Provisional chapter

Recent Research on Jensen's Inequality for Oparators

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1. Introduction

The self-adjoint operators on Hilbert spaces with their numerous applications play an important part in the operator theory. The bounds research for self-adjoint operators is a very useful area of this theory. There is no better inequality in bounds examination than Jensen's inequality. It is an extensively used inequality in various fields of mathematics.

Let I be a real interval of any type. A continuous function $f : I \to \mathbb{R}$ is said to be operator convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad \text{(id1)}$$

holds for each $\lambda \in [0, 1]$ and every pair of self-adjoint operators x and y (acting) on an infinite dimensional Hilbert space H with spectra in I (the ordering is defined by setting $x \leq y$ if $y - x$ is positive semi-definite).

Let f be an operator convex function defined on an interval I. Ch. Davis [1] provedThere is small typo in the proof. Davis states that ϕ by Stinespring's theorem can be written on the form $\phi(x) = P\rho(x)P$ where ρ is a $*$ -homomorphism to $B(H)$ and$P$ is a projection on H. In fact, H may be embedded in a Hilbert space K on which ρ and P acts. The theorem then follows by the calculation $f(\phi(x)) = f(P\rho(x)P) \leq Pf(\rho(x))P = P\rho(f(x)P = \phi(f(x))$, where the pinching inequality, proved by Davis in the same paper, is applied. a Schwarz inequality

$$f(\phi(x)) \leq \phi(f(x)) \quad \text{(id3)}$$

where ϕ: $\to B(K)$ is a unital completely positive linear mapping from a C^*-algebra to linear operators on a Hilbert space K, and x is a self-adjoint element in with spectrum in I. Subsequently M. D. Choi [2] noted that it is enough to assume that ϕ is unital and positive. In fact, the restriction of ϕ to the commutative C^*-algebra generated by x is automatically completely positive by a theorem of Stinespring.

F. Hansen and G. K. Pedersen [3] proved a Jensen type inequality

$$f\left(\sum_{i=1}^{n} a_i^* x_i a_i\right) \le \sum_{i=1}^{n} a_i^* f(x_i) a_i \tag{id4}$$

for operator convex functions f defined on an interval $I = [0, \alpha)$ (with $\alpha \le \infty$ and $f(0) \le 0$) and self-adjoint operators $x_1, \cdots, x_n$ with spectra in I assuming that $\sum_{i=1}^{n} a_i^* a_i = \mathbf{1}$. The restriction on the interval and the requirement $f(0) \le 0$ was subsequently removed by B. Mond and J. Pečarić in [4], cf. also [5].

The inequality (▭) is in fact just a reformulation of (▭) although this was not noticed at the time. It is nevertheless important to note that the proof given in [3] and thus the statement of the theorem, when restricted to $n \times n$ matrices, holds for the much richer class of $2n \times 2n$ matrix convex functions. Hansen and Pedersen used (▭) to obtain elementary operations on functions, which leave invariant the class of operator monotone functions. These results then served as the basis for a new proof of Löwner's theorem applying convexity theory and Krein-Milman's theorem.

B. Mond and J. Pečarić [6] proved the inequality

$$f\left(\sum_{i=1}^{n} w_i \phi_i(x_i)\right) \le \sum_{i=1}^{n} w_i \phi_i(f(x_i)) \tag{id5}$$

for operator convex functions f defined on an interval I, where $\phi_i : B(H) \to B(K)$ are unital positive linear mappings, $x_1, \cdots, x_n$ are self-adjoint operators with spectra in I and $w_1, \cdots, w_n$ are are non-negative real numbers with sum one.

Also, B. Mond, J. Pečarić, T. Furuta et al. [6], [7], [8], [9], [10], [11] observed conversed of some special case of Jensen's inequality. So in [10] presented the following generalized converse of a Schwarz inequality (▭)

$$F[\phi(f(A)), g(\phi(A))] \le \max_{m \le t \le M} F\left[f(m) + \frac{f(M) - f(m)}{M - m}(t - m), g(t)\right] 1_{\tilde{n}} \tag{id6}$$

for convex functions f defined on an interval $[m, M]$, $m < M$, where g is a real valued continuous function on $[m, M]$, $F(u, v)$ is a real valued function defined on $U \times V$, matrix non-decreasing in u, $U \supset f[m, M]$, $V \supset g[m, M]$, $\phi : H_n \to H_{\tilde{n}}$ is a unital positive linear mapping and A is a Hermitian matrix with spectrum contained in $[m, M]$.

