We are IntechOpen,
the world's leading publisher of
Open Access books
Built by scientists, for scientists

6,900
Open access books available

186,000
International authors and editors

200M
Downloads

154
Countries delivered to

Our authors are among the

TOP 1%
most cited scientists

12.2%
Contributors from top 500 universities

CLARIVATE ANALYTICS
BOOK CITATION INDEX
INDEXED

WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index
in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com

MATLAB Aided Option Replication

Vasilios N. Katsikis

Additional information is available at the end of the chapter

1. Introduction

In reality markets are incomplete in the sense that perfect replication of contingent claims using only the underlying asset and a riskless bond is impossible. In other words, that is perfect risk transfer is not possible since some payoffs cannot be replicated by trading in marketed securities. From the work of Ross in [21], it is evident that whenever the payoff of every call or put option can be replicated then the securities market is complete. In addition, an important implication of the aforementioned work of Ross, is the existence of options that cannot be replicated by the primitive securities when markets are incomplete. In [7], the authors came to the conclusion that Ross's result is, in fact, a negative result since it asserts that in an incomplete market one cannot expect to replicate the payoff of each option even if the underlying asset is traded. In the same paper, it is proved that if the number of securities is less than half the number of states of the world, then (generically) not a single option can be replicated by traded securities. In [10], the author extended the aforementioned result in [7], to accommodate cases where the condition on the number of primitive securities is not imposed. In particular, it is proved that if there exists no binary payoff vector in the asset span, then for each portfolio there exists a set of nontrivial exercise prices of full measure such that any option on the portfolio with an exercise price in this set is non-replicated. Furthermore, note that the results of Ross for two-date security markets with finitely many states holds for security markets with more than two dates, see [8, 9].

It is well accepted that the lattice theoretic ideas are the most important technical contributions of the large literature on infinite-dimensional modern mathematical finance (for example lattice theoretic ideas in general equilibrium theory). However, ordered vector spaces that are not lattice ordered arise naturally in models of portfolio trading. Moreover, if available securities have smooth payoffs, then the portfolio space is never a vector lattice. It should be pointed out that since call and put options are vector lattice operations in the space of contingent claims, their replication by available securities requires a vector lattice structure in the portfolio space. There is a large literature on vector lattice theory related to mathematical economics; see for instance [1–7, 17–20, 22, 23].

On the other hand, there is an obvious need for properly structured high performance computational methods on vector lattices. Moreover, the main concern is to describe, in

computational terms, and then solve problems arising from mathematical economics such as portfolio insurance and option replication. A lot of work in this area has been done in [11, 12, 14? –16].

In this chapter, we focus to the option replication problem. We consider an incomplete market of primitive securities, meaning that some call and put options need not be marketed and our objective is to describe an efficient method for computing maximal submarkets that replicate any option. Even though, there are several important results on option replication they cannot provide a method for the determination of the replicated options. By using the theory of vector-lattices and positive bases it is provided a procedure in order to determine the set of securities with replicated options. In particular, it is shown that the union of all maximal replicated submarkets (i.e., submarkets Y, such that any option written on the elements of Y can be replicated and Y is as large as possible) defines a set of elements such that any option written on these elements is replicated.

In [11–14, 16], it was shown that it is possible to construct computational methods in order to efficiently compute vector sublattices and lattice-subspaces of $\mathbb{R}^m$ as well as in the general case of $C[a, b]$. In addition, these methods has been successfully applied in portfolio insurance and in completion of security markets.

Here we consider a two-period security market X with a finite number m of states and a finite number of primitive securities with payoffs in $\mathbb{R}^m$ and we construct computational methods in order to determine maximal replicated submarkets of X by using the theory of vector sublattices and lattice-subspaces. Moreover, in the theory of security markets it is a usual practice to take call and put options with respect to the riskless bond $\mathbf{1} = (1, 1, ..., 1)$. Then, the completion $F_{\mathbf{1}}(X)$ of X by options is the subspace of $\mathbb{R}^m$ generated by all options written on the elements of $X \cup \{\mathbf{1}\}$. Since the payoff space is $\mathbb{R}^m$, which is a vector lattice, in the case where $\mathbf{1} \in X$ then $F_{\mathbf{1}}(X)$ is exactly the vector sublattice generated by X. If, in addition, X is a vector sublattice of $\mathbb{R}^m$ then $F_{\mathbf{1}}(X) = X$ therefore any option is replicated, unfortunately this situation is extremely rare.

A recent article, [15], provided a computationally efficient method for computing maximal submarkets that replicate any option. In particular, by using computational methods and techniques from [11–14] in order to determine vector sublattices and their positive bases, it is presented a procedure in order to calculate the set of securities with replicated options. The aforementioned article emphasizes the most important interrelationship between the theory of vector lattices, positive bases, projection bases and the problem of option replication.

The material in this chapter is spread out in 5 sections. Section 2 is divided in two subsections; in the first the fundamental properties of lattice-subspaces and vector sublattices are presented, whereas in the second we introduce the basic results for vector sublattices, positive bases and projection bases of $\mathbb{R}^k$ together with the solution to the problem of whether a finite collection of linearly independent, positive vectors of $\mathbb{R}^k$ generates a lattice-subspace or a vector sublattice. In section 3, there are three subsections where it is discussed the theoretical background for option replication. Also, section 3 emphasis the most important interrelationship between positive bases, projection bases and the problem of option replication. Section 4 presents an algorithm for calculating maximal submarkets that replicate any option followed by the corresponding computational approach. Conclusions and research directions are provided in Section 5.

In this chapter, all the numerical tasks have been performed using the Matlab 7.8 (R2009a) environment on an Intel(R) Pentium(R) Dual CPU T2310 @ 1.46 GHz 1.47 GHz 32-bit system with 2 GB of RAM memory running on the Windows Vista Home Premium Operating System.

2. Basic results in the theory of positive bases and projection bases of $\mathbb{R}^m$

In this section, a brief introduction is provided to the theory of vector lattices of $\mathbb{R}^m$. Furthermore, we present some basic results related to the theory of positive bases and projection bases of $\mathbb{R}^m$.

