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Learning Automation to Teach Mathematics

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1. Introduction

One of the recurrent controversies at our Engineering Faculty concerns the orientation of first year basic courses, particularly the subject area of mathematics, considering its role as an essential tool in technological disciplines. In order to provide the basic courses with technological applications, a mathematical engineering seminar was held at the Engineering Faculty of Barcelona. Sessions were each devoted to one technological discipline and aimed at identifying the most frequently used mathematical tools with the collaboration of guest speakers from mathematics and technology departments.

In parallel, the European Space of Higher Education process is presented as an excellent opportunity to substitute the traditional teaching-learning model with another where students play a more active role. In this case, we can use the Problem-Based Learning (PBL) method. This environment is a really useful tool to increase student involvement as well as multidisciplinarity. With PBL, before students increase their knowledge of the topic, they are given a real situation-based problem which will drive the learning process. Students will discover what they need to learn in order to solve the problem, either individually or in groups, using tools provided by the teacher or 'facilitator', or found by themselves.

Therefore, a collection of exercises and problems has been designed to be used in the PBL session of the first course. These exercises include the applications identified in the seminar sessions and would be considered as the real situation-based problems to introduce the different mathematic topics. Two conditions are imposed: availability for first year students and emphasis on the use of mathematical tools in technical subjects in later academic years. As additional material, guidelines for each technological area addressed to faculties without engineering backgrounds are defined.

Some material on Electrical Engineering has been already published ((Ferrer et al., 2010)). Here we focus on control and automation. The guidelines and some exercises will be presented in detail later on. As general references on linear algebra see for example (Puerta, 1976) and on system theory (Kalman et al., 1974) and (Chen, 1984). For other applications, see (Lay, 2007).

Here we describe some of the items regarding control and automation that are presented in the guideline (Section 2).

(1) The input-output description: black box, input-output signals, impulse response, linearity, causality, relaxedness, time invariance, transfer-function matrix, time domain, frequency domain, Laplace transform, Fourier transform, gain, phase, poles, zeros, Bode diagram, filters, resonances.

(2) The state-variable description: the concept of state, state equation, output equation, transfer-function matrix, linear changes, feedbacks, realizations, stability, reachable states, controllability, control canonical form, pole assignment, observability, Kalman decomposition.

The exercises (Section 3) related to this area cover the following topics:

• Matrices. Determinant. Rank: Composition of Systems (ex. 1), Controllability Matrix, Controllable Systems and Controllability Indices (ex. 2), Realizations (ex. 3).

• Vector Spaces. Bases. Coordinates: States in Discrete Systems (ex. 4), Control Functions in Discrete Systems (ex. 4).

• Vector Subspaces: Reachable States for one or several Controls and Sum and Intersection of these Subspaces (ex. 5).

• Linear Maps: Changes of Bases in the System Equations and Invariance of the Transfer-Function Matrix (ex. 6), Controllability Subspace and Unobservable Subspace (ex. 7), Kalman Decomposition (ex. 8).

• Diagonalization. Eigenvectors, Eigenvalues: Invariant Subspaces and Restriction to an Invariant Subspace (ex. 7), Controllable Subsystem (ex. 10), Poles and Pole assignment (ex. 9).

• Non-Diagonalizable Matrices: Control Canonical Form (ex. 11)

2. Guideline for teachers

(1) External system description

Systems are considered as "black boxes" in which each input $u(t)$ (input, control, cause,...) causes an output $y(t)$ (output, effect,...), both multidimensional vectors, in general. We consider only known inputs, ignoring other ones like, for example, disturbance, noise...
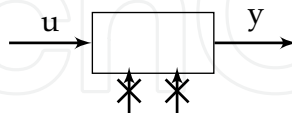
The most usual inputs will be piecewise continuous functions, built from the elementary functions or standard "signals" (impulse or delta, step, ramp, sawtooth, periodics...). Simple systems are adders, gain blocks, integrators, pure delays, filters...

In general, the aim is to analyse their behaviour without looking inside. Indeed, if we consider the "impulse response" $g(t; \tau)$ (that is, the output when the input is an "impulse" in τ, δ_τ) and the system is linear, it results

$$y(t) = \int_{-\infty}^{+\infty} g(t; \tau) u(\tau) d\tau.$$

The upper integration limit will be t if we assume that the system is "causal" (that is, if the current response does not depend on the future inputs, as in all physical systems) and the lower will be t_0 (or simply 0) if it is "relaxed" at t_0 (that is, $y(t) = 0$ for $t \geq t_0$, when $u(t) = 0$ for $t \geq t_0$). Finally, if it is "time-invariant", instead of $g(t; \tau)$ we can write $g(t - \tau)$, resulting in the following expression of $y(t)$ as a convolution product of $g(t)$ and $u(t)$

$$y(t) = \int_{-\infty}^{+\infty} g(t - \tau)u(\tau)d\tau = g(t) * u(t),$$

which is the general system representation in the "time domain".

Applying Laplace transform we get the general representation in the "s-domain"

$$\hat{y}(s) = G(s)\hat{u}(s),$$

where $G(s) = \hat{g}(s)$ is called "transfer function matrix". Indeed, it is the focus of study in this external representation.

If we do the change of variable $s = j\omega$ (imaginary axis of the "complex plane") we obtain the representation in the "frequency domain", (more usual in engineering)

$$y(\omega) = G(\omega)u(\omega),$$

where $G(\omega)$ is called "isochronous transfer function matrix". It can be obtained directly as a "Fourier" transform when appropriate hypotheses hold. This allows the use of tools such as Fourier transform, Parseval theorem... if we have some basic knowledge of functional analysis and complex variable: function spaces, norms, Hilbert spaces, integral transforms...

Its denomination shows that $G(\omega)$ indicates the system behaviour for each frequency. So, if $u(t) = A\sin(\omega t)$, then $y(t) = B\sin(\omega t + \varphi)$, being the "gain" B/A the module of $G(\omega)$, and the "phase" φ is its argument. A widely used tool in engineering is the "Bode diagrams" which represent these magnitudes on the ordinate (usually, the gain is in logarithmic scale, in decibels: $dB = 20log|G|$) as a function of the frequency as abscissa (also in log scale).

In generic conditions (see next section) the coefficients of G are "proper rational fractions", so that the system's behaviour is largely determined by its "degrees", "zeros" and "poles". So:

- The relative degree (denominator degree minus numerator degree) gives the "order of differentiability" at the origin of the response to the input step signal.

- As already mentioned, this difference must be strictly positive (if zero, the Parseval theorem would give infinite energy in the output signal; if negative, it would contradict the causality), so that the gain tends to 0 for high frequencies.

- Roughly speaking, the zeros indicate filtered frequencies (of gain 0, or in practice far below), so that in the Bode diagram place the "inverted comb pas" as "filters". On the contrary, the poles indicate dangerous frequencies because of resonance (infinite gain).

(2) Internal system description

In addition, state variables $x(t)$ (not univocally defined) are considered. They characterize the state in the sense that they accumulate all the information from the past, that is, future outputs are determined by the current state and the future inputs. Typically, the derivative $\dot{x}(t)$

functionally depends on $x(t)$, $u(t)$ and t (state equation), and $y(t)$ as well (output equation), although in this case we may obviate the dependence on $u(t)$. In the linear case:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t); \quad y(t) = C(t)x(t).$$

From elementary theory of ordinary differential equations, it holds that for every continuous (or piecewise continuous) control $u(t)$, there exists a unique "solution" $x(t)$ for every "initial condition" $x_0 = x(0)$.

In particular, it can be applied a "feedback control" by means of a matrix F

$$u(t) = Fx(t).$$

One of the first historical (non linear) examples is the Watt regulator which controlled the velocity of a steam engine acting on the admission valve in the function of the centrifugal force created in the regulator balls by this velocity. Nowadays this "automatic regulation" can be found in simple situations such as thermostats.

