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Biometric Solvency Risk for Portfolios of General Life Contracts (II) The Markov Chain Approach

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1. Introduction

The main theoretical goal of the present exposé is to extend the results presented in Hürlimann [1] to the Markov chain model of life insurance, which enables modeling all single life/multiple life traditional contracts subject to biometric risk with multiple causes of decrement. In particular, a complete risk modeling of single-life insurance products with mortality and disability risks requires the specification of a Markov model with three states. As novel illustration we offer to the interested practitioner an in-depth treatment of endowment contracts with waiver of premium by disability.

The present investigation is restricted to biometric risks encountered in traditional insurance contracts within a discrete time Markov chain model. The current standard requirements for the Solvency II life risk module have been specified in QIS5 [2], pp.147-163. QIS5 prescribes a solvency capital requirement (SCR), which only depends on the time of valuation (=time at which solvency is ascertained) but not on the portfolio size (=number of policies). It accounts explicitly for the uncertainty in both trends (=systematic risk) and parameters (=parameter risk) but not for the random fluctuations around frequency and severity of claims (=process risk). In fact, the process risk has been disregarded as not significant enough, and, in order to simplify the standard formula, it has been included in the systematic/parameter risk component. For the purpose of internal models and improved risk management, it appears important to capture separately or simultaneously all risk components of biometric risks. A more detailed account of our contribution follows.

As starting point, we recall in Section 2 the general solvency rule for the prospective liability risk derived in [1], Section 2, which has resulted in two simple liability VaR & CVaR target capital requirements. In both stochastic models, the target capital can be decomposed into a solvency capital component (liability risk of the current period) and a risk margin component (liability risk of future periods), where the latter must be included (besides the

best estimate liabilities) in the technical provisions. This general decomposition is in agreement with the current QIS5 specification. The proposed approach is then applied to determine the biometric solvency risk capital for a portfolio of general traditional life contracts within the Markov chain model of life insurance. For this, we assume that the best estimate liabilities of a general life contract coincide with the so-called "net premium reserves". After introduction of the Markov chain approach to life insurance in Section 3, we recall in Section 4 the ubiquitous backward recursive actuarial reserve formula and the theorem of Hattendorff. Based on this we determine in Section 5 the conditional mean and variance of a portfolio's prospective liability risk (=random present value of future cash-flows at a given time of valuation) and use a gamma distribution approximation to obtain the liability VaR & CVaR solvency capital as well as corresponding solvency capital ratios. These first formulas include only the process risk and not the systematic risk. To include the latter risk in solvency investigations we propose either to shift the biometric transition probabilities, as done in Section 6.2, or apply a stochastic model, which allows for random biometric transition probabilities, as explained in Section 6.3. Section 7 illustrates numerically and graphically the considered VaR & CVaR solvency capital models for a cohort of endowment contracts with waiver of premium by disability and compares them with the current Solvency II standard approach. Finally, Section 8 summarizes, concludes and provides an outlook for possible alternatives and extensions.

2. A general prospective approach to the liability risk solvency capital

Starting point is a multi-period discrete time stochastic model of insurance. Given is a time horizon T and a probability space (Ω, F, P) endowed with a filtration $(F_t)_{t=0}^T$ such that $F_0 = \{\Omega, \varnothing\}$ and $F_T = F$. Let $L^\infty(F_t)$ be the space of essentially bounded random variables on (Ω, F, P) and B^∞ the space of essentially bounded stochastic processes on (Ω, F, P) which are adapted to the filtration $(F_t)_{t=0}^T$. The basic discrete time stochastic processes are

A_t, L_t : the *assets* and *actuarial liabilities* at time t

In a total balance sheet approach, their values depend upon the stochastic processes in B^∞, which describe the random cash-in and cash-out flows of any type of insurance business:

P_{t-1} : *loaded premiums* to be paid at time $t-1$ (assumed invested at time $t-1$)

X_t : *insurance costs* to be paid at time t (includes insurance benefits, expenses and bonus payments paid during the time period $(t-1, t]$)

R_t : *accumulation factor* for return on investment for the time period $(t-1, t]$

We assume that X_t is F_t-measurable and R_t is F_{t-1}-measurable. The random *cumulated accumulation factor* for return over the period $(s, t]$, $0 \le s < t \le T$, is denoted by $R_{s,t} = \prod_{j=s+1}^t R_j$.

Since R_t is F_{t-1}-measurable $R_{s,t}$ is F_{t-1}-measurable, and therefore $\{R_{s,t}, t > s\}$ is a

predictable stochastic process. The quantity $D_{s,t} = R_{s,t}^{-1}$ is called random discount rate. Consider the F_{t+j+1}-measurable discrete time stochastic process $\{CF_{t+j}, j = 0,1,...,T-t-1\}$ of *future insurance cash-flows* defined by

$$CF_{t+j} = D_{t+j,t+j+1} \cdot X_{t+j+1} - P_{t+j} \quad j = 0,1,...,T-t-1. \tag{1}$$

The actuarial liabilities at time t, also called time-t *prospective insurance liability*, coincide with the random present value of all future insurance cash-flows at time t given by

$$L_t = \sum_{j=0}^{T-t-1} D_{t,t+j} CF_{t+j}, \quad t = 0,1,...,T-1. \tag{2}$$

Using (1)-(2) and the relationship $D_{t,t+1+k} = D_{t,t+1} \cdot D_{t+1,t+1+k}$, one obtains the recursive equation $L_{t+1} = (L_t + P_t) \cdot R_{t+1} - X_{t+1}, t = 1,...,T-1$. On the other hand, the random assets over the time horizon $[0,T]$ satisfy by definition the recursive equation $A_{t+1} = (A_t + P_t) \cdot R_{t+1} - X_{t+1}, t = 1,...,T-1$. Through subtraction it follows that

$$A_{t+1} - L_{t+1} = (A_t - L_t) \cdot R_{t+1}, \quad t = 1,...,T-1, \tag{3}$$

which implies the following equivalent probabilistic conditions (use that trivially $L_T = 0$)

$$P(A_T \geq 0 | F_t) \geq 1 - \varepsilon, \tag{4}$$

$$P(A_{t+\tau} \geq L_{t+\tau}, \tau = 1,2,...,T-t-1 | F_t) \geq 1 - \varepsilon, \tag{5}$$

$$P(A_t \geq L_t | F_t) \geq 1 - \varepsilon. \tag{6}$$

Given a default probability $\varepsilon > 0$, the *liability VaR solvency criterion* (6) says that at time t the initial (deterministic) capital requirement A_t should exceed the random present value of future cash-flows with a probability of at least $1 - \varepsilon$. By (4)-(5) this criterion automatically implies that assets will exceed liabilities with the same probability at each future time over the time horizon T. Let $A_t^{VaR} = VaR_{1-\varepsilon}[L_t | F_t]$ be a minimum solution to (6), and assume that the best estimate insurance liabilities at time t coincide with the *net premium reserves* (in the sense defined later in (35), that is let $E[L_t | F_t] = V_t^Z$. Then, the *liability VaR solvency capital* $SC_t^{VaR} = A_t^{VaR} - V_t^Z = VaR_{1-\varepsilon}[L_t | F_t] - V_t^Z$ represents the capital available at time t to meet the insurance risk liabilities with high probability. A *risk margin* is added to this capital requirement (recall that in Solvency II the sum of the best estimate insurance liabilities and the risk margin determines the Technical Provisions). The *liability VaR target capital* is the sum of the liability VaR solvency capital and the risk margin defined by

$$TC_t^{VaR} = SC_t^{VaR} + RM_t^{VaR}. \tag{7}$$

The cost-of-capital risk margin with cost-of-capital rate $i_{CoC} = 6\%$ is defined by

$$RM_t^{VaR} = i_{CoC} \cdot \sum_{k=1}^{T-t} v_f^k \cdot SC_{t+k}^{VaR},\tag{8}$$

where T denotes the time horizon, and v_f is the risk-free discount rate. For comparison with other solvency rules, one considers the *VaR solvency capital ratio* at time t defined by

$$SR_t^{VaR} = SC_t^{VaR} / V_t^Z.\tag{9}$$

Alternatively, let $CVaR_{1-\varepsilon}\left[L_t | F_t \right] = E\left[L_t | L_t > VaR_{1-\varepsilon}\left[L_t | F_t \right], F_t \right]$ be the conditional value-at-risk of the random present value of future cash-flows at the confidence level $1-\varepsilon$ given the information available at time t. The *liability CVaR target capital* $TC_t^{CVaR} = CVaR_{1-\varepsilon}\left[L_t | F_t \right] - V_t^Z + RM_t^{CVaR} = SC_t^{CVaR} + RM_t^{CVaR}$ also meets the insurance risk liabilities and it defines the *CVaR solvency capital ratio* at time t:

$$SR_t^{CVaR} = SC_t^{CVaR} / V_t^Z.\tag{10}$$

3. The Markov chain approach to general life contracts

Consider the Markov chain model of a *general life insurance* (GLIFE) contract with state space S and arbitrary payments. The *state space* S is the finite set of states a contract can be during its lifetime. Payments are induced by two kinds of events:

Type 1: payments induced by being in a certain state
Type 2: payments induced by a jump of state

The *payment function vector* of a contract at time $k = 0,1,2,...$ is expressed as a vector $a(k) = \left\{ a_i(k), a_{ij}(k) | i \neq j \in S \right\}$, where the *payment functions* are defined by

Type 1: $a_i(k), i \in S$, is the payment if the contract at time k is in state i.

Type 2: $a_{ij}(k), i \neq j \in S, k \geq 1$, is the payment if the contract was in state i at time $k-1$ and is in state state j at time k. For convenience set $a_{ij}(0) = 0$ for $i \neq j \in S$ and $a_{ij}(k) = 0$ for $i = j \in S$.

