

We are IntechOpen, the world's leading publisher of Open Access books Built by scientists, for scientists

6,900

Open access books available

186,000

International authors and editors

200M

Downloads

Our authors are among the

154

Countries delivered to

TOP 1%

most cited scientists

12.2%

Contributors from top 500 universities



WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index
in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com



Boundary-Value Problems for Second Order PDEs Arising in Risk Management and Cellular Neural Networks Approach

Rossella Agliardi, Petar Popivanov and Angela Slavova

Additional information is available at the end of the chapter

<http://dx.doi.org/10.5772/49936>

1. Introduction

This work deals with the Dirichlet problem for some PDEs of second order with non-negative characteristic form. One main motivation is to study some boundary-value problems for PDEs of Black-Scholes type arising in the pricing problem for financial options of barrier type. Barrier options on stocks have been traded since the end of the Sixties and the market for these options has been dramatically expanding, making barrier options the most popular ones among the exotic. The class of standard barrier options includes 'in' barriers and 'out' barriers, which are activated (knocked in) and, respectively, extinguished (knocked out) if the underlying asset price crosses the barrier before expiration. Moreover, each class includes 'down' or 'up' options, depending on whether the barrier is below or above the current asset price and thus can be breached from above or below. Therefore there are eight types of standard barrier options, depending on their 'in' or 'out', 'down' or 'up', and 'call' or 'put' attributes. It is possible to include a cash rebate, which is paid out at option expiration if an 'in' ('out') option has not been knocked in (has been knocked out, respectively) during its lifetime. One can consider barrier options with rebates of several types, terminal payoffs of different forms (e.g. power options), more than one underlying assets and/or barriers, and allow for time-dependent barriers, thus enriching this class still further. On the other hand, a large variety of new exotic barriers have been designed to accommodate investors' preferences. Another motivation for the study of such options is related to credit risk theory. Several credit-risk models build on the barrier option formalism, since the default event can be modeled throughout a signalling variable hitting a pre-specified boundary value (See [3],[8] among others). As a consequence, a substantial body of academic literature provides pricing methods for valuating barrier options, starting from the seminal work of [18], where an exact formula is offered for a down-and-out European call with zero rebate. Further extensions

are provided - among others - in [22] for the different types of standard barrier options, in [16] for simultaneous 'down' and 'up' barriers with exponential dependence on time, in [10] for two boundaries via Laplace transform, in [12] and [7] for partial barrier and rainbow options, in [17] for multi-asset options with an outside barrier, in [5] in a most comprehensive setting employing the image solution method. Many analytical formulas for barrier options are collected also in handbooks (see [11], for example).

For analytical tractability most literature assumes that the barrier hitting is monitored in continuous time. However there exist some works dealing with the discrete version, i.e. barrier crossing is allowed only at some specific dates -typically at daily closings. (See [1] and [15], for a survey). Furthermore, a recent literature relaxes the Brownian motion assumption and considers a more general Lévy framework. For example, [4] study barrier options of European type assuming that the returns of the underlying asset follows a Lévy process from a wide class. They employ the Wiener-Hopf factorization method and elements of pseudodifferential calculus to solve the related boundary problem. This book chapter adopts a classical Black-Scholes framework. The problem of pricing barrier options is reducible to boundary value problems for a PDE of Black-Scholes type and with pre-specified boundaries. The value at the terminal time T is assigned, specifying the terminal payoff which is paid provided that an 'in' option is knocked in or an 'out' option is not knocked out during its lifetime. The option holder may be entitled or not to a rebate. From a mathematical point of view, the boundary condition can be inhomogeneous or homogeneous. While there are several types of barrier options, in this work we will focus on 'up' barriers in view of the relationships between the prices of different types of vanilla options (see [25]). Moreover, the case of floating barriers of exponential form can be easily accommodated by substitution of the relevant parameters (see [25], Chapter 11), thus we confine ourselves to the case of constant barriers. On the other hand, we work within a general framework that allows for multi-asset options, a generic payoff and rebate. Furthermore, we tackle some regularity questions and the problem of existence of generalized solutions. In Section 2 the (initial) boundary value problem is studied in a multidimensional framework generalizing the Black-Scholes equation and analytical solutions are obtained, while a comparison principle is provided in Section 4. Section 3 presents some applications in Finance: our general setting incorporates several known pricing expressions and, at the same time, allows to generate new valuation formulas. Section 5 and the Appendix study the existence and regularity of generalized solutions to the boundary value problems for a class of PDEs incorporating the Black-Scholes type. We build on the approach of Oleinik and Radkevič and adapt the method to the PDEs of interest in the financial applications.

2. Generalizations of the Black-Scholes equation in the multidimensional case: (initial) boundary value problems

Consider in $\mathbf{R}_t^1 \times \mathbf{R}_x^n$ the following generalization of the Black-Scholes equation:

$$Lu = u_t + \sum_{i,j=1}^n a_{ij}x_i x_j u_{x_i x_j} + \sum_{i=1}^n b_i x_i u_{x_i} + cu = f(t, x), \quad (1)$$

where $0 \leq t \leq T$ and $x_j \geq 0, 1 \leq j \leq n$.

This is the Cauchy problem:

$$\begin{cases} Lu = f(t, x), \\ u|_{t=T} = u_0(x) \end{cases} \quad (2)$$

$$\begin{cases} x_j \geq 0, & 1 \leq j \leq n \\ 0 \leq t \leq T \end{cases}$$

and this is the boundary value problem:

$$\begin{cases} Lu = f \\ u|_{t=T} = u_0(x) \\ u|_{x_j=a_j} = g_j(t, x)|_{x_j=a_j}, 1 \leq j \leq n, \end{cases} \quad (3)$$

$$\begin{cases} 0 \leq t \leq T \\ 0 \leq x_j \leq a_j, a_j > 0 \\ 1 \leq j \leq n \end{cases}$$

In (1) $a_{ij} = a_{ji} = \text{const}, b_i = \text{const}, c = \text{const}$ and

$$\sum_{i,j=1}^n a_{ij} \xi_i \xi_j \geq c_0 |\xi|^2, c_0 = \text{const} > 0. \quad (4)$$

Our first step is to make in the non-hypoelliptic PDE L the change of the space variables:

$$y_j = \ln x_j, 1 \leq j \leq n, \tau = T - t \Rightarrow \frac{\partial u}{\partial t} = -\frac{\partial u}{\partial \tau}, y_j \in \mathbf{R}^1 \quad (5)$$

$$\frac{\partial u}{\partial x_i} = e^{-y_i} \frac{\partial u}{\partial y_i}, \frac{\partial^2 u}{\partial x_i \partial x_j} = e^{-y_i - y_j} \left[\frac{\partial^2 u}{\partial y_i \partial y_j} - \delta_{ij} \frac{\partial u}{\partial y_i} \right], \delta_{ij} \text{ being the Kronecker symbol.}$$

Thus, (1) takes the form:

$$\tilde{L}u = -\frac{\partial u}{\partial \tau} + \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial y_i \partial y_j} + \sum_{i=1}^n \frac{\partial u}{\partial y_i} (b_i - a_{ii}) + cu = f, \quad (6)$$

i.e.

$$\frac{\partial u}{\partial \tau} = \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial y_i \partial y_j} + \sum_{i=1}^n \tilde{b}_i \frac{\partial u}{\partial y_i} + cu - f; \tilde{b}_i = b_i - a_{ii}$$

In the case (2) we have

$$\begin{cases} \tilde{L}u = f, & 0 \leq \tau \leq T \\ u|_{\tau=0} = \tilde{u}_0(y) = u_0(e^{y_1}, \dots, e^{y_n}), y \in \mathbf{R}^n, \end{cases} \quad (7)$$

while in the case (3)

$$\begin{cases} \tilde{L}u = f, & 0 \leq \tau \leq T \\ u|_{\tau=0} = \tilde{u}_0(y) \\ u|_{y_j=\tilde{a}_j} = g_j|_{y_j=\tilde{a}_j} \end{cases} \quad (8)$$

Denote $D = \{0 \leq \tau \leq T, -\infty < y_j \leq \ln a_j = \tilde{a}_j, 1 \leq j \leq n\}$, $x_j = e^{y_j}, 1 \leq j \leq n \Rightarrow f(t, x) = f(T - \tau, e^{y_1}, \dots, e^{y_n})$.

In (6) we make the change of the unknown function $u : u = v(\tau, y)e^{\sum \alpha_i y_i + \beta \tau}$ in $(\tau, y) \in D$. Thus, after standard computations we get:

$$\begin{aligned} v_\tau + \beta v &= \sum_{i,j} a_{ij} v_{y_i y_j} + \sum_{i,j} a_{ij} (\alpha_i v_{y_j} + \alpha_j v_{y_i}) + \\ &+ \sum_{i=1}^n \tilde{b}_i v_{y_i} + \sum_{i,j=1}^n a_{ij} \alpha_i \alpha_j v + \sum_{i=1}^n \tilde{b}_i \alpha_i v + cv - f e^{-\sum \alpha_i y_i - \beta \tau}. \end{aligned} \quad (9)$$

Let us take

$$\beta = \sum_{i,j} a_{ij} \alpha_i \alpha_j + \sum_i \tilde{b}_i \alpha_i + c \quad (10)$$

and put $f_1 = -f e^{-\sum \alpha_i y_i - \beta \tau}$. Put $A = (a_{ij})_{i,j=1}^n$, $A^* = A$, $\alpha = (\alpha_1, \dots, \alpha_n)$. Then the scalar product $(A\alpha, \nabla_y v) = \sum_{i,j} a_{ij} \alpha_j \frac{\partial v}{\partial y_i} = \sum_{i,j} a_{ij} \alpha_j \alpha_i \frac{\partial v}{\partial y_j} = \sum_{i,j} a_{ij} \alpha_i \frac{\partial v}{\partial y_j}$, i.e. we assume that

$$2(A\alpha, \nabla_y v) + (\tilde{b}, \nabla_y v) = 0 \iff \quad (11)$$

$$2A\alpha + \tilde{b} = 0,$$

where $\tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_n)$ is given, $\det A \neq 0$.

In conclusion we solve the algebraic system (11): $\alpha = -\frac{1}{2}A^{-1}(\tilde{b})$ and then we define β by (10). This way (9) takes the form:

$$v_\tau = \sum_{i,j=1}^n a_{ij} v_{y_i y_j} + f_1(\tau, y) \quad (12)$$

The Cauchy problem (12) has initial condition

$$v_0(y) = v|_{\tau=0} = \tilde{u}_0(y) e^{-\sum \alpha_i y_i}; \tilde{u}_0 \equiv u_0(e^{y_1}, \dots, e^{y_n}), y \in \mathbf{R}^n.$$

To find a formula (Poisson type) for the solution of the Cauchy problem (12), $v|_{\tau=0} = v_0(y)$ we must use some auxiliary results from the linear algebra. So let $Mu = \sum_{i,j=1}^n a_{ij} v_{y_i y_j}$. Then the change of the independent variables $y = Bz \iff z = B^{-1}y$, $B^{-1} = (\beta_{li})_{l,i=1}^n$ leads to $\frac{\partial^2}{\partial y_i \partial y_j} = \sum_{k,l=1}^n \beta_{li} \beta_{kj} \frac{\partial^2}{\partial z_k \partial z_l}$, i.e.

$$Mu = \sum_{k,l=1}^n \left(\sum_i \left(\sum_j a_{ij} \beta_{kj} \right) \beta_{li} \right) \frac{\partial^2 u}{\partial z_k \partial z_l}.$$

One can easily guess that $\sum_i (\sum_j a_{ij} \beta_{kj}) \beta_{li} = \tilde{c}_{kl}$ are the elements of the matrix $B^{-1}A(B^{-1})^*$ and of course $(B^{-1})^* = (B^*)^{-1}$. On the other hand consider the elliptic quadratic form $(Ax, x) = (C^*ACy, y)$ after the nondegenerate change $x = Cy$. As we know one can find such a matrix C that

$$C^*AC = I_n, \quad (13)$$

I_n being the unit matrix. Put now $C = (B^{-1})^* \Rightarrow C^* = B^{-1}$. Then $C^*AC = I_n \Rightarrow B^{-1}A(B^{-1})^* = I_n \Rightarrow Mu = \sum_{k=1}^n \frac{\partial^2 u}{\partial z_k^2}$.