2.1. Preliminaries and notation

Initially, we recall some definitions and notation from the vector lattice theory. Let $\mathbb{R}^m = \{x = (x(1), x(2), ..., x(m)) | x(i) \in \mathbb{R}, \text{ for each } i\}$, where we view $\mathbb{R}^m$ as an ordered space. The *pointwise order* relation in $\mathbb{R}^m$ is defined by

$$x \leq y \text{ if and only if } x(i) \leq y(i), \text{ for each } i = 1, ..., m.$$

The positive cone of $\mathbb{R}^m$ is defined by $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m | x(i) \geq 0, \text{ for each } i\}$ and if we suppose that X is a vector subspace of $\mathbb{R}^m$ then X ordered by the pointwise ordering is an *ordered subspace* of $\mathbb{R}^m$, with positive cone $X_+ = X \cap \mathbb{R}^m_+$. By $\{e_1, e_2, ..., e_m\}$ we shall denote the usual basis of $\mathbb{R}^m$. A point $x \in \mathbb{R}^m$ is an *upper bound, (resp. lower bound)* of a subset $S \subseteq \mathbb{R}^m$ if and only if $y \leq x$ (resp. $x \leq y$), for all $y \in S$. For a two-point set $S = \{x, y\}$, we denote by $x \vee y$ (resp. $x \wedge y$) the *supremum* of S i.e., its least upper bound (resp. the *infimum* of S i.e., its greatest lower bound). Thus, $x \vee y$ (resp. $x \wedge y$) is the componentwise maximum (resp. minimum) of x and y defined by

$$(x \vee y)(i) = \max\{x(i), y(i)\} \,((x \wedge y)(i) = \min\{x(i), y(i)\}), \text{ for all } i = 1, ..., m.$$

For any $x = (x(1), x(2), ..., x(m)) \in \mathbb{R}^m$, the set $supp(x) = \{i | x(i) \neq 0\}$ is the *support* of x. The vectors $x, y \in \mathbb{R}^m$ have *disjoint supports* if $supp(x) \cap supp(y) = \emptyset$.

An ordered subspace Z of $\mathbb{R}^m$ is a *vector sublattice* or a *Riesz subspace* of $\mathbb{R}^m$ if for any $x, y \in Z$ the supremum and the infimum of the set $\{x, y\}$ in $\mathbb{R}^m$ belong to Z.

Assume that X is an ordered subspace of $\mathbb{R}^m$ and $B = \{b_1, b_2, ..., b_\mu\}$ is a basis for X. Then B is a *positive basis* of X if for each $x \in X$ it holds: x is positive if and only if its coefficients in the basis B are positive. In other words, B is a positive basis of X if the positive cone X_+ of X has the form,

$$X_+ = \{x = \sum_{i=1}^{\mu} \lambda_i b_i | \lambda_i \geq 0, \text{ for each } i\}.$$

Then, for any $x = \sum_{i=1}^{\mu} \lambda_i b_i$ and $y = \sum_{i=1}^{\mu} \varrho_i b_i$ we have $x \leq y$ if and only if $\lambda_i \leq \varrho_i$ for each $i = 1, 2, ..., \mu$. A positive basis $B = \{b_1, b_2, ..., b_\mu\}$ is a *partition of the unit* if the vectors b_i have disjoint supports and $\sum_{i=1}^{\mu} b_i = 1$.

Recall that a nonzero element x_0 of X_+ is an extremal point of X_+ if, for any $x \in X, 0 \leq x \leq x_0$ implies $x = \lambda x_0$, for a real number λ. Since each element b_i of the positive basis of X is an extremal point of X_+, a positive basis of X is unique in the sense of positive multiples.

The existence of positive bases is not always ensured, but in the case where X is a vector sublattice of $\mathbb{R}^m$ then X always has a positive basis. If $B = \{b_1, b_2, ..., b_\mu\}$ is a positive basis for a vector sublattice X the lattice operations in X, namely $x \vee y$ for the supremum and $x \wedge y$ for the infimum of the set $\{x, y\}$ in X, are given by

$$x \vee y = \sum_{i=1}^{\mu} \max\{\lambda_i, \varrho_i\}b_i \text{ and } x \wedge y = \sum_{i=1}^{\mu} \min\{\lambda_i, \varrho_i\}b_i,$$

for each $x = \sum_{i=1}^{\mu} \lambda_i b_i, y = \sum_{i=1}^{\mu} \varrho_i b_i \in X$.

Suppose that L is a finite dimensional subspace of $C(\Omega)$ generated by a set $\{z_1, z_2, ..., z_r\}$ of linearly independent positive vectors of $C(\Omega)$. If Z is the sublattice of $C(\Omega)$ generated by L and $\{b_1, ..., b_\mu\}$ is a positive basis for Z ($\mu = dim(Z)$) then, a *projection basis* $\{\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_r\}$ of Z is a basis for L such that its elements are projections of the elements of the positive basis $\{b_1, ..., b_\mu\}$. In our current work we consider that $\Omega = \{1, 2, ..., m\}$ hence $C(\Omega) = \mathbb{R}^m$.

For an extensive presentation of vector sublattices as well as for notation not defined here we refer to [11–13, 15?, 16] and the references therein.

2.2. Theoretical background

In this section we present theoretical results for positive bases and projection bases in $\mathbb{R}^m$. Given a collection $x_1, x_2, ..., x_n$ of linearly independent, positive vectors of $\mathbb{R}^m$ we define the function,

$$h : \{1, 2, ..., m\} \to \mathbb{R}^n \text{ such that } h(i) = (x_1(i), x_2(i), ..., x_n(i))$$

and the function,

$$\beta : \{1, 2, ..., m\} \to \mathbb{R}^n \text{ such that } \beta(i) = \frac{h(i)}{\|h(i)\|_1} \tag{1}$$

for each $i \in \{1, 2, ..., m\}$ with $\|h(i)\|_1 \neq 0$. We shall refer to β as the *basic function* of the vectors $x_1, x_2, ..., x_n$. The set

$$R(\beta) = \{\beta(i) | i = 1, 2, ..., m, \text{ with } \|h(i)\|_1 \neq 0\},$$

is the *range* of the basic function and the *cardinal number*, $cardR(\beta)$, of $R(\beta)$ is the number of different elements of $R(\beta)$.

Suppose that Z denotes the sublattice of $\mathbb{R}^m$ generated by $X = [x_1, x_2, ..., x_n]$. We shall denote by $P_1, P_2, ..., P_n, P_{n+1}, ..., P_\mu$ an enumeration of $R(\beta)$ such that the first n vertices $P_1, P_2, ..., P_n$ are linearly independent and $\mu = \dim(Z)$. Note that such an enumeration always exists. The notation, A^T stands for the transpose of a matrix A. A procedure in order to construct the sublattice Z is given by the following theorem.