For better interpretation one splits the payment $a_i(k)$ into a *benefit part* and a *premium part* such that $a_i(k) = b_i(k) - \pi_i(k), \quad i \in S, k = 0,1,2,...$, where $\pi_i(k) \geq 0$ denotes the non-negative premium paid at time k when the contract is in state i. Note that in most applications one has $\pi_i(k) = 0$ if the state i is different from the state of being "active" (premiums are only paid in this situation). Restricting the attention to biometric risk only, we assume throughout a flat term structure of interest rates with annual interest rate i and discount factor $v = 1/(1+i)$. The state of a GLIFE contract over time is described by the discrete time

stochastic process $(X_k)_{k=0,1,2,...}$ with values in S. The event $\{X_k = s\}$ means that the contract at time k is in state s. We assume that $(X_k)_{k=0,1,2,...}$ is a *Markov chain*, which implies that the joint distributions of the random states can be represented in terms of the *one-step transition probability matrix* $p(k) = (p_{ij}(k))_{i,j \in S}$, which is defined by

$$p_{ij}(k) = P(X_{k+1} = j | X_k = i), \quad k = 0,1,2,... \tag{11}$$

The set $(S, a(k), X_k, k = 0,1,2,...)$ defines the Markov chain model widely discussed in life insurance (Amsler [3]; Hoem [4], [5]; Koller [6]; Milbrodt & Helbig [7]; Wolthuis [8]; etc.). Now, using the indicator function $I(\cdot)$, consider the *random cash-flow* of the GLIFE contract in year $(k, k+1]$ valued at time $k = 0,1,2,...$, which is defined by

$$C_k = \sum_{i \in S} \{b_i(k) - \pi_i(k)\} \cdot I(X_k = i) + v \cdot \sum_{i \neq j \in S} a_{ij}(k+1) \cdot I(X_k = i \wedge X_{k+1} = j). \tag{12}$$

The *insurance loss* random variable of a GLIFE contract is defined by

$$L = \sum_{k=0}^{\infty} v^k C_k. \tag{13}$$

This identifies the insurance loss with the random present value of all future cash-flows. Furthermore, for an arbitrary non-negative integer $\tau = 0,1,...$, one defines the *time-τ prospective loss* random variable

$$L_\tau = (1+i)^\tau \cdot \sum_{k=\tau}^{\infty} v^k C_k, \tag{14}$$

whose (conditional) expected value defines the time-τ actuarial reserve

$$V_\tau = E[L_\tau | X_\tau] = \sum_{k \in S} V_\tau^k, \quad V_\tau^k = E[L_\tau | X_\tau = k]. \tag{15}$$

The quantity V_τ^k is called *state-k time-τ actuarial reserve*. In particular, one has $L_0 = L$ and $V_0 = E[L]$ is the *initial actuarial reserve*, which is not assumed to vanish.

4. Backward recursive reserve formula and the theorem of Hattendorff

In a first step, we derive a recursion formula for the actuarial reserves. Recall the recursion formula for the random prospective loss

$$L_\tau = C_\tau + vL_{\tau+1}. \tag{16}$$

Assume that the contract is in state $. k \in S$ at time τ. Inserting (16) into (15) yields

$$V_\tau^k = E\big[C_\tau\big|X_\tau = k\big] + v \cdot E\big[L_{\tau+1}\big|X_\tau = k\big]. \tag{17}$$

Using (12) the first expectation in (17) can be rewritten as

$$v \cdot \sum_{i \neq j \in S} a_{ij}(\tau + 1) \cdot P\big(X_\tau = i \wedge X_{\tau+1} = j\big|X_\tau = k\big) + \sum_{i \in S}\{b_i(\tau) - \pi_i(\tau)\} \cdot P\big(X_\tau = i\big|X_\tau = k\big)$$

$$= v \cdot \sum_{j \neq k \in S} a_{kj}(\tau + 1) \cdot p_{kj}(\tau) + b_k(\tau) - \pi_k(\tau).$$

The second expectation equals

$$E\big[L_{\tau+1}\big|X_\tau = k\big] = \sum_{j \in S} E\big[L_{\tau+1}\big|X_{\tau+1} = j\big] \cdot P\big(X_{\tau+1} = j\big|X_\tau = k\big) = \sum_{j \in S} V_{\tau+1}^j \cdot p_{kj}(\tau).$$

Inserting both expressions into (17) and using the made convention $a_{kk}(\tau + 1) = 0$ as well as the relationship $\sum_{j \in S} p_{kj}(\tau) = 1$, one obtains the backward recursive reserve formula

$$V_\tau^k = \sum_{j \in S} p_{kj}(\tau) \cdot \Big\{v \cdot V_{\tau+1}^j + v \cdot a_{kj}(\tau + 1) + b_k(\tau)\Big\} - \pi_k(\tau). \tag{18}$$

The actuarial reserve at time τ given the contract is in state $k \in S$ equals the one-year discounted sum over all possible states of the

- actuarial reserves at time $\tau + 1$,
- payments at time $\tau + 1$ due to a jump in states,
- payments at time τ due if being in a certain state,

which is weighted by the one-step transition probabilities and reduced by the premium paid at time τ when the contract is in state k. The representation (18) is a discrete version of *Thiele's differential equation*. Thiele's differential equation is a simple example of a Kolmogorov backward equation, which is a basic tool for determining conditional expected values in intensity-driven Markov processes, e.g. Norberg [9].

Let us rearrange (18) in order to obtain the Markov chain analogue of the classical decomposition of the premium into risk premium and saving premium (Gerber [10], [11]), Section 7.5, equation (5.3), and [1], equation (19).

Theorem 4.1 The premium $\pi_k(\tau)$ at time τ if the contract is in state $k \in S$ is the sum of a *saving premium* $\pi_k^S(\tau)$ and a *risk premium* $\pi_k^R(\tau)$, which are defined as follows:

$$\pi_k^S(\tau) = v \cdot V_{\tau+1}^k - V_\tau^k \tag{19}$$

$$\pi_k^R(\tau) = b_k(\tau) - v \cdot V_{\tau+1}^k + v \cdot \sum_{j \in S} p_{kj}(\tau) \cdot \Big\{a_{kj}(\tau + 1) + V_{\tau+1}^j\Big\} \tag{20}$$

Proof. Making use of the recursion (18) and the relationship $\sum_{j\in S} p_{kj}(\tau) = 1$, one obtains

$$\pi_k^S(\tau) + \pi_k^R(\tau) = v \cdot V_{\tau+1}^k - V_\tau^k + b_k(\tau) - v \cdot V_{\tau+1}^k + v \cdot \sum_{j\in S} p_{kj}(\tau) \cdot \left\{ a_{kj}(\tau+1) + V_{\tau+1}^j \right\}$$

$$= -V_\tau^k + \sum_{j\in S} p_{kj}(\tau) \cdot \left\{ v \cdot a_{kj}(\tau+1) + v \cdot V_{\tau+1}^j + b_k(\tau) \right\} = \pi_k(\tau),$$

which shows the desired decomposition. ◊

The saving premium represents the expected change in actuarial reserve at time τ for a contract in state k while the risk premium is the expected value at time τ of a contract in state k needed to cover the insurance risk in time period $(\tau, \tau+1]$. Rewrite the latter as

$$\pi_k^R(\tau) = b_k(\tau) + v \cdot \sum_{j\in S} p_{kj}(\tau) \cdot \left\{ a_{kj}(\tau+1) + V_{\tau+1}^j - V_{\tau+1}^k \right\}. \tag{21}$$

This is the sum of the benefit payment at time τ for a contract in state k and the probability weighted sum of the *sums at risk* $\left\{ a_{kj}(\tau+1) + V_{\tau+1}^j - V_{\tau+1}^k \right\}$ due to transitions from state k to state j at time $\tau+1$. The sum at risk is the amount credited to the insured's contract upon a transition, namely the lump sum payable immediately plus the adjustment of the actuarial reserve. The obtained results constitute a discrete time version of those mentioned in Norberg [12], p.10.

To evaluate the mean and variance of the random insurance loss (13) of a GLIFE contract, we follow the martingale approach to the *Theorem of Hattendorff* (Bühlmann [13]; Gerber [14]; Gerber et al. [15]; Hattendorff [16]; Kremer [17]; Patatriandafylou & Waters [18]; etc.). For this consider the set of $\tau+1$ contract states $S_\tau = \left\{ X_\eta, \eta = 0,1,...,\tau \right\}$ at time τ and the sequence of random variables

$$Y_\tau = E\left[L \middle| S_\tau \right], \quad \tau = 1,2,..., \quad Y_0 = E\left[L \right] = V_0. \tag{22}$$

The discrete time stochastic process $\{Y_\tau\}$ is a martingale with respect to $\{S_\tau\}$. The martingale differences $Y_{\tau+1} - Y_\tau = v^\tau \Lambda_\tau, \tau = 0,1,2,...$, represent the *discounted one-year insurance losses* and form a sequence of uncorrelated random variables such that

$$E\left[\Lambda_\eta \right] = 0, \quad Cov\left[\Lambda_\eta, \Lambda_\tau \right] = 0, \quad 0 \le \eta < \tau, \quad L - V_0 = \sum_{\tau=0}^\infty v^\tau \Lambda_\tau. \tag{23}$$

Through detailed calculation one obtains the following result.