This way the change $y = (C^{-1})^*z \Rightarrow z = B^{-1}y$ transforms the Cauchy problem (12) to:

$$\begin{cases} \frac{\partial v}{\partial \tau} = \sum_{k=1}^n \frac{\partial^2 v}{\partial z_k^2} + \tilde{f}_1(\tau, z), & 0 \leq \tau \leq T \\ v|_{\tau=0} = v_0((C^{-1})^*z) \equiv \tilde{v}_0(z), z \in \mathbf{R}^n. \end{cases} \quad (14)$$

The solution of the Cauchy problem (14) is given by the formula

$$\begin{aligned} v(\tau, z) = & \frac{1}{(2\sqrt{\pi\tau})^n} \int_{\mathbf{R}^n} \tilde{v}_0(\lambda) e^{-\frac{|z-\lambda|^2}{4\tau}} d\lambda + \\ & + \int_{\mathbf{R}^n} \int_0^\tau \frac{\tilde{f}_1(\Theta, \lambda)}{[2\sqrt{\pi(\tau-\Theta)}]^n} e^{-\frac{|z-\lambda|^2}{4(\tau-\Theta)}} d\lambda d\Theta, \end{aligned} \quad (15)$$

$z \in \mathbf{R}^n, \lambda \in \mathbf{R}^n \Rightarrow |z - \lambda|^2 = \sum_{i=1}^n (z_i - \lambda_i)^2$ (see [6] or [21]).

Going back to the old coordinates (τ, x) and the old function $u = ve^{\sum \alpha_i y_i + \beta \tau}$, we find $u(t, x)$ -the solution of (2); $t = T - \tau, y_j = \ln x_j, z = B^{-1}y = B^{-1}(\ln x_1, \dots, \ln x_n); u = vx_1^{\alpha_1} \dots x_n^{\alpha_n} e^{\beta(T-t)}$.

We shall concentrate now on (3), $n = 2$.

Remark 1. To simplify the things, consider the quadratic form (elliptic) $Q = a_{11}\xi^2 + 2a_{12}\xi\eta + a_{22}\eta^2, a_{11} > 0, a_{22} > 0, a_{12}^2 - a_{11}a_{22} < 0, Q = (A \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix})$.

Then $Q = \frac{1}{a_{11}}(a_{11}\xi + a_{12}\eta)^2 + b\eta^2; b = a_{22} - \frac{a_{12}^2}{a_{11}} > 0$. The change

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{a_{11}} & \frac{a_{12}}{\sqrt{a_{11}}} \\ 0 & \sqrt{b} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad (16)$$

leads to $Q = x^2 + y^2$. Moreover, the first quadrant $\xi \geq 0, \eta \geq 0$ is transformed under the linear transformation with matrix $D = \begin{pmatrix} \sqrt{a_{11}} & \frac{a_{12}}{\sqrt{a_{11}}} \\ 0 & \sqrt{b} \end{pmatrix}, D^{-1} = \begin{pmatrix} \frac{1}{\sqrt{a_{11}}} & -\frac{a_{12}}{a_{11}\sqrt{b}} \\ 0 & \frac{1}{\sqrt{b}} \end{pmatrix}$ into angle between

the rays (straight lines) $l_1 : \begin{cases} x \geq 0 \\ y = 0 \end{cases}$ and $l_2 : \begin{cases} x = \frac{a_{12}}{\sqrt{a_{11}}}\eta \\ y = \sqrt{b}\eta \geq 0 \end{cases}$ with opening φ_0 . Evidently, $(D^{-1})^*AD^{-1} = I_2$.

Consequently, the transformation D is not orthogonal for $a_{12} \neq 0$.

Let us now consider the boundary value problem (8). The above-proposed procedure yields:

$$\begin{cases} v_\tau = \sum_{i,j=1}^2 a_{ij} v_{y_i y_j} + f_1(\tau, y) \\ v|_{\tau=0} = v_0(y) = u_0(e^{y_1}, e^{y_2}) e^{-\sum \alpha_i y_i} \\ v|_{y_1=\tilde{a}_1} = g_1(T - \tau, e^{y_1}, e^{y_2})|_{y_1=\tilde{a}_1} e^{-\beta\tau} a_1^{-\alpha_1} e^{-\alpha_2 y_2} \equiv \tilde{g}_1(\tau, y_2) \\ v|_{y_2=\tilde{a}_2} = g_2(T - \tau, e^{y_1}, e^{y_2})|_{y_2=\tilde{a}_2} e^{-\beta\tau} a_2^{-\alpha_2} e^{-\alpha_1 y_1} \equiv \tilde{g}_2(\tau, y_1) \end{cases} \quad (17)$$

$$\begin{cases} -\infty < y_j < \ln a_j = \tilde{a}_j \\ 0 \leq \tau \leq T \end{cases}$$

The change $\begin{cases} \lambda_j = \tilde{a}_j - y_j \geq 0, j = 1, 2 \\ \tau = \tau \end{cases}$ in (17) yields:

$$\begin{cases} \tilde{v}_\tau = \sum_{i,j=1}^2 a_{ij} \tilde{v}_{\lambda_i \lambda_j} + \tilde{f}_1(\tau, \lambda) \\ \tilde{v}|_{\tau=0} = \tilde{v}_0(\lambda) = v_0(\tilde{a}_1 - \lambda_1, \tilde{a}_2 - \lambda_2) e^{-\sum_{i=1}^2 \alpha_i (\tilde{a}_i - \lambda_i)} \\ \tilde{v}|_{\lambda_1=0} = \tilde{g}_1(\tau, \tilde{a}_2 - \lambda_2) \\ \tilde{v}|_{\lambda_2=0} = \tilde{g}_2(\tau, \tilde{a}_1 - \lambda_1), \end{cases} \quad (18)$$

$\Omega = \{0 \leq \tau \leq T, \lambda_j \geq 0, j = 1, 2\}$, Ω is a wedge with opening $\frac{\pi}{2}$.

Now we use the linear transformation described in Remark 1, that maps the first quadrant $\lambda_1 \geq 0, \lambda_2 \geq 0$ onto the angle between the rays l_1 and l_2 in the plane $0_{z_1 z_2}$ and we obtain:

$$\begin{cases} w_\tau = w_{z_1 z_1} + w_{z_2 z_2} + f(\tau, z) \\ w|_{\tau=0} = w_0(z) \\ w|_{z_1=0} = \tilde{g}_1(\tau, z_1), (\tau, z) \in \tilde{\Omega} \\ w|_{l_2} = \tilde{g}_2(\tau, z_1, z_2)|_{(z_1, z_2)} \in l_2, \end{cases} \quad (19)$$

$l_1 : \begin{cases} z_1 = 0 \\ z_2 = \frac{\lambda_2}{\sqrt{b}} \end{cases}, l_2 : \begin{cases} z_1 = \frac{\lambda_1}{\sqrt{a_{11}}} \\ z_2 = \frac{-a_{12}}{\sqrt{b a_{11}}} \lambda_1 \end{cases}, \tilde{\Omega}$ is a wedge with opening φ_0 , i.e. $\tilde{\Omega} = [0, T] \times \Gamma$, Γ being the interior of the angle between l_1, l_2 .

In fact, $\lambda = Bz \iff z = B^{-1}\lambda$ and $B^{-1}A(B^{-1})^* = I_2$ implies that $\sum_{i,j=1}^2 a_{ij} \frac{\partial^2}{\partial \lambda_i \partial \lambda_j}$ is transformed in $\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2}$. According to Remark 1: $(D^{-1})^* A D^{-1} = I_2$. Taking $B^{-1} = (D^{-1})^*$, i.e. $B = D^*$ we obtain that $\{\lambda_1 \geq 0, \lambda_2 \geq 0\}$ is mapped onto the angle φ_0 between the rays l_1, l_2 . Of course, there are three possibilities: $\varphi_0 = \frac{\pi}{2}, 0 < \varphi_0 < \frac{\pi}{2}, \frac{\pi}{2} < \varphi_0 < \pi$.

From now on we shall make polar coordinates change in (19): $\begin{cases} z_1 = r \cos \varphi \\ z_2 = r \sin \varphi \end{cases}$ and to fix the

ideas let $0 < \varphi_0 < \frac{\pi}{2}, \begin{cases} r \geq 0 \\ \frac{\pi}{2} - \varphi_0 \leq \varphi \leq \frac{\pi}{2} \end{cases}$, φ_0 is the angle between l_2 and l_1 .

The new change $\Phi = \varphi - (\frac{\pi}{2} - \varphi_0) \Rightarrow 0 \leq \Phi \leq \varphi_0$ and $\frac{\partial}{\partial \Phi} = \frac{\partial}{\partial \varphi}$. To simplify the notation we shall write again (r, φ) instead of (r, Φ) , $0 \leq \Phi \leq \varphi_0$. Thus we have a wedge type

initial-boundary value problem for (19) with unknown function $w(\tau, r, \varphi)$:

$$\begin{cases} w_\tau = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \varphi^2} + f(\tau, r, \varphi) \\ w|_{\tau=0} = w_0(r, \varphi) \\ w|_{\varphi=0} = \tilde{g}_1(\tau, r) \\ w|_{\varphi=\varphi_0} = \tilde{g}_2(\tau, r) \end{cases} \quad (20)$$

$r \geq 0, 0 \leq \varphi \leq \varphi_0, l_1 : \{\varphi = 0, r \geq 0\}, l_2 : \{\varphi = \varphi_0, r \geq 0\}, r \leftrightarrow \xi, \varphi \leftrightarrow \eta, 0 \leq \Theta \leq \tau, 0 \leq \xi \leq \infty, 0 \leq \eta \leq \varphi_0, 0 < \varphi_0 < \pi$. Then

$$\begin{aligned} w(\tau, r, \varphi) = & \int_0^\tau \int_0^{\varphi_0} \int_0^\infty f(\Theta, \xi, \eta) G(r, \varphi, \xi, \eta, \tau - \Theta) \xi d\xi d\eta d\Theta + \\ & + \int_0^\tau \int_0^\infty \tilde{g}_1(\Theta, \xi) \frac{1}{\xi} \left[\frac{\partial}{\partial \eta} G(r, \varphi, \xi, \eta, \tau - \Theta) \right]_{\eta=0} d\xi d\Theta - \\ & - \int_0^\tau \int_0^\infty \tilde{g}_2(\Theta, \xi) \frac{1}{\xi} \left[\frac{\partial}{\partial \eta} G(r, \varphi, \xi, \eta, \tau - \Theta) \right]_{\eta=\varphi_0} d\xi d\Theta + \\ & + \int_0^{\varphi_0} \int_0^\infty w_0(\xi, \eta) G(r, \varphi, \xi, \eta, \tau) \xi d\xi d\eta, \end{aligned} \quad (21)$$

where $G(r, \varphi, \xi, \eta, \tau) = \frac{1}{\varphi_0 \tau} e^{-\frac{(r^2 + \xi^2)}{4\tau}} \sum_{n=1}^\infty I_{\frac{n\pi}{\varphi_0}} \left(\frac{r\xi}{2\tau} \right) \sin \frac{n\pi}{\varphi_0} \varphi \sin \frac{n\pi}{\varphi_0} \eta$ and the modified Bessel function $w = I_\nu(z)$ satisfies the equation:

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} - (z^2 + \nu^2) w = 0, \nu \geq 0, \quad (22)$$

$$I_\nu(z) = \sum_{m=0}^\infty \frac{\left(\frac{z}{2}\right)^{2m+\nu}}{m! \Gamma(m+\nu+1)} \quad (\text{see [2]}).$$

Remark 2. One can see that $\lim_{\tau \rightarrow +0} \int_0^{\varphi_0} \int_0^\infty w_0(\xi, \eta) G(r, \varphi, \xi, \eta, \tau) \xi d\xi d\eta = w_0(r, \varphi)$, i.e. formally $\lim_{\tau \rightarrow +0} \xi G(r, \varphi, \xi, \eta, \tau) = \delta(r - \xi, \varphi - \eta)$ in the sense of Schwartz distributions $D'(\mathbf{R}_+^1 \times [0, \varphi_0])$, $\mathbf{R}_+ = \{\xi \geq 0\}$. G is the corresponding Green function.

