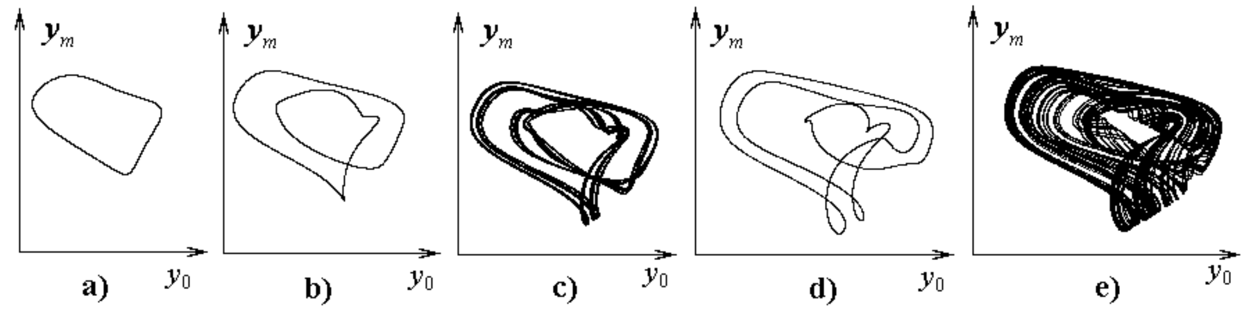

Figure 14. Projections of period one (a), two (b) and three (d) cycles, Feigenbaum attractor (c) and one of more complex singular attractor (e) in the Mackey-Glass equation (20).

3. Chaos in Hamiltonian and conservative systems

The modern classical theory of Hamiltonian systems reduces a problem of the analysis of dynamics of such system to the problem of its integralability, i.e. to a problem of construction of the canonical transformation reducing system to variables "action - angle" in which, as it is considered to be, movement occurs on a surface of n -dimensional torus and is periodic or quasiperiodic. Any nonintegrable nonlinear Hamiltonian system is considered as perturbation of integrable system, and the analysis of its dynamics is reduced to finding-out of a question on destruction or nondestruction some tori of nonperturbed system depending on value of perturbation.

In the present Section absolutely other bifurcation approach is considered for analysis of chaotic dynamics not only Hamiltonian, but also any conservative system of nonlinear differential equations. The method consists in consideration of approximating extended two-parametrical dissipative system of the equations, stable solutions (attractors) of which are as much as exact aproximations to solutions of original Hamiltonian (conservative) system. Attractors (stable cycles, tori and singular attractors) of extended dissipative system one can search by numerical methods with use the results of universal FSM (Feigenbaum-Sharkovskii-Magnitskii) theory, developed initially for nonlinear dissipative systems of ordinary differential equations and considered in detail in the previous Section of the chapter. It becomes clear what chaos is in Hamiltonian and simply conservative systems. And this chaos is not a result of destruction of some tori of nonperturbed system as it is considered to be in the modern literature, but, on the contrary, it is a result of bifurcation cascades of a birth of regular (cycles and tori) and singular attractors in extended dissipative system in accordance with the universal FSM theory when dissipation parameter tends to zero.

3.1. Bifurcation approach to analysis of Hamiltonian and conservative systems

The fact that the dynamics of any conservative system is a limit case of a dynamics of an extended dissipative system with weak dissipation as the dissipation parameter tends to zero was proved by author in (Magnitskii, 2008b; Magnitskii, 2011) and illustrated by numerous examples of Hamiltonian systems with one and a half, two and three degrees of freedom and by examples of simply conservative but not Hamiltonian systems. The stability domains of cycles of such a system with zero dissipation become tori of a conservative (Hamiltonian) system around its elliptic cycles into which the stable cycles themselves go. Complicated separatrix heteroclinic manifolds spanned by unstable singular cycles of the dissipative system become (for zero dissipation) even more complicated separatrix manifolds of the conservative (Hamiltonian) system along which the motion of a trajectory is treated as chaotic dynamics. Thus, it becomes clear why the order of the tori alternation in conservative (Hamiltonian) systems can differ from the Sharkovskii order existing in systems with strong dissipation.

3.1.1. Theoretical basis of bifurcation approach

Let's consider generally nonlinear conservative system of ordinary differential equations with a smooth right part

$$\dot{x} = f(x), x \in R^n, \ \ div\, f(x) = 0 \tag{21}$$

which variables are connected by some equation

$$H(x_1, ..., x_n) = \varepsilon. \tag{22}$$

Any Hamiltonian system is a special case of system of Eqs. (21)-(22) at even value of dimension n and at the given integral of movement (22) generating system of Eqs. (21). Movement in system of Eqs. (21) occurs in $n-1$ -dimensional subspace, set by the equation (22).

Theorem. Let two-parametrical system of ordinary differential equations

$$\dot{x} = g(x, \varepsilon, \mu), \ \ x \in R^n, \tag{23}$$

possesses following properties: 1) the only solutions of system of Eqs. (21)-(22) are solutions of system of Eqs. (23) with initial conditions $H(x_{10}, ..., x_{n0}) = \varepsilon$ at $\mu = 0$; 2) at all $\mu > 0$ the system of Eqs. (23) is dissipative system on its solutions laying in neighborhoods of solutions of system of Eqs. (21)-(22). Then attractors of dissipative system of Eqs. (23) at small $\mu > 0$ are as much as exact approximations of solutions of conservative system of Eqs. (21)- (22) (see proof in (Magnitskii, 2008; Magnitskii, 2011)).

So, for application of the offered approach to the analysis of conservative and, in particular, Hamiltonian systems it is necessary to construct an extended dissipative system, satisfying the properties 1) and 2). Then for everyone $\varepsilon > 0$ one should to find numerically all stable solutions and their cascades of bifurcations according to the FSM scenario in extended dissipative system of Eqs. (23) when μ tends to zero, starting from the various initial conditions, satisfying the equality (22). Areas of stability of the found simple regular solutions (simple cycles) will generate at $\mu = 0$ regular solutions (tori) of original conservative system of Eqs. (21)-(22), and areas of stability of complex cycles and singular attractors and also heteroclinic separatrix manifolds will generate chaotic solutions. By the same method in the area of parameters $\varepsilon > 0,\ \mu \geq 0$ one can construct bifurcation diagrams of all bifurcations existing in two-parametrical extended dissipative system of Eqs. (23) and smoothly passing to bifurcations in conservative system of Eqs. (21)-(22) on the boundary $\mu = 0$.

3.1.2. Subharmonic cascade of bifurcations in Hamiltonian and conservative systems

From theoretical positions of bifurcation approach to the analysis of Hamiltonian and any conservative systems it follows, that at enough great values of parameter $\varepsilon > 0$ transition to chaotic dynamics in system of Eqs. (21)-(22) occurs according to universal FSM scenario and that bifurcation diagram of this scenario can be received by limiting transition at $\mu \to 0$ from similar bifurcation diagram of two-parametrical extended dissipative system of Eqs. (23). Let's illustrate this position by the example of classical conservative Croquette equation

$$\ddot{x} + \alpha \sin x + \beta \sin(x - \omega t) = 0, \tag{24}$$

modeling a magnet rotary fluctuations in an external magnetic field in absence of friction. It is easy to see, that the equation (24) is equivalent to two-dimensional conservative system with periodic coefficients (Hamiltonian system with one and a half degrees of freedom) and also to four-dimensional conservative (not Hamiltonian) autonomous system of the equations

$$\dot{x} = y,\ \ \dot{y} = -(\alpha + r)\sin x + z\cos x,\ \ \dot{z} = \omega r,\ \ \dot{r} = -\omega z \tag{25}$$

with a condition $H = z^2 + r^2 = \varepsilon^2$, $z_0 = z(0) = 0.$ Extended dissipative system for the system of Eqs. (25) will be

$$\dot{x} = y,\ \ \dot{y} = -\mu y - (\alpha + r)\sin x + z\cos x,\ \ \dot{z} = \omega r,\ \ \dot{r} = -\omega z. \tag{26}$$

It is easy to check up numerically, that the two-parametrical system of Eqs. (26) with initial conditions $z_0 = z(0) = 0,\ \ r_0 = r(0) = \varepsilon$ has the subharmonic cascade of bifurcations at each value of parameter ε and at reduction of values of parameter μ . For each cycle of the cascade in a plane of parameters (ε, μ) it is possible to construct monotonously increasing bifurcation curve $\mu(\varepsilon)$ of births of the given cycle. Boundary values of such curves at $\mu = 0$ are bifurcation values of the subharmonic cascade of bifurcations in conservative Croquette system of the Eqs. (25) for parameter $\varepsilon > 0$ (see Fig. 15).

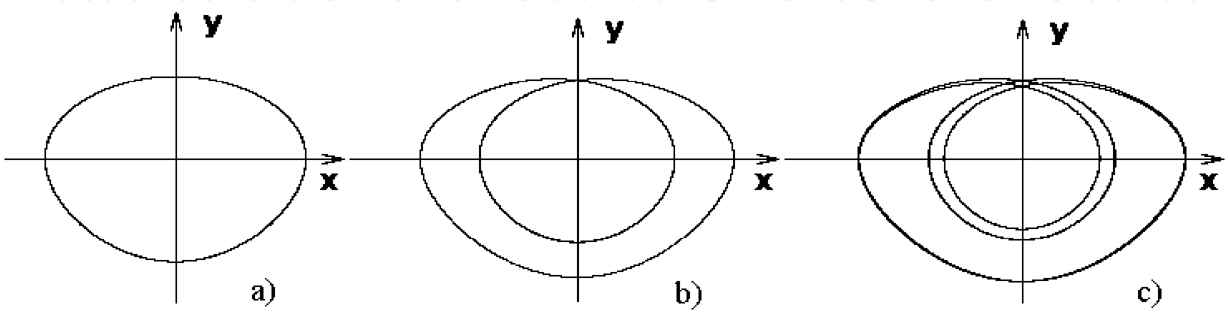

Figure 15. Projections on the plane (x, y) of the cycle (a) for $\varepsilon = 0.45$, period two cycle (b) for $\varepsilon = 0.48$ and period four cycle (c) for $\varepsilon = 0.497$ in conservative Croquette system of the Eqs. (25) for $\alpha = \omega = 1$.

3.1.3. Heteroclinic separatrix manifolds

Cascade of saddle-node bifurcations in extended dissipative system, consisting in a simultaneous birth of stable and saddle cycles, leads to formation in conservative (Hamiltonian) system of family of complex multiturnaround tori around of elliptic cycles and heteroclinic separatrix manifold which is tense on complex multiturnaround hyperbolic cycles of the system. In Poincare section it looks like a family of the hyperbolic singular points connected by separatrix contours. This picture at any shift in initial conditions passes into a family of so-called islands (points in Poincare section forming closed curves around of points of elliptic cycle).