Theorem 4.2 The variance of the random insurance loss of a GLIFE contract is determined by the following formulas

$$Var\left[L \right] = \sum_{\tau=0}^\infty v^{2\tau} \cdot Var\left[C_\tau \right], \tag{24}$$

$$Var\left[C_\tau\right] = \sum_{k \in S} E\left[C_\tau^2 \big| X_\tau = k\right] \cdot P\left(X_\tau = k\right) - \pi^S(\tau)^2, \quad \pi^S(\tau) = \sum_{k \in S} \pi_k^S(\tau), \tag{25}$$

$$E\left[C_\tau^2 \big| X_\tau = k\right] = \left[b_k(\tau) - \pi_k(\tau)\right]^2 + \sum_{j \in S} va_{kj}(\tau+1) \cdot \left[b_k(\tau) - \pi_k(\tau) + va_{kj}(\tau+1)\right] \cdot p_{kj}(\tau). \tag{26}$$

Proof. Similarly to Gerber et al. [15], formula (89), one has

$$Y_\tau = \sum_{j=0}^{\tau-1} v^j C_j + v^\tau E\left[L_\tau \big| S_\tau\right]. \tag{27}$$

Using (15) one obtains

$$v^\tau \Lambda_\tau = Y_{\tau+1} - Y_\tau = v^\tau \left(C_\tau + vE\left[L_{\tau+1} \big| S_{\tau+1}\right] - E\left[L_\tau \big| S_\tau\right]\right)$$
$$= v^\tau \left(C_\tau + \sum_{k \in S}\left\{vE\left[L_{\tau+1} \big| X_{\tau+1} = k\right] - E\left[L_\tau \big| X_\tau = k\right]\right\}\right) = v^\tau \left(C_\tau + \pi^S(\tau)\right), \tag{28}$$

$$\Lambda_\tau = C_\tau + \pi^S(\tau). \tag{29}$$

Since $E\left[\Lambda_\tau\right] = 0$ one gets $E\left[C_\tau\right] = -\pi^S(\tau)$ and further

$$Var\left[\Lambda_\tau\right] = Var\left[C_\tau\right] = E\left[C_\tau^2\right] - E\left[C_\tau\right]^2 = \sum_{k \in S} E\left[C_\tau^2 \big| X_\tau = k\right] \cdot P\left(X_\tau = k\right) - \pi^S(\tau)^2, \tag{30}$$

which is (25). To obtain (26) one uses (12) and the convention $a_{ii}(\tau+1) = 0$ to get

$$C_\tau^2 = \sum_{i \in S}\left[b_i(\tau) - \pi_i(\tau)\right]^2 \cdot I\left(X_\tau = i\right)$$
$$+ \sum_{i,j \in S} va_{ij}(\tau+1) \cdot \left[b_i(\tau) - \pi_i(\tau) + va_{ij}(\tau+1)\right] \cdot I\left(X_\tau = i \wedge X_{\tau+1} = j\right), \tag{31}$$

which implies that

$$E\left[C_\tau^2 \big| X_\tau = k\right] = \left[b_k(\tau) - \pi_k(\tau)\right]^2 + \sum_{j \in S} va_{kj}(\tau+1) \cdot \left[b_k(\tau) - \pi_k(\tau) + va_{kj}(\tau+1)\right] \cdot p_{kj}(\tau). \tag{32}$$

Remark 4.1 In the single life case, the variance formulas in Theorem 4.2 should be compared with the ones for the GLIFE contract with one and multiple causes of decrement in [1], formulas (24)-(26). One can ask if the formula (25) is equivalent to the following one (at least in the single life case)

$$Var\left[C_\tau\right] = \sum_{k \in S}\left[\sum_{j \in S}\left(a_{kj}(\tau+1) + V_{\tau+1}^j - V_{\tau+1}^k\right)^2 v^2 p_{kj}(\tau) - \pi_k^R(\tau)^2\right] \cdot P\left(X_\tau = k\right). \qquad (33)$$

5. The liability VaR & CVaR solvency capital for portfolios of GLIFE contracts

We begin with risk calculations for a single GLIFE contract, and use them to determine the liability VaR & CVaR solvency capital for a portfolio of GLIFE contracts.

5.1. Risk calculations for a single GLIFE contract

Given is a single GLIFE contract with random future cash-flows $\{C_k\}$ defined by (12). We assume that the state space contains a unique distinguished "void" state $\{X_k = \phi\}$ meaning that the contract has terminated at time k. We assume *contract survival*, i.e. a contract is still alive at time of valuation t, which implies that the conditional event $E_t = \{X_t \neq \phi\}$ is fulfilled. We note that the random present value of future cash-flows at time t defined by

$$Z_t = \sum_{j=0}^{\infty} v^j C_{t+j}, \quad t = 0,1,..., \qquad (34)$$

coincides with the time-t prospective loss defined in (14), that is $Z_t = L_t, t = 0,1,...$. Therefore, the expected value given contract survival equals

$$V_t^Z = E\left[Z_t \big| E_t\right] = \sum_{k \in S} V_t^k \cdot P\left(X_t = k \big| E_t\right) = \sum_{k \in S} V_t^k \cdot \frac{P\left(X_t = k\right)}{P\left(E_t\right)}. \qquad (35)$$

In contrast to (15) the reserve defined in (35) is state independent and called *net premium reserve*, see Bowers et al. [19], Chap.17.7, p. 500, for a special case. Following Section 2, this value can been chosen as best estimate of the contract liabilities.

Remarks 5.1 (i) The motivation for state-independent reserves is second-to-die life insurance, where during lifetime the insurer may not be informed about the first death. An endowment with waiver of premium during disability, which is our illustration in Section 7, seems to contradict this concept because it cannot be argued that the insurer is unaware of the state occupied while the premium is being waived. However, at a given arbitrary time of valuation (including starting dates of contracts) future states of contracts are unknown, and therefore it is reasonable in a first step to assume state independent reserves for the design of a general method. Later refinement might be necessary to cover all possible cases.

(ii) State independent reserves have been introduced by Frasier [20] for the last-survivor status, see also The Actuary [21] and Margus [22]. The choice between state independent and state dependent reserves depends upon loss recognition in the balance sheet (recognition or not of a status change). With state independent reserves, the insurance

company administers the contract as if it had no knowledge of any decrements, as long as the contract is not terminated. Only the latter situation is considered in the present work.

In a first step, we determine the mean and variance of the conditional distribution of Z_t given E_t. Similarly to [1], Section 5.1, the variance formulas (24)-(26) generalize to an arbitrary discrete time $t = 1, 2, \ldots$. Formula (23) generalizes as

$$L_t - V_t \cdot I(E_t) = \sum_{k=t}^{\infty} v^{k-t} \Lambda_k. \qquad (36)$$

Noting further that $Z_t = L_t, t = 0, 1, \ldots$, one obtains from (36) the following conditional variance formulas (conditional version of Theorem 4.2).

Theorem 5.1 The conditional variance is determined by the following formulas

$$Var\left[Z_t | E_t\right] = \sum_{\tau=0}^{\infty} v^{2\tau} \cdot Var\left[C_{t+\tau} | E_t\right] = \sum_{\tau=0}^{\infty} v^{2\tau} \cdot \left(\frac{E\left[C_{t+\tau}^2\right] \cdot P(E_t) - E\left[C_{t+\tau}\right]^2}{P(E_t)^2} \right),$$

$$E\left[C_{t+\tau}^2\right] = \sum_{k \in S} E\left[C_{t+\tau}^2 | X_{t+\tau} = k\right] \cdot P(X_{t+\tau} = k), \quad E\left[C_{t+\tau}\right] = \pi^S(t+\tau) = \sum_{k \in S} \pi_k^S(t+\tau) \qquad (37)$$

$$E\left[C_{t+\tau}^2 | X_{t+\tau} = k\right] = \left[b_k(t+\tau) - \pi_k(t+\tau)\right]^2$$

$$+ \sum_{j \in S} va_{kj}(t+\tau+1) \cdot \left[b_k(t+\tau) - \pi_k(t+\tau) + va_{kj}(t+\tau+1)\right] \cdot p_{kj}(t+\tau).$$

As shown in the next Subsection, these formulas can be used to determine the target capital and solvency capital ratio of a portfolio of GLIFE contracts using appropriate approximations for the distribution of the random present value of future cash-flows associated to this portfolio under the condition that the contracts are still alive.

5.2. Solvency capital and solvency capital ratio for a portfolio of GLIFE contracts

Towards the ultimate goal of solvency evaluation for an arbitrary life insurance portfolio, we consider now a set of n policyholders alive at time t. From Section 3 one knows that the i-th contract $i \in \{1, \ldots, n\}$ is characterized by the following data elements:

- contract duration t_i at time t

- state space $S^{(i)}$

- states $\left(X_k^{(i)}\right)_{k=0,1,2,\ldots}$ of the contract over time with values in $S^{(i)}$

- condition for contract survival $E_{t_i}^{(i)} = \left\{X_{t_i}^{(i)} \neq \phi\right\}$ at time t

- one-step transition probabilities $p_{rs}^{(i)}(k) = P\left(X_{k+1}^{(i)} = s \middle| X_k^{(i)} = r\right), \quad k = 0, 1, 2, \ldots$, defining the Markov chain model of the contract

- payment function vector $a^{(i)}(k) = \left\{ a_r^{(i)}(k), a_{rs}^{(i)}(k) \middle| r \neq s \in S^{(i)} \right\}$ of the contract at time $k = 0,1,2,\ldots$ with the two types of payment:

Type 1: $a_r^{(i)}(k), r \in S^{(i)}$, is the payment if the contract at time k is in state r.

Type 2: $a_{rs}^{(i)}(k), r \neq s \in S^{(i)}, k \geq 1$, is the payment if the contract was in state r at time $k-1$ and is in state state s at time k

- splitting $a_r^{(i)}(k) = b_r^{(i)}(k) - \pi_r^{(i)}(k), \quad r \in S^{(i)}, k = 0,1,2,\ldots$ into a benefit & premium part

To the i-th contract one associates its random future cash-flows $\left\{ C_k^{(i)} \right\}$ as defined in (12), the corresponding $L_{t_i}^{(i)}$ time-t_i random prospective loss (14) and time-t_i net premium reserve $V_{t_i}^{Z,(i)} = E\left[L_{t_i}^{(i)} \middle| E_{t_i}^{(i)} \right]$ obtained from (35). The random present value of future cash-flows of the portfolio is obtained by summing (34) over all contracts and is given by

$$Z_t = \sum_{i=1}^{n} \sum_{j=0}^{\infty} v^j C_{t_i+j}^{(i)} = \sum_{j=0}^{\infty} v^j \left(\sum_{i=1}^{n} C_{t_i+j}^{(i)} \right), \quad t = 0,1,\ldots . \tag{38}$$

Similarly, summing the individual net premium reserves, one gets the *portfolio reserve*

$$V_t^Z = \sum_{i=1}^{n} V_{t_i}^{Z,(i)}. \tag{39}$$

Following Section 2, one defines the *portfolio VaR solvency capital*

$$SC_t^{VaR} = VaR_{1-\varepsilon}\left[Z_t \middle| E_{t_i}^{(i)}, i = 1,\ldots,n \right] - V_t^Z, \tag{40}$$

as well as the *portfolio CVaR solvency capital*

$$SC_t^{CVaR} = CVaR_{1-\varepsilon}\left[Z_t \middle| E_{t_i}^{(i)}, i = 1,\ldots,n \right] - V_t^Z, \tag{41}$$

and the corresponding solvency capital ratios

$$SR_t^{VaR} = SC_t^{VaR} / V_t^Z, \quad SR_t^{CVaR} = SC_t^{CVaR} / V_t^Z. \tag{42}$$