Formula (21) is given in [21], pages 182 and 166 or in [6], pp.498. The proof of (21) is based on the properties of the Bessel functions and Hankel transform.

Remark 3. In the special case when $a_{12} = 0$ in (16) we obtain (18) and after the change $\tau = \tau$, $\lambda_j = \sqrt{a_{jj}} z_j, 1 \leq j \leq 2$ (18) takes the form:

$$\begin{cases} \frac{\partial \tilde{v}}{\partial \tau} = \frac{\partial^2 \tilde{v}}{\partial z_1^2} + \frac{\partial^2 \tilde{v}}{\partial z_2^2} + \tilde{f}_1(\tau, z) \\ \tilde{v}|_{\tau=0} = \tilde{v}_0(z) \\ \tilde{v}|_{z_1=0} = \tilde{g}_1(\tau, z_2) \\ \tilde{v}|_{z_2=0} = \tilde{g}_2(\tau, z_1) \end{cases} \quad (23)$$

$0 \leq \tau \leq T, z_j \geq 0, 1 \leq j \leq 2$. Certainly, $\varphi_0 = \frac{\pi}{2}$.

According to [21]:

$$\begin{aligned} \tilde{v}(\tau, z) = & \int_0^\tau \int_0^\infty \int_0^\infty \tilde{f}_1(\Theta, \xi, \eta) G(\tau - \Theta, z_1, z_2, \xi, \eta) d\xi d\eta d\Theta + \\ & + \int_0^\infty \int_0^\infty \tilde{v}_0(\xi, \eta) G(\tau, z_1, z_2, \xi, \eta) d\xi d\eta + \\ & + \int_0^\tau \int_0^\infty \tilde{g}_1(\Theta, \eta) \left[\frac{\partial}{\partial \xi} G(\tau - \Theta, z_1, z_2, \xi, \eta) \right]_{\xi=0} d\eta d\Theta + \\ & + \int_0^\tau \int_0^\infty \tilde{g}_2(\Theta, \xi) \left[\frac{\partial}{\partial \eta} G(\tau - \Theta, z_1, z_2, \xi, \eta) \right]_{\eta=0} d\xi d\Theta, \end{aligned} \quad (24)$$

where the Green function $G(\tau, z_1, z_2, \xi, \eta) = \frac{1}{4\pi\tau} \left[e^{-\frac{(z_1-\xi)^2}{4\tau}} - e^{-\frac{(z_1+\xi)^2}{4\tau}} \right] \times \left[e^{-\frac{(z_2-\eta)^2}{4\tau}} - e^{-\frac{(z_2+\eta)^2}{4\tau}} \right]$.

3. Applications to financial options and numerical results via CNN

Here the analysis of Section 2 is applied to some problems arising in option pricing theory. Some known pricing formulas are revisited in a more general setting and some new results are offered. We apply Cellular Neural Networks (CNN) approach [24] in order to obtain some numerical results. Let us consider a two-dimensional grid with 3×3 neighborhood system as it is shown on Figure 1.

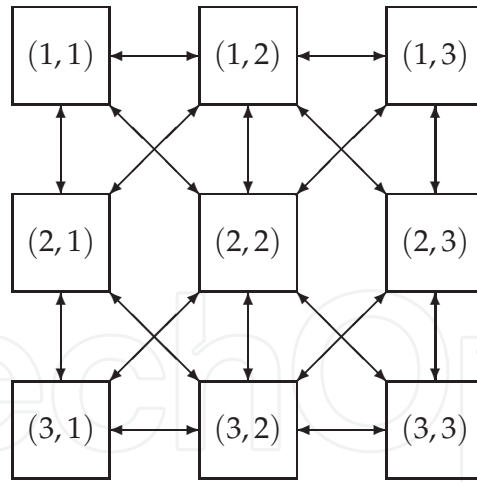


Figure 1. 3×3 neighborhood CNN.

[htb] One of the key features of a CNN is that the individual cells are nonlinear dynamical systems, but that the coupling between them is linear. Roughly speaking, one could say that these arrays are nonlinear but have a linear spatial structure, which makes the use of techniques for their investigation common in engineering or physics attractive.

We will give the general definition of a CNN which follows the original one:

Definition 1. *The CNN is a*

- a). 2-, 3-, or n - dimensional array of
- b). mainly identical dynamical systems, called cells, which satisfies two properties:
- c). most interactions are local within a finite radius r , and
- d). all state variables are continuous valued signals.

Definition 2. *An $M \times M$ cellular neural network is defined mathematically by four specifications:*

- 1). CNN cell dynamics;
- 2). CNN synaptic law which represents the interactions (spatial coupling) within the neighbor cells;
- 3). Boundary conditions;
- 4). Initial conditions.

Now in terms of definition 2 we can present the dynamical systems describing CNNs. For a general CNN whose cells are made of time-invariant circuit elements, each cell $C(ij)$ is characterized by its CNN cell dynamics :

$$\dot{x}_{ij} = -g(x_{ij}, u_{ij}, I_{ij}^s), \quad (25)$$

where $x_{ij} \in \mathbf{R}^m$, u_{ij} is usually a scalar. In most cases, the interactions (spatial coupling) with the neighbor cell $C(i+k, j+l)$ are specified by a CNN synaptic law:

$$\begin{aligned} I_{ij}^s = & A_{ij,kl}x_{i+k,j+l} + \\ & + \tilde{A}_{ij,kl} * f_{kl}(x_{ij}, x_{i+k,j+l}) + \\ & + \tilde{B}_{ij,kl} * u_{i+k,j+l}(t). \end{aligned} \quad (26)$$

The first term $A_{ij,kl}x_{i+k,j+l}$ of (26) is simply a linear feedback of the states of the neighborhood nodes. The second term provides an arbitrary nonlinear coupling, and the third term accounts for the contributions from the external inputs of each neighbor cell that is located in the N_r neighborhood.

It is known [24] that some autonomous CNNs represent an excellent approximation to nonlinear partial differential equations (PDEs). The intrinsic space distributed topology makes the CNN able to produce real-time solutions of nonlinear PDEs. There are several ways to approximate the Laplacian operator in discrete space by a CNN synaptic law with an appropriate A -template:

- one-dimensional discretized Laplacian template:

$$A_1 = (1, -2, 1),$$

- two-dimensional discretized Laplacian template:

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Example 1 (Single-asset inside barrier options) The case of single-barrier zero-rebate down-and-out options was already priced in [18], while the case with rebate is found in [22]. A simple method for obtaining analytical formulas for barrier options is the reflection principle that has a long history in Physics and is commonly used in Finance. Here we write down the pricing formula for a general payoff and rebate and study its analytical properties. Let us consider the following boundary value problem:

$$\begin{cases} Lu = 0 & \text{in } \Omega = \{(t, S); 0 < t < T, 0 < S < S^*\} \\ u|_{t=T} = u_0(S), 0 \leq S \leq S^* \\ u|_{S=S^*} = g(t), 0 \leq t \leq T \end{cases}$$

where $L = \partial_t + rS\partial_S + \frac{1}{2}\sigma^2 S^2 \partial_S^2 - r$, u_0 and g are continuous and $u_0(S^*) = g(T)$. Using the notation of Section 2 and taking $\alpha = \frac{1}{2} - \frac{r}{\sigma^2}$, $\beta = -r \left[\frac{1}{2} + \frac{r}{\sigma^2} \right]$, $C = \frac{\sqrt{2}}{\sigma}$ we straightforwardly obtain the following pricing formula (after changing to variables $\frac{\sigma}{\sqrt{2}}\lambda = \ln S^* - \xi$):

$$\begin{aligned} u(t, S) = & \left(\frac{S}{S^*} \right)^\alpha \frac{e^{\beta(T-t)}}{\sqrt{2\pi\sigma}} \left[\frac{1}{\sqrt{(T-t)}} \int_0^{+\infty} u_0(S^* e^{-\xi}) e^{\alpha\xi} \times \right. \\ & \times \left[\exp\left(-\frac{[\ln(S/S^*) + \xi]^2}{2\sigma^2(T-t)}\right) - \exp\left(-\frac{[\ln(S/S^*) - \xi]^2}{2\sigma^2(T-t)}\right) \right] d\xi + \\ & \left. + \ln \frac{S^*}{S} \int_0^{T-t} \frac{g(T-s)}{(T-t-s)^{3/2}} e^{-\frac{\beta\sigma^2 s}{2}} \exp\left(-\frac{\ln^2(S/S^*)}{2\sigma^2(T-t-s)}\right) ds \right] \end{aligned} \quad (27)$$

Let us study the properties of $u(t, S)$ analytically. Without loss of generality we can assume $S^* = 1$ and therefore $e^{-\beta(T-t)}u(t, S) = \tilde{u}(t, S)$ is written in the form $I_1 + I_2 + I_3$ with:

$$I_1(\tau, y) = \frac{-ye^{\alpha y}}{2\sqrt{\pi}} \int_0^\tau \frac{g(T-\frac{2\gamma}{\sigma^2})}{(\tau-\gamma)^{3/2}} e^{-\beta\gamma} \exp\left(-\frac{y^2}{4(\tau-\gamma)}\right) d\gamma$$

$$I_2(\tau, y) = \frac{e^{\alpha y}}{2\sqrt{\pi\tau}} \int_0^{+\infty} u_0(e^{-\xi}) e^{\alpha\xi} \exp\left(-\frac{[y+\xi]^2}{4\tau}\right) d\xi$$

$$I_3(\tau, y) = -\frac{e^{\alpha y}}{2\sqrt{\pi\tau}} \int_0^{+\infty} u_0(e^{-\xi}) e^{\alpha\xi} \exp\left(-\frac{[y-\xi]^2}{4\tau}\right) d\xi$$

where $y = \ln S$ and $\tau = \frac{\sigma^2}{2}(T-t)$. We shall examine the asymptotics of $\tilde{v}(\tau, y) = \tilde{u}(t, S)$ for $0 < \tau < \frac{\sigma^2}{2}T$ (i.e. $0 < t < T$) fixed and for $y \rightarrow -\infty$ (i.e. $S \rightarrow 0^+$). Put $h(\xi) = u_0(e^{-\xi})$, $\xi \geq 0$. Then:

$$I_2(\tau, y) = \frac{e^{\tau\alpha^2}}{2\sqrt{\pi\tau}} \int_0^{+\infty} h(\xi) \exp\left(-\frac{[y+\xi-2\alpha\tau]^2}{4\tau}\right) d\xi = \frac{e^{\tau\alpha^2}}{\sqrt{\pi}} \int_{\frac{-y-2\alpha\tau}{2\sqrt{\tau}}}^{+\infty} h(-y+2\alpha\tau+2\eta\sqrt{\tau}) e^{-\eta^2} d\eta.$$