At the same time, as follows from the theory, at enough great values of perturbation parameter $\varepsilon > 0$ in extended dissipative system there are cascades of bifurcations in accordance with scenario FSM. These cascades of bifurcations generate considered in the previous Section of the chapter infinitely folded heteroclinic separatrix manifolds having in Poincare section a kind of heteroclinic separatrix zigzag. These manifolds are tense on unstable singular cycles of FSM-cascade of dissipative system and they pass at zero dissipation in even more complex separatrix manifolds of conservative (Hamiltonian) system, movement of trajectories on which looks like as chaotic dynamics. Thus there is a stretching of an accordion of infinitely folded heteroclinic separatrix zigzag on some area of phase space of the conservative system. In the remained part of phase space elliptic cycles from the right part of subharmonic and homoclinic cascades can simultaneously coexist with tori around of them.

In Fig. 16a islands of solutions of conservative system of the Croquette Eqs. (25) are presented at $\varepsilon = 0.2$ in Poincare section $(z = 0, r = \varepsilon)$. Around of a picture presented in Fig. 16a there is not represented in figure an area of chaotic movement around of original separatrix contour of nonperturbed system connecting the points $(\pm\pi, 0)$. Development and complication of heteroclinic separatrix zigzag in the extended dissipative Croquette system of Eqs. (26) close to conservative system Eqs. (25) is presented in Fig. 16b,c for $\varepsilon = 0.55$. At reduction of values of dissipation parameter μ in system of Eqs. (26) the subharmonic cascade of bifurcations is observed. It generates the heteroclinic separatrix zigzag represented in Fig. 16b at $\mu = 0.1415$. At values of parameter $\mu < 0.138$ the accordion of heteroclinic separatrix zigzag starts to cover all phase space of the system merging with heteroclinic separatrix manifold which is tense on hyperbolic cycles from the cascade a saddle-node bifurcations.

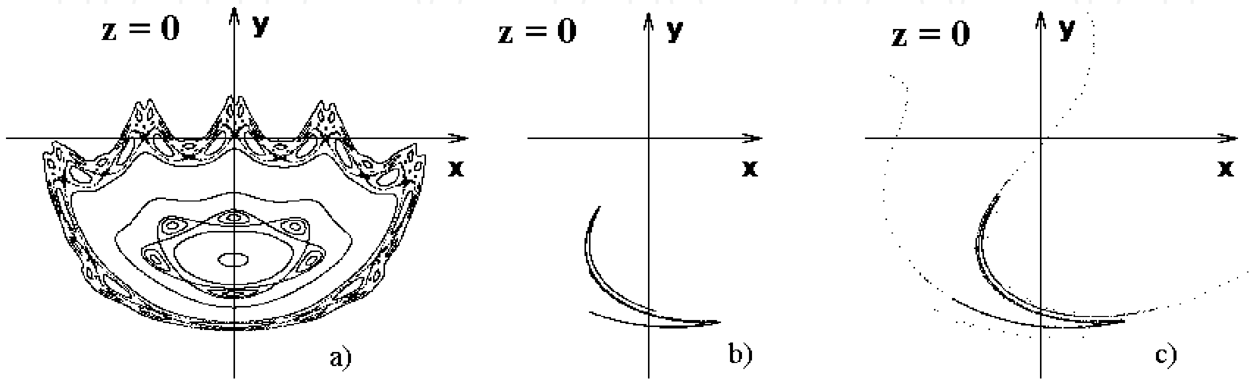

Figure 16. Projections on the plane (x, y) of Poincare section $(z = 0, r = \varepsilon)$ of solutions of conservative Croquette system of the Eqs. (25) for $\varepsilon = 0.2$ (a); development and complication of heteroclinic

separatrix zigzag in dissipative Croquette system of Eqs. (26), for $\varepsilon = 0.55$ and $\mu = 0.1415$ (b), $\mu = 0.138$ (c).

3.2. Hamiltonian systems with one and a half degrees of freedom

In modern scientific literature Hamiltonian systems with one and a half degrees of freedom refer to as nonautonomous conservative two-dimensional systems of ordinary differential equations with time-dependent Hamiltonian. Considered above Croquette system is an example of such a system. Let us analyze some other examples.

3.2.1. Hyperbolic nonautonomous concervative system

Consider nonautonomous conservative two-dimensional system of ordinary differential equations

$$\dot{x} = y,\ \ \dot{y} = (1 + \varepsilon \cos t)x - x^3. \tag{27}$$

Nonperturbed $(\varepsilon = 0)$ system of Eqs. (27) has in the plane (x, y) two homoclinic separatrix loops of zero saddle singular point around singular points $O^{\pm} = (\pm 1, 0)$ which are centers of nonperturbed system. System of Eqs. (27) is equivalent to the perturbed four-dimensional conservative autonomous system

$$\dot{x} = y,\ \ \dot{y} = (1 + z)x - x^3,\ \dot{z} = r,\ \dot{r} = -z. \tag{28}$$

with conditions $H = z^2 + r^2 = \varepsilon^2$, $z_0 = z(0) = \varepsilon$. The system

$$\dot{x} = y,\ \ \dot{y} = (1 + z)x - x^3 - \mu y,\ \dot{z} = r,\ \dot{r} = -z. \tag{29}$$

is the extended dissipative system for conservative system of Eqs. (28). For large enough values of perturbation parameter (for example, $\varepsilon > 1.5$) conservative system of Eqs. (28) has a chaotic dynamics, because at reduction of values of parameter μ in dissipative system of Eqs. (29) there are subharmonic cascades of bifurcations in full accordance with the theory FSM.

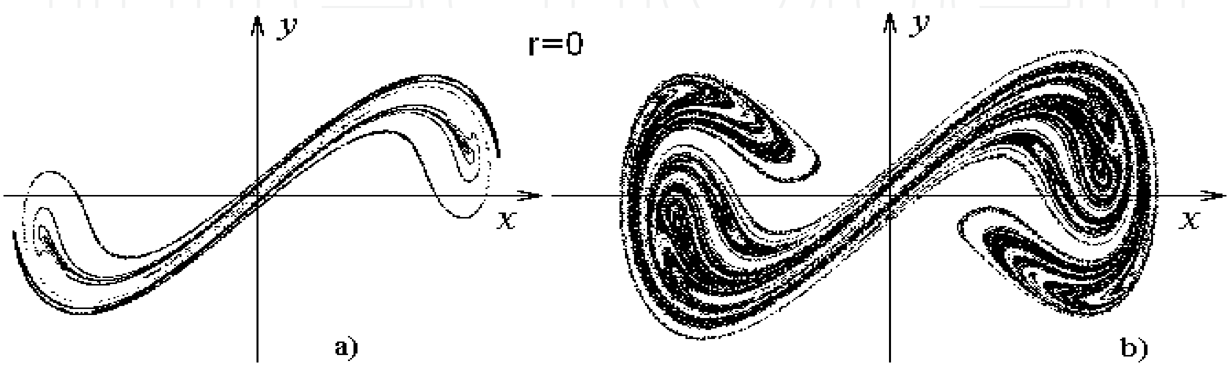

Figure 17. Projections on the plane (x, y) of Poincare section $(r = 0, z > 0)$ of solutions of dissipative system of Eqs. (29) for $\varepsilon = 1.5$ and $\mu = 0.25$ (a), $\mu = 0.04$ (b).

Then at $\mu \approx 0.251$ there is a merge of two tapes (separatrix manifolds) of singular attractors, accompanied formation of uniform heteroclinic separatrix zigzag. At the further reduction of values of parameter μ there is a development and complication of heteroclinic separatrix zigzag, accompanied a stretching of its accordion on all phase space of conservative system of Eqs. (28) at $\mu = 0$. In Fig. 17 accordions of infinitely folded heteroclinic separatrix zigzags in Poincare section of dissipative system of Eqs. (29) are shown at $\mu = 0.25$ and $\mu = 0.04$.

3.2.2. Standard example of a pendulum with oscillating point of fixing

Consider a standard example of a pendulum with vertically periodically oscillating point of fixing, that is a system with Hamiltonian

$$H(x,y,t,\varepsilon) = y^2/2 + (\omega^2 + \varepsilon\cos t)\cos x. \qquad (30)$$

Let us write down the system of equations with Hamiltonian (30) in the form of four-dimensional conservative system of the equations

$$\dot{x} = y,\ \ \dot{y} = (\omega^2 + z)\sin x,\ \ \dot{z} = r,\ \ \dot{r} = -z \qquad (31)$$

with conditions $H = z^2 + r^2 = \varepsilon^2$, $z_0 = z(0) = \varepsilon$. Let us consider alongside with system of Eqs. (31) the extended dissipative system of the equations

$$\dot{x} = y,\ \ \dot{y} = (\omega^2 + z)\sin x - \mu y,\ \ \dot{z} = r,\ \ \dot{r} = -z \qquad (32)$$

and analyze numerically transition from solutions of dissipative system of Eqs. (32) to solutions of conservative system of Eqs. (31) at the fixed values of parameters $\varepsilon, \omega = 1$ when parameter μ tends to zero. It is convenient to analyze solutions of systems of Eqs. (31)- (32) in coordinates $(\sin x, y)$.

At value of perturbation parameter $\varepsilon = 2$ the conservative system of Eqs. (31) already possesses chaotic dynamics in sense of theory FSM. It is easy to be convinced of it if parameter μ in dissipative extended system of Eqs. (32) tends to zero. At $\mu \approx 0.38$ the double period bifurcation of each of original singular stable limit cycles $C^{\pm}$ occurs, that gives rise to cascades of Feigenbaum period doubling bifurcations. The given cascades of bifurcations come to the end with a birth of two singular Feigenbaum attractors at $\mu \approx 0.348$.

At further reduction of values of parameter μ the cascades of bifurcations of births of stable cycles with the periods according to the Sharkovskii order begin. Cycles of period five, for example, can be observed at $\mu \approx 0.3428$. At $\mu \approx 0.34$ two homoclinic cascades of bifurcations begin, then, as well as in other systems, there is a merge of two tapes of singular attractors (two infinitely folded heteroclinic separatrix manifolds) and then process of formation of new stable cycles proceeds on uniform infinitely folded heteroclinic separatrix surface. Development and complication of infinitely folded heteroclinic separatrix zigzag in dissipative extended system of Eqs. (32) accompanied a stretching of its accordion on the most part of phase space of conservative system of Eqs. (31) at reduction of values of parameter μ is shown in Fig. 18.