Theorem 1. *Suppose that the above assumptions are satisfied. Then,*

(i) X is a vector sublattice of $\mathbb{R}^m$ if and only if $R(\beta)$ has exactly n points (i.e., $\mu = n$). Then a positive basis $b_1, b_2, ..., b_n$ for X is defined by the formula

$$(b_1, b_2, ..., b_n)^T = A^{-1}(x_1, x_2, ..., x_n)^T,$$

where A is the $n \times n$ matrix whose ith column is the vector P_i, for each $i = 1, 2, ..., n$. It is clear that in such a case Z and X coincide.

(ii) *Let* $\mu > n$. *If* $I_s = \beta^{-1}(P_s)$, *and*

$$x_s = \sum_{i \in I_s} \|h(i)\|_1 e_i, \quad s = n+1, n+2, ..., \mu,$$

then

$$Z = [x_1, x_2, ..., x_n, x_{n+1}, ..., x_\mu],$$

is the vector sublattice generated by $x_1, x_2, ..., x_n$ *and* $\dim Z = \mu$.

For a positive basis $\{b_1, b_2, ..., b_\mu\}$ *of* Z, *consider the basic function* γ *of* $\{x_1, x_2, ..., x_\mu\}$ *with range,* $R(\gamma) = \{P_1', P_2', ..., P_\mu'\}$. *Then, the relation*

$$(b_1, b_2, ..., b_\mu)^T = B^{-1}(x_1, x_2, ..., x_\mu)^T \tag{2}$$

where B *is the* $\mu \times \mu$ *matrix with columns the vectors* $P_1', P_2', ..., P_\mu'$, *defines a positive basis for* Z.

The notion of the projection basis is important for our study. Furthermore, in the following, we are interested for a projection basis that corresponds to a positive basis. Let $\{z_1, z_2, ..., z_r\}$ be a set of linearly independent and positive vectors of $\mathbb{R}^m$ then by using Theorem 1 we construct the sublattice Z of $\mathbb{R}^m$ generated by these vectors. If $\dim(Z) = \mu$, by Theorem 1, a positive basis $\{b_1, b_2, ..., b_\mu\}$ of Z can be determined. The basic result for calculating the projection basis that corresponds to the positive basis $\{b_1, b_2, ..., b_\mu\}$ of Z is the following theorem.

Theorem 2. *Suppose that* β *is the basic function of the vectors* $\{z_1, z_2, ..., z_r\}$ *and* $P_1, P_2, ..., P_r,$ $P_{r+1}, ..., P_\mu$ *is an enumeration of the range of* β *such that the first* r *vectors* $P_1, P_2, ..., P_r$ *are linearly independent and suppose also that* $z_{r+1}, ..., z_\mu$ *are the new vectors constructed in Theorem 1. If* $L = [z_1, z_2, ..., z_r]$ *is the subspace of* $\mathbb{R}^m$ *generated by the vectors* $z_1, z_2, ..., z_r$ *then,*

(i) $Z = L \oplus [z_{r+1}, ..., z_\mu]$,

(ii) $\{b_{r+1}, b_{r+2}, ..., b_\mu\} = \{2z_{r+1}, 2z_{r+2}, ..., 2z_\mu\}$,

(iii) *If* $b_i = \widetilde{b}_i + b_i'$, *with* $\widetilde{b}_i \in L$ *and* $b_i' \in [z_{r+1}, ..., z_\mu]$, *for each* $i = 1, 2, ..., r$, *then* $\{\widetilde{b}_1, \widetilde{b}_2, ..., \widetilde{b}_r\}$ *is a basis for* L *which is given by the formula*

$$(\widetilde{b}_1, \widetilde{b}_2, ..., \widetilde{b}_r)^T = A^{-1}(z_1, z_2, ..., z_r)^T,$$

where A *is the* $r \times r$ *matrix whose ith column is the vector* P_i, *for* $i = 1, 2, ..., r$. *This basis,* $\{\widetilde{b}_1, \widetilde{b}_2, ..., \widetilde{b}_r\}$ *is called the projection basis of* L *and has the property: The* r *first coordinates of any element* $x \in L$ *expressed in terms of the basis* $\{b_1, b_2, ..., b_\mu\}$ *coincide with the corresponding coordinates of* x *in the projection basis, i.e.,*

$$x = \sum_{i=1}^{\mu} \lambda_i b_i \in L \Rightarrow x = \sum_{i=1}^{r} \lambda_i \widetilde{b}_i$$

Suppose that Z is the sublattice generated by a collection $z_1, z_2, ..., z_r$ of linearly independent, positive vectors of $\mathbb{R}^m$ and $\{d_1, d_2, ..., d_\mu\}$ is a positive basis for Z.

Then, by Theorem 2, if

$$(\widetilde{d}_1, \widetilde{d}_2, ..., \widetilde{d}_r)^T = A^{-1}(z_1, z_2, ..., z_r)^T,$$

where A is the $r \times r$ matrix whose ith column is the vector P_i, for $i = 1, 2, ..., r$ then $\{\tilde{d}_1, \tilde{d}_2, ..., \tilde{d}_r\}$ is the projection basis of $L = [z_1, z_2, ..., z_r]$. The projection basis $\{\tilde{d}_1, \tilde{d}_2, ..., \tilde{d}_r\}$ is called the *projection basis of X corresponding to the basis* $\{d_1, d_2, ..., d_\mu\}$. The following proposition allows us to determine a positive basis and its corresponding projection basis. Moreover, the calculated positive basis is a partition of the unit.

Proposition 1. *Suppose that $\{d_i\}$ is the basis of Z given by equation (2) of Theorem 1 and $\{\tilde{d}_i\}$ is the projection basis of $L = [z_1, z_2, ..., z_r]$ corresponding to the basis $\{d_i\}$. Then $\{b_i = \frac{d_i}{\|d_i\|_\infty} | i = 1, 2, ..., \mu\}$ is the positive basis of Z which is a partition of the unit and $\{\tilde{b}_i = \frac{\tilde{d}_i}{\|d_i\|_\infty} | i = 1, 2, ..., n\}$ is the projection basis of L corresponding to the basis $\{b_i\}$ of Z.*

In the following, we shall denote by $\mathbf{1}$ the vector $\mathbf{1} = (1, 1, ..., 1)$. A vector x is a *binary vector* if $x \neq \mathbf{0} = (0, 0, ..., 0)$, $x \neq \mathbf{1}$ and $x(i) = 0$ or $x(i) = 1$, for any i. Let $\{b_i | i = 1, 2, ..., \mu\}$ be a positive basis of Z which is a partition of the unit and let $\{\tilde{b}_i | i = 1, 2, ..., n\}$ be the projection basis of L corresponding to the basis $\{b_i\}$. A partition $\delta = \{\sigma_i | i = 1, 2, ..., k\}$ of $\{1, 2, ..., n\}$ is *proper* if for any $r = 1, 2, ..., k$, the vector $w_r = \sum_{i \in \sigma_r} \tilde{b}_i$ is a binary vector with $\sum_{r=1}^{k} w_r = \mathbf{1}$. If there is no proper partition of $\{1, 2, ..., n\}$ strictly finer than δ, then we say that δ is a *maximal proper partition* of $\{1, 2, ..., n\}$.