According to Lebesgue's dominated convergence theorem, since $\lim_{y \rightarrow -\infty} h(-y+2\alpha\tau+2\eta\sqrt{\tau}) = u_0(0)$ for each fixed η and τ , one has $\lim_{y \rightarrow -\infty} I_2(\tau, y) = e^{\tau\alpha^2} u_0(0)$. On the other hand:

$$\begin{aligned} |I_3(\tau, y)| & \leq \frac{\text{const}}{2\sqrt{\pi\tau}} \int_0^{+\infty} e^{\alpha(y+\xi)} \left(-\frac{(\xi-y)^2}{4\tau}\right) d\xi = \\ & = \text{const.} e^{\tau\alpha^2} \int_{\frac{-y-2\alpha\tau}{2\sqrt{\tau}}}^{+\infty} \exp[-\mu^2 + 2\alpha y - 4\tau\alpha^2 + 2\alpha\mu\sqrt{\tau}] d\mu = \\ & = \text{const.} e^{2\alpha y - 2\tau\alpha^2} \int_{\frac{-y}{2\sqrt{\tau}}}^{+\infty} e^{-\varepsilon^2} d\varepsilon. \end{aligned}$$

Thus, for fixed τ , $0 < \tau < \frac{\sigma^2}{2}T$, and $y \ll -1$, we have

$|I_3(\tau, y)| \leq \text{const.} e^{2\alpha y - 2\tau\alpha^2} \frac{\sqrt{\tau}}{-y} e^{-\frac{y^2}{4\tau}}$, which implies that $\lim_{y \rightarrow -\infty} I_3(\tau, y) = 0$. Finally, we observe that:

$|I_1(\tau, y)| \leq \frac{\max|g|}{2\sqrt{\pi}} |y| e^{\alpha y} \int_0^\tau \frac{e^{-\beta\tau}}{(\tau-\gamma)^{3/2}} \exp(-\frac{y^2}{4(\tau-\gamma)}) d\gamma$ as $\beta \leq 0$ implies $0 \leq -\beta\gamma \leq -\beta\tau$. The change $\theta = \frac{-y}{2\sqrt{\tau-\gamma}}$ yields

$|I_1(\tau, y)| \leq \text{const.} e^{\alpha y} \int_{\frac{-y}{2\sqrt{\tau}}}^{+\infty} e^{-\theta^2} d\theta$, that is

$|I_1(\tau, y)| \leq \text{const.} e^{\alpha y} \frac{2\sqrt{\tau}}{\sqrt{\pi}|y|} e^{-\frac{y^2}{4\tau}}$ for $y \rightarrow -\infty$, τ fixed. Therefore we get:

$\lim_{S \rightarrow 0^+} u(t, S) = u_0(0) e^{-r(T-t)}$, $0 < t < T$.

Remark 4. Assume that $u \in C^2(\overline{\Omega})$. Then, putting $S = 0$, $U(t) = u(t, 0)$, we get $U'(t) = rU$, $U(T) = u_0(0)$. Evidently, $U(t) = u_0(0) e^{-r(T-t)}$ is the only solution of that Cauchy problem. So $u|_{\Sigma_0}$, with $\Sigma_0 = \{0 < t < T, S = 0^+\}$, is uniquely determined by $u_0(0)$.

For this example our CNN model is the following:

$$\frac{dS_{ij}}{dt} + rS_{ij}A_1 * S_{ij} + \frac{1}{2}\sigma^2 S_{ij}^2 A_2 * S_{ij} - r = 0, \quad (28)$$

where $*$ is the convolution operator [24], $M \leq i, j \leq M$. We shall consider this model with free-boundary conditions:

$$u_{ij}(x, t) = x - k, \frac{\partial u_{ij}(x, t)}{\partial t} = +1,$$

$$u_{ij}(x, t) = k - x, \frac{\partial u_{ij}(x, t)}{\partial t} = -1.$$

These are classical first-order contact free-boundary conditions for obstacle problems.

Based on the above CNN model (28) we obtain the following simulations for different values of the parameters:

Example 2. (Multi-asset option with single barrier) Analytic valuation formulas for standard European options with single external barrier have been provided in Heynen-Kat (1994), Kwok-Wu-Yu (1998) and Buchen (2001). Here we give a slightly more general formula in that we allow for any payoff and for both an internal and an external barrier. We confine ourselves to the case of an upstream barrier and zero rebate for simplicity of exposition. Consider the following boundary value problem in $\Omega = \{(t, S_1, S_2); 0 < t < T, 0 < S_1, 0 < S_2 < S^*\}$:

$$\begin{cases} Lu = 0 \\ u|_{t=T} = u_0(S_1, S_2) \quad 0 \leq S_2 \leq S^* \\ u|_{S_2=S^*} = 0 \quad 0 \leq t \leq T \end{cases}$$

where $L = \partial_t + \sum_{i=1}^2 \frac{\sigma_i^2 S_i^2}{2} \partial_{S_i}^2 + \rho \sigma_1 \sigma_2 S_1 S_2 \partial_{S_1 S_2}^2 + r \sum_{i=1}^2 S_i \partial_{S_i} - r$, u_0 is continuous and $u_0(S_1, S^*) = 0$. Assume that $\sigma_1, \sigma_2 > 0$, $\rho^2 < 1$. Using the notation of Section 2 and

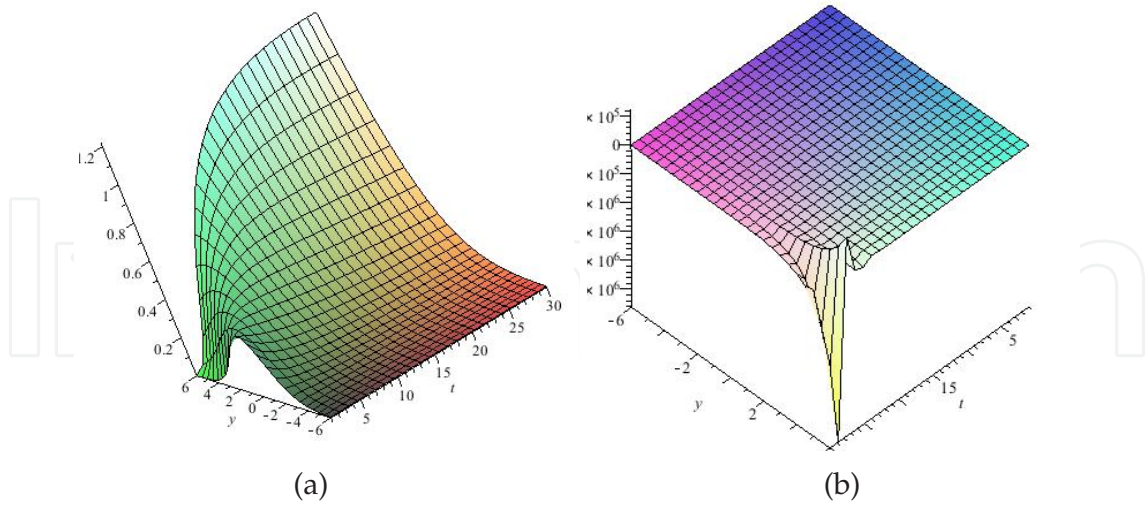


Figure 2. CNN simulations for Example 1. (a) $r = 1, 1 \leq t \leq 30, \sigma = 1$; (b) $r = 0.5, 1 \leq t \leq 30, \sigma = 1.5$.

taking $\mu_i = r - \frac{\sigma_i^2}{2}$ for $i, j = 1, 2$, we have $\alpha_i = \frac{-\mu_i + \rho\mu_i\sigma_i/\sigma_j}{\sigma_i^2(1-\rho^2)}$ for $i, j = 1, 2$ and $i \neq j$, $\beta = \sum_{i,j=1,2} \frac{\sigma_i\sigma_j}{2} \alpha_i\alpha_j + \sum_{i=1,2} \mu_i\alpha_i - r$. Then we have the following pricing formula:

$$u(t, S_1, S_2) = S_1^{\alpha_1} S_2^{\alpha_2} \frac{e^{\beta\tau}}{4\pi\tau} \int_{\mathbb{R}^2} w_0(\lambda_1, \lambda_2) \exp\left[-\frac{\left|\frac{\sqrt{2}\ln S_1}{\sigma_1\sqrt{1-\rho^2}} - \rho\frac{\sqrt{2}\ln S_2}{\sigma_2\sqrt{1-\rho^2}} - \lambda_1\right|^2}{4\tau}\right] \left\{ \exp\left[-\frac{(\sqrt{2}\ln S_2/\sigma_2 - \lambda_2)^2}{4\tau}\right] - \exp\left[-\frac{(\sqrt{2}\ln S_2/\sigma_2 + \lambda_2)^2}{4\tau}\right] \right\} d\lambda_1 d\lambda_2$$

where

$$w_0(\lambda_1, \lambda_2) = \exp\left[-\frac{\alpha_1\sigma_1}{\sqrt{2}}(\lambda_1\sqrt{1-\rho^2} + \rho\lambda_2) - \frac{\alpha_2\sigma_2}{\sqrt{2}}\lambda_2\right] u_0\left(\frac{\sigma_1}{\sqrt{2}}(\lambda_1\sqrt{1-\rho^2} + \rho\lambda_2), \frac{\sigma_2\lambda_2}{\sqrt{2}}\right) 1_{\lambda_2} < \frac{\sqrt{2}\ln S^*}{\sigma_2}.$$

Splitting the integral into two integrals and changing to variables $\eta_1 = \frac{\lambda_1\sqrt{1-\rho^2} + \rho\lambda_2 - \sqrt{2}\ln S_1}{\sigma_1\sqrt{2\tau}}$, $\eta_2 = \frac{\lambda_2 - \sqrt{2}\ln S_2}{\sigma_2\sqrt{2\tau}}$ ($\eta_2 = \frac{\lambda_2 + \sqrt{2}\ln S_2}{\sigma_2\sqrt{2\tau}}$) in the first (second) integral, one gets:

$$u(t, S_1, S_2) = I_1 - I_2$$

where

$$I_1 = \frac{e^{\beta\tau}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{\frac{\ln(S^*/S_2)}{\sigma_2\sqrt{\tau}}} \exp[-(\alpha_1\sigma_1\eta_1 + \alpha_2\sigma_2\eta_2)\sqrt{\tau}] u_0(S_1 e^{\sigma_1\sqrt{\tau}\eta_1}, S_2 e^{\sigma_2\sqrt{\tau}\eta_2})$$

$$\exp\left[-\frac{(\eta_1 - \rho\eta_2)^2}{2(1-\rho^2)} - \frac{\eta_2^2}{2}\right] d\eta_1 d\eta_2$$

$$I_2 = \frac{S_2 e^{\beta\tau}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{\frac{\ln(S^*/S_2)}{\sigma_2\sqrt{\tau}}} \exp[-(\alpha_1\sigma_1\eta_1 + \alpha_2\sigma_2\eta_2)\sqrt{\tau}] u_0(S_1 e^{\sigma_1\sqrt{\tau}\eta_1}, S_2^{-1} e^{\sigma_2\sqrt{\tau}\eta_2})$$

$$\exp\left[-\frac{(-\eta_1 + \rho\eta_2 - 2\rho\ln S_2/(\sigma_2\sqrt{\tau}))^2}{2(1-\rho^2)} - \frac{\eta_2^2}{2}\right] d\eta_1 d\eta_2.$$

Note that $(\beta + r)(1 - \rho^2) + \frac{\mu_1^2}{2\sigma_1^2} + \frac{\mu_2^2}{2\sigma_2^2} - \rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2} = 0$. Then the first integral (after changing to variables $X_1 = -\eta_1 + \frac{\mu_1}{\sigma_1}\sqrt{\tau}$, $X_2 = \eta_2 - \frac{\mu_2}{\sigma_2}\sqrt{\tau}$) is written in the form:

$$I_1 = \frac{e^{-r\tau}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{-\frac{\ln(S_2/S^*)+\mu_2\tau}{\sigma_2\sqrt{\tau}}} \exp\left[-\frac{1}{2(1-\rho^2)}(X_1^2 + X_2^2 + 2\rho X_1 X_2)\right] u_0(S_1 e^{\mu_1\tau - \sigma_1\sqrt{\tau}X_1}, S_2 e^{\mu_2\tau + \sigma_2\sqrt{\tau}X_2}) dX_1 dX_2.$$