To determine these quantities it is necessary to determine the distribution of Z_t conditional on contract survivals at time t, and under the assumption that the remaining lifetimes of all contracts are independent of each other. From Theorem 5.1 we have

$$E\left[Z_t \middle| E_{t_i}^{(i)}, i = 1,\ldots,n \right] = V_t^Z, \quad Var\left[Z_t \middle| E_{t_i}^{(i)}, i = 1,\ldots,n \right] = \sum_{\tau=0}^{\infty} v^{2\tau} \cdot \sum_{i=1}^{n} Var\left[C_{t_i+\tau}^{(i)} \right]$$

$$= \sum_{\tau=0}^{\infty} v^{2\tau} \cdot \sum_{i=1}^{n} \frac{E\left[(C_{t_i+\tau}^{(i)})^2 \right] \cdot P(E_{t_i}^{(i)}) - E\left[C_{t_i+\tau}^{(i)} \right]^2}{P(E_{t_i}^{(i)})^2}, \tag{43}$$

$$E\left[C_{t_i+\tau}^{(i)}\right] = \pi^{S,(i)}(t_i+\tau) = \sum_{k \in S^{(i)}} \pi_k^{S,(i)}(t_i+\tau),$$

$$E\left[(C_{t_i+\tau}^{(i)})^2\right] = \sum_{k \in S^{(i)}} E\left[(C_{t_i+\tau}^{(i)})^2 \middle| X_{t_i+\tau}^{(i)} = k\right] \cdot P\left(X_{t_i+\tau}^{(i)} = k\right),$$

$$E\left[(C_{t_i+\tau}^{(i)})^2 \middle| X_{t_i+\tau}^{(i)} = k\right] = \left[b_k^{(i)}(t_i+\tau) - \pi_k^{(i)}(t_i+\tau)\right]^2 \tag{44}$$

$$+ \sum_{j \in S^{(i)}} va_{kj}^{(i)}(t_i+\tau+1) \cdot \left[b_k^{(i)}(t_i+\tau) - \pi_k^{(i)}(t_i+\tau) + va_{kj}^{(i)}(t_i+\tau+1)\right] \cdot p_{kj}^{(i)}(t_i+\tau).$$

Based on the conditional mean and variance we approximate the distribution function of Z_t by a gamma distribution as in [1], Section 5. Denote this approximation by $F_t(x) = \Pr\left(Z_t \leq x \middle| E_{t_i}^{(i)}, i = 1, ..., n\right)$. Then, recalling the gamma distribution function, one has

$$F_t(x) = G(\beta_t x; \alpha_t) = \frac{1}{\Gamma(\alpha_t)} \cdot \int_0^{\beta_t x} t^{\alpha_t - 1} e^{-t} dt, \quad \alpha_t = \frac{1}{k_t^2}, \quad \beta_t = \frac{1}{k_t^2 \mu_t}, \tag{45}$$

where μ_t, k_t are the conditional mean and coefficient of variation of Z_t (obtained from (43)-(44)). In this setting, the solvency capital ratio formulas (42) take the forms

$$SR_t^{VaR} = z_{1-\varepsilon}\left(k_t^{-2}\right) \cdot k_t^2 - 1, \quad SR_t^{CVaR} = z_{1-\varepsilon}\left(k_t^{-2}\right) \cdot k_t^2 \cdot \frac{g\left(z_{1-\varepsilon}\left(k_t^{-2}\right); k_t^{-2}\right)}{\varepsilon}, \tag{46}$$

where $z_{1-\varepsilon}(\alpha) := G^{-1}(1-\varepsilon; \alpha)$ denotes the $(1-\varepsilon)$-quantile of the standardized gamma $G(x; \alpha)$ and $g(x; \alpha) = G'(x; \alpha)$ denotes its probability density. The limiting results for a portfolio of infinitely growing size are similar to those in [1], Remark 5.1. If the coefficients of variation tend to zero, the gamma distributions converge to normal distributions and the solvency capital ratios converge to zero. This holds under the following assumption. Whenever insured contracts are independent and identically distributed, and if the portfolio size is large enough, then the ratio of observed state transitions to portfolio size is close to the given rates of transition with high probability. This assumption is related to the *process risk*, which describes the random fluctuations in the biometric transition probability matrix. However, if the ratio of observed state transitions to portfolio size is not close to the given rates of transition, even for large portfolio sizes, *systematic risk* exists, e.g. Olivieri & Pitacco [23], Section 2.1. In this situation, the rates of transition are uncertain and assumed to be random, and we consider stochastic models that include the process and systematic risk components. This is the subject of Section 6.3.

6. Comparing the standard approach with variants of the stochastic approach

Since the present Section has some overlap with [1], Section 6, it is treated more briefly, but can be read independently. Facts peculiar to the Markov chain approach are added whenever felt necessary. Recall that *biometric risks* in QIS5 accounts for the uncertainty in

trends and parameters, the so-called *systematic/parameter risk*, but not for the *process risk*. We note that the solvency capital models of Section 5.2 only apply to the process risk. For full coverage of the process and systematic risk components, these solvency models are revised and extended. For this, we either shift the biometric transition probability matrix (see Section 6.2) or apply a stochastic biometric model with random biometric rates of transition (see Section 6.3). For completeness we briefly recall the QIS5 standard approach.

6.1. Solvency II standard approach

To value the net premium reserves a biometric "best estimate" life table is chosen. In the Markov chain model the life table is replaced by the one-step transition probabilities $p_{ij}(k) = P(X_{k+1} = j | X_k = i), \quad k = 0, 1, 2, \ldots$. Given is a single life policy at time of valuation t with net premium reserve V_t^Z. Denote by $V_t^{Z,\Delta}$ the value of the reserves subject to a biometric shock Δ. The one-year *solvency capital requirement* (SCR) for this single policy is

$$SCR_t = V_t^{Z,\Delta} - V_t^Z. \tag{47}$$

Similarly to the decomposition (7) the *Solvency II target capital* (upper index S2 in quantities) is understood as the sum of the SCR and a risk margin defined by

$$TC_t^{S2} = SCR_t + RM_t. \tag{48}$$

$$RM_t = i_{CoC} \cdot \sum_{k=1}^{T-t} v_f^k \cdot SCR_{t+k}, \quad i_{CoC} = 6\%, \tag{49}$$

where T denotes the time horizon, which may depend on the life policy, and v_f is the risk-free discount rate. Since Solvency II uses a total balance sheet approach, the defined single policy quantities must be aggregated on a portfolio and/or line of business level. For comparison with internal models it is useful to consider the solvency capital ratio at time t under the Solvency II standard approach defined by the quotient

$$SR_t^{S2} = SCR_t / V_t^Z. \tag{50}$$

By using a matrix of transition shocks $\Delta = (\Delta_{ij})$, we obtain formulas for the Markov chain model. Consider the *shifted biometric transition probabilities* defined by

$$p_{ij}^{\Delta_{ij}}(k), \tag{51}$$

which is associated to $p_{ij}(k)$, for a permanent shift of amount Δ_{ij} over all contracts and years $k = 0, 1, 2, \ldots$. In the current specification one has $\Delta_{AD} = 0.15$ (permanent 15% increase in mortality rates at each age for jumping from the alive state "A" to the dead state "D" for the mortality risk), $\Delta_{AD} = -0.20$ (permanent 20% decrease in mortality rates at each age for jumping from "A" to "D" for the longevity risk), and $\Delta_{AJ} = 0.35$ respectively $\Delta_{AJ} = 0.25$ (increase of 35% in disability rates for the next year, respectively a permanent 25% increase

in disability rates at each age in following years for jumping from "A" to the disability state "J" for the disability risk). To calculate the portfolio reserve V_t^Z in (39) and the corresponding shifted value $V_t^{Z,\Delta}$ under a matrix of transition shocks $\Delta = \left(\Delta_{ij} \right)$, we use (34) and the backward recursion formula (18) to get

$$
\begin{aligned}
V_t^Z &= \sum_{i=1}^{n} V_{t_i}^{Z,(i)} = \sum_{i=1}^{n} \sum_{k \in S} V_{t_i}^{k,(i)} \cdot P\left(X_{t_i}^{(i)} = k \middle| E_{t_i}^{(i)} \right), \\
V_{t_i}^{k,(i)} &= \sum_{j \in S} p_{kj}(t_i) \cdot \left\{ v \cdot V_{t_i+1}^{j,(i)} + v \cdot a_{kj}^{(i)}(t_i+1) + b_k^{(i)}(t_i) \right\} - \pi_k^{(i)}(t_i),
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
V_t^{Z,\Delta} &= \sum_{i=1}^{n} V_{t_i}^{Z,(i),\Delta} = \sum_{i=1}^{n} \sum_{k \in S} V_{t_i}^{k,(i),\Delta} \cdot P\left(X_{t_i}^{(i),\Delta} = k \middle| E_{t_i}^{(i),\Delta} \right), \\
V_{t_i}^{k,(i),\Delta} &= \sum_{j \in S} p_{kj}^{\Delta_{kj}}(t_i) \cdot \left\{ v \cdot V_{t_i+1}^{j,(i),\Delta} + v \cdot a_{kj}^{(i)}(t_i+1) + b_k^{(i)}(t_i) \right\} - \pi_k^{(i),\Delta}(t_i).
\end{aligned}
\tag{53}
$$

Similarly to (47)-(50) and using (52)-(53) we obtain the risk capital formulas

$$
\begin{aligned}
SCR_t &= V_t^{Z,\Delta} - V_t^Z, \quad TC_t^{S2} = SCR_t + RM_t, \\
RM_t &= i_{CoC} \sum_{k=1}^{\max\{\omega-(x_i+t_i)\}} v_f^k \cdot SCR_{t+k}, \quad SR_t^{S2} = \frac{SCR_t}{V_t^Z}.
\end{aligned}
\tag{54}
$$