Example 1. Let $\{b_i | i = 1, 2, 3\}$ be a positive basis such that the corresponding projection basis is the following

$$\tilde{b}_1 = (\frac{1}{2}, 1, 0, 1, 0), \ \tilde{b}_2 = (\frac{1}{2}, 0, 0, 0, 1), \ \tilde{b}_3 = (0, 0, 1, 0, 0).$$

We calculate the partitions of $\{1, 2, 3\}$ which are the following:

$$\delta_1 = \{\sigma_1, \sigma_2\}, \ where \ \sigma_1 = \{1\}, \ \sigma_2 = \{2 \ 3\}$$

$$\delta_2 = \{\sigma_1, \sigma_2\}, \ where \ \sigma_1 = \{2\}, \ \sigma_2 = \{3 \ 4\}$$

$$\delta_3 = \{\sigma_1, \sigma_2\}, \ where \ \sigma_1 = \{3\}, \ \sigma_2 = \{1 \ 2\}$$

$$\delta_4 = \{\sigma_1, \sigma_2, \sigma_3\}, \ where \ \sigma_1 = \{1\}, \ \sigma_2 = \{2\}, \ \sigma_3 = \{3\}.$$

Then δ_1 is not a proper partition since $w_1 = \sum_{i \in \sigma_1} \tilde{b}_i = \tilde{b}_1$ and $\tilde{b}_1$ is not a binary vector. Similarly, δ_2 is not a proper partition. On the other hand δ_3 is a proper partition since $w_1 = \sum_{i \in \sigma_1} \tilde{b}_i = \tilde{b}_3$ and $\tilde{b}_3$ is a binary vector, $w_2 = \sum_{i \in \sigma_2} \tilde{b}_i = \tilde{b}_1 + \tilde{b}_2 = (1, 1, 0, 1, 1)$ which is a binary vector and $w_1 + w_2 = (1, 1, 1, 1, 1)$. Notice that δ_3 is strictly finer than δ_4, hence δ_3 is a maximal proper partition of $\{1, 2, 3\}$.

3. Option replication

In this section we shall discuss the economic model of our study. Moreover, first we discuss an inductive method for calculating the completion of security markets. So, if $\mathbf{1} \in X$ then it is possible to determine a basic set of marketed securities i.e., a set of linearly independent and positive vectors of X and the sublattice of $\mathbb{R}^m$ generated by a basic set of marketed securities is $F_{\mathbf{1}}(X)$. Finally, $F_{\mathbf{1}}(X)$ has a positive basis which is a partition of the unit. Under these observations we present an algorithmic procedure in order to determine maximal submarkets that replicate any option.

3.1. The economic model

In our economy there are two time periods, $t = 0, 1$, where $t = 0$ denotes the present and $t = 1$ denotes the future. We consider that at $t = 1$ we have a finite number of states indexed by $s = 1, 2, ..., m$, while at $t = 0$ the state is known to be $s = 0$.

Suppose that, agents trade $x_1, x_2, ..., x_n$ non-redundant securities in period $t = 0$, then the future payoffs of $x_1, x_2, ..., x_n$ are collected in a matrix

$$A = \left[x_i(j) \right]_{i=1,2,...n}^{j=1,2,...,m} \in \mathbb{R}^{m \times n}$$

where $x_i(j)$ is the payoff of one unit of security i in state j. In other words, A is the matrix whose columns are the non-redundant security vectors $x_1, x_2, ..., x_n$. It is clear that the matrix A is of full rank and the *asset span* is denoted by $X = Span(A)$. So, X is the vector subspace of $\mathbb{R}^m$ generated by the vectors x_i. That is, X consists of those income streams that can be generated by trading on the financial market. A *portfolio* is a column vector $\theta = (\theta_1, \theta_2, ..., \theta_n)^T$ of $\mathbb{R}^n$ and the *payoff* of a portfolio θ is the vector $x = A\theta \in \mathbb{R}^m$ which offers payoff $x(i)$ in state i, where $i = 1, ..., m$. A vector in $\mathbb{R}^m$, is said to be *marketed* or *replicated* if x is the payoff of some portfolio θ (called the *replicating portfolio* of x), or equivalently if $x \in X$. If $m = n$, then markets are said to be *complete* and the asset span coincides with the space $\mathbb{R}^m$. On the other hand, if $n < m$, the markets are *incomplete* and some state contingent claim cannot be replicated by a portfolio.

In the following, we shall denote by $\mathbf{1}$ the *riskless bond* i.e., the vector $\mathbf{1} = (1, 1, ..., 1)$. A vector x is a *binary vector* if $x \neq \mathbf{0} = (0, 0, ..., 0)$, $x \neq \mathbf{1}$ and $x(i) = 0$ or $x(i) = 1$, for any i. The *call option* written on the vector $x \in \mathbb{R}^m$ with exercise price α is the vector $c(x, a) = (x - \alpha\mathbf{1})^+ = (x - \alpha\mathbf{1}) \vee \mathbf{0}$. The *put option* written on the vector $x \in \mathbb{R}^m$ with exercise price α is the vector $p(x, a) = (\alpha\mathbf{1} - x)^+ = (\alpha\mathbf{1} - x) \vee \mathbf{0}$. If y is an element of a Riesz space then the following lattice identities hold, $y = y^+ - y^-$ and $y^- = (-y)^+$. It is clear that $x - \alpha\mathbf{1} = (x - \alpha\mathbf{1})^+ - (x - \alpha\mathbf{1})^- = (x - \alpha\mathbf{1})^+ - (\alpha\mathbf{1} - x)^+ = c(x, \alpha) - p(x, \alpha)$. Therefore we have the identity

$$x - a\mathbf{1} = c(x, a) - p(x, a),$$

which is called *put-call parity*.