Changing to the variables $X_1 = -\eta_1 + \frac{\mu_1}{\sigma_1}\sqrt{\tau} - \frac{2\rho\ln S_2}{\sigma_2\sqrt{\tau}}$, $X_2 = \eta_2 - \frac{\mu_2}{\sigma_2}\sqrt{\tau}$, the second integral becomes:

$$I_2 = \frac{e^{-r\tau}}{2\pi\sqrt{1-\rho^2}} (S_2)^{-\frac{2\mu_2}{\sigma_2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{\frac{\ln(S_2 S^*) - \mu_2\tau}{\sigma_2\sqrt{\tau}}} \exp\left[-\frac{1}{2(1-\rho^2)}(X_1^2 + X_2^2 + 2\rho X_1 X_2)\right] u_0(S_1 S_2^{-2\rho\frac{\sigma_1}{\sigma_2}} e^{\mu_1\tau - \sigma_1\sqrt{\tau}X_1}, S_2^{-1} e^{\mu_2\tau + \sigma_2\sqrt{\tau}X_2}) dX_1 dX_2.$$

In the special case of standard options one has: $u_0(S_1, S_2) = \max(\omega(S_1 - K), 0)$, $\omega = \pm 1$. Then I_1 can be written in the form:

$$\omega S_1 N_2(\omega d^+, e^+; -\rho\omega) - \omega K e^{-r\tau} N_2(\omega d^-, e^-; -\rho\omega)$$

where N_2 is the bivariate cumulative normal distribution function, $d^\pm = \frac{\ln(\frac{S_1}{K}) + (r \pm \frac{\sigma_1^2}{2})\tau}{\sigma_1\sqrt{\tau}}$, $e^- = -\frac{\ln(\frac{S_2}{S^*}) + \mu_2\tau}{\sigma_2\sqrt{\tau}}$, $e^+ = e^- - \rho\sigma_1\sqrt{\tau}$. Similarly I_2 is written in the form:

$$\omega e^{-2\frac{\mu_2}{\sigma_2} \ln(\frac{S_2}{S^*})} [e^{-2\rho\frac{\sigma_1}{\sigma_2} \ln(\frac{S_2}{S^*})} S_1 N_2(\omega \hat{d}^+, \hat{e}^+; -\rho\omega) - K e^{-r\tau} N_2(\omega \hat{d}^-, \hat{e}^-; -\rho\omega)]$$

where $\hat{d}^\pm = d^\pm - \frac{2\rho}{\sigma_2\sqrt{\tau}} \ln(\frac{S_2}{S^*})$, $\hat{e}^\pm = e^\pm + \frac{2}{\sigma_2\sqrt{\tau}} \ln(\frac{S_2}{S^*})$.

Simulating CNN for multi-asset option with single barrier model, we obtain the following figure with different values of the parameter set:

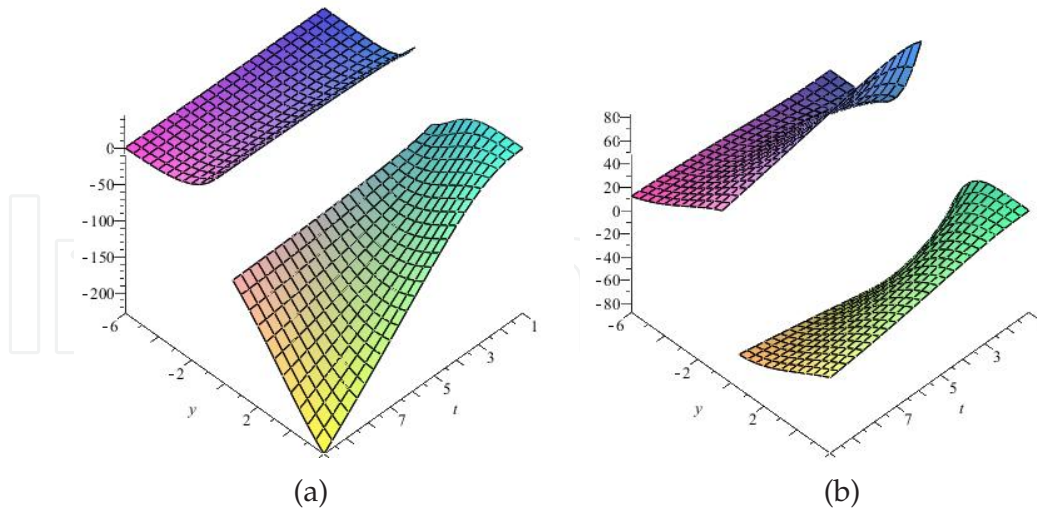


Figure 3. CNN simulations for Example 2. (a) $r = 1$, $T = 60$ days, $\sigma = 1$, $\rho = 0.05$; (b) $r = 0.5$, $T = 120$ days, $\sigma = 1.5$, $\rho = 0.06$.

Example 3. (Two-asset barrier options with simultaneous barriers) While single-asset barrier options have received substantial coverage in the literature, multi-asset options with several barriers have been discussed only in some special cases (e.g. sequential barriers, radial

options, etc.). Here we show how the case of two simultaneous barriers can be valued straightforwardly from the arguments in Section 2. Let us confine ourselves to zero-rebate options for simplicity's sake, although Section 2 deals with the general case too. Then the boundary value problem takes the form:

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u|_{t=T} = u_0(S_1, S_2) \\ u|_{S_1=S_1^*} = 0 \quad \text{and} \quad u|_{S_2=S_2^*} = 0 & 0 \leq t \leq T \end{cases}$$

where $L = \partial_t + \sum_{i=1}^2 \frac{\sigma_i^2 S_i^2}{2} \partial_{S_i}^2 + \rho \sigma_1 \sigma_2 S_1 S_2 \partial_{S_1 S_2}^2 + r \sum_{i=1}^2 S_i \partial_{S_i} - r$, $\Omega = \{(t, S_1, S_2); 0 < t < T, 0 < S_1 < S_1^*, 0 < S_2 < S_2^*\}$. Arguing as in the last part of Section 2 and taking

$$D = \begin{pmatrix} \sigma_1 & \rho \sigma_2 \\ 0 & \sqrt{1 - \rho^2} \sigma_2 \end{pmatrix}, \rho^2 < 1, \sigma_1 > 0, \sigma_2 > 0$$

and φ_0 as the opening of the angle between $\begin{cases} x \leq 0 \\ y = 0 \end{cases}$ and $\begin{cases} x = \rho \sigma_2 \eta, \eta \geq 0 \\ y = \sqrt{1 - \rho^2} \sigma_2 \eta \end{cases}$, from (21) we have

$$w(\tau, r, \varphi) = \int_0^{\varphi_0} \int_0^\infty w_0(\xi, \eta) G(r, \varphi, \xi, \eta, \tau) \xi d\xi d\eta, \quad (29)$$

where $G(r, \varphi, \xi, \eta, \tau) = \frac{1}{\varphi_0 \tau} e^{-\frac{(r^2 + \xi^2)}{4\tau}} \sum_{n=1}^\infty I_{\frac{n\pi}{\varphi_0}} \left(\frac{r\xi}{2\tau} \right) \sin \frac{n\pi}{\varphi_0} \varphi \sin \frac{n\pi}{\varphi_0} \eta$ and I_v is the modified Bessel function satisfying (22). Here $w_0(r, \varphi) = \tilde{w}_0(D^* z) |_{z_1=r \cos \varphi, z_2=r \sin \varphi}$ where $\tilde{w}_0(\lambda) = u_0(S_1^* e^{-\lambda_1}, S_2^* e^{-\lambda_2}) e^{-\sum \alpha_i (\ln S_i^* - \lambda_i)}$. Changing back the variables one obtains $u(t, S_1, S_2)$.

Simulating CNN for two-asset barrier options with simultaneous barriers model, we obtain the following figure with different values of the parameter set:

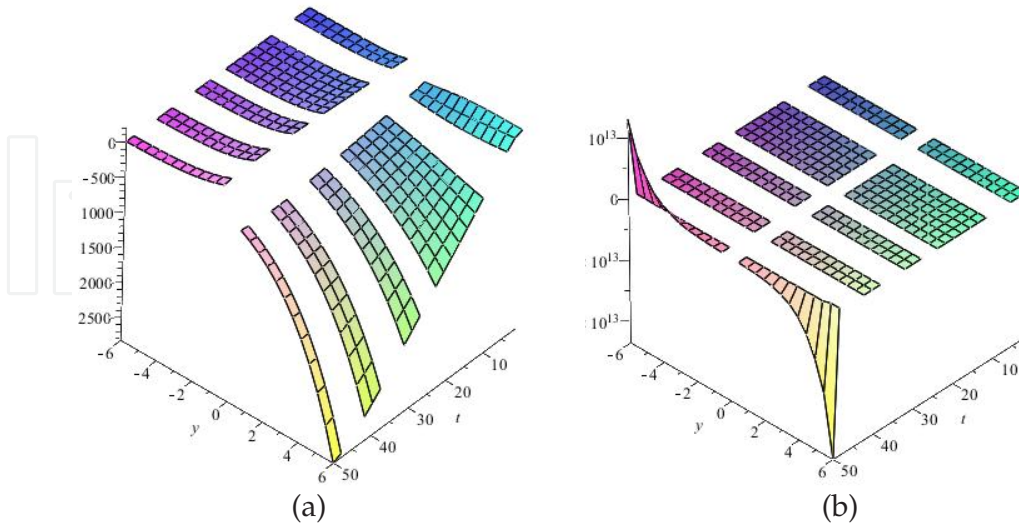


Figure 4. CNN simulations for Example 3. (a) $r = 1$, $T = 120$ days, $\sigma = 1$, $\rho = 0.05$; (b) $r = 0.5$, $T = 180$ days, $\sigma = 1.5$, $\rho = 0.06$.

4. Comparison principle for multi-asset Black-Scholes equations

For the sake of simplicity consider

$$u_t + \sum_{i,j=1}^2 a_{ij} x_i x_j u_{x_i x_j} + \sum_{i=1}^2 b_i x_i u_{x_i} + cu = f, \quad (30)$$

where $(a_{ij})^* = (a_{ij})$, $(a_{ij}) > 0$, a_{ij}, b_i, c are real constants and $c < 0$ in the domain $D : \left\{ \begin{array}{l} 0 < t < T \\ 0 < x_j < a_j, \end{array} j = 1, 2 \right\}$, $a_j = \text{const} > 0$. The boundary of the parallelepiped D is split into two parts: Parabolic $\Gamma = \{x_1 = a_1, 0 < x_2 < a_2, 0 < t < T\} \cup \{x_2 = a_2, 0 < x_1 < a_1, 0 < t < T\} \cup \{t = T, 0 < x_j < a_j, j = 1, 2\}$ and free of boundary data part $\Gamma_1 = I \cup II \cup III$, where $I = \{0 < x_j < a_j, j = 1, 2; t = 0\}$, $II = \{x_1 = 0, 0 < x_2 < a_2, 0 < t < T\}$, $III = \{x_2 = 0, 0 < x_1 < a_1, 0 < t < T\}$. The Dirichlet data are prescribed on Γ :

$$u|_{\Gamma} = g \quad (31)$$

Theorem 1. (Comparison principle)

Assume that u is a classical solution of (30), (31), i.e. $u \in C^2(D \cup \bar{\Gamma}_1) \cap C^0(\bar{D})$. Let v be another solution of (30), (31) belonging to $C^2(D \cup \bar{\Gamma}_1) \cap C^0(\bar{D})$. Suppose that $u|_{\Gamma} \leq v|_{\Gamma}$. Then $u \leq v$ everywhere in \bar{D} .