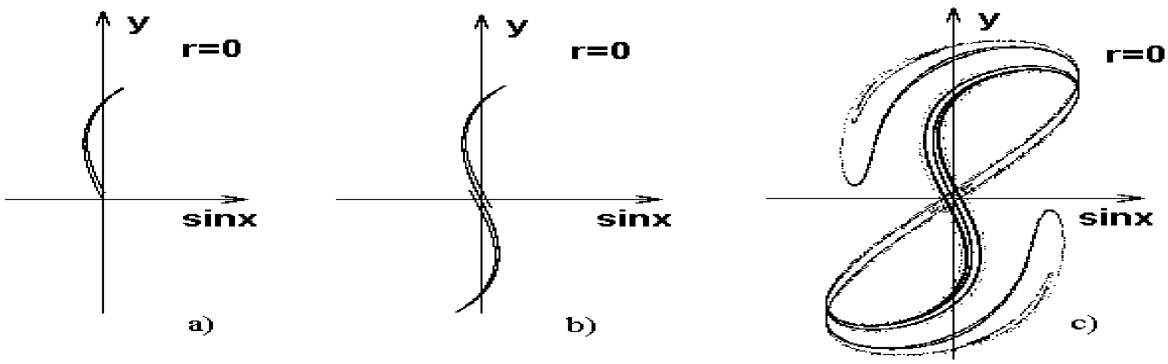

Figure 18. Projections on the plane $(\sin x, y)$ of Poincare section $(r = 0, z > 0)$ of solutions of dissipative system of Eqs. (32) for $\varepsilon = 2$ and $\mu \approx 0.337$ (a), $\mu \approx 0.33$ (b) and $\mu \approx 0.29$ (c).

3.2.3. *Conservative Duffing-Holmes equation*

Rewrite conservative Duffing-Holmes equation in the form of two-dimensional nonautonomous conservative system of the equations

$$\dot{x} = y,\ \dot{y} = \delta x - x^3 + \varepsilon \cos \omega t. \tag{33}$$

Nonperturbed $(\varepsilon = 0)$ system of Eqs. (33) has in the plane (x, y) two homoclinic separatrix loops of zero saddle singular point around singular points $O^{\pm} = (\pm\delta^{1/2}, 0)$ which are centers of nonperturbed system.

As other above considered systems, system of Eqs. (33) is equivalent to the perturbed four-dimensional conservative autonomous system

$$\dot{x} = y,\ \dot{y} = \delta x - x^3 + z,\ \dot{z} = \omega r,\ \dot{r} = -\omega z \tag{34}$$

with conditions $H = z^2 + r^2 = \varepsilon^2$, $z_0 = z(0) = \varepsilon$. The system

$$\dot{x} = y,\ \dot{y} = \delta x - x^3 + z - \mu y,\ \dot{z} = \omega r,\ \dot{r} = -\omega z \tag{35}$$

is the extended dissipative system for conservative system of Eqs. (34).

In the work (Dubrovsky, 2010) the two-parametrical bifurcation diagram of system of Eqs. (35) in space of parameters (ε, μ) is constructed. All cycles of the subharmonic cascade of bifurcations up to the cycle of period three, stable in dissipative system of Eqs. (35) at the some values of parameters $(\varepsilon, \mu > 0)$, are continued in a plane of parameters up to the value $\mu = 0$ (when the system becomes conservative) by the modified Magnitskii method of stabilization (Magnitskii & Sidorov, 2006). Thus, it is proved an existence of full subharmonic cascade of bifurcations of cycles of any period according to Sharkovskii order in conservative system of Duffing-Holmes equations (34). For large enough values of perturbation parameter ε conservative system of Eqs. (34) has also homoclinic cascade of bifurcations in full accordance with the FSM theory.

Note in conclusion of this item that the FSM scenario of transition to chaos takes place also in many other nonautonomous two-dimensional nonlinear conservative systems and, in particular, in classical generalized conservative Mathieu system

$$\dot{x} = y,\ \dot{y} = -(\delta + z)x - \alpha x^3,\ \dot{z} = \omega r,\ \dot{r} = -\omega z \tag{36}$$

which is equivalent to conservative generalized Mathieu equation (7) with $\mu = 0$ (Magnitskii , 2008b; Magnitskii , 2011)).

3.3. More complex Hamiltonian and conservative systems

In modern scientific literature Hamiltonian systems with two degrees of freedom refer to as autonomous Hamiltonian four-dimensional systems of ordinary differential equations, Hamiltonian systems with two and a half degrees of freedom refer to as nonautonomous conservative four-dimensional systems of ordinary differential equations with time-dependent Hamiltonian and Hamiltonian systems with three degrees of freedom refer to as autonomous Hamiltonian six-dimensional systems of ordinary differential equations. We consider examples of such systems and show that all such conservative systems satisfy the universal FSM theory of transition to chaos.

3.3.1. Hamiltonian systems with two degrees of freedom

Consider generalized Hamiltonian-Mathieu system with two degrees of freedom

$$\dot{x} = y,\ \ \dot{y} = -(\delta + z)x - x^3,\ \ \dot{z} = r,\ \ \dot{r} = -z - x^2/2 \tag{37}$$

wth Hamiltonian

$$H(x,y,z,r) = (\delta x^2 + y^2 + z^2 + r^2)/2 + zx^2/2 + x^4/4 = \varepsilon.$$

The system of Eqs. (37) contains additional composed $-x^2/2$ in the fourth equation of the conservative four-dimensional generalized Mathieu system of Eqs. (36). In this case extended dissipative system can have a kind of

$$\dot{x} = y,\ \ \dot{y} = -(\delta + z)x - x^3 - \mu y,\ \ \dot{z} = r,\ \ \dot{r} = -z - x^2/2 + (\varepsilon - H(x,y,z,r))r. \tag{38}$$

Let's consider a case $\delta = 0.5$ at which the cycle $z^2 + r^2 = \varepsilon^2$ $(x = y = 0)$ of Hamiltonian system of Eqs. (37) is an elliptic cycle at enough small ε. At $\varepsilon \approx 0.185$ period doubling bifurcation of the elliptic cycle occurs giving rise to various cascades of period doubling bifurcations and subharmonic cascades of bifurcations, generating infinitely folded heteroclinic separatrix manifolds both in extended dissipative system of Eqs. (38) and in Hamiltonian system of Eqs. (37) when $\mu \to 0$. Development and complication of infinitely folded heteroclinic separatrix zigzag in dissipative extended system of Eqs. (38) at $\varepsilon = 1$ accompanied a stretching of its accordion on all phase space of conservative system of Eqs. (37) at reduction of values of parameter $\mu \to 0$ is shown in Fig. 19.

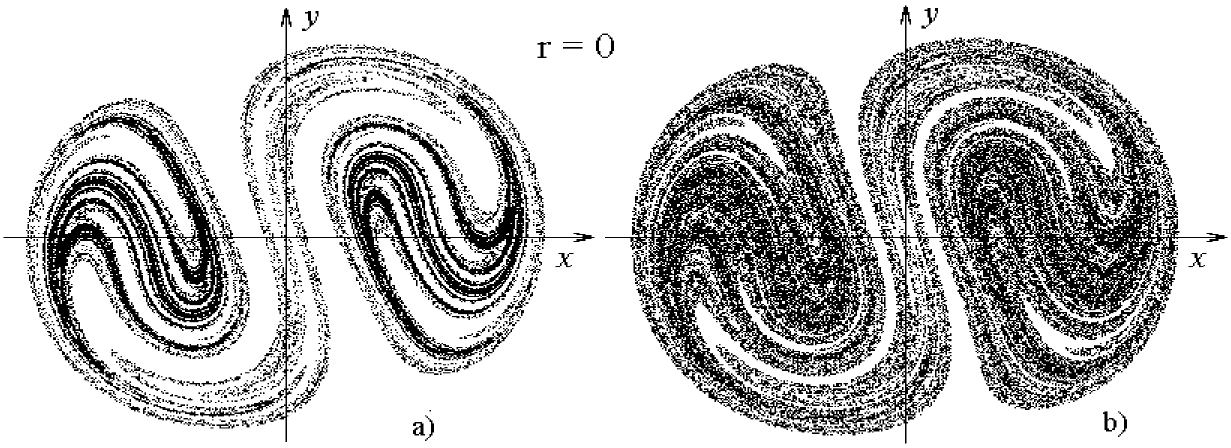

Figure 19. Projections on the plane (x, y) of the Poincare section $(r = 0, z > 0)$ of solutions of dissipative system of Eqs. (38) for $\delta = 0.5$, $\varepsilon = 1$ and $\mu \approx 0.029$ (a), $\mu \approx 0.01$ (b).

In conclusion of this item note that the FSM scenario of transition to chaos takes place also in classical Henon-Heiles system with Hamiltonian

$$H(x, y, z, r) = (x^2 + y^2 + z^2 + r^2) / 2 + zx^2 - z^3 / 3$$

and in Yang-Mills-Higgs system (Magnitskii, 2008b; Magnitskii , 2009) with two degrees of freedom and with Hamiltonian

$$H = (\dot{x}^2 + \dot{z}^2) / 2 + x^2 z^2 / 2 + \nu (x^2 + z^2) / 2 .$$

3.3.2. Hamiltonian systems with two and a half degrees of freedom

It is considered to be in modern literature that in case of systems with one and a half and two degrees of freedom, conservation of energy limits divergence of trajectories along all power surface, and in case of systems with two and a half and more degrees of freedom trajectories form in phase space uniform everywhere dense network named by Arnold web. Trajectories thus, as it is considered, for large enough time cover all power surface of system, approaching as much as close to its any point.

About inadequacy of the first part of this statement to the real situation all considered above examples of Hamiltonian systems with one and a half and two degrees of freedom testify. It follows from the established fact that chaotic dynamics in conservative systems is not consequence of tori resonances in nonperturbed systems, but is consequence of infinite cascades of bifurcations of births of new elliptic and hyperbolic cycles, not being cycles of nonperturbed systems. Thus the accordion of heteroclinic separatrix zigzag can be stretched on all phase space of perturbed conservative system (on all power surface), and this process is not connected in any way with tori of nonperturbed system.

Let's show now that the second part of the above mentioned statement does not correspond also to the real situation, and that in Hamiltonian systems with two and a half degrees of freedom trajectories are not obliged to cover all power surface even at the large perturbations. Thus, areas with regular, local chaotic and global chaotic dynamics can exist simultaneously on power surface of such systems even at large values of perturbation parameter.