6.2. Stochastic approach: Shifting the biometric transition probability matrix

Following the Sections 5.2 and 6.1, we consider the "shifted" random present value Z_t^Δ of future cash-flows of the portfolio at time t with conditional mean and variance

$$
\begin{aligned}
E\left[Z_t^\Delta \middle| E_{t_i}^{(i),\Delta}, i=1,...,n \right] &= V_t^{Z,\Delta}, \quad Var\left[Z_t^\Delta \middle| E_{t_i}^{(i),\Delta}, i=1,...,n \right] = \sum_{\tau=0}^{\infty} v^{2\tau} \cdot \sum_{i=1}^{n} Var\left[C_{t_i+\tau}^{(i),\Delta} \right] \\
&= \sum_{\tau=0}^{\infty} v^{2\tau} \cdot \sum_{i=1}^{n} \frac{E\left[(C_{t_i+\tau}^{(i),\Delta})^2 \right] \cdot P(E_{t_i}^{(i),\Delta}) - E\left[C_{t_i+\tau}^{(i),\Delta} \right]^2}{P(E_{t_i}^{(i),\Delta})^2},
\end{aligned}
\tag{55}
$$

$$
\begin{aligned}
E\left[C_{t_i+\tau}^{(i),\Delta} \right] &= \pi^{S,(i),\Delta}(t_i+\tau) = \sum_{k \in S^{(i)}} \pi_k^{S,(i),\Delta}(t_i+\tau), \\
E\left[(C_{t_i+\tau}^{(i),\Delta})^2 \right] &= \sum_{k \in S^{(i)}} E\left[(C_{t_i+\tau}^{(i),\Delta})^2 \middle| X_{t_i+\tau}^{(i),\Delta} = k \right] \cdot P\left(X_{t_i+\tau}^{(i),\Delta} = k \right), \\
E\left[(C_{t_i+\tau}^{(i),\Delta})^2 \middle| X_{t_i+\tau}^{(i),\Delta} = k \right] &= \left[b_k^{(i)}(t_i+\tau) - \pi_k^{(i),\Delta}(t_i+\tau) \right]^2 \\
&+ \sum_{j \in S^{(i)}} va_{kj}^{(i)}(t_i+\tau+1) \cdot \left[b_k^{(i)}(t_i+\tau) - \pi_k^{(i),\Delta}(t_i+\tau) + va_{kj}^{(i)}(t_i+\tau+1) \right] \cdot p_{kj}^{(i),\Delta_{kj}}(t_i+\tau).
\end{aligned}
\tag{56}
$$

The distribution of Z_t^Δ conditional on contract survivals at time t is again approximated by a gamma distribution denoted by

$$F_t^\Delta(x) = G(\beta_t^\Delta x; \alpha_t^\Delta), \quad \alpha_t^\Delta = (k_t^\Delta)^{-2}, \beta_t^\Delta = (k_t^\Delta)^{-2}/\mu_t^\Delta, \tag{57}$$

where the conditional mean and coefficient of variation μ_t^Δ, k_t^Δ of Z_t^Δ are obtained from the formulas (55)-(56). Making use of (46) and (47) one sees that the portfolio VaR & CVaR solvency capitals under the shifted biometric transition probability matrix are given by the expressions

$$SC_t^{\Delta,VaR} = VaR_{1-\varepsilon}\left[Z_t^\Delta \middle| E_{t_i}^{(i),\Delta}, i=1,...,n \right] - V_t^{Z,\Delta} = SCR_t + \left(z_{1-\varepsilon}\left(\left[k_t^\Delta\right]^{-2} \right) \cdot \left[k_t^\Delta\right]^2 - 1 \right) \cdot V_t^{Z,\Delta}, \tag{58}$$

$$SC_t^{\Delta,CVaR} = CVaR_{1-\varepsilon}\left[Z_t^\Delta \middle| E_{t_i}^{(i),\Delta}, i=1,...,n \right] - V_t^{Z,\Delta}$$

$$= SCR_t + z_{1-\varepsilon}\left(\left[k_t^\Delta\right]^{-2} \right) \cdot \left[k_t^\Delta\right]^2 \cdot \frac{g\left(z_{1-\varepsilon}\left(\left[k_t^\Delta\right]^{-2} \right); \left[k_t^\Delta\right]^{-2} \right)}{\varepsilon} \cdot V_t^{Z,\Delta}. \tag{59}$$

The observations in [1], Section 6.2, hold for the Markov chain model. By small coefficients of variation the gamma distributions converge to normal distributions, and the corresponding solvency capitals converge to those of normal distributions such that

$$SC_t^{\Delta,VaR} = SCR_t + \Phi^{-1}(1-\varepsilon)k_t^\Delta V_t^{Z,\Delta}, \quad SC_t^{\Delta,CVaR} = SCR_t + \frac{\varphi\left[\Phi^{-1}(1-\varepsilon)\right]}{\varepsilon} k_t^\Delta V_t^{Z,\Delta}. \tag{60}$$

Asymptotically, the solvency capital ratios tend to the following minimum values

$$\lim_{k_t^\Delta \to 0} SR_t^{\Delta,VaR} = \lim_{k_t^\Delta \to 0} SR_t^{\Delta,CVaR} = \frac{SCR_t}{V_t^Z}. \tag{61}$$

By vanishing coefficients of variation the VaR & CVaR solvency capital ratios converge to the Solvency II solvency capital ratio. In this situation, the process risk has been fully diversified away, and, as expected, only the parameter/systematic risks remain.

6.3. Stochastic approach: Poisson-gamma model of biometric transition

For simplicity let us fix the states i,j of the transition probabilities $p_{ij}(k), k=0,1,2,....$. In case the ratio of observed state transitions to portfolio size is not close to the given rates of transition, even for large portfolio sizes, systematic risk exists. In this situation, the transition rates are uncertain and assumed to be random. This situation is modelled similarly to [1], Section 6.3. We assume a Bayesian Poisson-Gamma model such that the number of transitions is conditional Poisson distributed with a Gamma distributed random transition probability, which results in a negative binomial distribution for the unconditional distribution of the number of transitions. Then, we consider a Poisson-Gamma model with time-dependence of the type introduced in Olivieri & Pitacco [23], which up-dates its parameters to experience. Given is a fixed time t and biometric

transition probabilities $p_{ij}(k), k = 0,1,2,\ldots$, for the given fixed states, which is based on an initial cohort of size ℓ_t at time t. Let $D_{t+\tau-1}$ denote the random number of transitions produced by the cohort in the time period $(t+\tau-1, t+\tau], \tau = 1,2,\ldots$. For the first time period $\tau = 1$, we assume that there is no experience available and that the random number of transitions is conditional Poisson distributed such that

$$D_t \sim Po(\ell_t Q_t), \quad Q_t \sim Gamma\left(\alpha, \frac{\beta}{p_{ij}(t)}\right). \tag{62}$$

It follows that the unconditional distribution of the number of transitions in the first time period is negative binomially distributed such that

$$D_t \sim NB\left(\alpha, \frac{\theta_1}{\theta_1 + 1}\right), \quad \theta_1 = \frac{\beta}{p_{ij}(t)}. \tag{63}$$

In contrast to the expected number of transitions $\ell_t p_{ij}(t)$ predicted by the biometric transition probability matrix, one has

$$E[D_t] = \frac{\alpha}{\beta} \cdot \ell_t p_{ij}(t).. \tag{64}$$

To model a systematic deviation from the expectation, one assumes that the quotient α/β is different from one, for example greater than one for transitions produced by the mortality and disability risks and less than one for those produced by the longevity risk. Suppose that at time $t+1$, the number of transitions d_t observed in the cohort over the first time period is available, and let $\ell_{t+1} = \ell_t - d_t$ be the up-dated cohort size. A calculation shows that the posterior distribution of Q_t conditional on the information $D_t = d_t$ is Gamma distributed

$$Q_t | d_t \sim Gamma\left(\alpha + d_t, \frac{\beta + \ell_t p_{ij}(t)}{p_{ij}(t)}\right), \tag{65}$$

which shows that the initial structural systematic risk parameters are up-dated as follows:

$$(\alpha, \beta) \quad \rightarrow \quad (\alpha + d_t, \beta + \ell_t p_{ij}(t)). \tag{66}$$

Passing to the second time period $(t+1, t+2]$, we assume similarly to the first period that

$$D_{t+1} | d_t \sim Po(\ell_{t+1} Q_{t+1}), \quad Q_{t+1} | d_t \sim Gamma\left(\alpha + d_t, \frac{\beta + \ell_t p_{ij}(t)}{p_{ij}(t+1)}\right). \tag{67}$$

This implies a so-called predictive distribution of negative binomial type

$$D_{t+1}|d_t \sim NB\left(\alpha + d_t, \frac{\theta_2}{\theta_2 + 1}\right), \quad \theta_2 = \frac{\beta + \ell_t p_{ij}(t)}{\ell_{t+1} p_{ij}(t)}. \tag{68}$$

Iterating the above Bayesian scheme, one generalizes as follows. At time $t+\tau-1, \tau \geq 2$, having observed the annual number of transitions $d_t, d_{t+1}, ..., d_{t+\tau-2}$, the up-dated cohort size for the next time period $(t+\tau-1, t+\tau]$ is obtained from the recursion $\ell_{t+k-1} = \ell_{t+k-2} - d_{t+k-2}$, $k = 2, 3, ..., \tau$. The corresponding predictive negative binomial distribution of the number of transitions is then given by

$$D_{t+\tau-1}|d_t, d_{t+1}, ..., d_{t+\tau-2} \sim NB\left(\alpha + \sum_{k=0}^{\tau-2} d_{t+k}, \frac{\theta_\tau}{\theta_\tau + 1}\right), \quad \theta_\tau = \frac{\beta + \sum_{k=1}^{\tau-1} \ell_{t+k-1} p_{ij}(t+k-1)}{\ell_{t+\tau-1} p_{ij}(t+\tau-1)}. \tag{69}$$

The conditional expected number of transitions compares with the given expected number as follows:

$$E\left[D_{t+\tau-1}|d_t, d_{t+1}, ..., d_{t+\tau-2}\right] = \frac{\alpha + \sum_{k=1}^{\tau-1} d_{t+k-1}}{\beta + \sum_{k=1}^{\tau-1} \ell_{t+k-1} p_{ij}(t+k-1)} \cdot \ell_{t+\tau-1} p_{ij}(t+\tau-1) \tag{70}$$

$$\underset{\geq}{\overset{\leq}{}} E\left[D_{t+\tau-1}\right] = \ell_t \cdot {}_{\tau-1}p_{ij}(t) p_{ij}(t+\tau-1).$$

If biometric experience is consistent with what is expected, the quotient of both expected values remains constant over time. On the other hand, if experience is better (worse) than expected, the same quotient will increase (decrease) over time.