Proof. Put $w = u - v$. Assume that $\max w = w(t_0, x_0) = M > 0$, $P_0 = (t_0, x_0) \in \bar{D}$. Evidently, $(t_0, x_0) \in D \cup \Gamma_1$ as $w|_{\Gamma} \leq 0$.

Case a). $(t_0, x_0) \in D$. Having in mind that $\sum a_{ij} x_i x_j w_{x_i x_j}$ is a strictly elliptic operator in the open rectangle $\{0 < x_j < a_j, j = 1, 2\}$ we shall apply the interior parabolic maximum principle (see A.Friedman, Partial Differential equations of parabolic type, Prentice Hall, Inc.

(1964), Chapter II). To do this we shall work in the domain $D_1 : \left\{ \begin{array}{l} 0 < t < T \\ 0 < \varepsilon_j < x_j < a_j, \end{array} j = 1, 2 \right\}$,

such that $x_0 \in \Pi = (\varepsilon_1, a_1) \times (\varepsilon_2, a_2)$, $0 < t_0 < T$. Then Th1 from Chapter II of the above mentioned book gives: $w \equiv M > 0$ for $T \geq t \geq t_0$, $x \in \bar{\Pi}$ and this is a contradiction with $w \leq 0$ on $t = T$.

Case b). $(t_0, x_0) \in I \Rightarrow t_0 = 0$, (1) $\left\{ \begin{array}{l} 0 < x_{10} < a_1 \\ 0 < x_{20} < a_2 \end{array} \right.$, (2) $\left\{ \begin{array}{l} 0 < x_{10} < a_1 \\ x_{20} = 0 \end{array} \right.$, (3) $\left\{ \begin{array}{l} x_{10} = 0 \\ x_{20} = 0 \end{array} \right.$ and a similar case with respect to $x_{20} \in [0, a_2)$, $x_{10} = 0$. Thus,

b). (1) x_0 is interior point of $(0, a_1) \times (0, a_2)$ and therefore $\frac{\partial w}{\partial x_j}(P_0) = 0$, $j = 1, 2$, while $\sum_{i,j}^2 a_{ij} x_{i0} x_{j0} \frac{\partial^2 w}{\partial x_i \partial x_j}(P_0) \leq 0$ as it is shown in Friedman book. Obviously, $w_t(P_0) \leq 0$, as $w(0, x_0) = M = \max_{\bar{D}} w$. As we know, (30) is satisfied on $I \Rightarrow \sum_1^2 a_{ij} x_{i0} x_{j0} \frac{\partial^2 w}{\partial x_i \partial x_j}(P_0) + cw(P_0) + w_t(P_0) = 0$ -contradiction with $c < 0$, $w(P_0) > 0$.

b). (2) Again $w_t(P_0) \leq 0$ and $w_{x_1}(P_0) = 0$, $w_{x_1 x_1}(P_0) \leq 0$ as P_0 is interior point for the interval $(0, a_1)$. According to (30): $a_{11} x_{10}^2 \frac{\partial^2 w}{\partial x_1^2}(P_0) + b_1 \frac{\partial w}{\partial x_1}(P_0) + cw(P_0) + w_t(P_0) = 0 \rightarrow$ contradiction.

b). (3) Then (30) takes the form: $cw(P_0) + w_t(P_0) = 0$ - contradiction.

Case c). $(t_0, x_0) \in II \Rightarrow 0 \leq t_0 < T, x_{10} = 0$; (1) $\begin{cases} 0 < t_0 < T \\ 0 < x_{20} < a_2 \end{cases}$, (2) $\begin{cases} t_0 = 0 \\ 0 < x_2^0 < a_2 \end{cases}$,
(3) $\begin{cases} t_0 = 0 \\ x_2^0 = 0 \end{cases}$, (4) $\begin{cases} T > t_0 > 0 \\ x_2^0 = 0 \end{cases}$.

Certainly, $w_t(P_0) \leq 0$ in each case (1) -(4).

c). (1) As P_0 is interior point in the rectangle $\{0 < t < T\} \times \{0 < x_2 < a_2\} \Rightarrow w_t(P_0) = 0$, $w_{x_2}(P_0) = 0$, $w_{x_2x_2}(P_0) \leq 0$. According to (30) $a_2x_{20}^2w_{x_2x_2}(P_0) + b_2x_{20}w_{x_2}(P_0) + cw(P_0) + w_t(P_0) = 0$ - contradiction.

c). (2) As $x_2^0 \in (0, a_2) \Rightarrow w_{x_2}(P_0) = 0$, $w_{x_2x_2}(P_0) \leq 0$. The contradiction is obvious.

c). (3) The equation (30) takes the form:

$$cw(P_0) + w_t(P_0) = 0 \quad (32)$$

and again a contradiction.

c). (4). Then $w_t(P_0) = 0$ and according to (30) $cw(P_0) + w_t(P_0) = 0$ - contradiction.

We conclude that $M = \sup_{\bar{D}} w \leq 0 \Rightarrow u - v \leq 0$ in $\bar{D} \Rightarrow u \leq v$ in \bar{D} .

The comparison principle is proved.

Remark 5. The operator

$$Lu = u_t + \sum_{i,j=1}^n a_{ij}x_i x_j u_{x_i x_j} + \sum_{i=1}^n b_i x_i u_{x_i} + cu$$

is non-hypoelliptic. The constants a_{ij} , b_i , c are arbitrary. To verify this we recall that the function $s_+^a = \begin{cases} s^a, & s > 0 \\ 0, & s \leq 0 \end{cases}$ considered as a Schwartz distribution in $D'(\mathbf{R}^1)$ satisfies for $\text{Re } a > 1$ the following identities:

$$ss_+^a = s_+^{a+1}, \frac{d}{ds}s_+^a = as_+^{a-1}, \frac{d^2}{ds^2}s_+^a = a(a-1)s_+^{a-2}.$$

Consider now the distribution $u = e^{\lambda t} u_1(x_1) \otimes \dots \otimes u_n(x_n)$, where $\lambda = \text{const}$, $u_j(x_j) = x_j^{d_j} \in D'(\mathbf{R}_{x_j}^1)$, $\text{Re } d_j > 1$. Then $u \in D'(\mathbf{R}^{n+1})$ satisfies in distribution sense $Lu = 0$ if

$$\lambda + \sum_{i \neq j}^n a_{ij} d_i d_j + \sum_{i=j}^n a_{ii} d_i (d_i - 1) + \sum_{i=1}^n b_i d_i + c = 0$$

Of course, $\text{sing supp } u = \partial\{x \in \mathbf{R}^n : x_j \geq 0, 1 \leq j \leq n\}$, i.e. $\text{sing supp } u$ is the boundary of the first octant of \mathbf{R}_x^n multiplied by \mathbf{R}_t^1 . The nonhypoellipticity is proved. Evidently, under (4) L is hypoelliptic in the open domain $\{x_j > 0, 1 \leq j \leq n\}$ as it is strictly parabolic there.

5. The approach of Fichera-Oleinik-Radkevič

In this section we revise the results of [9] and [20] for the Dirichlet problem for PDEs of second order having non-negative characteristic form; then the method is applied to some PDEs of Black-Scholes type.

To begin with consider the following equation in a bounded domain $\Omega \subset \mathbf{R}^m$ with piecewise smooth boundary Σ :

$$L(u) = \sum_{k,j=1,\dots,m} a^{kj}(x) u_{x_k x_j} + \sum_{k=1,\dots,m} b^k(x) u_{x_k} + c(x) u = f(x) \quad (33)$$

where $\sum_{k,j=1,\dots,m} a^{kj}(x) \xi_k \xi_j \geq 0$, $\forall x \in \overline{\Omega}$, $\forall \xi \in \mathbf{R}^m$; $a^{kj}(x) = a^{jk}(x)$, $\forall x \in \Omega$. Moreover, $a^{kj} \in C^2(\overline{\Omega})$, $b^k \in C^1(\overline{\Omega})$, $c \in C^0(\overline{\Omega})$. Denote the unit inner normal to Σ by $\vec{n} = (n_1, \dots, n_m)$ and let $\Sigma_3 = \{x \in \Sigma; \sum_{k,j=1,\dots,m} a^{kj}(x) n_k n_j > 0\}$ be the non-characteristic part of Σ . Define $\Sigma^0 = \{x \in \Sigma; \sum_{k,j=1,\dots,m} a^{kj}(x) n_k n_j = 0\}$, i.e. $\Sigma = \Sigma^0 \cup \Sigma_3$ and Σ^0 is the characteristic part of Σ . Following Fichera (1956) we introduce on Σ^0 the Fichera function:

$$\beta(x) = \sum_{k=1,\dots,m} (b^k(x) - \sum_{j=1,\dots,m} a_{x_j}^{kj}(x)) n_k, x \in \Sigma^0 \quad (34)$$

Then we split Σ^0 into three parts, namely

$$\Sigma_1 = \{x \in \Sigma^0; \beta(x) > 0\},$$

$$\Sigma_2 = \{x \in \Sigma^0; \beta(x) < 0\},$$

$$\Sigma_0 = \{x \in \Sigma^0; \beta(x) = 0\}.$$

As it is proved in Oleinik and Radkevič (1971) the sets Σ_0 , Σ_1 , Σ_2 , Σ_3 are invariant under smooth non-degenerate changes of the variables. More precisely, let $L(u) = f$ in Ω ; after the change $y = F(x)$ it takes the form $\tilde{L}(\tilde{u}) = \tilde{f}$ in $\tilde{\Omega}$. Denote the Fichera function for $\tilde{L}(\tilde{u}) = \tilde{f}$ by $\tilde{\beta}$. Then $\tilde{\beta} = \beta \cdot A$ where $A > 0$ and A is continuous.

Assume now that $u \in C^2(\Omega)$ and $v \in C_0^\infty(\Omega)$. Then

$$\int_\Omega L(u) v dx = \int_\Omega u L^*(v) dx,$$

where

$$L^*(v) = \sum_{k,j=1,\dots,m} a^{kj}(x) v_{x_k x_j} + \sum_{k=1,\dots,m} b^{*k}(x) v_{x_k} + c^*(x) v \quad (35)$$

and $b^{*k} = 2 \sum_{j=1,\dots,m} a_{x_j}^{kj} - b^k$, $c^* = \sum_{k=1,\dots,m} (\sum_{j=1,\dots,m} a_{x_k x_j}^{kj} - b_{x_k}^k) + c$. One can easily see that if we denote the Fichera function for $L^*(v)$ by β^* , then $\beta^* = -\beta$ and β is defined by (34).

Assume now that $u \in C^2(\overline{\Omega})$, $u = 0$ at $\Sigma_2 \cup \Sigma_3$, and define the following set of test functions: $\mathcal{V} = \{v \in C^2(\overline{\Omega}); v = 0 \text{ at } \Sigma_1 \cup \Sigma_3\}$. In view of the Green formula for L we get:

$$\int_\Omega (L(u) v - L^*(v) u) dx = 0 \Leftrightarrow \int_\Omega L(u) v dx = \int_\Omega u L^*(v) dx \quad (36)$$

for any u and $v \in \mathcal{V}$. Let us now recall the definitions of generalized solution.

Definition 3. The function $u \in L^p(\Omega)$, $p \geq 1$, is called a generalized solution of the boundary value problem

$$\begin{cases} L(u) = f & \text{in } \Omega \\ u = 0 & \text{at } \Sigma_2 \cup \Sigma_3 \end{cases} \quad (37)$$

if for each test function $v \in \mathcal{V}$ the following integral identity holds:

$$\int_{\Omega} f v dx = \int_{\Omega} u L^*(v) dx. \quad (38)$$

Theorem 2. (See [20], Th.1.3.1).