If both $c(x, \alpha) > 0$ and $p(x, \alpha) > 0$, we say that the call option $c(x, \alpha)$ and the put option $p(x, \alpha)$ are *non trivial* and the exercise price α is a *non trivial exercise price* of x. If $c(x, \alpha)$ and $p(x, \alpha)$ belong to X then we say that $c(x, \alpha)$ and $p(x, \alpha)$ are *replicated*. If we suppose that $\mathbf{1} \in X$ and at least one of $c(x, a), p(x, a)$ is replicated, then both of them are replicated since, $x - \alpha\mathbf{1} = c(x, \alpha) - p(x, \alpha)$.

3.2. Completion of security markets

We shall discuss the problem of completion by options of a two-period security market in which the space of marketed securities is a subspace of $\mathbb{R}^m$. The present study involves vector sublattices generated by a subset B of $\mathbb{R}^m$ of positive, linearly independent vectors. A computational solution to this problem is provided by using the `SUBlat` Matlab function from [16].

Let us assume that in the beginning of a time period there are n securities traded in a market. Let $\mathcal{S} = \{1, ..., m\}$ denote a finite set of states and $x_j \in \mathbb{R}^m_+$ be the payoff vector of security j in m states. The payoffs $x_1, x_2, ..., x_n$ are assumed linearly independent so that there are no redundant securities. If $\theta = (\theta_1, \theta_2, ..., \theta_n) \in \mathbb{R}^n$ is a non-zero portfolio then its payoff is the vector

$$T(\theta) = \sum_{i=1}^{n} \theta_i x_i.$$

The set of payoffs of all portfolios is referred as the space of *marketed securities* and it is the linear span of the payoffs vectors $x_1, x_2, ..., x_n$ in $\mathbb{R}^m$ which we shall denote it by X, i.e.,

$$X = [x_1, x_2, ... x_n].$$

For any $x, u \in \mathbb{R}^m$ and any real number a the vector $c_u(x, a) = (x - au)^+$ is the *call option* and $p_u(x, a) = (au - x)^+$ is the *put option* of x with respect to the *strike vector* u and *exercise price* a.

Let U be a fixed subspace of $\mathbb{R}^m$ which is called *strike subspace* and the elements of U are the *strike vectors*. Then, the *completion by options* of the subspace X with respect to U is the space $F_U(X)$ which is defined inductively as follows:

- X_1 is the subspace of $\mathbb{R}^m$ generated by $\mathcal{O}_1$, where $\mathcal{O}_1 = \{c_u(x, a) | x \in X, u \in U, a \in \mathbb{R}\}$, denotes the set of call options written on the elements of X,

- X_n is the subspace of $\mathbb{R}^m$ generated by $\mathcal{O}_n$, where $\mathcal{O}_n = \{c_u(x, a) | x \in X_{n-1}, u \in U, a \in \mathbb{R}\}$, denotes the set of call options written on the elements of X_{n-1},

- $F_U(X) = \cup_{n=1}^{\infty} X_n$.

The completion by options $F_U(X)$ of X with respect to U is the vector sublattice of $\mathbb{R}^m$ generated by the subspace $Y = X \cup U$. The details are presented in the next theorem,

Theorem 3. *In the above notation, we have*

(i) $Y \subseteq X_1$,

(ii) $F_U(X)$ *is the sublattice $S(Y)$ of $\mathbb{R}^m$ generated by Y, and*

(iii) *if $U \subseteq X$, then $F_U(X)$ is the sublattice of $\mathbb{R}^m$ generated by X.*

Any set $\{y_1, y_2, \ldots, y_r\}$ of linearly independent positive vectors of $\mathbb{R}^m$ such that $F_U(X)$ is the sublattice of $\mathbb{R}^m$ generated by $\{y_1, y_2, \ldots, y_r\}$ is a *basic set* of the market.

Theorem 4. *Any maximal subset $\{y_1, y_2, \ldots, y_r\}$ of linearly independent vectors of $\mathcal{A}$ is a basic set of the market, where $\mathcal{A} = \{x_1^+, x_1^-, \ldots, x_n^+, x_n^-\}$, if $U \subseteq X$ and $\mathcal{A} = \{x_1^+, x_1^-, \ldots, x_n^+, x_n^-, u_1^+, u_1^-, \ldots, u_d^+, u_d^-\}$, if $U \subsetneq X$*

The space of marketed securities X is *complete by options* with respect to U if $X = F_U(X)$.

From theorem 1 it follows,

Theorem 5. *The space X of marketed securities is complete by options with respect to U if and only if $U \subseteq X$ and $card R(\beta) = n$. In addition, the dimension of $F_U(X)$ is equal to the cardinal number of $R(\beta)$. Therefore, $F_U(X) = \mathbb{R}^k$ if and only if $card R(\beta) = k$.*

Example 2. Suppose that in a security market, the payoff space is $\mathbb{R}^{12}$ and the primitive securities are:

$$x_1 = (1,2,2,-1,1,-2,-1,-3,0,0,0,0)$$
$$x_2 = (0,2,0,0,1,2,0,3,-1,-1,-1,-2)$$
$$x_3 = (1,2,2,0,1,0,0,0,-1,-1,-1,-2)$$

and that the strike subspace is the vector subspace U generated by the vector

$$u = (1,2,2,1,1,2,1,3,-1,-1,-1,-2).$$

Then, a maximal subset of linearly independent vectors of $\{x_1^+, x_1^-, x_2^+, x_2^-, x_3^+, x_3^-, u_1^+, u_1^-\}$ can be calculated by using the following code:

```
>>XX = [max(X,zeros(size(X)));max(-X,zeros(size(X)))];
>>S = rref(XX');
>>[I,J] = find(S);
>>Linearindep = accumarray(I,J,[rank(XX),1],@min)';
>>W = XX(Linearindep,:);
```

where X denotes a matrix whose rows are the vectors x_1, x_2, x_3, u. We can determine the completion by options of X i.e., the space $F_U(X)$, with the SUBlat function from [16] by using the following code:

```
>>[VectorSublattice,Positivebasis]=SUBlat(W')
```