When this kind of feedback is applied, we obtain an autonomous dynamical system

$$\dot{x}(t) = Ax(t) + BFx(t) = (A + BF)x(t)$$

called a "closed loop" system. It is natural to consider if we can adequately choose F such that this system has suitable dynamic properties. For example, for being "stable", that is, that the real parts of the eigenvalues of $A + BF$ are negative. Or more in general, that these eigenvalues have some prefixed values. As we will see later, one of the main results is that this feedback "pole assignment" is possible if the initial system is "controllable".

If it is time-invariant (that is, A, B, and C are constant) and we assume $x(0) = 0$, the Laplace transform gives

$$\hat{y}(t) = G(s)\hat{u}(s); \quad G(s) = C(sI - A)^{-1}B$$

recovering the previous transfer matrix. Reciprocally, the "realization" theory constructs triples (A, B, C) giving a prefixed G, formed by proper rational fractions: it can be seen that it is always possible; the uniqueness conditions will be seen later.

It is a simple exercise to check that when introducing a "linear change" S in the state variables, the system matrices become $S^{-1}AS$, $S^{-1}B$ and CS, respectively, and $G(s)$ do not change.

In this description it is clear that the coefficients of $G(s)$ are proper rational fractions and in particular its poles are the eigenvalues of A.

One of the main results is that the set of "reachable states" (the possible $x(t)$ starting from the origin, when varying the controls $u(t)$) is the image subspace $\operatorname{Im} K(A, B)$ of the so-called "controllability matrix"

$$\operatorname{Im} K(A, B); \quad K(A, B) = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix},$$

that is, the subspace spanned by the columns of B and successive images for A, called "controllability subspace". It is an interesting exercise to justify that we can truncate in $n - 1$, since the columns of higher powers are linearly dependent with the previous ones. In fact we can consider each control individually (each column of B separately), in such a way that the

"sum and intersection of subspaces" give the reachable states when using different controls at the same time or for each one of them.

The system is "controllable" if all the states are reachable, that is, if and only if $K(A, B)$ has full "rank". Hence, it is a generic condition (that is, the subset of controllable pairs (A, B) is open and dense).

In the single input case (A, b), it can be seen that if the system is controllable it can be transformed, by means of a suitable basis change, in the so-called "control canonical form"

$$\bar{A} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ & & \ldots & \ldots & \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ * & * & * & \ldots & * & * \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Observe that $\bar{A}$ is a "companion" (or Sylvester) matrix. For these kinds of matrices, it can be easily seen that the coefficients of the characteristic polynomial are the ones of the last row (with opposite sign and in reverse order), and they are "non derogatory matrices", that is, for each eigenvalue there is a unique linearly independent eigenvector, and therefore only one Jordan block. Hence, if some eigenvalue is multiple, the matrix is non diagonalizable.

In the multi input case, another reduced form is used (Brunovsky, or Kronecker form), which is determined by the so-called "controllability indices" that can be computed as a conjugated partition from the one of the ranks of: B, (B, AB),...,$K(A, B)$.

From these reduced forms it is easy to prove that the pole assignment is feasible, as well as to compute the suitable feedbacks.

If the system is not controllable, it is easy to see that the subspace Im $K(A, B)$ is "A-invariant", and that in any "adapted basis" (of the state space) the matrices of the system are of the form:

$$\begin{pmatrix} A_c & * \\ 0 & * \end{pmatrix}, \quad \begin{pmatrix} B_c \\ 0 \end{pmatrix},$$

being (A_c, B_c) controllable. It is also easy to deduce that if $x(0)$ belongs to Im $K(A, B)$, $x(t)$ also belongs, for all time t and all control $u(t)$, and that its trajectory is determined for the pair (A_c, B_c), which enables considering the "restriction" of the system (A, B) to Im $K(A, B)$, called the "controllable subsystem" of the initial one.

In the same direction, the "Kalman decomposition" is obtained by considering "(Grassman) adapted bases to the pair of subspaces" Im $K(A, B)$ and Ker $L(C, A)$, being

$$L(C, A) = \begin{pmatrix} C \\ CA \\ \ldots \\ CA^{n-1} \end{pmatrix}$$

that is, the "transposed matrix" of $K(A^t, C^t)$. In fact there are interesting properties of "duality" between the systems (A, B, C) and (A^t, C^t, B^t).

We remark the equivalence between the controllability of (A, B, C) and the "observability" of (A^t, C^t, B^t), in the sense that the initial conditions are computable if the outputs for determined inputs are known.

On the other hand it is interesting to note that the transfer matrix of the initial system is the same as that of the controllable subsystem, as well as that of the "complete subsystem" (that is, controllable and observable) obtained by means of the Kalman decomposition.

This means that, given any transfer matrix (formed by proper rational fractions), not only it is possible to find realizations, but also "controllable realizations" and even complete ones. In fact it can be seen that all the complete realizations are equivalent. In particular, they have the same number of state variables. Moreover, a realization is complete if and only if it is a "minimal realization", in the sense that there are not realizations with a smaller number of state variables. This minimal number of state variables of the realizations is called "McMillan degree", which coincides with the dimension of the complete realizations.
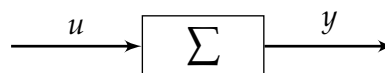
3. Exercises for students

We will present here the guideline of proposed exercises for the students and their solutions:

3.1 Proposed exercises

1. Composition of systems

A control system Σ

is defined by the equations

$$\dot{x}(t) = Ax(t) + Bu(t); \qquad y(t) = Cx(t),$$

or simply by the "triple of matrices" (A, B, C). This triple determines the "transfer matrix"

$$G(s) = C(sI - A)^{-1}B$$

which relates the Laplace transforms of u, y:
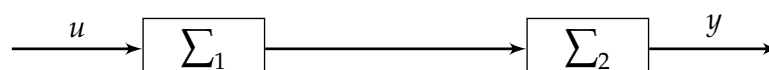
$$\hat{y}(s) = G(s)\hat{u}(s).$$

Given two systems:

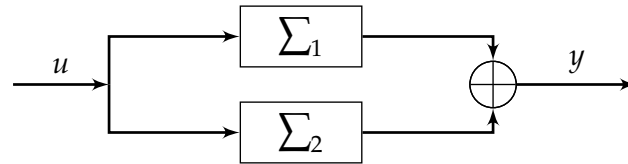$$\Sigma_1: \quad \dot{x}_1 = A_1 x_1 + B_1 u_1; \quad y_1 = C_1 x_1$$
$$\Sigma_2: \quad \dot{x}_2 = A_2 x_2 + B_2 u_2; \quad y_2 = C_2 x_2$$

they can be composed of different ways to obtain a new system. For example in:
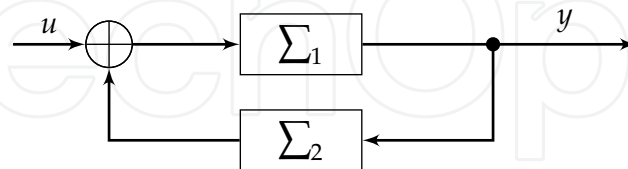
(i) Series

(ii) Parallel

(iii) Feedback

In all these cases the new state variables are:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(1) Deduce the following relations between the different input and output variables:

(1.i) $u = u_1; y_1 = u_2; y_2 = y$

(1.ii) $u = u_1 = u_2; y = y_1 + y_2$

(1.iii) $u + y_2 = u_1; y_1 = y = u_2$

(2) Deduce the expression of the triple of matrices of each composed system, in terms of $A_1, B_1, C_1, A_2, B_2, C_2$.