Suppose that $c < 0$, $c^* < 0$ in Ω and $p > 1$. Then for each $f \in L^p(\Omega)$ there exists a generalized solution $u \in L^p(\Omega)$ of (37) in the sense of (38) and such that

$$\inf_{u_0 \in Z} \|u + u_0\|_{L^p(\Omega)} \leq K \|f\|_{L^p(\Omega)} \quad (39)$$

$K = \text{const} > 0$. The set $Z = \{u_0 \in L^p(\Omega) : \int_{\Omega} u_0 L^*(v) dx = 0, \forall v \in \mathcal{V}\}$.

Theorem 3. (See [20], Th. 1.3.2).

Let $c < 0$ in $\overline{\Omega}$, $\frac{1}{p} + \frac{1}{q} = 1$ and $-c + (1 - q)c^* > 0$ in $\overline{\Omega}$. Then for each $f \in L^p(\Omega)$ there exists a generalized solution u of (37) satisfying the a-priori estimate (39).

Theorem 4. (See [20], Th. 1.3.3).

Let $c^* < 0$ in $\overline{\Omega}$ and $-c + (1 - q)c^* > 0$ in $\overline{\Omega}$, $\frac{1}{p} + \frac{1}{q} = 1$. Then for each $f \in L^p(\Omega)$ there exists a generalized solution u of (37) satisfying the estimate (39).

Conclusion. Assume that $c < 0$. Then (37) is solvable in the sense of Definition 1 for $p \gg 1$ as $p \rightarrow +\infty \Rightarrow q \rightarrow 1$. On the other hand, $c^* < 0$ implies the solvability of (40) for $p \geq 1$, $p \approx 1$ as $p \rightarrow 1 \Rightarrow q \rightarrow +\infty$.

We shall now discuss the problem for existence of a generalized solution of (37) in the Sobolev space $H^1(\Omega)$ with an appropriate weight. Define the following set of test functions:

$$\mathcal{W} = \{v \in C^1(\overline{\Omega}); v|_{\Sigma_3} = 0\}$$

and equip \mathcal{W} with the scalar product: $(u, v)_{\mathcal{H}} = \int_{\Omega} (\sum_{k,j} a^{kj} u_{x_j} v_{x_k} + uv) dx + \int_{\Sigma_1 \cup \Sigma_3} uv |\beta| d\sigma$. The completion of \mathcal{W} with respect to the norm $\|u\|_{\mathcal{H}}$ is a real Hilbert space denoted by \mathcal{H} . For each two functions $u, v \in \mathcal{W}$ we consider the bilinear form $B(u, v) = - \int_{\Omega} [\sum_{k,j} a^{kj} u_{x_j} v_{x_k} + \sum_k (u \ell^k v_{x_k} + (\ell^k_{x_k} - c) uv)] dx - \int_{\Sigma_1} uv \beta d\sigma$, where $\ell^k = b^k - \sum_j a^{kj}_{x_j}$. According to the Cauchy-Schwartz inequality $|B(u, v)| \leq \text{const} [\|v\|_{H^1(\Omega)} + \|v\|_{L^2(\Sigma_1)}] \|u\|_{\mathcal{H}}$. Therefore, $B(u, v)$ is well defined for $v \in \mathcal{W}$ and $u \in \mathcal{H}$.

Definition 4. Let $f \in L^2(\Omega)$. We shall say that the function $u \in \mathcal{H}$ is a generalized solution of (37) if for each $v \in \mathcal{W}$ the following identity is satisfied:

$$\int_{\Omega} v f dx = B(u, v). \quad (40)$$

Theorem 5. (See [20], Th. 1.4.1).

Assume that $f \in L^2(\Omega)$ and $\frac{1}{2} \sum_k (b_{x_k}^k - \sum_j a_{x_k x_j}^{kj}) - c \geq c_0 > 0$ in $\overline{\Omega}$. Then the boundary value problem (37) possesses a generalized solution $u \in \mathcal{H}$ (i.e. a weak solution) in the sense of (40).

Finally we propose the existence of a generalized solution of (37) in the space $L^\infty(\Omega)$. To fix the ideas we assume that the coefficients of L and L^* belong to $C^1(\overline{\Omega})$ and Σ is thrice piecewise smooth (i.e. Σ can be split into several parts and each of them is C^3 smooth). Consider the boundary value problem:

$$\begin{cases} L(u) = f & \text{in } \Omega \\ u = g & \text{on } \Sigma_2 \cup \Sigma_3 \end{cases} \quad (41)$$

If $u \in C^2(\overline{\Omega})$ is a classical solution of (41) and $v \in \mathcal{V}$ then according to the Green formula

$$\int_{\Omega} L^*(v) u dx = \int_{\Omega} f v dx - \int_{\Sigma_3} g \frac{\partial v}{\partial \vec{\nu}} d\sigma + \int_{\Sigma_2} \beta g v d\sigma, \quad (42)$$

where $\vec{\nu} = (v_1, \dots, v_m)$, $v_k = \sum_j a^{kj} n_j$, $\frac{\partial}{\partial \vec{\nu}} = \sum_k v_k \frac{\partial}{\partial x_k}$.

Definition 5. We shall say that the function $u \in L^\infty(\Omega)$ is a generalized solution of (41) if for each test function $v \in \mathcal{V}$ the identity (42) is fulfilled.

We point out that $f \in L^\infty(\Omega)$ and $g \in L^\infty(\Sigma_2 \cup \Sigma_3)$.

Theorem 6. (See [20], Th. 1.5.1).

Assume that the coefficient $c(x)$ of L is such that $c(x) \leq -c_0 < 0$ in $\overline{\Omega}$, $f \in L^\infty(\Omega)$, $g \in L^\infty(\Sigma_2 \cup \Sigma_3)$ and $\beta(x) \leq 0$ in the interior points of $\Sigma_2 \cup \Sigma_0$. Then there exists a generalized solution of (41) in the sense of Definition 5. Moreover, $|u(x)| \leq \max(\sup \frac{|f|}{c_0}, \sup |g|)$.

Remark 6. In Th.6 it is assumed that $\sum_{k,j=1,\dots,m} a^{kj}(x) \xi_k \xi_j \geq 0$ in an m -dimensional neighbourhood of Σ^0 , $\forall \xi \in \mathbb{R}^m$.

Theorem 7. (See [20], Th. 1.5.2).

Suppose that g is continuous in the interior points of $\Sigma_2 \cup \Sigma_3$. Then the generalized solution u of (41) constructed in Th. 6 is continuous at those points and, moreover, $u = g$ there.

As we shall deal with (degenerate) parabolic PDEs we shall have to work in cylindrical domains (rectangles in \mathbb{R}^2). Therefore $\Sigma = \partial\Omega$ will be piecewise smooth. Consider now the bounded domain Ω having piecewise C^3 smooth boundary Σ . The corresponding boundary value problem is:

$$\begin{cases} L(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma_2 \cup \Sigma_3 \end{cases} \quad (43)$$

We shall say that the point $P \in \Sigma$ is regular if locally near to P the surface Σ can be written in the form $x_k = \varphi_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_m)$, $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_m)$ describing some

neighborhood of the projection of P onto the plane $x_k = 0$. The set of the boundary points which do not possess such a representation will be denoted by B .

Definition 6. The function $u \in L^\infty(\Omega)$ is called a generalized solution of (43) for $f \in L^\infty(\Omega)$ if for each function $v \in C^2(\overline{\Omega})$, $v = 0$ at $\Sigma_1 \cup \Sigma_3 \cup B$ the following identity holds:

$$\int_{\Omega} u L^*(v) dx = \int_{\Omega} f v dx.$$

Theorem 8. (See [20], Th. 1.5.5).

Suppose that the boundary Σ of the bounded domain Ω is C^3 piecewise smooth, $f \in L^\infty(\Omega)$, $g = 0$, $c(x) \leq -c_0 < 0$ in $\overline{\Omega}$ and $\beta \leq 0$ in the interior points of $\Sigma_0 \cup \Sigma_2$. Then there exists a generalized solution u of (43) in the sense of Definition 6 and such that $|u| \leq \sup \frac{|f|}{c_0}$.

We shall not discuss here in details the problems of uniqueness and regularity of the generalized solutions. Unicity results are given by Theorems 1.6.1.-1.6.2. in [20]. For domains with C^3 smooth boundary under several restrictions on the coefficients, including $c(x) \leq -c_0 < 0$, $c^* < 0$ in $\overline{\Omega}$, $\beta \leq 0$ in the interior points of $\Sigma_0 \cup \Sigma_2$, $\beta^* = -\beta < 0$ at Σ_1 , the maximum principle is valid for each generalized solution u in the sense of Definition 5:

$$|u| \leq \max \left\{ \sup_{\Omega} \frac{|f|}{c_0}, \sup_{\Sigma_3 \cup \Sigma_2} |g| \right\}.$$

In Th. 1.6.9. uniqueness result is proved for the boundary value problem (43) in the class $L^\infty(\Omega)$. The existence result is given Th. 8. Regularity result is given in the Appendix.

Remark 7. Backward parabolic and parabolic operators satisfy the conditions: $a^{km} = 0$, $k = 1, \dots, m$, and $b^m = \pm 1$ if $x = (x_1, \dots, x_{m-1}, t)$, i.e. $t = x_m$. Put now $u = v e^{\alpha t}$ in (33). Then

$$L_1(v) = \sum_{k,j=1,\dots,m} a^{kj} v_{x_k x_j} + \sum_{k=1,\dots,m} b^k v_{x_k} + (c + \alpha) v = f e^{-\alpha t}$$

and

$$L_1^*(w) = \sum_{k,j=1,\dots,m} a^{kj} w_{x_k x_j} + \sum_{k=1,\dots,m} b^{*k} w_{x_k} + c_1^* w$$

$$\text{where } c_1 = c + \alpha, b^{*k} = 2 \sum_{j=1,\dots,m} a_{x_j}^{kj} - b^k, c_1^* = \sum_{k,j=1,\dots,m} a_{x_k x_j}^{kj} - \sum_{k=1,\dots,m} b_{x_k}^k + c + \alpha.$$

Having in mind that $|c| \leq \tilde{c} = \text{const}$ we conclude that for $b^m = \pm 1$ and $\alpha \rightarrow \mp \infty$ then $c_1 \rightarrow -\infty$, $c_1^* \rightarrow -\infty$ uniformly in $(x_1, \dots, x_{m-1}, t) \in \Omega$. So for parabolic (backward parabolic) equations the conditions of Theorems 2, 5 are fulfilled.

We shall illustrate the previous results by the backward parabolic equations:

$$L(u) = \frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + r x \frac{\partial u}{\partial x} - r u = f(t, x) \quad (44)$$

which is the famous Black-Scholes equation, and

$$M(u) = \frac{\partial u}{\partial t} + x^2 \frac{\partial^2 u}{\partial x^2} + b(x) \frac{\partial u}{\partial x} + c(x) u = f(t, x) \quad (45)$$

We shall work in the following rectangles: $\Omega_1 = \{(t, x) : 0 < t < T, 0 < x < a_1\}$, $\Omega_2 = \{(t, x) : 0 < t < T, a_2 < x < 0\}$, $\Omega = \{(t, x) : 0 < t < T, a_2 < x < a_1\}$. Under the previous notation for Ω we have: $\Sigma_1 = \{t = 0\}$, $\Sigma_2 = \{t = T\}$, $\Sigma_3 = \{x = a_1\} \cup \{x = a_2\}$. Certainly, for Ω_1, Ω_2 another part of the boundary appears, $\Sigma_0 = \{x = 0\}$.

As we know from [20] there exists an $L^p(\Omega_1)$ solution of the boundary value problem

$$\begin{cases} L(u_1) = f & \text{in } \Omega_1 \\ u_1 = 0 & \text{on } \Sigma_2^{(1)} \cup \Sigma_3^{(1)} \end{cases} \quad (46)$$

According to the Definition 3: $\int_{\Omega_1} u_1 L^*(v_1) dx = \int_{\Omega_1} f v_1 dx$ for each test function $v_1 \in C^2(\overline{\Omega}_1)$, $v_1|_{\Sigma_1^{(1)} \cup \Sigma_3^{(1)}} = 0$.