Let's consider the system consisting from two nonlinear oscillators with weak periodic nonlinear connection. Hamiltonian of this system looks like

$$H = (\dot{x}^2 + x^2 + x^4 / 2 + \dot{z}^2 + z^2 + z^4 / 2) / 2 + \varepsilon x z \cos t. \tag{39}$$

Hamiltonian (39) generates so called Hamiltonian system with two and a half degrees of freedom, i.e. four-dimensional system of ordinary differential equations with periodic coefficients

$$\dot{x} = y,\ \dot{y} = -x - x^3 - \varepsilon z \cos t,\ \ \dot{z} = r,\ \ \dot{r} = -z - z^3 - \varepsilon x \cos t. \tag{40}$$

Having designated $\varepsilon \cos t = u$ we shall receive from the system of Eqs. (40) the conservative six-dimensional autonomous system of ordinary differential equations

$$\dot{x} = y,\ \ \dot{y} = -x - x^3 - zu,\ \ \dot{z} = r,\ \ \dot{r} = -z - z^3 - xu,\ \ \dot{u} = v,\ \ \dot{v} = -u \tag{41}$$

with the condition $H = u^2 + v^2 = \varepsilon^2$, $u(0) = \varepsilon, v(0) = 0$. In this case extended dissipative system can have a kind of

$$\dot{x} = y,\ \ \dot{y} = -x - x^3 - zu - \mu y,\ \ \dot{z} = r,\ \ \dot{r} = -z - z^3 - xu - \mu r,\ \ \dot{u} = v,\ \ \dot{v} = -u. \tag{42}$$

It is easy to see, that solutions of conservative system of Eqs. (41) with initial conditions $z_0 = x_0, r_0 = y_0$ are solutions of four-dimensional conservative system

$$\dot{x} = y,\ \ \dot{y} = -x - x^3 - xu,\ \ \dot{u} = v,\ \ \dot{v} = -u \tag{43}$$

The right part of last system coincides with the right part of the considered above conservative generalized Mathieu system of Eqs. (36) with $\delta = 1$. At large enough values of parameter ε (for example, $\varepsilon \ge 1.8$) conservative system of Eqs. (41) possesses chaotic dynamics even on solutions of system of Eqs. (43), as at reduction of values of parameter μ the subharmonic cascade of bifurcations of stable cycles exists in dissipative system of Eqs. (42) giving rise complex heteroclinic separatrix manifolds in four-dimensional subspace of solutions of conservative system of Eqs. (41) being solutions of system of Eqs. (43). However, chaotic dynamics of solutions of system of Eqs. (41) is local even inside this four-dimensional subspace of solutions and is limited by area of regular movements on two-dimensional tori (see in Fig. 20a). At the same time for solutions, not satisfying conditions $z_0 = x_0, r_0 = y_0$ or $z_0 = -x_0, r_0 = -y_0$ conservative system of Eqs. (41) has areas of complex global chaotic dynamics and areas of regular movement on three-dimensional tori even at such large values of perturbation parameter (see in Fig. 20b).

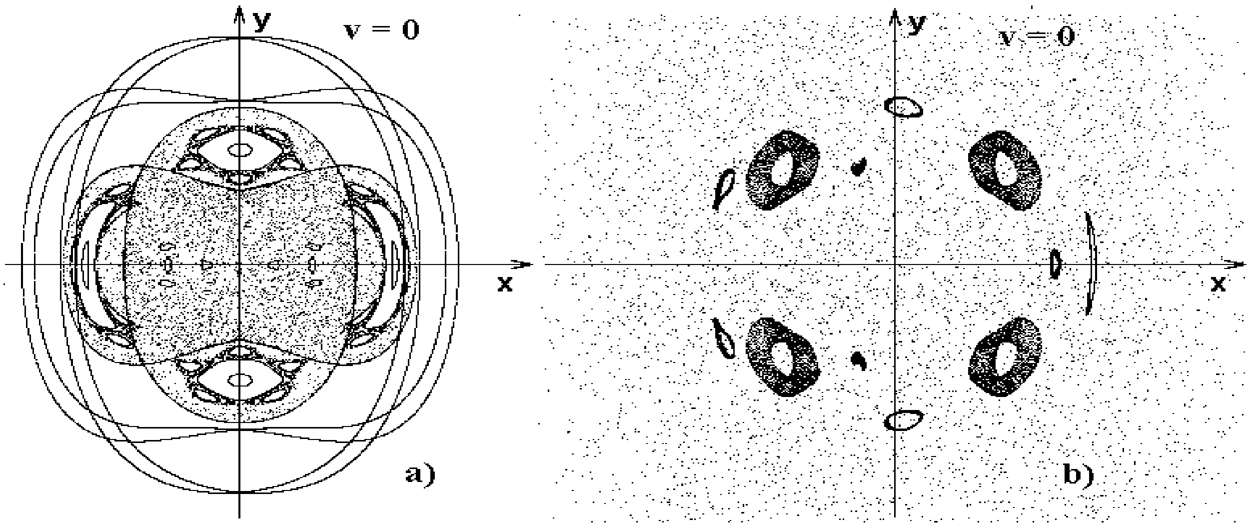

Figure 20. Projections of the section $v = 0$ of system of Eqs. (41) on the plane (x, y) for $\varepsilon = 1.8$, $z_0 = x_0, r_0 = y_0$ (a) and $z_0 \neq \pm x_0, r_0 \neq \pm y_0$ (b).}

So, in conservative system of Eqs. (41) even at enough large values of parameter ε there exist simultaneously areas of regular movement on two-dimensional tori around of basic cycles of the system, areas of regular movement on three-dimensional tori around of mentioned above two-dimensional tori, areas of local chaotic behaviour of trajectories of the system in four-dimensional subspace of five-dimensional phase space and areas of global chaotic behavior of trajectories of the system in the other part of phase space. All tori of the system are not tori of nonperturbed system, and are born as a result of various bifurcations in accordance with FSM theory. Global chaos in the system is not consequence of destruction of any mythical tori of nonperturbed system as this phenomenon is treated by the modern classical Hamiltonian mechanics and KAM (Kolmogorov-Arnold-Mozer) theory, and it is extreme consequence of complication of infinitely folded heteroclinic separatrix manifold of extended dissipative system of Eqs. (42) when dissipation parameter μ tends to zero (Magnitskii , 2011).

3.3.3. Hamiltonian system with three degrees of freedom

Let's consider a complex Hamiltonian system with three degrees of freedom

$$\dot{x} = y,\ \dot{y} = -(\delta + z)x - x^3,\ \dot{z} = r,\ \dot{r} = -z - x^2/2 - u^2/2,\ \dot{u} = v,\ \dot{v} = -(\gamma + z)u - u^3 \tag{44}$$

with Hamiltonian

$$H(x,y,z,r,u,v) = (\delta x^2 + y^2 + z^2 + \gamma u^2 + v^2)/2 + z(x^2 + u^2)/2 + (x^4 + u^4)/4 = \varepsilon.$$

Extened dissipative two-parametrical system in this case can look like

$$\begin{gathered}\dot{x} = y,\ \dot{y} = -(\delta + z)x - x^3 - \mu y,\ \dot{z} = r, \dot{r} = -z - x^2/2 - u^2/2 \\ +(\varepsilon - H)r,\ \dot{u} = v,\ \dot{v} = -(\gamma + z)u - u^3 - \mu v.\end{gathered} \tag{45}$$

The system of Eqs. (44) is interesting to those, that character of its dynamics contradicts practically to all propositions of the modern classical theory of Hamiltonian systems. In system of Eqs. (44) there exist simultaneously areas of regular movement on two-dimensional tori around of basic cycles of the system, areas of regular movement on three-dimensional tori around of mentioned above two-dimensional tori, areas of local chaotic behavior of trajectories of the system in four-dimensional subspace of a five-dimensional power surface and areas of global chaotic behavior of trajectories of the system in other part of a power surface even at enough large value of the perturbation parameter ε . All tori of the system are not tori of so-called nonperturbed system, but they are born as a result of various bifurcations. Global chaos in the system is not consequence of destruction of any mythical tori of nonperturbed system as this phenomenon is treated by the modern classical Hamiltonian mechanics and KAM theory. It is extreme consequence of complication of infinitely folded heteroclinic separatrix manifold of extended dissipative system of Eqs. (45) when dissipation parameter μ tends to zero. Corresponding heteroclinic separatrix zigzags in projections to the plane (x,y) of the section $r=0$ of solutions of extended dissipative system of Eqs. (45) at $\varepsilon=3$, $u_0=x_0, v_0=y_0$ and $\mu=0.125$, $\mu=0.095$ and $\mu=0.005$ are presented in Fig. 21 (see (Magnitskii , 2008b; Magnitskii , 2011)).

Thus we can make a conclusion that universal bifurcation Feigenbaum-Sharkovskii-Magnitskii theory describes also transition to dynamical chaos in nonlinear conservative and, in particular, Hamiltonian systems of ordinary differential equations at large enough values of perturbation parameter. Note that for small values of perturbation parameter the key role in complication of dynamics of any conservative system is played by nonlocal effect of duplication of hyperbolic and elliptic cycles and tori in a neighborhood of separatrix contour (or surface) of nonperturbed system opened and analyzed by the author in (Magnitskii , 2009b; Magnitskii , 2011).

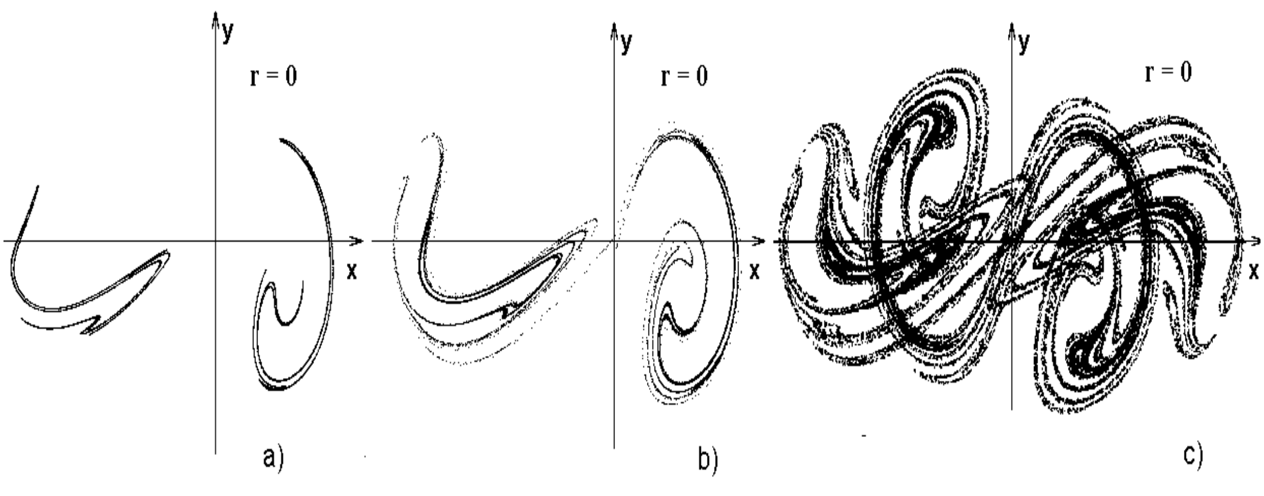

Figure 21. Development and complication of heteroclinic separatrix zigzag in dissipative system of Eqs. (45) for $\varepsilon=3$, $u_0=x_0, v_0=y_0$ and $\mu=0.125$ (a), $\mu=0.095$ (b) and $\mu=0.005$ (c).