There are a lot of new research on the classical Jensen inequality (▭) and its reverse inequalities. For example, J.I. Fujii et all. in [12], [13] expressed these inequalities by externally dividing points.

2. Classic results

In this section we present a form of Jensen's inequality which contains (▭), (▭) and (▭) as special cases. Since the inequality in (▭) was the motivating step for obtaining converses of Jensen's inequality using the so-called Mond-Pečarić method, we also give some results pertaining to converse inequalities in the new formulation.

We recall some definitions. Let T be a locally compact Hausdorff space and let be a C^*-algebra of operators on some Hilbert space H. We say that a field $(x_t)_{t\in T}$ of operators in is continuous if the function $t \mapsto x_t$ is norm continuous on T. If in addition μ is a Radon measure on T and the function $t \mapsto \| x_t \|$ is integrable, then we can form *the Bochner integral* $\int_T x_t \, d\mu(t)$, which is the unique element in such that

$$\varphi\left(\int_T x_t \, d\mu(t)\right) = \int_T \varphi(x_t) \, d\mu(t) \tag{}$$

for every linear functional φ in the norm dual *.

Assume furthermore that there is a field $(\phi_t)_{t\in T}$ of positive linear mappings $\phi_t : \to \mathcal{B}$ from to another *-algebra $\mathcal{B}$ of operators on a Hilbert space K. We recall that a linear mapping $\phi_t : \to \mathcal{B}$ is said to be a positive mapping if $\phi_t(x_t) \geq 0$ for all $x_t \geq 0$. We say that such a field is continuous if the function $t \mapsto \phi_t(x)$ is continuous for every $x \in$. Let the *-algebras include the identity operators and the function $t \mapsto \phi_t(1_H)$ be integrable with $\int_T \phi_t(1_H) \, d\mu(t) = k 1_K$ for some positive scalar k. Specially, if $\int_T \phi_t(1_H) \, d\mu(t) = 1_K$, we say that a *field* $(\phi_t)_{t\in T}$ is *unital*.

Let $B(H)$ be the C^*-algebra of all bounded linear operators on a Hilbert space H. We define bounds of an operator $x \in B(H)$ by

$$m_x = \inf_{\|\xi\|=1} \langle x\xi, \xi \rangle \quad \text{and} \quad M_x = \sup_{\|\xi\|=1} \langle x\xi, \xi \rangle \tag{id7}$$

for $\xi \in H$. If (x) denotes the spectrum of x, then $(x) \subseteq [m_x, M_x]$.

For an operator $x \in B(H)$ we define operators $|x|$, x^+, x^- by

$$|x| = (x^* x)^{1/2}, \qquad x^+ = (|x| + x)/2, \qquad x^- = (|x| - x)/2 \tag{}$$

Obviously, if x is self-adjoint, then $|x| = (x^2)^{1/2}$ and $x^+, x^- \geq 0$ (called positive and negative parts of $x = x^+ - x^-$).

2.1. Jensen's inequality with operator convexity

Firstly, we give a general formulation of Jensen's operator inequality for a unital field of positive linear mappings (see [14]).

Theorem 1 Let $f : I \to \mathbb{R}$ be an operator convex function defined on an interval I and let and $\mathcal{B}$ be unital C^*-algebras acting on a Hilbert space H and K respectively. If $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : \to \mathcal{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ, then the inequality

$$f\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \le \int_T \phi_t(f(x_t))\, d\mu(t) \qquad \text{(id10)}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in with spectra contained in I.

We first note that the function $t \mapsto \phi_t(x_t) \in \mathcal{B}$ is continuous and bounded, hence integrable with respect to the bounded Radon measure μ. Furthermore, the integral is an element in the multiplier algebra $M(\mathcal{B})$ acting on K. We may organize the set $CB(T,)$ of bounded continuous functions on T with values in as a normed involutive algebra by applying the point-wise operations and setting

$$\| (y_t)_{t\in T} \| = \sup_{t\in T} \| y_t \| \qquad (y_t)_{t\in T} \in CB(T,) \qquad ()$$

and it is not difficult to verify that the norm is already complete and satisfy the C^*-identity. In fact, this is a standard construction in C^*-algebra theory. It follows that $f((x_t)_{t\in T}) = (f(x_t))_{t\in T}$. We then consider the mapping

$$\pi : CB(T,) \to M(\mathcal{B}) \subseteq B(K) \qquad ()$$

defined by setting

$$\pi((x_t)_{t\in T}) = \int_T \phi_t(x_t)\, d\mu(t) \qquad ()$$

and note that it is a unital positive linear map. Setting $x = (x_t)_{t\in T} \in CB(T,)$, we use inequality (▭) to obtain

$$f(\pi((x_t)_{t\in T})) = f(\pi(x)) \le \pi(f(x)) = \pi(f((x_t)_{t\in T})) = \pi((f(x_t))_{t\in T}) \qquad ()$$

but this is just the statement of the theorem.