In a similar way there exists $u_2 \in L^p(\Omega_2)$ such that

$$\begin{cases} L(u_2) = f & \text{in } \Omega_2 \\ u_2 = 0 & \text{on } \Sigma_2^{(2)} \cup \Sigma_3^{(2)} \end{cases} \quad (47)$$

Therefore: $\int_{\Omega_2} u_2 L^*(v_2) dx = \int_{\Omega_2} f v_2 dx$ for each test function $v_2 \in C^2(\overline{\Omega}_2)$, $v_2|_{\Sigma_1^{(2)} \cup \Sigma_3^{(2)}} = 0$.

Certainly, there exists $u \in L^p(\Omega)$ such that $\int_{\Omega} u L^*(v) dx = \int_{\Omega} f v dx$ for each test function $v \in C^2(\overline{\Omega})$, $v|_{\Sigma_1 \cup \Sigma_3} = 0$. Evidently, $v \in C^2(\overline{\Omega})$, $v|_{\Sigma_1 \cup \Sigma_3} = 0 \Rightarrow v \in C^2(\overline{\Omega}_i)$, $v|_{\Sigma_1^{(i)} \cup \Sigma_3^{(i)}} = 0$, $i = 1, 2$. Consequently, $\int_{\Omega_1} u_1 L^*(v) dx = \int_{\Omega_1} f v dx$ and $\int_{\Omega_2} u_2 L^*(v) dx = \int_{\Omega_2} f v dx$, and thus the function

$$W = \begin{cases} u_1 & \text{in } \Omega_1 \\ u_2 & \text{in } \Omega_2 \end{cases} \in L^p(\Omega) \quad (48)$$

satisfies the identity $\int_{\Omega} f v dx = \int_{\Omega_1} f v dx + \int_{\Omega_2} f v dx = \int_{\Omega} W L^*(v) dx$, i.e. W is a generalized $L^p(\Omega)$ solution of

$$\begin{cases} L(W) = f & \text{in } \Omega \\ W = 0 & \text{on } \Sigma_2 \cup \Sigma_3 \end{cases} \quad (49)$$

We conclude as follows: (a) If u_i satisfies

$$\begin{cases} L(u_i) = f & \text{in } \Omega_i \\ u_i = 0 & \text{on } \Sigma_2^{(i)} \cup \Sigma_3^{(i)} \end{cases} \quad (50)$$

$i = 1, 2$, then (48) satisfies (49).

(b) In the special case when $f \in L^\infty(\Omega)$, $u_i \in L^\infty(\Omega_i)$, $i = 1, 2$, $u \in L^\infty(\Omega)$, u satisfies the identity $\int_{\Omega} f v dx = \int_{\Omega} u L^*(v) dx$, we have a uniqueness theorem and therefore $u = W$.

The set Σ_0 is called interior boundary of Ω .

Acknowledgments

The authors gratefully acknowledge financial support from CNR/BAS.

Author details

Rossella Agliardi

Department of Mathematics for Economic and Social Sciences, University of Bologna, Italy

Petar Popivanov and Angela Slavova
Institute of Mathematics, Bulgarian Academy of Sciences, Bulgaria

Appendix

One can find results concerning regularity of the generalized solutions of degenerate parabolic operators in cylindrical domains in [14] and [19]. For the sake of simplicity we shall consider only one example from II' in as the conditions are simple and clear. Consider

$$N(u) = \frac{\partial u}{\partial t} + h(t, x) \frac{\partial^2 u}{\partial x^2} + g(t, x) \frac{\partial u}{\partial x} + c(t, x)u = F(t, x) \quad (51)$$

in the rectangle $Q = \{(t, x) : 0 < t < T, a_2 < x < a_1\}$ and $h, g, c, F \in C^3(\bar{Q})$. Moreover, we assume that in some domain

- (i) $Q' \supset \bar{Q}$ the function $h \geq 0$ and $h \in C^2(Q')$.
- (ii) Suppose that if $h(t, a_1) = 0$ ($h(t, a_2) = 0$), $0 \leq t \leq T$, then $g(t, a_1) > 0$ ($g(t, a_2) < 0$).

Moreover, we assume that the following compatibility conditions hold:

- (iii) $D_{t,x}^\alpha F(T, a_1) = D_{t,x}^\alpha F(T, a_2) = 0, |\alpha| \leq 2$.

Define now the following parts of the boundary ∂Q :

$$I = \{(t, x) : 0 < t < T, x = a_2\}, II = \{(t, x) : 0 < t < T, x = a_1\}, \\ III = \{(t, x) : a_2 < x < a_1, t = 0\} \text{ and } IV = \{(t, x) : a_2 < x < a_1, t = T\}.$$

One can easily see that: $\Sigma_3 = \{(t, x) \in I \cup II : h(t, x) > 0\}$, $\Sigma^0 = \{(t, x) \in I \cup II : h(t, x) = 0\} \cup \{(t, x) \in III \cup IV\}$, $\beta = gn_1 + n_2 - \frac{\partial h}{\partial x}n_1$, i.e. $(t, x) \in \Sigma^0, (t, x) \in I \cup II \Rightarrow h(t, x) = 0 \Rightarrow \frac{\partial h}{\partial x} = 0$ and $\vec{n} = (1, 0)$ on I , $\vec{n} = (-1, 0)$ on II . Thus $\beta|_{I \cap \Sigma^0} = gn_1 = g < 0$, while $\beta|_{II \cap \Sigma^0} = -g < 0$. Therefore, $I \cap \Sigma^0 \subset \Sigma_2$, $II \cap \Sigma^0 \subset \Sigma_2$. Evidently, $\beta|_{III} = n_2 = 1 \Rightarrow III \subset \Sigma_1$, while $IV \subset \Sigma_2$; $\Sigma_0 = \emptyset$.

In conclusion, III is free of data as it is of the type Σ_1 ; $(I \cup II) \cap \Sigma^0$ and IV are of the type Σ_2 , while $\Sigma_3 = (I \cup II) \cap \{h > 0\}$. Part of $I \cup II$ is non-characteristic, part of $I \cup II$ is of Σ_2 type. Data are prescribed on $\Sigma_2 \cup \Sigma_3$, i.e. on $I \cup II \cup IV$.

Theorem 9. (see [14]).

There exists a unique classical solution u of (51), $u|_{I \cup II \cup IV} = 0$ under the conditions (i), (ii), (iii). More specifically, there exists Lipschitz continuous derivatives: $u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \in C^{0,\alpha}(\bar{Q})$, $0 < \alpha < 1$.

In [19] it is mentioned that under several restrictions on the coefficients the boundary value problem

$$\begin{cases} N(u) = 0 \\ u|_{I \cup II \cup IV} = 0 \end{cases} \quad (52)$$

possesses a unique generalized bounded solution which is Lipschitz continuous in \bar{Q} . The proof relies on the method of elliptic regularization.

Remark 8. If $a_2 < x < a_1$, $a_2 < 0$, $a_1 > 0$, the Black-Scholes equation (44) is with $h(t, x) = \frac{\sigma^2}{2}x^2 > 0$ on $I \cup II$, i.e. $\Sigma_3 = I \cup II$ and the equation

$$\begin{cases} L(u) = f & \text{in } Q \\ u|_{I \cup II \cup IV} = 0 \end{cases}$$

possesses a unique classical solution. As we know, $u|_{x=0} = U(t)$ satisfies in classical sense the ODE:

$U'(t) - rU(t) = f(t, 0)$, $U(T) = 0$. Therefore, we can consider the restrictions: $u|_{x>0}$, $u|_{x<0}$ and conclude that they are classical solutions of the respective boundary value problems with 0 data at $\Sigma_2^{(1)} \cup \Sigma_3^{(1)}$, respectively at $\Sigma_3^{(2)} \cup \Sigma_2^{(2)}$.

6. References

- [1] Agliardi, R. (2011) A comprehensive mathematical approach to exotic option pricing. forthcoming in *Mathematical Methods in the Applied Sciences*, 19 pp.
- [2] Bateman, H., Erdélyi, A. (1953) *Higher transcendental functions* vol.2. NY-Toronto-London, McGraw-Hill book Company.
- [3] Black, F., Cox, J. C. (1976) Valuing corporate securities: some effects of bond indenture provisions. *Journal of Finance*. 31: 351-367.
- [4] Boyarchenko, S. I., and S. Z. Levendorskiĭ (2002) Barrier options and touch-and-out options under regular Lévy processes of exponential type. *Annals of Applied Probability*. 12: 1261-1298.
- [5] Buchen, P. (2001) Image options and the road to barriers. *Risk*. (Sept.)
- [6] Budak, B., Samarskii, A., Tihonov, A. (1972) *A Problem Book on the Equations of Mathematical Physics*. Nauka, Moscow.
- [7] Carr, P. (1995) Two extensions to barrier option valuation. *Applied Mathematical Finance*. 2: 173-209.
- [8] Ericsson, J., Reneby, J. (1998) A framework for valuing corporate securities. *Applied Mathematical Finance*. 5: 143-163.
- [9] Fichera, G. (1956) Sulle equazioni differenziali lineari ellittico-paraboliche del secondo ordine. *Atti dell'Accademia Nazionale dei Lincei. Cl. Sci. Fis. Mat. Natur. Rend. Lincei*. 5: 1-30.
- [10] Geman, H., Yor, M. (1996) Pricing and hedging double barrier options: a probabilistic approach. *Mathematical Finance*. 6: 365-378.
- [11] Haug, E. G. (1997) *The Complete Guide to Option Pricing Formulas*. Mc-Graw Hill, N.Y.
- [12] Heynen, R., Kat, H. (1994) Crossing the barrier. *Risk*. 7: 46-51.
- [13] Heynen, R., Kat, H. (1996) Discrete partial barrier options with a moving barrier. *Journal of Financial Engineering*. 5: 199-209.
- [14] Il'in, A. (1960) Degenerate Elliptic and Parabolic Equations. *Russian Math. Sbornik*. vol. 50. 4: 443-498
- [15] Kou, S. G. (2008) Discrete barrier and lookback options. in J. R. Birge and V. Linetsky (Eds.). *Handbooks in OR & MS*, Elsevier. vol. 15: 343-373.
- [16] Kunitomo, N., Ikeda, M. (1992) Pricing options with curved boundaries. *Mathematical Finance*. 2: 275-298.

- [17] Kwok, Y., Wu, L., Yu, H. (1998) Pricing multi-asset options with an external barrier. *International Journal of Theoretical and Applied Finance*. 1: 523-541.
- [18] Merton, R. (1973) A rational theory of option pricing. *Bell Journal of Economics and Management Science*. 4: 141-183.
- [19] Oleinik, O. A. (1965) Regularity of the solutions of degenerate elliptic and parabolic equations. *Dokladi Acad. Nauk USSR*. 163. 3: 577-580.
- [20] Oleinik, O. A., Radkevič, E. V. (1971) *Second Order Equations with Non-negative Characteristic Form*. Itogi Nauki, Moscow [English translation in Plenum Press, N.Y.-London (1973)].
- [21] Poljanin, A. (2001) *Handbook-Linear Equations of Mathematical Physics*. Fizmatlit, Moscow (In Russian).
- [22] Rubinstein, M., Reiner, E. (1991) Breaking down the barriers. *Risk*. 4: 28-35.
- [23] Skipper, M. (2007) The analytic valuation of multi-asset, one-touch barrier options. *Oxford Centre of Industrial and Applied Mathematics Research Report*. 28 p.
- [24] Slavova A. (2003) *Cellular Neural Networks: Dynamics and Modeling*. Kluwer Academic Publisher. 221p.
- [25] Zhang, P. G. (2001) *Exotic Options*. World Scientific, Singapore. 2nd edition.