(3) Deduce the expression of the transfer matrix of each composed system, in terms of $G_1(s)$ and $G_2(s)$.

2. Controllable systems; controllability indices

A control system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad A \in M_n, \quad B \in M_{n,m}$$

is "controllable" (that is, any change in the values of x is possible by means of a suitable control $u(t)$), if and only if the rank of the so-called "controllability matrix" is full

$$K(A,B) = (B, AB, A^2B, \ldots, A^{n-1}B).$$

(a) Discuss for which values of $\alpha, \beta \in \mathbb{R}$ the system defined by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & \beta \\ 0 & 1 \end{pmatrix}$$

is controllable.

(b) Discuss for which values of $\alpha, \beta \in \mathbb{R}$ the system is controllable with only the second control, that is, when instead of the initial matrix B we consider only the column matrix

$$b_2 = \begin{pmatrix} 0 \\ \beta \\ 1 \end{pmatrix}.$$

(c) In general, the "controllability indices" are determined by the rank of the matrices

$$B, \quad (B, AB), \quad (B, AB, A^2 B), \ldots, \quad K(A, B).$$

Compute these ranks, in terms of parameters $\alpha, \beta \in \mathbb{R}$, for the system defined by

$$A = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & 0 \\ & & & & 0 & \\ \hline & 0 & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \gamma \\ 0 & \delta \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

3. Realizations

Given a matrix $G(s)$ of proper rational fractions in the variable s, it is called a "realization" any linear control system

$$\dot{x}(t) = Ax(t) + Bu(t); \quad y(t) = Cx(t)$$

which has $G(s)$ as "transfer matrix", that is,

$$G(s) = C(sI - A)^{-1} B.$$

In particular, the so-called "standard controllable realization" is obtained in the following way, assuming that $G(s)$ has p rows and m columns.

(i) We determine the least common multiple polynomial of the denominators

$$P(s) = p_0 + p_1 s + \ldots + p_r s^r.$$

(ii) Then, $G(s)$ can be written

$$G(s) = \frac{1}{P(s)} \begin{pmatrix} G_{11}(s) & \cdots & G_{1m}(s) \\ \cdots & \cdots & \cdots \\ G_{p1}(s) & \cdots & G_{pm}(s) \end{pmatrix},$$

where $G_{ij}(s)$ are polynomials with degree strictly lower than $r (= \deg P(s))$.

(iii) Grouping the terms of the same degree we can write $G(s)$ in the form

$$G(s) = \frac{1}{P(s)} (R_0 + R_1 s + \ldots + R_{r-1} s^{r-1}),$$

where $R_0, \ldots, R_{r-1} \in M_{p,m}(\mathbb{R})$.

(iv) Then, the standard controllable realization is given by the triple of matrices (where 0_m is the null matrix of $M_m(\mathbb{R})$):

$$A = \begin{pmatrix} 0_m & I_m & 0_m & \cdots & 0_m \\ 0_m & 0_m & I_m & \cdots & 0_m \\ & \cdots & \cdots\cdots \\ 0_m & 0_m & 0_m & \cdots & I_m \\ -p_0 I_m & -p_1 I_m & \cdots\cdots & \cdots & -p_{r-1} I_m \end{pmatrix} \in M_{mr}(\mathbb{R}), \quad B = \begin{pmatrix} 0_m \\ \vdots \\ 0_m \\ I_m \end{pmatrix} \in M_{mr,m}(\mathbb{R}),$$

$$C = \begin{pmatrix} R_0 & \cdots & R_{r-1} \end{pmatrix} \in M_{p,mr}(\mathbb{R}).$$

We consider, for example,

$$G(s) = \begin{pmatrix} 1/s \\ 1/(s-1) \end{pmatrix}.$$

(1) Following the above paragraphs, compute the triple of matrices (A, B, C) which give the standard controllable realization.

(2) Check that it is controllable, that is, that the matrix

$$K(A, B) = (B, AB, \ldots, A^{mr-1}B)$$

has full rank.

(3) Check that it is a realization of $G(s)$, that is,

$$\begin{pmatrix} 1/s \\ 1/(s-1) \end{pmatrix} = C(sI - A)^{-1}B.$$

4. Reachable states; control functions

Given a linear control system

$$x(k+1) = Ax(k) + Bu(k); \quad A \in M_n(\mathbb{R}), \quad B \in M_{n,m}(\mathbb{R}),$$

the h-step "reachable states", from $x(0)$, are:

$$x(h) = A^h x(0) + A^{h-1} Bu(0) + \ldots + ABu(h-2) + Bu(h-1)$$

where all possible "control functions" $u(0), u(1), \ldots$ are considered. More explicitly, if we write $B = (b_1 \ldots b_m)$ and $u(k) = (u_1(k), \ldots, u_m(k))$, we have that:

$$\begin{aligned} x(h) = {}& A^h x(0) \\ & + (A^{h-1} b_1 u_1(0) + \ldots + A^{h-1} b_m u_m(0)) + \ldots \\ & + (Ab_1 u_1(h-2) + \ldots + Ab_m u_m(h-2)) + (b_1 u_1(h-1) + \ldots + b_m u_m(h-1)) \end{aligned}$$

Let

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 1 & -1 \\ 2 & 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}.$$

(1) Assume that only the first control acts, that is: $u_2(0) = u_2(1) = \ldots = 0$. Show that in this case, the state $x = (-1, 2, 1)$ is not 3-step reachable from the origin.

(2) Assume now that only the second control acts.

(2.1) Prove that $x = (-1, 2, 1)$ is 2-step reachable from the origin.

(2.2) Compute the corresponding control function.

(3) Assume that both controls act.

(3.1) Determine the control functions set to reach $x = (-1, 2, 1)$ from the origin, at second step.

(3.2) In particular, check if it is possible to choose positive controls.

(3.3) Idem for $u_1(0) = u_2(0)$.

5. Subspaces of reachable states

Given a linear control system

$$x(k+1) = Ax(k) + Bu(k); \quad A \in M_n(\mathbb{R}), \quad B \in M_{n,m}(\mathbb{R}),$$

the h-step "reachable states", from $x(0)$, are

$$x(h) = A^h x(0) + A^{h-1} Bu(0) + \ldots + ABu(h-2) + Bu(h-1),$$

where all possible control functions $u(0), u(1), \ldots$ are considered.

(1) We write $K(h)$ the set of these states when $x(0) = 0$. Show that:

(1.1) $K(h) = [B, AB, \ldots, A^{h-1}B] \subset \mathbb{R}^n$

(1.2) $K(1) \subset K(2) \subset \ldots \subset K(j) \subset \ldots$

(1.3) $K(h) = K(h+1) \Rightarrow K(h+1) = K(h+2) = \ldots$

(1.4) $K(n) = K(n+1) = \ldots$

This maximal subspace of the chain is called "subspace of reachable states":

$$K = [B, AB, \ldots, A^{n-1}B] \subset \mathbb{R}^n$$

(2) Analogous results hold when only the control $u_i(k)$ acts. In particular, the subspace of reachable states, from the origin, with only this control is:

$$K_i = [b_i, Ab_i, \ldots, A^{n-1}b_i] \subset \mathbb{R}^n; \quad 1 \leq i \leq m$$

where $B = (b_1, \ldots, b_m)$.

(2.1) Reason that the reachable states by acting the controls $u_i(k)$ and $u_j(k)$ are $K_i + K_j$.

(2.2) Reason that $K_i \cap K_j$ is the subspace of reachable states by acting any of the controls $u_i(k)$ or $u_j(k)$.

(3) Let us consider the linear control system defined by the matrices

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 1 & -1 \\ 2 & 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ -1 & 1 \\ 1 & 0 \end{pmatrix}.$$

(3.1) Determine the subspaces K, K_1 and K_2, and construct a basis of each one.