The results then are as follows

```
VectorSublattice =
1    2    2    0    1    0    0    0    0    0    0    0
0    2    0    0    1    2    0    3    0    0    0    0
1    2    2    1    1    2    1    3    0    0    0    0
0    0    0    0    0    0    0    0    1    1    1    2
2    0    4    0    0    0    0    0    0    0    0    0

Positivebasis =
0    0    0    0    0    0    0    0    1    1    1    2
0    0    0    1    0    0    1    0    0    0    0    0
0    0    0    0    0    4    0    6    0    0    0    0
4    0    8    0    0    0    0    0    0    0    0    0
0    6    0    0    3    0    0    0    0    0    0    0
```

3.3. Computation of maximal submarkets that replicate any option

We consider a two-period security market X with a finite number m of states and a finite number of primitive securities with payoffs in $\mathbb{R}^m$ and we construct computational methods in order to determine maximal submarkets of X that replicate any option by using the results provided in subsection 2.2. In particular, in the theory of security markets it is a usual practice

to take call and put options with respect to the riskless bond $\mathbf{1} = (1, 1, ..., 1)$. Then, the completion $F_{\mathbf{1}}(X)$ of X by options (see subsection 3.2) is the subspace of $\mathbb{R}^m$ generated by all options written on the elements of $X \cup \{\mathbf{1}\}$. Since the payoff space is $\mathbb{R}^m$, which is a vector lattice, in the case where $\mathbf{1} \in X$ then $F_{\mathbf{1}}(X)$ is exactly the vector sublattice generated by X. If, in addition, X is a vector sublattice of $\mathbb{R}^m$ then $F_{\mathbf{1}}(X) = X$ therefore any option is replicated.

A basic set of marketed securities (i.e., a set of linearly independent and positive vectors) of X always exist and the sublattice of $\mathbb{R}^m$ generated by a basic set of marketed securities is $F_{\mathbf{1}}(X)$. In addition, $F_{\mathbf{1}}(X)$ has a positive basis which is a partition of the unit.

Let us assume that X is generated by a basic set of marketed securities, then from Theorem 1 it is possible to determine a positive basis $\{b_1, b_2, ..., b_\mu\}$ of $F_{\mathbf{1}}(X)$.

The sublattice Z, generated by a basic set of marketed securities, is exactly $F_{\mathbf{1}}(X)$ and $F_{\mathbf{1}}(X)$ has a positive basis which is a partition of the unit, i.e., $\sum_{i=1}^{\mu} b_i = \mathbf{1}$. This is possible since the notion of a positive basis is unique in the sense of positive multiples therefore we are able to extract from the positive basis $\{b_1, b_2, ..., b_\mu\}$ another positive basis $\{d_1, d_2, ..., d_\mu\}$ of $F_{\mathbf{1}}(X)$ which is a partition of the unit. Therefore, let us denote by $\{d_1, d_2, ..., d_\mu\}$ a positive basis of $F_{\mathbf{1}}(X)$ which is a partition of the unit. Then, by Theorem 2, if

$$(\widetilde{d}_1, \widetilde{d}_2, ..., \widetilde{d}_r)^T = A^{-1}(z_1, z_2, ..., z_r)^T,$$

where A is the $r \times r$ matrix whose ith column is the vector P_i, for $i = 1, 2, ..., r$ then $\{\widetilde{d}_1, \widetilde{d}_2, ..., \widetilde{d}_r\}$ is a projection basis of $F_{\mathbf{1}}(X)$. The projection basis $\{\widetilde{d}_1, \widetilde{d}_2, ..., \widetilde{d}_r\}$ is the projection basis of X corresponding to the basis $\{d_1, d_2, ..., d_\mu\}$. For $Z = F_{\mathbf{1}}(X)$ proposition 1 takes the following form.

Proposition 2. *Suppose that $\{d_i\}$ is the basis of $F_{\mathbf{1}}(X)$ given by equation (2) of theorem 1 and $\{\widetilde{d}_i\}$ is the projection basis of X corresponding to the basis $\{d_i\}$. Then $\{b_i = \frac{d_i}{\|d_i\|_\infty} | i = 1, 2, ..., \mu\}$ is the positive basis of $F_{\mathbf{1}}(X)$ which is a partition of the unit and $\{\widetilde{b}_i = \frac{\widetilde{d}_i}{\|d_i\|_\infty} | i = 1, 2, ..., n\}$ is the projection basis of X corresponding to the basis $\{b_i\}$ of $F_{\mathbf{1}}(X)$.*

Suppose that Y is a subspace of X, then if $F_{\mathbf{1}}(Y) \subseteq X$ we say that Y is *replicated*. If, in addition, for any subspace Z of X with $Y \subsetneq Z$ we have that $X \subsetneq F_{\mathbf{1}}(Z)$ then Y is a *maximal replicated subspace* or a *maximal replicated submarket* of X. Note that, the *replicated kernel of the market*, i.e., the union of all maximal replicated subspaces of the market is the set of all elements x of X so that any option written on x is replicated.

Let $\{b_i \ i = 1, 2, ..., \mu\}$ be a positive basis of $F_{\mathbf{1}}(X)$ which is a partition of the unit and let $\{\widetilde{b}_i \ i = 1, 2, ..., n\}$ be the projection basis of X corresponding to the basis $\{b_i\}$. Recall that, a partition $\delta = \{\sigma_i | i = 1, 2, ..., k\}$ of $\{1, 2, ..., n\}$ is *proper* if for any $r = 1, 2, ..., k$, the vector $w_r = \sum_{i \in \sigma_r} \widetilde{b}_i$ is a binary vector with $\sum_{r=1}^k w_r = \mathbf{1}$. If there is no proper partition of $\{1, 2, ..., n\}$ strictly finer than δ, then we say that δ is a *maximal proper partition* of $\{1, 2, ..., n\}$.

The following theorem provides the development of a method in order to determine the set of securities with replicated options by using the theory of positive bases and projection bases.

Theorem 6. *Let $\{b_i, i = 1, 2, ..., \mu\}$ be the positive basis of $F_1(X)$ which is a partition of the unit and let $\{\tilde{b}_i, i = 1, 2, ..., n\}$ be the projection basis of X corresponding to the basis $\{b_i\}$. If Y is a subspace of X, the following are equivalent:*

(i) Y is a maximal replicated subspace of X,

(ii) there exists a maximal proper partition $\delta = \{\sigma_i | i = 1, 2, ..., k\}$ of $\{1, 2, ..., n\}$ so that Y is the sublattice of $\mathbb{R}^m$ generated by δ.

The set of maximal replicated submarkets of X is nonempty.