4. Spatio-temporal chaos in nonlinear partial differential equations

4.1. Diffusion chaos in reaction-diffusion systems

Wide class of physical, chemical, biological, ecological and economic processes is described by reaction-diffusion systems of partial differential equations

$$u_t = D_1 u_{xx} + f(u,v,\mu), \quad v_t = D_2 v_{xx} + g(u,v,\mu), \; 0 \le x \le l, \tag{46}$$

depending on scalar or vector parameter μ. Such system is very complex system. Behavior of its solutions depends on coefficients of diffusion and their ratio, length of space area and edge conditions. As a rule, there exists a value of the parameter μ_0, such that for all $\mu < \mu_0$ reaction-diffusion system has a stable stationary and space homogeneous solution (U,V), denoted as thermodynamic branch. When $\mu > \mu_0$, then thermodynamic branch loses its stability and after that reaction-diffusion system can have quite different solutions such as periodic oscillations, stationary dissipative structures, spiral waves and nonstationary nonperiodic nonhomogeneous solutions. Last solutions are known as diffusion or spatio-temporal chaos.

4.1.1. *Diffusion chaos in the brusselator model*

Considered on a segment $[0,l]$ the system of the brusselator equations offered for the first time by the Brussels school of I. Prigoging as a model of some self-catalyzed chemical reaction with diffusion

$$u_t = D_1 u_{xx} + A - (\mu+1)u + u^2 v, \; v_t = D_2 v_{xx} + \mu u - u^2 v. \tag{47}$$

It is easily to see, that stationary spatially-homogeneous solution (a thermodynamic branch) of the system of Eqs. (47) is the solution $u = A, v = \mu / A$. Therefore the first boundary problem for brusselator should satisfy the boundary conditions

$$u(0,t) = u(l,t) = A, \; v(0,t) = v(l,t) = \mu / A.$$

A more detailed analysis shows (Hassard *et al.*, 1981; Magnitskii & Sidorov, 2006) that at $\mu > \mu_0$ stable periodic spatially inhomogeneous solutions of the system of Eqs. (47) have the following asymptotic representations for small $\varepsilon = (\mu - \mu_0)^{1/2}$:

$$u(x,t) = A + \varepsilon \cos \omega t \cdot \sin\frac{\pi x}{l} + O(\varepsilon^2), \; v(x,t) = \frac{\mu}{A} + \varepsilon\gamma \cos \omega t \cdot \sin\frac{\pi x}{l} + \varepsilon\delta \sin \omega t \cdot \sin\frac{\pi x}{l} + O(\varepsilon^2),$$

where $\omega = \omega(\varepsilon) = \omega_0(1 + O(\varepsilon^2))$, γ, δ are some constants, and a kind of spatial harmonics is defined by boundary conditions of a problem. Points of a segment make fluctuations with identical frequency and a constant gradient of a phase. The effect of "wave" running on a segment is created. In the case of the second boundary value problem on a segment with the free ends, the periodic solutions born at $\mu > \mu_0$ will be spatially homogeneous (Magnitskii & Sidorov, 2006).

Let us show that the further complication of solutions of brusselator equations (47) at growth of values of parameter μ occurs according to the universal FSM theory both for the first and the second boundary value problems. In the beginning we shall consider the first boundary value problem on the segment $[0,\pi]$ for brusselator system with diffusion coefficients $D_1 = 0.15, D_2 = 0.3$. The Feigenbaum period doubling cascade of bifurcations of stable limit cycles and then the Sharkovskii subharmonic cascade of bifurcations exist in infinitely-dimensional phase space of solutions of the problem. Some main cycles and singular attractors of these cascades of bifurcations are presented in Fig. 22.

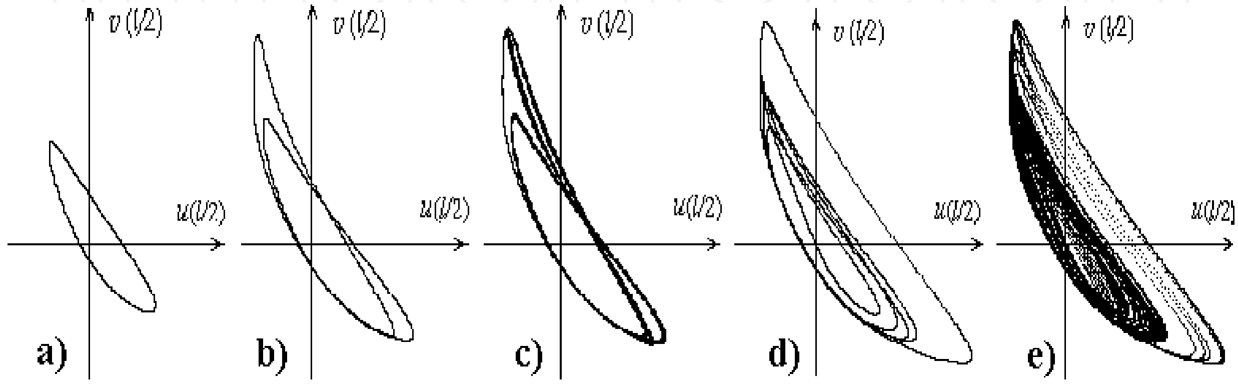

Figure 22. Singular cycle (a), cycle of the double period (b), the Feigenbaum attractor (c), cycle of period five (d) and one of the singular attractors in the first boundary value problem for the brusselator equations (47).

In the second boundary value problem singular toroidal attractors were found out in the brusselator equations for the parameter values $A = 4, l = \pi$ and for coefficients of diffusion $D_1 = 0.1, D_2 = 0.02$. At these fixed values of parameters a two-dimensional stable invariant torus is born from the stable limit cycle in the infinitely-dimensional phase space of the system of Eqs. (47). This torus begins the Feigenbaum period doubling cascade of bifurcations of stable tori on internal frequency generating by the end of cascade the Feigenbaum singular toroidal attractor (see Fig. 23).

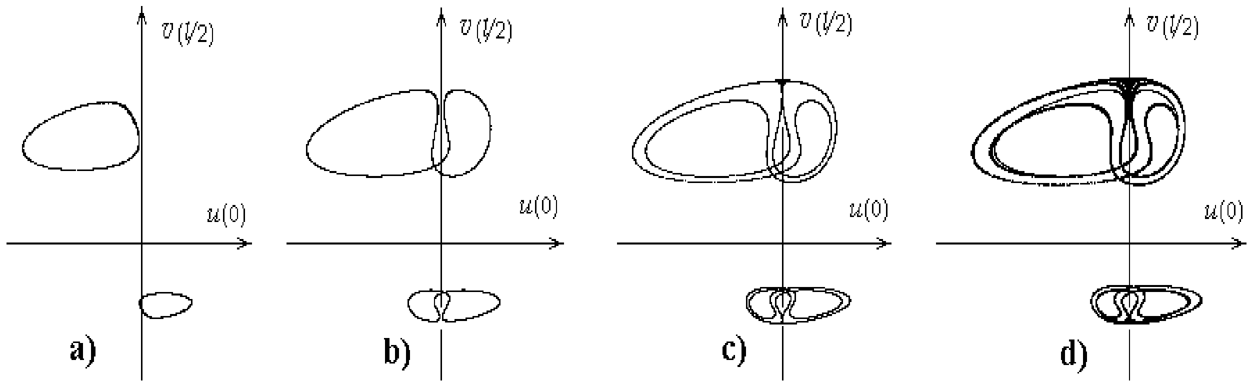

Figure 23. Projections of the section $u(l/2) = 0$ on the plane $(u(0), v(l/2))$ of two-dimensional torus (a), two-dimensional torus of double period on internal frequency (b), two-dimensional torus of period 4 (c) and the Feigenbaum singular toroidal attractor (d) in the second boundary value problems for the brusselator equations (47).

4.1.2. Running waves, impulses and diffusion chaos in excitable mediums

Special case of reaction-diffusion systems is the case of systems of the FitzHugh-Nagumo equations describing nonlinear processes occurring in so-called excitable mediums. Examples of such processes are distribution of impulse on a nervous fiber and a cardiac muscle and also various kinds of autocatalytic chemical reactions. The basic property describing a class of excitable mediums is slow diffusion of one variable in comparison with other variable in system of reaction-diffusion (46). Therefore the system of the FitzHugh-Nagumo equations can be written down in the following general form

$$u_t = Du_{xx} + f(u,v,\mu), \;\; v_t = g(u,v,\mu). \tag{48}$$

It is well-known, that in system of Eqs. (48) in one-dimensional spatial case there can be switching waves, running waves and running impulses, dissipative spatially nongomogeneous stationary structures, and also diffusion chaos - irregular nonperiodic nonstationary structures named sometimes as biological (or chemical) turbulence.

The analysis of solutions of system of Eqs. (48) on a straight line can be carried out by replacement $\xi = x - ct$ and transition to three-dimensional system of ordinary differential equations

$$\dot{u} = y, \;\; \dot{y} = -(cy + f(u,v,\mu))/D, \;\;\; \dot{v} = -g(u,v,\mu)/c, \tag{49}$$

where the derivative undertakes on a variable ξ. Thus the switching wave in system of Eqs. (48) is described by separatrix of the system (49) going from its one singular point into another singular point, running wave and running impulse of system of Eqs. (48) are described by limit cycle and separatrix loop of a singular point of the system (49).

Let's show, that diffusion chaos in the system of FitzHugh-Nagumo equations (48) is described by singular attractors of the system of ordinary differential equations (49) in accordance with the Feigenbaum-Sharkovskii-Magnitskii (FSM) theory. For this purpose consider the system of Eqs. (48)- (49) with nonlinearities

$$f(u,v,\mu) = -(u-1)(u-\delta v)/\varepsilon, \quad g(u,v,\mu) = arctg(\alpha u) - v, \tag{50}$$

where parameter ε is a small parameter. Note, that system of Eqs. (48) with polynomial function $f(u,v,\mu)$ and function $g(u,v,\mu)$ having at everyone v final limiting values at $u \to \pm\infty$, describes some kinds of autocatalytic chemical reactions (Zimmermann et al., 1997). It is easy to see that system of Eqs. (49)-(50) has singular point $O(0,0,0)$ for any values of parameters. Besides that, for $\alpha > 1/\delta$ system of Eqs. (49)-(50) has two more singular points $O_{\pm}(\pm u_*, 0, \pm u_*/\delta)$, where value u_* is a positive solution of the equation $\delta\, arctg(\alpha u_*) = u_*$.

A case of greatest interest is, naturally, a case when bifurcation parameter is the parameter c, not entering obviously in system of the Eqs. (48) and being the value of velocity of perturbations distribution along an axis x. For $c < \sqrt{1 + (\alpha\delta - 1)/(1-\varepsilon)}$ the limit stable cycle is born from a zero singular point as a result of Andronov-Hopf bifurcation. The singular point O becomes a saddle - focus. At the further reduction of values of parameter c the cascade of Feigenbaum double period bifurcations of stable limit cycles takes place in system of Eqs. (49)-(50) up to formation of the first singular attractor - Feigenbaum attractor. At the further reduction of values of parameter c in system of Eqs. (49)-(50) the full subharmonic cascade of bifurcations of stable cycles is realized according to the Sharkovskii order and then incomplete homoclinic cascade of bifurcations of stable cycles is realized. Last cycles converge to the homoclinic contour - the separatrix loop of the saddle-focus O (Fig. 24).