(3.2) Idem for $K_1 + K_2$, $K_1 \cap K_2$.

6. Change of state variables in control systems

In the linear control system

$$\dot{x}(t) = Ax(t) + Bu(t); \quad y(t) = Cx(t)$$

$$A \in M_n, \quad B \in M_{n,m}, \quad C \in M_{p,n},$$

we consider a linear change in the state variables given by:

$$\bar{x} = S^{-1}x.$$

(1) Prove that in the new variables the equations of the system are:

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t); \quad y(t) = \bar{C}\bar{x}(t)$$

$$\bar{A} = S^{-1}AS, \quad \bar{B} = S^{-1}B, \quad \bar{C} = CS.$$

(2) The "controllability indices" of the system are computed from the ranks:

$$\text{rank}(B, AB, \ldots, A^h B), \quad h = 1, 2, 3, \ldots$$

Deduce from (1) that they are invariant under linear changes in the state variables.

(3) The "transfer matrix" of the system is:

$$G(s) = C(sI - A)^{-1}B.$$

Deduce from (1) that it is invariant under linear changes in the state variables.

7. Controllability subspaces and unobservability subspaces

Given a linear control system

$$\dot{x}(t) = Ax(t) + Bu(t); \quad y(t) = Cx(t)$$

$$A \in M_n(\mathbb{R}), \quad B \in M_{n,m}(\mathbb{R}), \quad C \in M_{p,n}(\mathbb{R}),$$

the following subspaces are called "controllability subspace" and "unobservability subspace", respectively:

$$K = \text{Im} \begin{pmatrix} B & AB & \ldots & A^{n-1}B \end{pmatrix}$$

$$L = \text{Ker} \begin{pmatrix} C \\ CA \\ \ldots \\ CA^{n-1} \end{pmatrix}.$$

(1) Show that they are A-invariant subspaces.

(2) Let us consider

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = (0\ 1\ 0\ 1).$$

  (2.1) Compute the dimensions of K and L.
  (2.2) Construct a basis of each one.
  (2.3) Obtain the matrices, in these bases, of the restrictions $A|_K$ and $A|_L$.
  (2.4) Idem for the subspace $K \cap L$.

8. Kalman decomposition

Given a linear control system

$$\dot{x}(t) = Ax(t) + Bu(t); \quad y(t) = Cx(t)$$

$$A \in M_n(\mathbb{R}), \quad B \in M_{n,m}(\mathbb{R}), \quad C \in M_{p,n}(\mathbb{R}),$$

we consider the subspaces

$$K = \operatorname{Im} K(A, B), \quad L = \operatorname{Ker} L(C, A),$$

where

$$K(A, B) = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix},$$

$$L(C, A) = \begin{pmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{pmatrix}.$$

A (Grassman) adapted basis to both subspaces is called a "Kalman basis". More specifically, the basis change matrix is of the form

$$S = \begin{pmatrix} S_1 & S_2 & S_3 & S_4 \end{pmatrix}$$
$$S_2 : \text{basis of } K \cap L$$
$$(S_1 \quad S_2) : \text{basis of } K$$
$$(S_2 \quad S_4) : \text{basis of } L$$

where some of the submatrices $S_1, \dots, S_4$ can be empty.

(1) Prove that with a basis change of this form, the matrices of the system become of the form

$$\bar{A} = S^{-1}AS = \begin{pmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{pmatrix}$$

$$\bar{B} = S^{-1}B = \begin{pmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{pmatrix}$$

$$\bar{C} = CS = \begin{pmatrix} C_1 & 0 & C_3 & 0 \end{pmatrix}.$$

It is called "Kalman decomposition" of the given system.

(2) Consider the system given by

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}.$$

(2.1) Determine a Kalman basis.

(2.2) Determine a Kalman decomposition.

(2.3) Check that

(A_{11}, B_1, C_1) is controllable and observable,

$\left(\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \begin{pmatrix} C_1 \\ 0 \end{pmatrix} \right)$ is controllable,

$\left(\begin{pmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{pmatrix}, \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \begin{pmatrix} C_1 \\ C_3 \end{pmatrix} \right)$ is observable,

that is, that the following matrices have full rank:

$K(A_{11}, B_1) = \begin{pmatrix} B_1 & A_{11}B_1 & \dots & (A_{11})^{n-1}B_1 \end{pmatrix}$,

$(L(C_1, A_{11}))^t = \begin{pmatrix} C_1^t & A_{11}^t C_1^t & \dots & (A_{11}^t)^{n-1}C_1^t \end{pmatrix}$,

$K \left(\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right) = \left(\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \dots \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}^{n-1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right)$,

$\left(L \left(\begin{pmatrix} C_1 & C_3 \end{pmatrix}, \begin{pmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{pmatrix} \right) \right)^t = \left(\begin{pmatrix} C_1^t \\ C_3^t \end{pmatrix} \dots \begin{pmatrix} A_{11}^t & 0 \\ A_{13}^t & A_{33}^t \end{pmatrix}^{n-1} \begin{pmatrix} C_1^t \\ C_3^t \end{pmatrix} \right).$

9. Pole assignment by state feedback

Given a linear control system
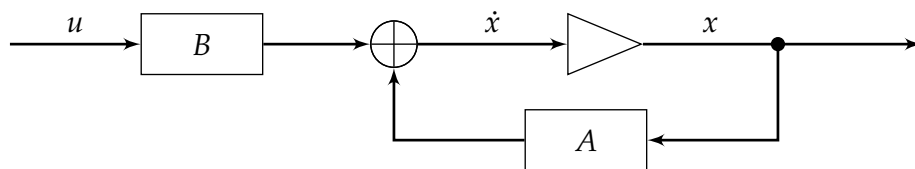
$$\dot{x}(t) = Ax(t) + Bu(t); \quad A \in M_n, \quad B \in M_{n,m},$$

the eigenvalues of the matrix A are called the "poles" of the system and play an important role in its dynamic behaviour. For example, they are the resonance frequencies; the system is "BIBO stable" if and only if the real part of its poles is negative.

If we apply an automatic control by means of a "feedback" F (for example, a thermostat, a Watt regulator,...)
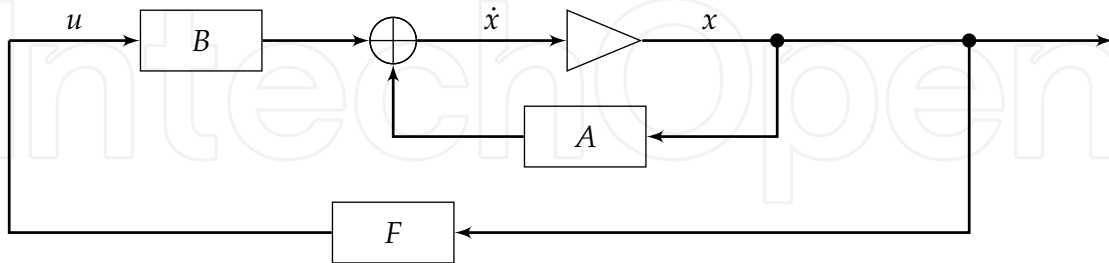
we obtain an autonomous dynamic system (called a "closed loop" system):

$$\dot{x}(t) = Ax(t) + BFx(t) = (A + BF)x(t)$$

with matrix $A + BF$ instead of the initial matrix A. We ask if it is possible to choose adequately F such that the new poles (that is, the eigenvalues of $A + BF$) have some prefixed desired values. For example, with negative real part, so that the automatic control is stable.

A main result of control theory ensures that this pole assignment by feedback is possible if the initial system is controllable, that is, if $\text{rank}(B, AB, A^2B, \dots, A^{n-1}B) = n$.