In practice one proceeds as follows. Given a fixed time t, consider for each pair of fixed states the Poisson-Gamma transition probabilities obtained from (64) and (70) defined by

$$p_{ij}^{PG}(t) = \frac{\alpha}{\beta} \cdot p_{ij}(t), \quad p_{ij}^{PG}(t+\tau) = \frac{\alpha + \sum_{k=1}^{\tau} d_{t+k-1}}{\beta + \sum_{k=1}^{\tau} \ell_{t+k-1} p_{ij}(t+k-1)} \cdot p_{ij}(t+\tau), \quad \tau = 1, 2, ..., \tag{71}$$

$${}_k p_{ij}^{PG}(t+\tau) = {}_{k-1} p_{ij}^{PG}(t+\tau) \cdot \left(1 - p_{ij}^{PG}(t+\tau+k-)\right), \quad {}_0 p_{ij}^{PG}(0) = 1, \quad \tau = 0, 1, ...$$

Replacing everywhere in the formulas (55)-(59) the superscripts Δ_{ij} by the superscript PG for the relevant transition probabilities and using (71) in calculations, we obtain portfolio VaR & CVaR solvency capital formulas under the Poisson-Gamma model of biometric transition similar to (58) and (59). Similar limiting results apply. An implementation requires a detailed specification. To be consistent with the Solvency II standard approach, one can assume that future transitions deviate systematically from the biometric life table according to the prescribed shock $\Delta = \Delta_{ij}$ for the given fixed states, such that

$$\ell_{s+t+k} = \ell_{s+t+k-1} - d_{s+t+k-1}, \quad d_{s+t+k-1} = (1-\Delta) \cdot \ell_{s+t+k-1} q_{s+t+k-1}, \quad k = 1,2,\dots \quad (72)$$

This choice is consistent with the expected number of transitions in the first period $E[D_{s+t}] = d_{s+t}$ if in (64) one sets $\alpha = (1-\Delta)\beta$. Assume further that $\beta = 100$, which implies a coefficient of variation for D_{s+t} equal to 10%. One shows that the choice (72) with $\alpha = (1-\Delta)\beta$ implies that the transition probabilities (71) coincide with the corresponding shifted entries in the biometric life table. In this special case, we observe that the stochastic model of Section 6.3 provides the same results as the shift method of Section 6.2. In general, the stochastic model of Section 6.3 is more satisfactory and flexible because it allows the use of effective observed numbers of transitions as time elapses.

7. The endowment contract with waiver of premium by disability

For a clear and simple Markov chain illustration we restrict the attention to a single cohort of identical n-year endowment contracts with waiver of premium in the event of disability and fixed one-unit of sum insured payable upon death or survival at maturity date. The treatment of other similarly complex disability contracts is left to future research. For some further possibilities consult Example 2.1 in Christiansen et al. [24].

7.1. Markov model for mortality and disability risks

A complete risk model for single-life insurance products with mortality and disability risks requires the specification of a Markov model with three states. A policyholder aged x at contract issue changes state at time $t > 0$ according to the following diagram
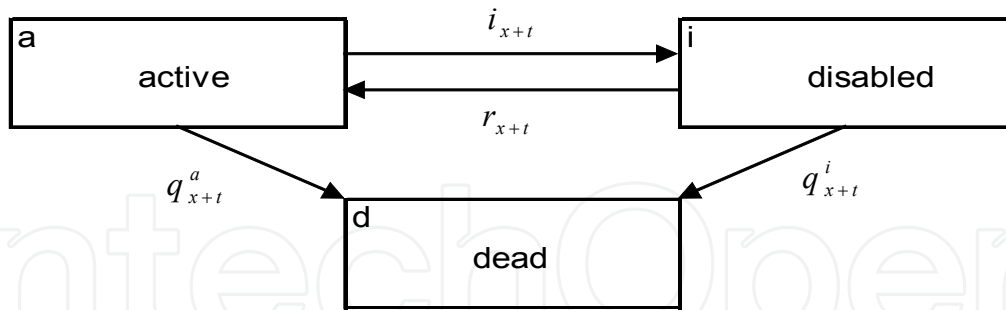
Figure 1. Markov chain states and their jump probabilities

The possible state changes occur with the following probabilities

i_{x+t} : one-year probability of disability at time t

q^a_{x+t} : one-year probability of active mortality at time t

q^i_{x+t} : one-year probability of disabled mortality at time t

r_{x+t} : one-year probability of recovery at time t

The states are $S = \{1,2,3\} = \{a,i,d\}$. The one-step transition probabilities are given by

$$p_{ij}(k) = \begin{pmatrix} 1 - i_{x+k} - q^a_{x+k} & i_{x+k} & q^a_{x+k} \\ r_{x+k} & 1 - r_{x+k} - q^i_{x+k} & q^i_{x+k} \\ 0 & 0 & 1 \end{pmatrix}, \ i,j \in \{1,2,3\}, k = 0,1,2,... \quad (73)$$

For a n-year endowment contract with waiver of premium by disability without recovery from disability, one has $r_{x+k} = 0, k = 1,2,...,n-1$. This simplifying assumption is sometimes made in practice and justified in economic environments with a small number of disabled persons, for which the probability of recovery can be neglected. For example, the Swiss Federal Insurance Pension applies such a model and uses a biometric life table called "EVK Table", where EVK is the abbreviation for "**E**idgenössische **V**ersicherungs**K**asse", e.g. Koller [6], p.129, or Chuard [25] for a detailed historical background.

7.2. State dependent actuarial reserves and net level premiums

The net level premium of the n-year endowment with waiver of premium and one unit of sum insured for a life in the active state at age x is denoted by $P^a = P^a(x:n)$, where the upper index indicates that the premium is only due if the contract remains in the active life state. In our notations the *payment functions* of this contract are defined by

Type 1:

$$a_1(k) = \begin{cases} -P^a, k = 0,1,...,n-1 \\ 1, k = n \end{cases}, a_2(k) = \begin{cases} 0, k = 1,2,...,n-1 \\ 1, k = n \end{cases}, \ a_3(k) = 0$$

Type 2:

$$a_{12}(k) = \begin{cases} 0, k = 1,2,...,n-1 \\ 1, k = n \end{cases}, \ a_{13}(k) = 1, k = 1,2,...,n, \ a_{21}(k) = 0.$$

To describe the state and time dependent actuarial reserves $V^i_k, i \in \{1,2,3\}, k = 0,1,...,n$, one needs the survival probabilities of staying in the active or disabled state. Denote by ω the maximum attainable age. Then, the *active survival probabilities* (probability a life in the active state at age x will attain age $x+k$ in the active state without disablement) are given by

$$_kp^a_x = {}_{k-1}p^a_x \cdot \left(1 - q^a_{x+k-1} - i_{x+k-1}\right), \quad _0p^a_x = 1, \quad k = 1,...,\omega-x. \quad (74)$$

Similarly, the *disabled survival probabilities* (probability a life in the disabled state at age x will attain age $x+k$ in the disabled state without recovery) are given by

$$_kp^i_x = {}_{k-1}p^i_x \cdot \left(1 - q^i_{x+k-1}\right), \quad _0p^i_x = 1, \quad k = 1,...,\omega-x \quad (75)$$

Corresponding to these survival probabilities one associates n-year life annuities for a life aged x being in the active or disabled state whose actuarial present values are defined by

$$a^a(x:n) = \sum_{k=0}^{n-1} v^k \cdot {}_k p_x^a, \text{ for the } n-\text{year active life annuity} \tag{76}$$

$$a^i(x:n) = \sum_{k=0}^{n-1} v^k \cdot {}_k p_x^i, \text{ for the } n-\text{year disabled life annuity} \tag{77}$$

The actuarial present value (APV) of future benefits for the n-year endowment with waiver of premium and one unit of sum insured for a life in the active (respectively disabled) state at age x is denoted by $A^a(x:n)$ (respectively $A^i(x:n)$). Using the backward recursive formulas for the state dependent actuarial reserves let us determine formulas for the evaluation of the introduced APVs. In particular, an explicit formula for the net level premium is derived. The backward recursive reserve formulas are given by

$$V_k^1 = v \cdot \left(q_{x+k}^a + p_{x+k}^a V_{k+1}^1 + i_{x+k} V_{k+1}^2 \right) - P^a, \quad k = 0, 1, \ldots, n-1, \quad V_n^1 = 1,$$
$$V_k^2 = v \cdot \left(q_{x+k}^i + p_{x+k}^i V_{k+1}^2 \right), \quad k = 2, 3, \ldots, n-1, \quad V_n^2 = 1. \tag{78}$$

One has $V_0^2 = 0$ because the life is in the state "a" at contract issue, $V_1^2 = 0$ because the life can only be in the state "i" after at least one year and then no actuarial reserve is available, and $V_k^3 = 0, \quad k = 0, 1, \ldots, n$, because no actuarial reserve is required in case the insured life has died. Since actuarial reserves represent differences between APVs of future benefits and future premiums one has further the relationships

$$V_k^1 = A^a(x+k:n-k) - P^a(x:n) \cdot a^a(x+k:n-k), \quad k = 0, 1, \ldots, n-1,$$
$$V_k^2 = A^i(x+k:n-k), \quad k = 2, 3, \ldots, n-1. \tag{79}$$

On the other hand the APVs of the active life annuities in (79) satisfy the recursions

$$a^a(x+k:n-k) = 1 + v \cdot p_{x+k}^a \cdot a^a(x+k+1:n-k-1), \quad k = 0, 1, \ldots, n-1. \tag{80}$$