4. The computational method

We present the proposed computational method that enables us to determine maximal submarkets that replicate any option. Our numerical method is based on the introduction of the `mrsubspace` function, from [15], that allow us to perform fast testing for a variety of dimensions and subspaces.

4.1. Algorithm for calculating maximal submarkets that replicate any option

Recall that X is the security market generated by a collection $\{x_1, x_2, ..., x_n\}$ of linearly independent vectors (not necessarily positive) of $\mathbb{R}^m$. If $\mathbf{1} \in X$ then it is possible to determine a basic set of marketed securities i.e., a set of linearly independent and positive vectors of X. This is possible through the following easy proposition:

Proposition 3. *If $a = \max\{\|x_i\|_\infty | i = 1, 2, ..., n\}$, then at least one of the two sets of positive vectors of X*

$$\{y_i = a\mathbf{1} - x_i | i = 1, 2, ..., n\}, \ \{z_i = 2a\mathbf{1} - x_i | i = 1, 2, ..., n\},$$

consists of linearly independent vectors.

The main steps of the underlying algorithmic procedure that enables us to determine maximal submarkets that replicate any option are the following:

(1) Use proposition 3 in order to determine a basic set $\{y_1, y_2, ..., y_n\}$ of marketed securities.

(2) Use Equation (1) in order to determine the basic curve β of the vectors y_i.

(3) Determine the range $R(\beta)$ of β.

(4) Use Theorem 1 in order to construct the vector sublattice generated by $y_1, y_2, ..., y_n$, which is exactly the completion by options $F_1(X)$ of X. Then, determine a positive basis $\{d_1, d_2, ..., d_\mu\}$ for $F_1(X)$.

(5) Use Theorem 2 in order to determine a projection basis $\{\tilde{d}_1, \tilde{d}_2, ..., \tilde{d}_n\}$ of X.

(6) Use Proposition 1 in order to determine a positive basis $\{b_1, b_2, ..., b_\mu\}$ of $F_1(X)$ which is a partition of the unit and the corresponding projection basis $\{\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_n\}$.

(7) Calculate all the possible proper partitions of the set $\{1, 2, ..., n\}$.

(8) Decide which of the proper partitions created in step (7) are maximal proper partitions and determine the corresponding maximal replicated submarkets.

In [15], it is presented the translation followed by the implementation of the aforementioned algorithm in $\mathbb{R}^m$ within a Matlab-based function named mrsubspace. Moreover, in the same paper, computational experiments assessing the effectiveness of this function and lead us to the conclusion that the mrsubspace function provides an important tool in order to investigate replicated subspaces.

4.2. The computational approach - Code features

We shall discuss the proposed computational method that enables us to determine maximal submarkets that replicate any option. The standard method used currently to determine the maximal replicated submarkets is based on a manual processing. From section 3, it is evident that the required number of verifications for this process can be of significant size even in a relatively low-dimensional space, thus rendering the problem too difficult to solve. Our numerical method is based on the introduction of the mrsubspace function, from [15] that allow us to perform fast testing for a variety of dimensions and subspaces.

The structure of the code ensures flexibility, meaning that it is convenient for applications as well as for research and educational purposes. The given security market X, generated by the linearly independent vectors $x_1, x_2, ..., x_n$, must be given under a matrix notation with columns the vectors $x_1, x_2, ..., x_n$. The mrsubspace function must be stored in a Matlab-accessible directory and then the input data, i.e., the matrix X, can be typed directly in the Matlab's environment. Under the following command,

```
mrsubspace(X);
```

the program solves the problem of option replication and prints out the maximal proper partitions as well as the corresponding maximal replicated subspaces. If X is a vector sublattice, then $X = F_1(X)$ and any option is replicated. In the case where the initial space X is not a vector sublattice, it is possible to produce the normalized positive basis and the corresponding projection basis with the following code,

```
[Npb,Cprb] = mrsubspace(X)
```

Inside the code there are several explanations that indicate the implemented part of the algorithm. A user proficient in Matlab can easily use the code and modify it if needed. Especially, the user can isolate a part of the code according to his/her special needs to solve different problems like

- Determine a basic set of marketed securities.
- Find the completion $F_1(X)$ of X by options in $\mathbb{R}^m$ or find the vector sublattice generated by a finite collection of linearly independent vectors of $\mathbb{R}^m$.
- Calculate a positive basis and a projection basis for a finite dimensional vector sublattice.

In the last part of the code, entitled Maximal proper partitions - Maximal replicated subspaces, the user can change the way that the mrsubspace function understands the values 0 and 1, according to his/her knowledge and needs.

Example 3. Consider the following 5 vectors $x_1, x_2, ..., x_5$ in $\mathbb{R}^{10}$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0\ 1\ 0\ 1\ 1\ 1\ 1\ 2\ 2\ 1 \\ 1\ 1\ 1\ 2\ 1\ 1\ 1\ 2\ 1\ 2 \\ 1\ 1\ 1\ 2\ 1\ 1\ 1\ 2\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 1 \\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \end{bmatrix}$$

and $X = [x_1, x_2, ..., x_5]$ is the marketed space.

Note that $\mathbf{1} = x_5 - x_4 + x_1$. In order to determine the maximal replicated subspaces for the above collection of vectors we use the following simple code:

```
>> X = [0,1,0,1,1,1,1,2,2,1;1,1,1,2,1,1,1,2,1,2;...
1,1,1,2,1,1,1,2,1,1;1,1,1,1,1,1,1,2,2,1;2,1,2,1,1,1,1,1,1,1];
>> [Npb,Cprb] =mrsubspace(X)
```

as a result and after removing irrelevant Matlab output one gets

```
The 1 partition(s) are:
    {1} {2 3} {4} {5}

ReplicatedSubspace =

    1    0    1    0    0    0    0    0    0    0
    0    1    0    1    1    1    1    0    0    0
    0    0    0    0    0    0    0    1    1    0
    0    0    0    0    0    0    0    0    0    1

The 1 partition(s) are:
    {1} {2} {3 4} {5}

ReplicatedSubspace =

    1    0    1    0    0    0    0    0    0    0
    0    1    0    0    1    1    1    0    1    0
    0    0    0    1    0    0    0    1    0    0
    0    0    0    0    0    0    0    0    0    1

Npb =

    1    0    1    0    0    0    0    0    0    0
    0    1    0    0    1    1    1    0    0    0
    0    0    0    1    0    0    0    0    0    0
    0    0    0    0    0    0    0    1    0    0
    0    0    0    0    0    0    0    0    0    1
    0    0    0    0    0    0    0    0    1    0
```

```
Cprb =

    1    0    1    0    0    0    0    0    0    0
    0    1    0    0    1    1    1    0    1    0
    0    0    0    1    0    0    0    0   -1    0
    0    0    0    0    0    0    0    1    1    0
    0    0    0    0    0    0    0    0    0    1
```