We will prove and apply that in the one-parameter case ($m = 1$). We will do it by transforming the initial system in the so-called "control canonical form" by means of a suitable basis change. In this form, the feedback computation is trivial, and finally it will only be necessary to undo the transformation.

(1) Consider a one-parameter control system

$$\dot{x}(t) = Ax(t) + bu(t); \quad A \in M_n, \quad b \in M_{n,1},$$

and let $\mu_1, \dots, \mu_n$ be the desired eigenvalues for the feedback system (each one repeated so many times as its algebraic multiplicity). The main hypothesis is that the system is controllable, that is, $\text{rank}(b, Ab, A^2b, \dots, A^{n-1}b) = n$.

Then it is possible to find a linear change S in the state variables such that the new matrices of the system have the so-called "control canonical form":

$$A_c = S^{-1}AS = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & \dots & \dots & & \\ 0 & 0 & 1 & \dots & 0 & 1 \\ a_n & a_{n-1} & a_{n-2} & \dots & a_2 & a_1 \end{pmatrix}, \quad b_c = S^{-1}b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

for certain coefficients $a_1, \dots, a_n$.

(1.1) Check that these coefficients are the same (with opposite sign) than the ones of the characteristic polynomial:

$$Q(t) = (-1)^n (t^n - a_1 t^{n-1} - \ldots - a_{n-1} t - a_n).$$

(1.2) Deduce that it is straightforward to find F_c such that:

$$\text{eigenvalues}(A_c + bcF_c) = \{\mu_1, \ldots, \mu_n\}.$$

(1.3) Prove that $F = F_c S^{-1}$ is the sought feedback, that is, that:

$$\text{eigenvalues}(A + bF) = \{\mu_1, \ldots, \mu_n\}.$$

(2) Consider the particular case:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(2.1) Check that the basis change

$$S = \begin{pmatrix} -2 & 1 & 1 \\ -2 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

transform the initial matrices into the control canonical form.

(2.2) Compute a feedback F such that:

$$\text{eigenvalues}(A + bF) = \{\mu_1, \mu_2, \mu_3\}.$$

10. Controllable subsystem

Given a linear control system

$$\dot{x}(t) = Ax(t) + Bu(t); \quad y(t) = Cx(t) \tag{1}$$
$$A \in M_n, \quad B \in M_{n,m}, \quad C \in M_{p,n},$$

the following matrix is called its "controllability matrix"

$$K(A, B) = \begin{pmatrix} B & AB & A^2 B & \ldots & A^{n-1} B \end{pmatrix}.$$

It can be seen that the subspace spanned by its columns $K = \text{Im } K(A, B)$ is the set of reachable states from the origin, and it is called "controllability subspace". We denote $d = \dim K$.

(1) Prove that K is an A-invariant subspace.

We consider a basis change S in the state space adapted to K, and we denote by x_c the first d new coordinates and by x_{uc} the remaining ones:

$$\bar{x} = S^{-1} x = \begin{pmatrix} x_c \\ x_{uc} \end{pmatrix}, \quad x_c \in M_{d,1}.$$

(2) Check that:
$$x \in K \Leftrightarrow x_{uc} = 0$$

that is:
$$x \in K \Leftrightarrow \bar{x} = S^{-1}x = \begin{pmatrix} x_c \\ 0 \end{pmatrix}.$$

(3) Prove that the equations of the system in the new variables are:

$$\begin{pmatrix} \dot{x}_c(t) \\ \dot{x}_{uc}(t) \end{pmatrix} = \bar{A} \begin{pmatrix} x_c(t) \\ x_{uc}(t) \end{pmatrix} + \bar{B}u(t)$$

$$\bar{A} = S^{-1}AS = \begin{pmatrix} A_c & * \\ 0 & * \end{pmatrix}, \quad A_c \in M_d$$

$$\bar{B} = S^{-1}B = \begin{pmatrix} B_c \\ 0 \end{pmatrix}, \quad B_c \in M_{d,m}$$

$$\bar{C} = CS = \begin{pmatrix} C_c \\ C_{uc} \end{pmatrix}, \quad C_c \in M_{p,d}.$$

(4) Deduce that if the initial state belongs to K, it also belongs all the trajectory, for any applied control $u(t)$:

$$x(0) \in K \Rightarrow x(t) \in K, \quad \forall t, \quad \forall u(t).$$

Therefore, it makes sense to consider the "restriction" to K of the initial system:

$$\dot{x}_c(t) = A_c x_c(t) + B_c u(t); \quad y_c(t) = C_c x_c(t). \tag{2}$$

The following paragraphs justify the interest of this subsystem and its denomination as "controllable subsystem" of the initial one.

(5) Justify that the trajectories of system (1) in K can be computed by means of equations (2) of the subsystem and relation (2).

(6) Prove that this subsystem is "controllable", that is, that the following matrix has full rank:

$$\begin{pmatrix} B_c & A_c B_c & \dots & A_c^{d-1} B_c \end{pmatrix}.$$

(7) Prove that the controllable subsystem has the same "transfer matrix" (which, we recall, reflects the input/output behaviour) than the initial system, that is, that:

$$C(sI - A)^{-1}B = C_c(sI - A_c)^{-1}B_c.$$

11. Control canonical form

For controllable systems, the so-called "control canonical form" simplifies the computations, for example, for the pole assignment by feedback. We are going to obtain it for the one-parameter case.

(1) Consider the system

$$\dot{x}(t) = Ax(t) + bu(t)$$

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(1.1) Check that it is controllable, that is, that:

$$\text{rank} \begin{pmatrix} b & Ab & A^2b \end{pmatrix} = 3.$$

(1.2) Check that doing the basis change

$$S_1 = \begin{pmatrix} A^2b & Ab & b \end{pmatrix}$$

we get

$$\bar{A} = S_1^{-1} A S_1 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \bar{b} = S^{-1}b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(1.3) Check that with the additional change

$$S_2 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

we get the control canonical form

$$A_c = S_2^{-1} \bar{A} S_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix}, \quad b_c = S_2^{-1}\bar{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(2) Consider the system

$$\dot{x}(t) = Ax(t) + bu(t); \quad A \in M_n, \quad b \in M_{n,1}$$

which is assumed to be controllable:

$$\text{rank} \begin{pmatrix} b & Ab & \ldots & A^{n-1}b \end{pmatrix} = n$$

and let $Q(t)$ be the characteristic polynomial of A

$$Q(t) = (-1)^n (t^n - a_1 t^{n-1} - \ldots - a_{n-1}t - a_n).$$

(2.1) Check that doing the basis change

$$S_1 = \begin{pmatrix} A^{n-1}b & \ldots & Ab & b \end{pmatrix}$$

we get

$$\bar{A} = S_1^{-1} A S_1 = \begin{pmatrix} a_1 & 1 & 0 & \ldots & 0 \\ a_2 & 0 & 1 & \ldots & 0 \\ & & \ldots & & \\ a_{n-1} & 0 & 0 & \ldots & 1 \\ a_n & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad \bar{b} = S_1^{-1}b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

(2.2) Check that with the additional change

$$S_2 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ -a_1 & 1 & \dots & 0 & 0 \\ -a_2 & -a_1 & \dots & 0 & 0 \\ & & \dots & & \\ -a_{n-1} & -a_{n-2} & \dots & -a_1 & 1 \end{pmatrix}$$

we get the control canonical form

$$A_c = S_2^{-1}\bar{A}S_2 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_n & a_{n-1} & a_{n-2} & \dots & a_2 & a_1 \end{pmatrix}, \quad b_c = S_2^{-1}\bar{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

(3) As an application, we use the control canonical form to check that A is non-derogatory:

(3.1) Prove that if λ_i is an eigenvalue of A, then:

$$\mathrm{rank}(A_C - \lambda_i I) = n - 1.$$

(3.2) Deduce that A is non-derogatory, and so it does not diagonalize if some of its eigenvalues are multiple.