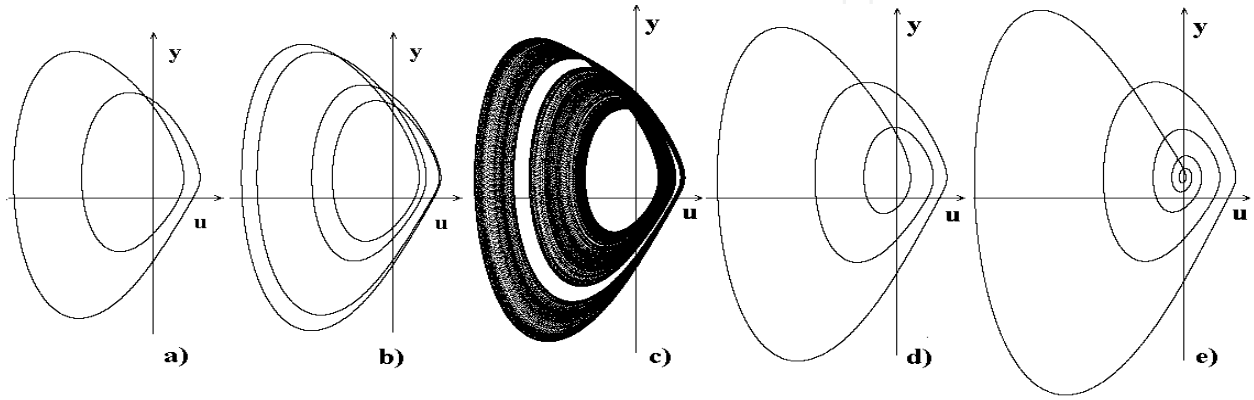

Figure 24. Projections of period two cycle (a), period four cycle (b), singular attractor (c), period three cycle (d) and homolcinic period four cycle (e) in the system of Eqs. (49).

Obtained result means, that the system of the FitzHugh-Nagumo equations (48) with fixed values of parameters can have infinite number of various autowave solutions of any period running along a spatial axis with various velocities and also infinite number of various regimes of spatio-temporal (diffusion) chaos.

4.1.3. Cycles and chaos in distributed market economy

Another, essentially different example of formation of spatio-temporal chaos in the nonlinear mediums is the distributed model of a market self-developing economy offered by the author and developed then in (Magnitskii & Sidorov, 2006). The model is a system of three nonlinear differential equations, two of which describe the change and intensity of motion (diffusion) of capital and consumer demand in a technology space under the influence of change of profit rate. The last is described by the third ordinary differential equation.

Self-development of market economy is characterized by spontaneous growth of capital and its movement in the technology space in response to differences in profitability. The model describes formation of social wealth, including production, distribution, exchange, and consumption. A distinctive feature of the model is that distribution of profitability (profit rates) determines the direction and the intensity of motion (diffusion) of capital and its spontaneous growth through generation of added value. Three economic agents having their own interests take part in economic processes that are employers, workers and government. In the model, being based on rigorous rules of Karl Marx's theory of added value, self-development of a market economy involves movement and spontaneous growth of capital of employers, which is the result of creation of added value by workers in the circulation process of capital under government control.

We show that the market economy system can exist in periodic or chaotic regimes only. Periodic regime can have any period in accordance with the theory FSM and any chaotic regime (economic crisis) can be described by some complex cycle or singular attractor.

The model assumes an unstructured closed economic system that is developing in a finite-dimensional Euclidean space R^n, called the technology space. Each point $c \in R^n$ corresponds to a certain production technology of some commodity and its coordinate $c_i, i = 1,...,n$ is the consumption of resource i per unit output.

System of market self-developing economy has the form (Magnitskii & Sidorov, 2005; Magnitskii & Sidorov, 2006):

$$\begin{aligned}
\frac{\partial x(t,c)}{\partial t} &= -div(d_1(c,x,z)grad\,z) + bx((1-\sigma)z - \delta y), \\
\frac{\partial y(t,c)}{\partial t} &= -div(d_2(c,y,z)grad\,z) + x(1-(1-\delta)y + \sigma z), \\
\frac{\partial z(t,c)}{\partial t} &= a(y - dx).
\end{aligned} \tag{51}$$

where $x(t,c)$ is a normalized distribution of capital density, $y(t,c)$ is a normalized distribution of total consumer demand density and $z(t,c)$ is a distribution of profit rate at time t in the technology space; δ is government portion of added value (taxis, custom duties, etc.), σ is employers personal consumption portion of added value and a,b,d are structural economic parameters. Note that the system of Eqs. (51) is a particular case of systems with multicomponent diffusion, where the activator (the variable providing positive feedback) is the capital and the inhibitor (the variable suppressing capital growth) is the consumer demand.

System of equations describing the variation of macroeconomic variables can be similarly reduced to the form

$$\dot{x}(t) = bx((1-\sigma)z - \delta y),\ \dot{y}(t) = x(1-(1-\delta)y + \sigma z),\ \dot{z}(t) = a(y - dx). \tag{52}$$

Parameters δ and σ are bifurcation parameters in system of Eqs. (52). Increase in values of parameter σ as well as reduction of values of parameter δ generate the Feigenbaum cascade of period-doubling bifurcations and then the Sharkovskii subharmonic cascade and chaotic dynamics in system of Eqs. (52) (cycle of period three is presented in Fig.10d). These results gave us possibility to draw the first important conclusion: uncontrolled growth of personal consumption of the employers as well as low government demand for consumer goods (government orders, government support to business, etc.) lead to various crisis phenomena and destroy the economic system.

Consider now the second boundary-value problem for the system of Eqs. (51) on an interval and the thermodynamic branch of this problem

$$(x^*, y^*, z^*) = \left(\frac{1-\sigma}{d(1-\delta-\sigma)}, \ \frac{1-\sigma}{1-\delta-\sigma}, \ \frac{1-\delta}{1-\delta-\sigma} \right)$$

Linearize the considered problem in the neighborhood of the thermodynamic branch, one can obtain that it is stable only when $d_1 \geq d_2 / d$ (Magnitskii & Sidorov, 2006). Thus, we can draw the second important conclusion: high inertia of the capital, slowing down its response to changes in profit rates and consumer demand, also makes the economic system unstable and lead to its destruction.

4.2. Spatio-temporal chaos in autooscillating mediums

It is well-known that any solution of the reaction-diffusion system (46) in a neighborhood $\mu > \mu_0$ of the thermodynamic branch can be approximated by some complex-valued solution $W(r,\tau) = u(r,\tau) + iv(r,\tau)$ of the Kuramoto-Tsuzuki (or Time Dependent Ginzburg-Landau) equation (Kuramoto & Tsuzuki, 1975):

$$W_\tau = W + (1+ic_1)W_{rr} - (1+ic_2)|W|^2 W, \tag{53}$$

where $r = \varepsilon x$, $\tau = \varepsilon^2 t$, $0 \leq r \leq R$, $\varepsilon = \sqrt{\mu - \mu_0}$, c_1, c_2 - some real constants. It is evident that for arbitrary phase ϕ the equation (53) has a space homogeneous solution $W(\tau) = \exp(-i(c_2\tau + \phi))$. Hence, each element of the medium (53) makes harmonious oscillations with frequency c_2 and this solution is stable in some area of parameters c_1 and c_2. Such mediums refer to as **autooscillating mediums**.

4.2.1. Transition to chaos in Kuramoto-Tsuzuki (Ginzburg-Landau) equation

In other area of parameters c_1 and c_2 the Kuramoto-Tsuzuki (Ginzburg-Landau) equation (53) has a stable automodel solution $W(r,\tau) = F(r)\exp(i(\omega\tau + a(r)))$. If $a(r) = kr$ then oscillations of the next elements occur with a constant phase lag, that corresponds to movement on space of a phase wave. In a two-dimensional case the equation (53) has also solutions in a kind of leading centers - sequences of running up concentric phase waves, and spiral waves. But equation (53) has also nonperiodic nonhomogeneous solutions in some areas of parameters - spatio-temporal or diffusion chaos.

From an opinion of most of researchers analysis of such solutions can be successfully fulfilled by using the Galerkin small-mode approximations for reducing the equation (53) to a nonlinear three-dimensional chaotic system of ordinary differential equations. As it was shown in (Magnitskii & Sidorov, 2005b; Magnitskii & Sidorov, 2006), all irregular attractors of reductive three-dimensional system are also singular attractors, and transition to chaos in this system occurs also in accordance with the Feigenbaum-Sharkovskii-Magnitskii (FSM) theory.

But further investigations of solutions of the Kuramoto-Tsuzuki (Ginzburg-Landau) equation (53) directly in its phase space showed that in reality subharmonic cascade of bifurcations of stable two-dimensional tori with arbitrary period in accordance with the Sharkovskii order in every frequency and in two frequencies simultaneously takes place in this equation.

It was considered the second boundary value problem on a segment $[0,l]$ for equation (53) and it was constructed four-dimensional subspace $(u(0),v(0),u(l/2),v(l/2))$ of infinitely-dimensional phase space of the problem. Then for different values of bifurcation parameters c_1 and c_2 the section of four-dimensional subspace has been carried out by the plane $u(l/2)=0$ and there were considered projections of this section on the plane $(u(0),v(l/2))$. Such method of the analysis of phase space of solutions of Kuramoto-Tsuzuki (Ginzburg-Landau) equation (53) appeared extremely fruitful and has enabled to find in the equation all cascades of bifurcations of two-dimensional tori in accordance with the theory FSM (see Figs. 25-26).

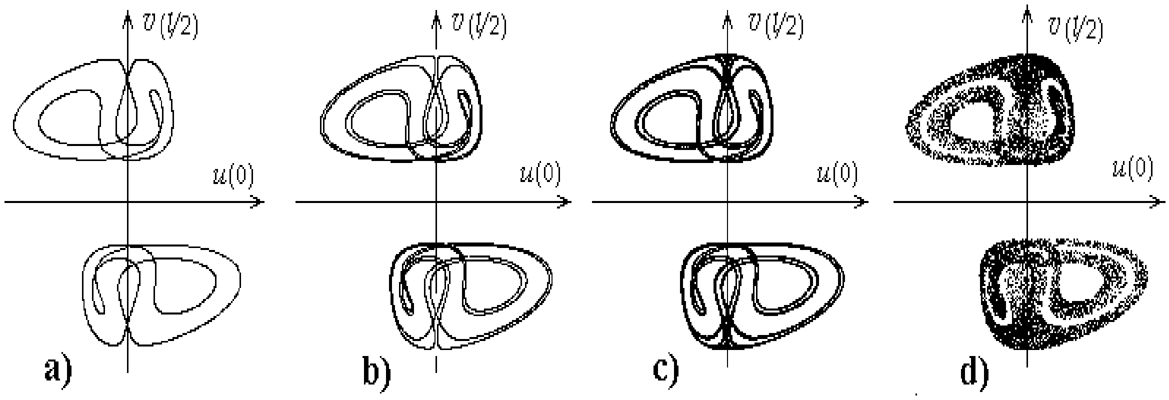

Figure 25. Bifurcation cascade on internal frequency in the equation (53). Projections of section $u(l/2)=0$ on the plane $(u(0),v(l/2))$ of two-dimensional invariant tori: period four torus (a), period eight torus (b), Feigenbaum toroidal singular attractor (c) and more complex toroidal singular attractor (d).