Inserting (79) and (80) into (78) one obtains the backward recursions for APVs

$$A^a(x+n-k:k) = v \cdot \left(\begin{array}{l} q_{x+n-k}^a + p_{x+n-k}^a A^a(x+n-k+1:k-1) \\ + i_{x+n-k} A^i(x+n-k+1:k-1) \end{array} \right), \quad k = 2, 3, \ldots, n,$$
$$A^i(x+n-k:k) = v \cdot \left(q_{x+n-k}^i + p_{x+n-k}^i A^i(x+n-k+1:k-1) \right), \quad k = 2, 3, \ldots, n-1, \tag{81}$$

with the starting values $A^a(x+n-1:1) = A^i(x+n-1:1) = v$. One sees that the second relationship in (81) is satisfied by the formula

$$A^i(x+n-k:k) = 1 - d \cdot a^i(x+n-k:k), \quad k = 1, 2, \ldots, n-1, \tag{82}$$

which reminds one of the usual formula for an endowment insurance with disability as single cause of decrement, e.g. Gerber [10], formula (2.15), p.37. Inserting (77) and rearranging one obtains the corresponding explicit sum representation

$$A^i(x+n-k:k) = \sum_{j=1}^{k-1} v^j \cdot {}_{j-1}p^i_{x+n-k} \, q^i_{x+n-k+j-1} + v^k \cdot {}_{k-1}p^i_{x+n-k}, \quad k=2,3,\ldots,n-1. \tag{83}$$

Using these results and proceeding through backward induction, one obtains the following explicit formula for the evaluation of the APV of future benefits for a life in the active state

$$A^a(x:n) = \sum_{k=1}^{n-1} v^k \cdot {}_{k-1}p^a_x \cdot q^a_{x+k-1} + v^n \cdot {}_{n-2}p^a_x \cdot (1-q^a_{x+n-2})$$

$$+ \sum_{k=2}^{n-1} v^k \cdot \left(\sum_{j=0}^{k-2} {}_jp^a_x \cdot i_{x+j} \cdot {}_{k-2-j}p^i_{x+j+1} \right) \cdot q^i_{x+k-1} + v^n \cdot \sum_{j=0}^{n-3} {}_jp^a_x \cdot i_{x+j} \cdot {}_{n-2-j}p^i_{x+j+1} \tag{84}$$

The net level premium is determined by the actuarial equivalence principle, which states that at contract issue $V_0^1 = 0$. Using (79) one obtains the explicit formula (use (76), (84))

$$P^a(x:n) = A^a(x:n) / a^a(x:n). \tag{85}$$

7.3. Conditional mean and variance of the prospective insurance loss

We determine the conditional mean and variance given survival of the time-t prospective insurance loss $Z_t = \sum_{k=0}^{n-1-t} v^k C_{t+k}$, $t=0,1,\ldots,n-1$. Besides the state dependent actuarial reserves (79), we consider the net premium reserve (35), which coincides with the conditional mean

$$V_t^Z := E[Z_t|E_t] = V_t^1 \cdot P(X_t = a|E_t) + V_t^2 \cdot P(X_t = i|E_t)$$

$$= V_t^1 \cdot \frac{P(X_t = a)}{P(E_t)} + V_t^2 \cdot \frac{P(X_t = i)}{P(E_t)} = \frac{V_t^1 \cdot {}_tp^a_x + V_t^2 \cdot i_x \cdot {}_{t-1}p^i_{x+1}}{{}_tp^a_x + i_x \cdot {}_{t-1}p^i_{x+1}}, \quad t=0,1,\ldots,n-1, \tag{86}$$

$$P(X_t = a) = {}_tp^a_x, \quad t=0,1,2,\ldots,n-1,$$

$$P(X_t = i) = i_x \cdot {}_{t-1}p^i_{x+1}, \quad t=1,2,\ldots,n-1, \quad P(X_0 = i) = 0, \tag{87}$$

$$P(E_t) = {}_tp^a_x + i_x \cdot {}_{t-1}p^i_{x+1}, \quad t=1,2,\ldots,n-1, \quad P(E_0) = 0.$$

Furthermore, a calculation based on Theorem 5.1 yields the following conditional variances

$$(\sigma_t^Z)^2 := Var[Z_t|E_t] = \sum_{k=0}^{n-1-k} v^{2k} \cdot \left(\frac{E[C_{t+k}^2] \cdot P(E_k) - \pi^S(t+k)^2}{P(E_k)^2} \right), \quad t=0,1,\ldots,n-1,$$

$$E[C_0^2] = P^a(x:n)^2 + v(v + P^a(x:n))q^a_x, \tag{88}$$

$$E[C_k^2] = \left[P^a(x:n)^2 + v(v + P^a(x:n))q^a_{x+k} \right] {}_kp^a_x + v^2 i_x \cdot {}_{k-1}p^i_{x+1}q^i_{x+k}, \quad k=1,\ldots,n-2,$$

$$E[C_{n-1}^2] = \left[P^a(x:n)^2 + v(v + P^a(x:n))(1-p^a_{x+n-1}) \right] {}_{n-1}p^a_x + v^2 i_x \cdot {}_{n-2}p^i_{x+1}q^i_{x+n-1},$$

where the savings premiums are determined by the formulas

$$
\pi^S(0) = vV_1^1,
$$
$$
\pi^S(k) = v\left(V_{k+1}^1 + A^i(x+k+1:n-k-1)\right) - \left(V_k^1 + A^i(x+k:n-k)\right), \quad k = 1, ..., n-1, \tag{89}
$$

and the probabilities $P(E_k)$ are defined in (87). Neglecting the probabilistic terms of second order $i_x \cdot {}_{k-1}p_{x+1}^i q_{x+k}^i$, one obtains the following simpler approximations to (88):

$$
E\left[C_k^2\right] = \left\{P^a(x:n)^2 + v\left(v + P^a(x:n)\right)q_{x+k}^a\right\} \cdot {}_kp_x^a, \quad k = 0, ..., n-2,
$$
$$
E\left[C_{n-1}^2\right] = \left\{P^a(x:n)^2 + v\left(v + P^a(x:n)\right)(1 - p_{x+n-1}^a)\right\} \cdot {}_{n-1}p_x^a. \tag{90}
$$

7.4. Numerical illustration

The Markov chain parameterization of the present contract type has been given at the beginning of Section 7.2. We assume that all the policyholders are aged x at time $t = 0$. Our construction of the biometric life table with mortality and disability risk factors is based on the classical textbook Saxer [26], Section 2.5. Besides the one-year probabilities introduced in Section 7.1, one considers further the partial or independent rates of decrement, see Saxer [26], Section 2.4, or Bowers et al. [19], Section 9.5, denoted by

${}^*q_{x+t}^a$: one-year independent rate of active mortality at time t

${}^*i_{x+t}$: one-year independent rate of disability at time t

The independent rates of decrement are linked to the probabilities of active mortality and disability through the relationship, e.g. Saxer [26], formulas (2.5.1) and (2.5.2),

$$
1 - q_{x+t}^a - i_{x+t} = \left(1 - {}^*q_{x+t}^a\right) \cdot \left(1 - {}^*i_{x+t}\right). \cdot \tag{91}
$$

For the purpose of illustration only and by lack of another reference, we base our calculations on Table 1, which is obtained by combining the Tables 4 and 5 in Saxer [26], p.240. The entries ${}^*q_{x+t}^a$, ${}^*i_{x+t}$ and q_{x+t}^i are taken from the "EVK Table 1950" and the entry i_{x+t} is taken from the "VZ Table 1950", where VZ stands for "**V**ersicherungskasse **Z**ürich". The missing entries between the 5-year ages are linearly interpolated.

While the standard solvency capital ratio does not depend on the initial cohort size, this is the case for the stochastic approaches. The age at contract issue is either $x = 30$ or $x = 40$ and the contract duration is $n = 20$. We compare the stochastic approach with the standard approach for the contract times $t \in \{0, 1, 2, 3, 4, 5, 10, 15, 18\}$. We use the shifted biometric life table with Solvency II standard like specifications, namely at each age 20% decrease for the probability to die as active (longevity risk) respectively 15% increase for the probability to die as disabled (mortality risk), 35% increase for the first year probability to disable and then 25% increase at each future age (disability risk). The interest rate and the risk-free interest

rate is 3%. Table 2 displays shifted coefficients of variation under varying cohort sizes. The values are sufficiently small so that the normal approximation to the gamma distribution can be applied. Table 3, which is based on (60), displays the cohort size dependent solvency capital ratios and their limiting values (61) for a portfolio of infinitely growing size. The chosen confidence level is 99.5% for VaR and 99% for CVaR (the accepted level, which corresponds to a 99.5% Solvency II calibration).

x	*qax	qax	*ix	ix	qix
30	0.183%	0.106%	0.064%	0.141%	4.348%
31	0.181%	0.097%	0.064%	0.148%	4.324%
32	0.178%	0.087%	0.064%	0.154%	4.300%
33	0.176%	0.078%	0.063%	0.161%	4.276%
34	0.173%	0.069%	0.063%	0.167%	4.252%
35	0.171%	0.060%	0.063%	0.174%	4.228%
36	0.174%	0.059%	0.072%	0.186%	4.204%
37	0.177%	0.059%	0.080%	0.198%	4.180%
38	0.180%	0.059%	0.089%	0.210%	4.156%
39	0.183%	0.058%	0.097%	0.222%	4.132%
40	0.186%	0.058%	0.106%	0.234%	4.108%
41	0.207%	0.079%	0.129%	0.257%	4.084%
42	0.227%	0.099%	0.153%	0.280%	4.061%
43	0.248%	0.120%	0.176%	0.304%	4.037%
44	0.268%	0.141%	0.200%	0.327%	4.014%
45	0.289%	0.161%	0.223%	0.350%	3.990%
46	0.335%	0.218%	0.285%	0.401%	3.966%
47	0.382%	0.275%	0.347%	0.452%	3.942%
48	0.428%	0.331%	0.408%	0.504%	3.919%
49	0.475%	0.388%	0.470%	0.555%	3.895%
50	0.521%	0.444%	0.532%	0.606%	3.871%
51	0.586%	0.532%	0.694%	0.743%	3.848%
52	0.650%	0.620%	0.856%	0.881%	3.824%
53	0.715%	0.708%	1.019%	1.018%	3.801%
54	0.779%	0.795%	1.181%	1.156%	3.777%
55	0.844%	0.883%	1.343%	1.293%	3.754%
56	0.909%	0.892%	1.765%	1.766%	3.731%
57	0.974%	0.900%	2.187%	2.240%	3.707%
58	1.040%	0.908%	2.609%	2.713%	3.684%
59	1.105%	0.916%	3.031%	3.187%	3.660%
60	1.170%	0.923%	3.453%	3.660%	3.637%

Table 1. One-step transition probabilities for the mortality and disability Markov chain

In the present case study, we observe that for all cohort sizes and contract times, the current standard approach prescribes almost negligible solvency capital ratios. For small cohort sizes and early contract times, the discrepancies between the stochastic and standard approach increase with age and contract duration attaining solvency capital ratios above 200% for small cohort sizes with 100 insured lives. In fact, as already explained, the current QIS5 specification neglects the process risk. Moreover, we note

that the chosen results for the normal distribution are only approximate, especially for small cohort sizes. In this respect, we think that the displayed figures are most likely lower bounds due to the fact that often a normal approximation rather underestimates than overestimates risk. A more detailed analysis of this point is left as open issue for further investigation (however, the use of the gamma approximation makes no big difference). On the other hand, solvency capital ratios of cohort sizes exceeding 10'000 policyholders and late contract times tend more and more to the lower limiting bound as expected from the central limit theorem. Fig. 2 visualizes these findings. In virtue of the made confidence level calibration, the VaR & CVaR solvency capital ratios are of the same order of magnitude. Finally, the considered example points out to another difficulty. Though almost negligible in absolute value, we note that the standard solvency capital ratios change their signs repeatedly over the time axis. In this respect, one can ask whether fixed transition shifts are the "crucial scenarios". As a response to this "biometric worst- and best-case scenarios" are proposed in Christiansen [27], [28].