3.2 Solutions

1. Solution

(1) It follows immediately from the observation of the diagrams.

(2.i)

$$\dot{x}_1 = A_1 x_1 + B_1 u_1 = A_1 x_1 + B_1 u$$
$$\dot{x}_2 = A_2 x_2 + B_2 u_2 = A_2 x_2 + B_2 y_1 = A_2 x_2 + B_2 C_1 x_1$$
$$y = y_2 = C_2 x_2$$

Hence:

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(2.ii) Reasoning in the same way, it results:

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad \begin{pmatrix} C_1 & C_2 \end{pmatrix}$$

(2.iii) Analogously:

$$\begin{pmatrix} A_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{pmatrix}, \quad \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} C_1 & 0 \end{pmatrix}$$

(3.i) $\hat{y} = \hat{y}_2 = G_2 \hat{u}_2 = G_2 \hat{y}_1 = G_2 G_1 \hat{u}_1 = G_2 G_1 \hat{u}$. Hence, the transfer matrix is:

$$G_2 G_1$$

(3.ii) Analogously:

$$G_1 + G_2$$

(3.iii) $\hat{y} = \hat{y}_1 = G_1 \hat{u}_1 = G_1 (\hat{u} + \hat{y}_2) = G_1 \hat{u} + G_1 G_2 \hat{u}_2 = G_1 \hat{u} + G_1 G_2 \hat{y}$

$(I - G_1 G_2) \hat{y} = G_1 \hat{u}$

$\hat{y} = (I - G_1 G_2)^{-1} G_1 \hat{u}$

2. Solution

(a) The controllability matrix is

$$K(A, B) = \begin{pmatrix} 1 & 1 & 0 & \beta & -2 & \beta \\ 0 & \beta & -2 & \beta & -2 & -\beta \\ 0 & 1 & \alpha & 1 & \alpha & \alpha\beta + 1 \end{pmatrix},$$

which clearly has full rank for all α, β. Hence, the given system is controllable for all α, β.

(b) When only the second control acts, that is to say, when $u_1(t) = 0$ for all t, the system can be written

$$\dot{x}(t) = Ax(t) + b_2 u_2(t).$$

Then, the controllability matrix is reduced to

$$K(A, b_2) = \begin{pmatrix} 1 & \beta & \beta \\ \beta & \beta & -\beta \\ 1 & 1 & \alpha\beta + 1 \end{pmatrix},$$

which has full rank for $\beta \neq 0$, $\alpha \neq \frac{-2}{\beta}$.

(c) Now one has

$$K(A, B) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \gamma & 0 & \delta & 1 & 0 & & & \\ 0 & 0 & 0 & \gamma & 0 & \delta & 1 & 0 & 0 & 0 & & & \\ 0 & \gamma & 0 & \delta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \end{pmatrix}.$$

The "controllability indices" can be computed from the ranks of the $2, 4, 6, \ldots$ first columns:

$$\text{rank}(B) = 2, \text{for all } \gamma, \delta$$
$$\text{rank}(B, AB) = 4, \text{for all } \gamma, \delta$$
$$\text{rank}(B, AB, A^2B) = 5 \text{ if } \gamma = \delta = 0; \quad \text{rank}(B, AB, A^2B) = 6 \text{ otherwise}$$
$$\text{rank}(B, \ldots, A^3B) = 6 \text{ if } \gamma = \delta = 0; \quad \text{rank}(B, \ldots, A^3B) = 7 \text{ otherwise}$$
$$\text{rank}(B, \ldots, A^4B) = 7$$

3. Solution

(1) Let

$$G(s) = \begin{pmatrix} 1/s \\ 1/(s-1) \end{pmatrix}.$$

Then: $p = 2, m = 1$. Following the given pattern:

(i) $P(s) = s(s-1) = -s + s^2$
$r = 2, p_0 = 0, p_1 = -1, p_2 = 1$

(ii) $G(s) = \frac{1}{-s+s^2} \begin{pmatrix} s-1 \\ s \end{pmatrix}$

(iii) $G(s) = \frac{1}{-s+s^2} \left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s \right)$

$R_0 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, R_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(iv) $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

(2) $\text{rank } K(A, B) = \text{rank } \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = 2$

(3) $(sI - A)^{-1} = \left(s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right)^{-1} = \begin{pmatrix} s & -1 \\ 0 & s-1 \end{pmatrix}^{-1} = \frac{1}{s^2 - s} \begin{pmatrix} s-1 & 1 \\ 0 & s \end{pmatrix}$

$$C(sI - A)^{-1} B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \frac{1}{s^2 - s} \begin{pmatrix} s-1 & 1 \\ 0 & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{s^2 - s} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ s \end{pmatrix}$$

$$= \frac{1}{s^2 - s} \begin{pmatrix} -1+s \\ s \end{pmatrix} = G(s)$$

4. Solution

(1) $x = (-1, 2, 1)$ is 3-step reachable, from the origin, if there are $u_2(0), u_2(1), u_2(2)$ such that:

$$x = A^2 b_1 u_1(0) + A b_1 u_1(1) + b_1 u_1(2)$$

$$= \begin{pmatrix} 10 \\ -8 \\ 8 \end{pmatrix} u_1(0) + \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} u_1(1) + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} u_1(2)$$

or equivalently::

$$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 & -2 & 0 \\ -8 & -2 & -1 \\ 8 & 2 & 1 \end{pmatrix} \begin{pmatrix} u_1(0) \\ u_1(1) \\ u_1(2) \end{pmatrix}$$

No solutions exist because

$$\text{rank} \begin{pmatrix} 10 & -2 & 0 \\ -8 & -2 & -1 \\ 8 & 2 & 1 \end{pmatrix} = 2 \qquad \text{rank} \begin{pmatrix} 10 & -2 & 0 & -1 \\ -8 & -2 & -1 & 2 \\ 8 & 2 & 1 & 1 \end{pmatrix} = 3$$

(2) Analogously:

$$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_2(0) \\ u_2(1) \end{pmatrix}$$

whose solution is: $u_2(0) = 1$, $u_2(1) = 1$.

(3.1) Analogously:

$$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 1 & -1 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_2(0) \\ u_2(0) \\ u_1(1) \\ u_2(1) \end{pmatrix}$$

The solutions can be parameterized by $u_1(0)$ as follows:

$$u_2(0) = 1 + u_1(0)$$
$$u_1(1) = 1 - 2u_1(0) - u_2(0) = -4u_1(0)$$
$$u_2(1) = \ldots = 1 - 4u_1(0)$$

(3.2) It is not possible, because $u_1(0)$ and $u_1(1)$ have opposite signs.

(3.3) It is possible $u_1(0) = u_2(0) = -1$. Then $u_1(1) = 4$, $u_2(1) = 5$.

5. Solution

(1.1) When $x(0) = 0$, one has:

$$x(k) = (A^{k-1}b_1u_1(0) + \ldots + A^{k-1}b_mu_m(0))$$
$$\vdots$$
$$+ (Ab_1u_1(k-2) + \ldots + Ab_mu_m(k-2))$$
$$+ (b_1u_1(k-1) + \ldots + b_mu_m(k-1))$$

where $B = (b_1, \ldots, b_m)$ and $u(0), \ldots, u(k-1)$ run over all possible control functions. Therefore

$$K(h) = [A^{k-1}b_1, \ldots, A^{k-1}b_m, \ldots, Ab_1, \ldots, Ab_m, b_1, \ldots, b_m] = [B, AB, \ldots, A^{k-1}B].$$

(1.2) It is obvious from (1.1).