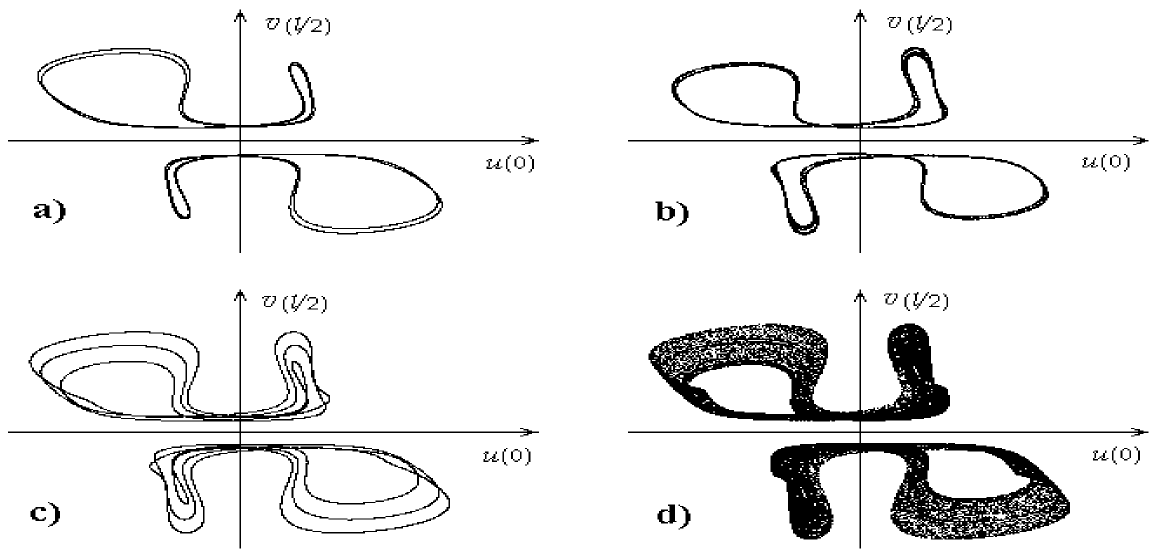

Figure 26. Bifurcation cascade on external frequency in the equation (53). Projections of section $u(l/2)=0$ on the plane $(u(0),v(l/2))$ of two-dimensional invariant tori: period two torus (a),

Feigenbaum toroidal singular attractor (b), period three torus (c) and more complex toroidal singular attractor (d).

Note that in monograph (Magnitskii & Sidorov, 2006) one can find full bifurcation diagram of existence of various subharmonic cascades of bifurcations of two-dimensional invariant tori in the second boundary value problem for the Kuramoto-Tsuzuki (Ginzburg-Landau) equation (53) in the space of parameters (c_1, c_2).

4.2.2. Running waves and chaos in autooscillating mediums

For the analysis of running waves and spatio-temporal chaos in autooscillating active mediums we apply the method used in the Section 4.1.2 for the analysis of mechanisms of formation of running waves, impulses and diffusion chaos in nonlinear excitable mediums. Let's show, that in case of autooscillating active mediums role of cascades of bifurcations of limit cycles converging to a separatrix loop of singular point is plaid by cascades of bifurcations of two-dimensional tori of four-dimensional system of ordinary differential equations converging to singular two-dimensional homoclinic structure, being the Cartesian product of a singular limit cycle on a separatrix loop of singular point. Thus the four-dimensional system has infinite number of subharmonic and homoclinic toroidal singular attractors, generating spatio-temporal chaos in original autooscillating system of partial differential equations. The solutions of four-dimensional system specifying movement on the singular homoclinic structure, tend to the periodic singular solution at $\xi \to \pm\infty$. Thus, formation of running waves and spatio-temporal chaos in autooscillating active mediums also is described by the universal bifurcation Feigenbaum-Sharkovskii-Magnitskii theory.

Rewrite the Cauchy problem on a straight line for the Kuramoto-Tsuzuki (Ginzburg-Landau) equation with complex-valued function $W(x,t) = u(x,t) + iv(x,t)$ as system of two parabolic equations with real variables $u(x,t)$ and $v(x,t)$

$$\begin{gathered} u_t = u + u_{xx} - c_1 v_{xx} - (u - c_2 v)(u^2 + v^2),\ v_t = v + c_1 u_{xx} + v_{xx} - (c_2 u + v)(u^2 + v^2), \\ -\infty < x < \infty,\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ 0 \le t < \infty. \end{gathered} \tag{54}$$

We shall search a solution of system of Eqs. (54) as a running wave $u(x,t) = u(x - ct),\ v(x,t) = v(x - ct)$. Let's enter an automodel variable $\xi = x - ct$ and write down the system of (54) as the system of two ordinary differential equations of the second order

$$-c\dot{u} = u + \ddot{u} - c_1\ddot{v} - (u - c_2 v)(u^2 + v^2),\ -c\dot{v} = v + c_1\ddot{u} + \ddot{v} - (c_2 u + v)(u^2 + v^2), \tag{55}$$

where the derivative undertakes on a variable ξ. Resolving the system of Eqs. (55) concerning the second derivatives $\ddot{u}$ and $\ddot{v}$ and passing to phase variables $u, \dot{u} = z, v, \dot{v} = r$ we shall receive four-dimensional system of ordinary differential equations

$$\begin{aligned} \dot{u} &= z, \quad \dot{z} = (-u - cz - c_1(v + cr) + ((c_1c_2 + 1)u + (c_1 - c_2)v)(u^2 + v^2)) / (1 + c_1^2), \\ \dot{v} &= r, \quad \dot{r} = (-v - cr + c_1(u + cz) + ((c_1c_2 + 1)v + (c_2 - c_1)u)(u^2 + v^2)) / (1 + c_1^2), \end{aligned} \tag{56}$$

The greatest interest, as well as in the case of excitable mediums, represents presence in the system of (56) cascades of bifurcations on parameter c, not entering obviously to system of the equations (54) and being the value of velocity of perturbation distribution along a spatial axis x. This case means, that the system of the Kuramoto-Tsuzuki (Ginzburg-Landau) equations (54) with the fixed parameters c_1 and c_2 can have infinite number of various autowave solutions of any period running along a spatial axis with various velocities, and also infinite number of various regimes of spatio-temporal chaos.

Let's illustrate the last statement with an example of system of Eqs. (56) with the fixed values of parameters $c_1 = 2$ and $c_2 = -0.1$. At these values of parameters the singular periodic solution

$$u = k\cos(\alpha\xi), \; v = k\sin(\alpha\xi), \;\; \alpha = (c + \sqrt{c^2 - 4c_2(c_1 - c_2)}) / (2(c_1 - c_2)), \;\; k = \sqrt{1 - \alpha^2}$$

of the system of Eqs. (56) is a stable cycle for $c > 1.306$. At smaller values of parameter c a stable two-dimensional torus is born from the singular cycle as a result of Andronov-Hopf bifurcation. At the further reduction of values of parameter c in system of Eqs. (56) the Feigenbaum cascade of period doubling bifurcations of stable two-dimensional tori on external frequency is realized. Then in system of Eqs. (56) the full subharmonic cascade of bifurcations of stable two-dimensional tori is realized according to the Sharkovskii order and then Magnitskii homoclinic cascade of bifurcations of stable tori is realized converging to the singular homoclinic structure being the Cartesian product of the original singular limit cycle on the separatrix loop of the singular point. Projections of Poincare section $(u = 0, z < 0)$ of some basic two-dimensional tori and singular toroidal attractors on the plane (r, v) are presented in Fig. 27.

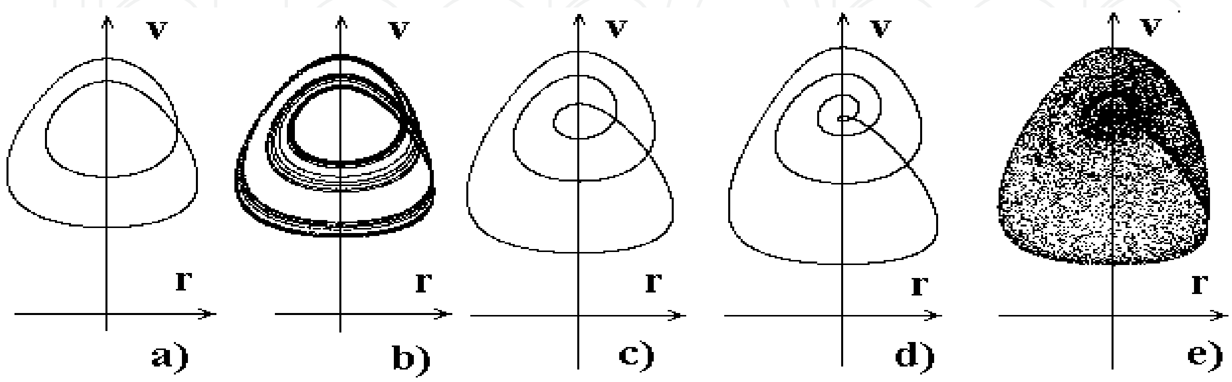

Figure 27. Projections of Poincare section $(u = 0, z < 0)$: period two torus (a), toroidal singular Feigenbaum attractor (b), period three torus (c), period four torus from homoclinic cascade (d) and more complex singular toroidal attractor (e) in the system of Eqs. (56).

4.2.3. Spiral waves and chaos in two-dimensional autooscillating mediums

Let's consider the second boundary value problem for Kuramoto-Tsuzuki (Ginzburg-Landau) equation in spatially two-dimensional area:

$$\begin{aligned} &W_t = W + (1+ic_1)(W_{xx}+W_{yy}) - (1+ic_2)\left|W\right|^2 W,\ 0 \le x \le l,\ 0 \le y \le l, \\ &W(x,y,0) = W_0(x,y),\ W_x(0,y,t) = W_x(l,y,t) = W_y(x,0,t) = W_y(x,l,t) = 0 \end{aligned} \tag{57}$$

with complex-valued function $W(\mathrm{x,y},t) = u(x,y,t) + iv(x,y,t)$. Well-known that solutions of the problem of Eqs. (57) can be plane waves, concentric phase waves (peasmakers) and also spiral waves, that are functions of a kind

$$W = R(r)e^{i(\omega t + a(r) + m\phi)},\ x = r\cos\phi,\ y = r\sin\phi.$$

Solutions with $m=1$ correspond to one-coil spiral waves, with $m>1$ - many-coils spiral waves. Spiral waves can be represented on a plane (x,y) by two kinds of areas, in one of which (shaded) $u(x,y,t) = \mathrm{Re}\,W(x,y,t) \ge 0$, and in another (not shaded) $u(x,y,t) = \mathrm{Re}\,W(x,y,t) < 0$. It is known also, that in some areas of change of values of parameters (c_1,c_2) the quantity of spiral waves starts to increase, that results finally in their destruction and to a forming in the active autooscillating medium, described by the equation (57), chaotic or turbulent regimes.