Cohort size	Contract Time								
	0	1	2	3	4	5	10	15	18
(x,n)=(30,20)									
100	53.815%	26.466%	17.602%	13.150%	10.483%	8.714%	4.628%	2.525%	1.464%
500	24.067%	11.836%	7.872%	5.881%	4.688%	3.897%	2.070%	1.129%	0.655%
1'000	17.018%	8.369%	5.566%	4.158%	3.315%	2.756%	1.463%	0.798%	0.463%
10'000	5.381%	2.647%	1.760%	1.315%	1.048%	0.871%	0.463%	0.252%	0.146%
100'000	1.702%	0.837%	0.557%	0.416%	0.332%	0.276%	0.146%	0.080%	0.046%
(x,n)=(40,20)									
100	91.895%	45.158%	30.663%	23.298%	18.838%	15.846%	8.844%	5.223%	3.176%
500	41.097%	20.195%	13.713%	10.419%	8.425%	7.087%	3.955%	2.336%	1.421%
1'000	29.060%	14.280%	9.696%	7.368%	5.957%	5.011%	2.797%	1.652%	1.004%
10'000	9.190%	4.516%	3.066%	2.330%	1.884%	1.585%	0.884%	0.522%	0.318%
100'000	2.906%	1.428%	0.970%	0.737%	0.596%	0.501%	0.280%	0.165%	0.100%

Table 2. Coefficients of variation of the shifted prospective insurance loss

Cohort size	Contract Time								
	0	1	2	3	4	5	10	15	18
(x,n)=(30,20)									
100	138.7%	68.9%	45.7%	34.1%	27.1%	22.5%	11.8%	6.4%	3.8%
500	62.0%	31.1%	20.6%	15.3%	12.2%	10.1%	5.2%	2.8%	1.7%
1'000	43.9%	22.1%	14.6%	10.9%	8.6%	7.1%	3.7%	2.0%	1.2%
10'000	13.9%	7.3%	4.8%	3.5%	2.8%	2.3%	1.1%	0.6%	0.4%
100'000	4.4%	2.6%	1.7%	1.2%	0.9%	0.7%	0.3%	0.1%	0.1%
Limiting QIS5 ratio SCRt/VZt	0.0%	0.5%	0.3%	0.1%	0.1%	0.0%	-0.1%	-0.1%	0.0%
QIS5 TC ratio = (SCRt + RMt)/VZt	-0.3%	0.3%	0.1%	0.0%	0.0%	-0.1%	-0.1%	-0.1%	0.0%
(x,n)=(40,20)									
100	235.6%	117.2%	79.3%	60.1%	48.5%	40.7%	22.7%	13.4%	8.1%
500	105.2%	52.6%	35.5%	26.9%	21.7%	18.2%	10.1%	6.0%	3.6%
1'000	74.3%	37.4%	25.2%	19.0%	15.3%	12.8%	7.1%	4.2%	2.5%
10'000	23.3%	12.1%	8.1%	6.0%	4.8%	4.0%	2.2%	1.3%	0.8%
100'000	7.1%	4.1%	2.7%	1.9%	1.5%	1.2%	0.7%	0.4%	0.2%
Limiting QIS5 ratio SCRt/VZt	-0.3%	0.4%	0.2%	0.0%	0.0%	-0.1%	-0.1%	0.0%	0.0%
QIS5 TC ratio = (SCRt + RMt)/VZt	-0.7%	0.2%	0.0%	-0.1%	-0.1%	-0.1%	-0.1%	0.0%	0.0%

Table 3. VaR solvency capital ratios for the endowment with waiver of premium
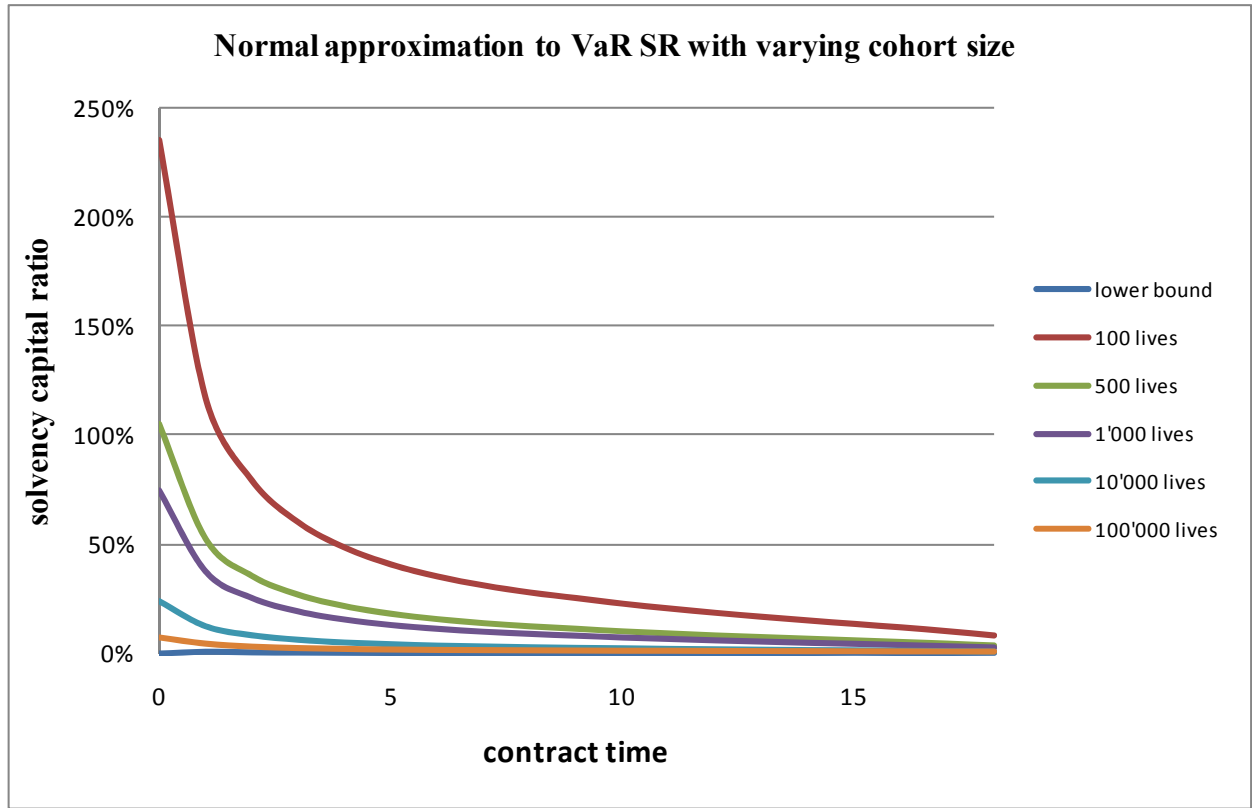
Figure 2. Time evolution of VaR solvency capital ratios, $(x, n) = (40, 20)$

8. Conclusions and outlook

Let us summarize the present work. We have derived a general solvency rule for the prospective liability, which has resulted in two simple liability VaR & CVaR target capital requirements. The proposed approach has been applied to determine the biometric solvency risk capital for a portfolio of general traditional life contracts within the Markov chain model of life insurance. Our main actuarial tools have been the backward recursive actuarial reserve formula and the theorem of Hattendorff. Based on this we have determined the conditional mean and variance of a portfolio's prospective liability risk and have used a gamma approximation to obtain the liability VaR & CVaR solvency capital. Since our first formulas include only the process risk and do not take into account the possibility of systematic risk, we have proposed either to shift the biometric transition probabilities, or apply a stochastic model, which allows for random biometric transition probabilities.

Similarly to [1], Section 8, the adopted general methodology is in agreement with several known facts as (i) the process risk is negligible for portfolios with increasing size and has a small impact on medium to large insurers (ii) all else equal, process risk will increase (decrease) with higher (lower) coefficients of variation (aggregated effect of both decrement rates and sums at risk). Another interesting observation has been made at the end of Section 6.3 that the model with shifted biometric transitions can be

viewed as a sub-model of the model with Poisson-Gamma time dependent biometric transitions.

Moreover, a detailed analysis for a single cohort of identical endowment contracts with waiver of premium by disability has been undertaken in Section 7. Besides a complete Markov chain specification, which seems to be missing in the literature, the numerical illustration has shown, as expected, that the cohort size is a main driving factor of process risk. Due to the statistical law of large numbers, the larger the cohort size the less solvency capital is actually required. In contrast to the life annuity "longevity risk" study in [1], the stochastic approach penalizes almost all insurers (except the very large ones) because the current standard approach prescribes almost negligible solvency capital ratios and does not measure explicitly the process risk effects.

The interested actuary might challenge the proposed approach with alternatives from other regulatory environments than Solvency II. Moreover, it is important to point out that a lot of technical issues remain to be settled properly. They are not only regulatory specific but also related to the complex mathematics of related software products and go beyond the Markov chain model. Today's life insurance contracts include many embedded options and are henceforth even more complex. A challenging issue is the definition of capital requirements for unit-linked contracts without and with guarantee and variable annuities with guaranteed minimum benefits (so-called variable GMXB annuities).

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