(1.3) $K(h) = K(h+1)$ if and only if $A^h b_1, \ldots, A^h b_m \in K(h)$.
Then $A^{h+1} b_1, \ldots, A^{h+1} b_m \in K(h+1)$, so that $K(h+2) = K(h+1)$.

(1.4) As $\dim K(h) \leq n$, the length of the increasing chain is n at most.

(2)

$$K_i + K_j = [b_i, Ab_i, \ldots, A^{n-1} b_i] + [b_j, Ab_j, \ldots, A^{n-1} b_j]$$
$$= [(b_i, b_j), A(b_i, b_j), \ldots, A^{n-1}(b_i, b_j)]$$

(3.1) $K = \mathrm{Im} \begin{pmatrix} 0 & 0 & 2 & -1 & 10 & -2 \\ -1 & 1 & -2 & 1 & -8 & 2 \\ 1 & 0 & 2 & 1 & 8 & 2 \end{pmatrix} = \mathbb{R}^3$

$K_1 = \mathrm{Im} \begin{pmatrix} 0 & 2 & 10 \\ -1 & -2 & -8 \\ 1 & 2 & 8 \end{pmatrix} = \mathrm{Im} \begin{pmatrix} 0 & 2 \\ -1 & -2 \\ 1 & 2 \end{pmatrix}$

A basis of K_1: $(0, -1, 1), (1, 0, 0)$

$K_2 = \mathrm{Im} \begin{pmatrix} 0 & -1 & -2 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} = \mathrm{Im} \begin{pmatrix} 0 & -1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$

A basis of K_2: $(0, 1, 0), (-1, 0, 1)$

(3.2) $K_1 + K_2 = \mathrm{Im} \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \mathbb{R}^3$

$K_1 = \{y + z = 0\}$
$K_2 = \{x + z = 0\}$
$K_1 \cap K_2 = \{y + z = x + z = 0\}$
A basis of $K_1 \cap K_2$: $(1, 1, -1)$

6. Solution

(1) If $\bar{x} = S^{-1} x$, then:

$$\dot{\bar{x}} = S^{-1} \dot{x} = S^{-1}(Ax + Bu) = S^{-1} A S \bar{x} + S^{-1} Bu$$
$$y = Cx = CS\bar{x}$$

(2) $\mathrm{rank}(\bar{B}, \bar{A}\bar{B}, \ldots, \bar{A}^h \bar{B}) = \mathrm{rank}(S^{-1} B, S^{-1} A S S^{-1} B, \ldots, (S^{-1} A S)^h S^{-1} B)$
$= \mathrm{rank}\, S^{-1}(B, AB, \ldots, A^h B) = \mathrm{rank}(B, AB, \ldots, A^h B)$

(3) $\bar{G}(s) = \bar{C}(sI - \bar{A})^{-1} \bar{B} = CS(sI - S^{-1} A S)^{-1} S^{-1} B = CS(S^{-1}(sI - A)S)^{-1} S^{-1} B$
$= CS S^{-1}(sI - A)^{-1} S S^{-1} B = G(s)$

7. Solution

(1) From Cayley-Hamilton theorem:

$$A^n = a_0 I + a_1 A + \ldots + a_{n-1} A^{n-1}$$

Therefore, if $x \in \text{Im} \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$, then

$$Ax \in \text{Im} \begin{pmatrix} AB & A^2B & \dots & A^nB \end{pmatrix} \subset \text{Im} \begin{pmatrix} B & AB & \dots & A^nB \end{pmatrix},$$

and

$$\text{Im} \begin{pmatrix} B & AB & \dots & A^nB \end{pmatrix} = \text{Im} \begin{pmatrix} B & AB & \dots & (a_0B + a_1AB + \dots + a_{n-1}A^{n-1}B) \end{pmatrix}$$

$$= \text{Im} \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$$

$$= K.$$

If $x \in L$, then:

$$Cx = CAx = \dots = CA^{n-1}x = 0$$

Clearly:

$$C(Ax) = \dots = CA^{n-2}(Ax) = 0$$

It is sufficient to prove that $CA^{n-1}(Ax) = 0$, but

$$CA^{n-1}(Ax) = CA^n x$$

$$= C(a_0I + a_1A + \dots + a_{n-1}A^{n-1})x$$

$$= a_0Cx + a_1CAx + \dots + a_{n-1}CA^{n-1}x = 0$$

(2.1) $K = \text{Im} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 \\ 1 & -1 & 1 & -1 \end{pmatrix}$; $\quad \dim K = 3$

$L = \text{Ker} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$; $\quad \dim L = 4 - 2 = 2$

(2.2) Basis of K: (u_1, u_2, u_3)
$u_1 = (1, 0, 0, 1)$, $\quad u_2 = (1, 0, 1, -1)$, $\quad u_3 = (1, 0, -2, 1)$
Basis of L: (v_1, v_2)
$v_1 = (1, 0, 0, 0)$, $v_2 = (0, 0, 1, 0)$

(2.3) $Au_1 = u_2$; $\quad Au_2 = u_3$; $\quad Au_3 = \begin{pmatrix} 1 \\ 0 \\ 3 \\ -1 \end{pmatrix} = u_1 + u_2 - u_3$

$\text{Mat } A|_K = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

$Av_1 = v_1$; $\quad Av_2 = -v_2$

$\text{Mat } A|_L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(2.4) $K = \{(x_1, x_2, x_3, x_4) : x_2 = 0\}$
$L = \{(x_1, x_2, x_3, x_4) : x_2 = x_4 = 0\}$
$K \cap L = L$

8. Solution

(1) We recall (see ex. 7) that K and L are A-invariant subspaces. Therefore:

$$A(S_2) \subset [S_2]$$
$$A(S_1) \subset [S_1, S_2]$$
$$A(S_4) \subset [S_2, S_4]$$

so that $\bar{A} = S^{-1}AS$ has the stated form.
Moreover $\operatorname{Im} B \subset K$. Hence, $\bar{B} = S^{-1}B \subset [S_1, S_2]$.
Finally, as $L \subset \operatorname{Ker} C$, we have $C(S_2) = C(S_4) = 0$.

(2.1) According to the solution of ex. 7, we can take:

$$S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad S_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad S_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and S_4 is empty. Hence:

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(2.2)

$$\bar{A} = S^{-1}AS = \left(\begin{array}{c|cc|c} -1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array} \right)$$

$$\bar{B} = S^{-1}B = \begin{pmatrix} 1 \\ \hline 1 \\ 0 \\ \hline 0 \end{pmatrix} \quad \bar{C} = CS = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 1 \end{array} \right)$$

That is to say:

$$A_{11} = (-1) \quad A_{13} = (0\ 0) \quad A_{21} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad A_{22} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad A_{23} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A_{33} = (1) \quad B_1 = (1) \quad B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C_1 = (1) \quad C_3 = (1)$$

(2.3) $\operatorname{rank} K(A_{11}, B_1) = \operatorname{rank} (1\ -1) = 1$
$\operatorname{rank}(L(C_1, A_{11}))^t = \operatorname{rank} (1\ -1) = 1$

$$\text{rank } K \left(\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right) = \text{rank } \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix} = 3$$

$$\text{rank } \left(L \left(\begin{pmatrix} C_1 & C_3 \end{pmatrix}, \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{33} \end{pmatrix} \right) \right)^t = \text{rank } \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 2$$

9. Solution

(1.1) It is easy to check that there is only one main minor of each sign. Then:

$$a_1 = \text{tr } A_c, \quad a_2 = -\det \begin{pmatrix} 0 & 1 \\ a_2 & a_1 \end{pmatrix}, \quad a_3 = \det \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_3 & a_2 & a_1 \end{pmatrix}, \dots, \quad a_n = \det A_c.$$