We show, that the mechanism of formation of spiral waves and turbulent regimes (spatio-temporal chaos) in the boundary value problem (57) for two-dimensional Kuramoto-Tsuzuki (Ginzburg-Landau) equation is subharmonic and homoclinic cascades of bifurcations of two-dimensional and many-dimensional tori in infinitely-dimensional phase space of variables $(u(x,y),v(x,y))$ that also satisfy the universal bifurcation Feigenbaum-Sharkovskii-Magnitskii (FSM) theory.

Detailed numerical analysis of the problem with initial conditions

$$\mathrm{W}_0 = u_0 + iv_0 = 0.1\sum_{m,n=0}^{4}\cos\frac{\pi m x}{l}\cos\frac{\pi n y}{l}[1 + i/(m+1)]$$

was carried out in the paper (Karamisheva, 2010) (see also (Magnitskii, 2011)) by the method of Poincare sections of finite-dimensional subspaces of infinitely-dimensional phase space. It was shown that for $c_1 = 0.5, l = 2$ spiral waves in the plane (x,y) appear at $c_2 <$ -0.65 (see Fig. 28a for $c_2 = -0.68$). Then for four pairs of points (x_1,y_1) and (x_2,y_2), laying near the centers of four spiral waves, projections of sections $u(x_1,y_1) = 0$ on the plane of coordinates $(v(x_1,y_1),u(x_2,y_2))$ were constructed. The projection corresponding to a neighborhood of the center of the bottom spiral wave is represented in Fig. 28b. Thus, the Fig. 28 specifies that stable two-dimensional invariant torus is an image of a simple one-coil spiral wave in phase space of solutions of the problem (57).

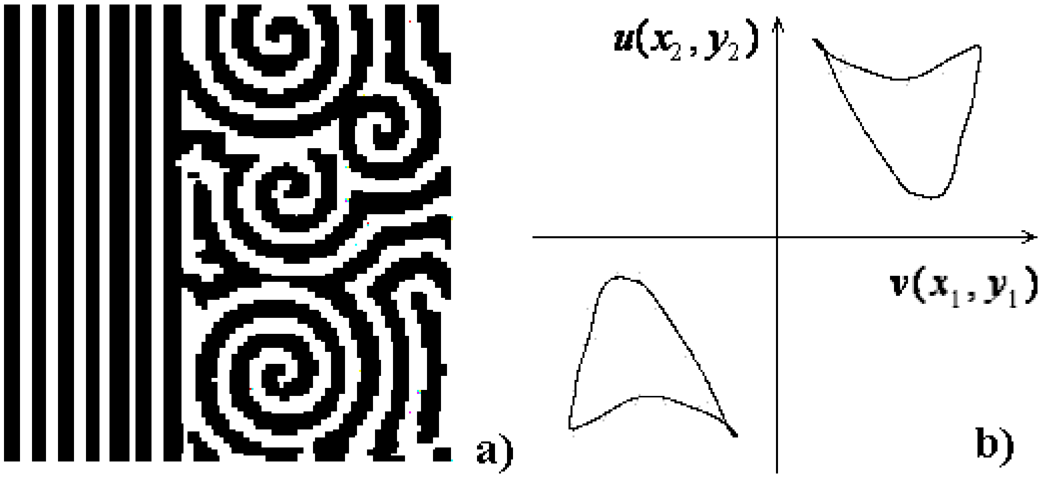

Figure 28. Spiral waves in the plane (x,y) at $c_1 = 0.5$, $c_2 = -0.68$ (a) and projection of the section $u(x_1, y_1) = 0$ of four-dimensional subspace of phase space near center of bottom spiral wave.

At the reduction of negative values of parameter c_2 there is a complication of structure of spiral waves and solutions corresponding to them in phase space of a boundary value problem (57). In Fig. 29a the picture of spiral waves on a plane (x,y) is shown at value $c_2 = -0.7$, and in Fig. 29b the projection of one of two parts of section $u(x_1, y_1) = 0$ on a plane of coordinates $(v(x_1, y_1), u(x_2, y_2))$ for two points from a neighborhood of the center of a spiral wave of the greatest radius from Fig. 29a is shown in the increased scale. It is visible, that in phase space of solutions complex two-dimensional torus of the period three from Sharkovskii subharmonic cascade corresponds to a neighborhood of the center of this spiral wave. In Fig. 29c the projection of section $u(x_1, y_1) = 0$ in a neighborhood of the other spiral wave located in a right bottom corner in Fig. 29a is presented. The projection represents the shaded ring area. But the second section by the plane $u(x_2, y_2) = -28$ of three-dimensional space of points received after carrying out the first section, gives in coordinates $(v(x_1, y_1), v(x_2, y_2))$ two closed curves. These curves testify the existence of three-dimensional torus in phase subspace of solutions in a neighborhood of the center of the second spiral wave.

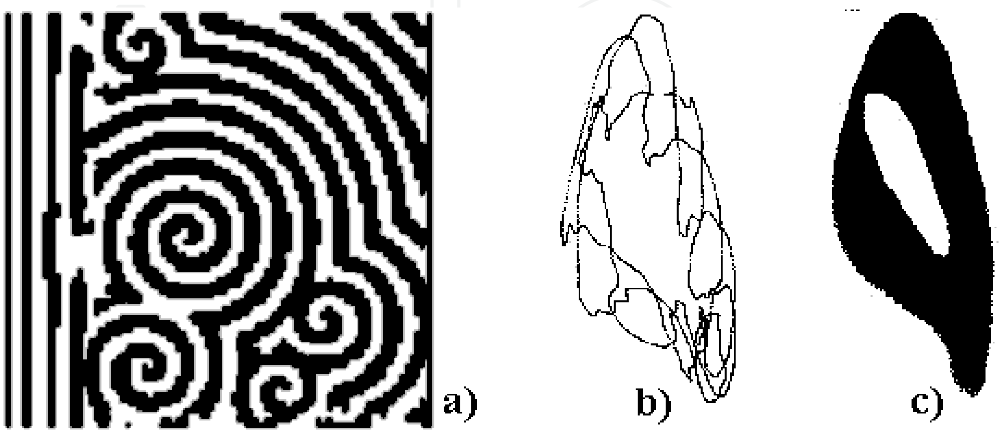

Figure 29. Spiral waves in the plane (x,y) at $c_1 = 0.5$, $c_2 = -0.7$ (a) and projections of parts of sections of four-dimensional subspace of phase space of solutions of the problem (57) in neighborhoods of two spiral waves (b), (c).

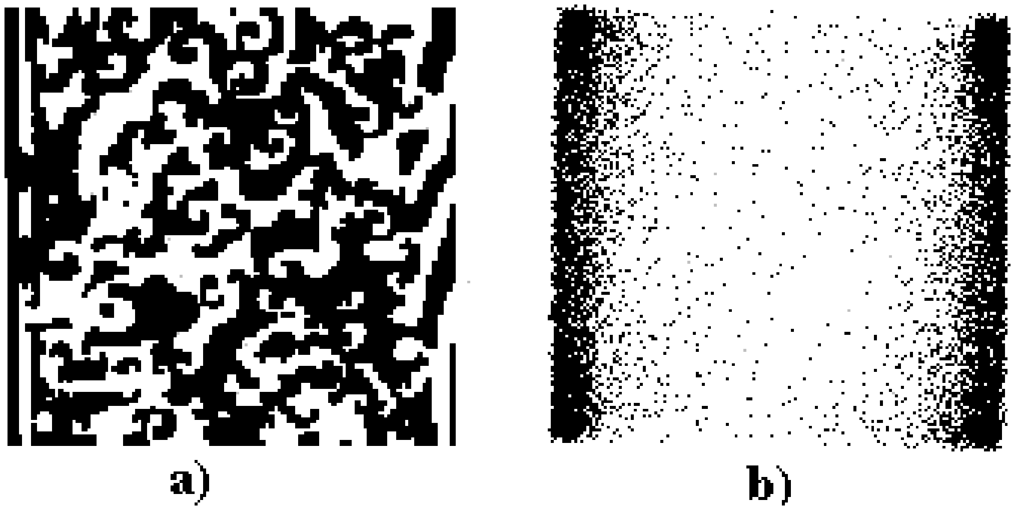

Figure 30. Spatio-temporal chaos at $c_1 = 0.5$, $c_2 = -0.9$ in the plane (x, y) (a) and in projection of section of four-dimensional subspace of phase space of solutions of the problem (57) (b).

At values of parameters $c_1 = 0.5$, $c_2 = -0.9$ already there are no stable spiral waves on a plane (x, y), and in projections of section $u(x_1, y_1) = 0$ of anyone four-dimensional subspace of phase space of solutions the continuous spatio-temporal chaotic regime is observed (Fig. 30).

5. Conclusion

In the chapter it is proved and illustrated with numerous analytical and numerical examples that there exists a uniform universal bifurcation mechanism of transition to dynamical chaos in all kinds of nonlinear systems of differential equations including dissipative and conservative, ordinary and partial, autonomous and non-autonomous differential equations and differential equations with delay arguments. This mechanism is working for all nonlinear continuous models describing both natural and social phenomena of a macrocosm surrounding us, including various physical, chemical, biological, medical, economic and sociological processes and laws. And this universal mechanism is described by the Feigenbaum-Sharkovskii-Magnitskii theory - the theory of development of complexity in nonlinear systems through subharmonic and homoclinic cascades of bifurcations of stable limit cycles or stable two-dimensional or many-dimensional invariant tori.

Notice, that theory FSM is also applicable for solutions of Navier-Stokes equations, i.e. it solves a problem of turbulence describing various bifurcation scenarios of transition from laminar to turbulent regimes in spatially three-dimensional problem of motion of a viscous incompressible liquid (Evstigneev *et al.*, 2009a,b; Evstigneev *et al.*, 2010; Evstigneev & Magnitskii, 2010). The solution of this super complex problem is presented in the separate chapter in the present book. Similar scenarios with classical Feigenbaum scenario and

Sharkovskii windows of periodicity where recently found also in (Awrejcewicz *et al.*, 2012} for initial-boundary value problems in continuous mechanical systems such as flexible plates and shallow shells. As to the processes occurring in a microcosm they, in opinion of the author, also can be successfully described by nonlinear systems of differential equations and their bifurcations. The first results in this direction are received by the author in (Magnitskii, 2010b; Magnitskii, 2011b; Magnitskii, 2012) where the basic equations and formulas of classical electrodynamics, quantum field theory and theory of gravitation are deduced from the nonlinear equations of dynamics of physical vacuum (ether).

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