Therefore, the marketed space X has two maximal replicated subspaces, $\{1\}$ $\{2\,3\}$ $\{4\}$ $\{5\}$ and $\{1\}$ $\{2\}$ $\{3\,4\}$ $\{5\}$ are maximal proper partitions with corresponding maximal replicated subspaces the subspaces

$$Y_1 = [(1,0,1,0,0,0,0,0,0,0), (0,1,0,1,1,1,1,0,0,0), (0,0,0,0,0,0,0,1,1,0),$$
$$(0,0,0,0,0,0,0,0,0,1)]$$

and

$$Y_2 = [(1,0,1,0,0,0,0,0,0,0), (0,1,0,0,1,1,1,0,1,0), (0,0,0,1,0,0,0,1,0,0),$$
$$(0,0,0,0,0,0,0,0,0,1)],$$

respectively. The replicated kernel of the market is $Y = Y_1 \cup Y_2$.

5. Conclusions

In this chapter, computational methods for option replication are presented based on vector lattice theory. It is well accepted that the lattice theoretic ideas are one of the most important technical contributions of the large literature on modern mathematical finance. In this chapter, we consider an incomplete market of primitive securities, meaning that some call and put options need not be marketed. Our objective is to describe an efficient method for computing maximal submarkets that replicate any option. Even though, there are several important results on option replication they cannot provide a method for the determination of the replicated options. By using the theory of vector-lattices and positive bases it is provided a procedure in order to determine the set of securities with replicated options. Moreover, we determine those subspaces of the marketed subspace that replicate any option by introducing a Matlab function, namely `mrsubspace`. The results of this work can give us an important tool in order to study the interesting problem of option replication of a two-period security market in which the space of marketed securities is a subspace of $\mathbb{R}^m$. This work is based on a recent work, [15], regarding computational methods for option replication.

Author details

Vasilios N. Katsikis
General Department of Mathematics, Technological Education Institute of Piraeus, 12244 Athens, Greece

6. References

[1] Aliprantis, C.D. & Brown, D.J. (1983). Equilibria in markets with a Riesz space of commodities, *J. Math. Econom.*, 11, pp. 189–207.

[2] Aliprantis, C.D., Brown, D.J. & Burkinshaw, O. (1987a). Edgeworth equilibria, *Econometrica*, 55, pp. 1109–1137.

[3] Aliprantis, C.D., Brown, D.J. & Burkinshaw, O. (1987b). Edgeworth equilibria in production economies, *J. Econom. Theory*, 43, pp. 253–291.

[4] Aliprantis, C.D., Brown, D.J. & Burkinshaw, O. (1990). Existence and optimally of competitive equilibria, *Springer–Verlag*.

[5] Aliprantis, C.D. & Burkinshaw, O. (1991). When is the core equivalence theorem valid?, *Econom. Theory*, 1, pp. 169–182.

[6] Aliprantis, C.D., Tourky, R. & Yannelis, N. C. (2001). A theory of value with non–linear prices. Equilibrium analysis beyond vector lattices, *J. Econom. Theory*, 100, pp. 22–72.

[7] Aliprantis, C.D. & Tourky, R. (2002). Markets that don't replicate any option, *Economics Letter*, 76, pp.443–447.

[8] Baptista A.M. (2003). Spanning with american options, *Journal of Economic Theory*, 110, pp.264-289.

[9] Baptista A.M. (2005). Options and efficiency in multidate security markets, *Mathematical Fi- nance*, 15, No.4, pp.569-587.

[10] Baptista A.M. (2007). On the non-existence of redundant options, *Economic Theory*, 31, pp.205-212.

[11] Katsikis, V.N. (2007). Computational methods in portfolio insurance, *Applied Mathematics and Computation*, 189, pp.9-22.

[12] Katsikis, V.N. (2008). Computational methods in lattice-subspaces of $C[a,b]$ with applications in portfolio insurance, *Applied Mathematics and Computation*, 200, pp.204-219.

[13] Katsikis, V.N. (2009). A Matlab-based rapid method for computing lattice-subspaces and vector sublattices of $\mathbb{R}^n$: Applications in portfolio insurance, *Applied Mathematics and Computation*, 215, pp.961-972.

[14] Katsikis, V.N. (2010). Computational and Mathematical Methods in Portfolio Insurance. A MATLAB-Based Approach., Matlab - Modelling, Programming and Simulations, Emilson Pereira Leite(Ed.), ISBN: 978-953-307-125-1, InTech, Available from: http://www.intechopen.com/articles/show/title/computational-and-mathematical-methods-in-portfolio-insurance-a-matlab-based-approach-

[15] Katsikis, V.N. (2011). Computational methods for option replication, *International Journal of Computer Mathematics*, 88, No. 13, pp. 2752-2769.

[16] Katsikis V.N. & Polyrakis I. (2012). Computation of vector sublattices and minimal lattice-subspaces. Applications in finance, *Applied Mathematics and Computation*, 218, pp.6860–6873.

[17] Mas-Colell, A. (1986). The price equilibrium existence problem in topological vector lattices, *Econometrica*, 54, pp. 1039–1053.

[18] Mas-Colell, A. & Richard, S.F. (1991). A new approach to the existence of equilibrium in vector lattices, *J. Econom. Theory*, 53, pp. 1–11.

[19] Podczeck, K. (1996). Equilibria in vector lattices without ordered preferences or uniform properness, *J. Math. Econom.*, 25, pp. 465–484.

[20] Richard, S.F. (1989). A new approach to production equilibria in vector lattices, *J. Math. Econom.*, 18, pp. 231–247.

[21] Ross, S.A. (1976). Options and efficiency, *Quaterly Journal of Economics*, 90, pp.75-89.

[22] Tourky, R. (1998). A new approach to the limit theorem on the core of an economy in vector lattices, *J. Econom. Theory*, 78, pp. 321–328.

[23] Yannelis, N.C. & Zame, W.R. (1986). Equilibria in Banach lattices without ordered preferences, *J. Math. Econom.*, 15, pp. 85–110.