(1.2) If $F_c = \begin{pmatrix} f_1 & \dots & f_n \end{pmatrix}$, then

$$A_c + b_c F_c = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & 1 \\ a_n + f_1 & a_{n-1} + f_2 & \dots & a_2 + f_{n-1} & a_1 + f_n \end{pmatrix}$$

Therefore, $f_1, \dots, f_n$ are the solutions of:

$$t^n - (a_1 + f_n)t^{n-1} - \dots - (a_{n-1} + f_2)t - (a_n + f_1) = (t - \mu_1)(t - \mu_2) \cdots (t - \mu_n).$$

(1.3)

$$\begin{aligned} \text{eigenvalues}(A + bF) &= \text{eigenvalues}(SA_c S^{-1} + Sb_c F_c S^{-1}) \\ &= \text{eigenvalues}(S(A_c + b_c F_c)S^{-1}) \\ &= \text{eigenvalues}(A_c + b_c F_c) = \{\mu_1, \dots, \mu_n\}. \end{aligned}$$

(2.1) It is straightforward to check that:

$$S^{-1}AS = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix}, \quad S^{-1}b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(2.2) First, we look for $F_c = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix}$ such that:

$$t - f_3 t^2 - (2 + f_2) - (-1 + f_1) = (t - \mu_1)(t - \mu_2)(t - \mu_3)$$

Therefore:

$$\begin{aligned} f_3 &= \mu_1 + \mu_2 + \mu_3 \\ f_2 &= \mu_1 \mu_2 + \mu_2 \mu_3 + \mu_3 \mu_1 - 2 \\ f_1 &= \mu_1 \mu_2 \mu_3 + 1 \end{aligned}$$

Finally: $F = F_c S^{-1}$.

10. Solution

(1) From Cayley-Hamilton theorem, if the characteristic polynomial of A is

$$Q(t) = (-1)^n (t^n - a_1 t^{n-1} - \ldots - a_{n-1} t - a_n),$$

then

$$A^n = a_n I + a_{n-1} A + \ldots + a_1 A^{n-1}.$$

Therefore, if $x \in K = \text{Im} \begin{pmatrix} B & AB & \ldots & A^{n-1}B \end{pmatrix}$, then

$$Ax \in \text{Im} \begin{pmatrix} AB & A^2 B & \ldots & A^n B \end{pmatrix}$$

$$\subset \text{Im} \begin{pmatrix} B & AB & \ldots & A^{n-1}B & (a_0 B + \ldots + a_{n-1} A^{n-1}B) \end{pmatrix} = K$$

(2) If $(u_1, \ldots, u_d, \ldots, u_m)$ is a basis of the state space adapted to K, then a state x belongs to K if and only if the last $n - d$ coordinates are 0.

(3) In the conditions of (2), the d first columns of $\bar{A}$ are $Au_1, \ldots, Au_d$, which belong to K (see (1)). Hence, again from (2), their last $n - d$ coordinates are 0. The same argument works for B, because its columns belong to K.

(4) From (3), it is clear that $x_{uc}(0) = 0$ implies $x_{uc}(t) = 0$ for any control $u(t)$.

(5) Again from (3), if $x_{uc}(t) = 0$, the remainder coordinates $x_c(t)$ are determined by system (2).

(6) By hypothesis

$$\text{rank} \begin{pmatrix} B & AB & \ldots & A^{n-1}B \end{pmatrix} = d.$$

The rank is preserved under changes of bases. Hence:

$$d = \text{rank} \begin{pmatrix} \bar{B} & \bar{A}\bar{B} & \ldots & \bar{A}^{n-1}\bar{B} \end{pmatrix} = \text{rank} \left(\begin{pmatrix} B_c \\ 0 \end{pmatrix} \begin{pmatrix} A_c B_c \\ 0 \end{pmatrix} \ldots \begin{pmatrix} A_c^{n-1} B_c \\ 0 \end{pmatrix} \right)$$

$$= \text{rank} \begin{pmatrix} B_c & A_c B_c & \ldots & A_c^{n-1} B_c \end{pmatrix} = \text{rank} \begin{pmatrix} B_c & A_c B_c & \ldots & A_c^{d-1} B_c \end{pmatrix}$$

where the last equality follows from Cayley-Hamilton theorem.

(7) Recall that the transfer matrix is preserved under changes of bases:

$$C(sI - A)^{-1} B = \bar{C} S^{-1}(sI - S\bar{A}S^{-1})^{-1} S\bar{B} = \bar{C} S^{-1}(S(sI - \bar{A})S^{-1})^{-1} S\bar{B} =$$

$$= \bar{C} S^{-1}(S^{-1})^{-1}(sI - \bar{A}) S^{-1} S\bar{B} = \bar{C}(sI - \bar{A})\bar{B}$$

From (2):

$$(sI - \bar{A})^{-1} = \left(sI - \begin{pmatrix} A_c & * \\ * & * \end{pmatrix} \right)^{-1} = \begin{pmatrix} sI_d - A_c & * \\ 0 & * \end{pmatrix}^{-1} = \begin{pmatrix} (sI_d - A_c)^{-1} & * \\ 0 & * \end{pmatrix}$$

where it does not matter the form of the blocks *:

$$\bar{C}(sI - \bar{A})^{-1}\bar{B} = \begin{pmatrix} C_c & C_{uc} \end{pmatrix} \begin{pmatrix} (sI_d - A_c)^{-1} & * \\ 0 & * \end{pmatrix} \begin{pmatrix} B_c \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} C_c & C_{uc} \end{pmatrix} \begin{pmatrix} (sI_d - A_c)^{-1}B_c \\ 0 \end{pmatrix}$$

$$= C_c(sI_d - A_c)^{-1}B_c$$

11. Solution

(1)

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}; \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(1.1)

$$\text{rank} \begin{pmatrix} b & Ab & A^2b \end{pmatrix} = \text{rank} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & -2 \\ 1 & 2 & 3 \end{pmatrix} = 3$$

(1.2)

$$S_1 = \begin{pmatrix} 2 & 1 & 1 \\ -2 & 0 & 1 \\ 3 & 2 & 1 \end{pmatrix}; \quad S_1^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -5 & 1 & 4 \\ 4 & 1 & -2 \end{pmatrix}$$

$$\bar{A} = \frac{1}{3} \begin{pmatrix} 6 & 3 & 0 \\ 0 & 0 & 3 \\ -3 & 0 & 0 \end{pmatrix} \quad \bar{b} = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -5 & 1 & 4 \\ 4 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

(1.3)

$$S_2 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}; \quad S_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix}$$

$$A_c = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \end{pmatrix}; \quad b_c = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(2) The computations are analogous to those in (1).

(3.1) If λ_i is an eigenvalue of A_c, then:

$$\text{rank}(A_c - \lambda_i I) < n.$$

On the other hand,

$$\text{rank}(A_c - \lambda_i I) = \text{rank} \begin{pmatrix} -\lambda_i & 1 & 0 & \dots & 0 & 0 \\ 0 & -\lambda_i & 1 & \dots & 0 & 0 \\ & & \dots & & \dots & \\ 0 & 0 & 0 & \dots & -\lambda_i & 1 \\ * & * & * & \dots & * & * \end{pmatrix} \geq n - 1.$$

(3.2) If λ_i is an eigenvalue of A, then λ_i is also en eigenvalue of A_c. Then, from (3.1):

$$\dim \operatorname{Ker}(A - \lambda_i I) = n - \operatorname{rank}(A - \lambda_i I) = n - \operatorname{rank}(A_c - \lambda_i I) = n - (n - 1) = 1.$$

Therefore, A is non-diagonalizable if the algebraic multiplicity of λ_i is greater than 1.

4. References

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