2.2. Converses of Jensen's inequality

In the present context we may obtain results of the Li-Mathias type cf. Chapter 3[15] and [16], [17].

Theorem 2 Let T be a locally compact Hausdorff space equipped with a bounded Radon measure μ. Let $(x_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra with spectra in $[m, M]$, $m < M$. Furthermore, let $(\phi_t)_{t\in T}$ be a field of positive linear mappings $\phi_t : \to \mathcal{B}$ from to another unital C^* - algebra $\mathcal{B}$, such that the function $t \mapsto \phi_t(1_H)$ is integrable with $\int_T \phi_t(1_H)\, d\mu(t) = k1_K$ for some positive scalar k. Let m_x and M_x, $m_x \le M_x$, be the bounds of the self-adjoint operator $x = \int_T \phi_t(x_t)\, d\mu(t)$ and $f : [m, M] \to \mathbb{R}$, $g : [m_x, M_x] \to \mathbb{R}$, $F : U \times V \to \mathbb{R}$ be functions such that $(kf)([m, M]) \subset U$, $g([m_x, M_x]) \subset V$ and F is bounded. If F is operator monotone in the first variable, then

$$\begin{aligned}\inf_{m_x\le z\le M_x} F\left[k\cdot h_1\left(\frac{1}{k}z\right), g(z)\right]1_K &\le F\left[\int_T \phi_t(f(x_t))d\mu(t), g\left(\int_T \phi_t(x_t)d\mu(t)\right)\right]\\ &\le \sup_{m_x\le z\le M_x} F\left[k\cdot h_2\left(\frac{1}{k}z\right), g(z)\right]1_K\end{aligned} \qquad \text{(id13)}$$

holds for every operator convex function h_1 on $[m, M]$ such that $h_1 \le f$ and for every operator concave function h_2 on $[m, M]$ such that $h_2 \ge f$.

We prove only RHS of (▭). Let h_2 be operator concave function on $[m, M]$ such that $f(z) \le h_2(z)$ for every $z \in [m, M]$. By using the functional calculus, it follows that $f(x_t) \le h_2(x_t)$ for every $t \in T$. Applying the positive linear mappings ϕ_t and integrating, we obtain

$$\int_T \phi_t(f(x_t))d\mu(t) \le \int_T \phi_t(h_2(x_t))d\mu(t) \qquad ()$$

Furthermore, replacing ϕ_t by $\frac{1}{k}\phi_t$ in Theorem ▭, we obtain $\frac{1}{k}\int_T \phi_t(h_2(x_t))\, d\mu(t) \le h_2\left(\frac{1}{k}\int_T \phi_t(x_t)\, d\mu(t)\right)$, which gives $\int_T \phi_t(f(x_t))d\mu(t) \le k\cdot h_2\left(\frac{1}{k}\int_T \phi_t(x_t)\, d\mu(t)\right)$. Since $m_x 1_K \le \int_T \phi_t(x_t)d\mu(t) \le M_x 1_K$, then using operator monotonicity of $F(\cdot, v)$ we obtain

$$\begin{aligned}&F\left[\int_T \phi_t(f(x_t))d\mu(t), g\left(\int_T \phi_t(x_t)d\mu(t)\right)\right]\\ &\le F\left[k\cdot h_2\left(\frac{1}{k}\int_T \phi_t(x_t)\, d\mu(t)\right), g\left(\int_T \phi_t(x_t)d\mu(t)\right)\right] \le \sup_{m_x\le z\le M_x} F\left[k\cdot h_2\left(\frac{1}{k}z\right), g(z)\right]1_K\end{aligned} \qquad \text{(id14)}$$

Applying RHS of (▭) for a convex function f (or LHS of (▭) for a concave function f) we obtain the following generalization of (▭).

Theorem 3 Let $(x_t)_{t\in T}$, m_x, M_x and $(\phi_t)_{t\in T}$ be as in Theorem ▭. Let $f:[m,M]\to\mathbb{R}$, $g:[m_x,M_x]\to\mathbb{R}$, $F:U\times V\to\mathbb{R}$ be functions such that $(kf)([m,M])\subset U$, $g([m_x,M_x])\subset V$ and F is bounded. If F is operator monotone in the first variable and f is convex on the interval $[m,M]$, then

$$\begin{aligned} &F\left[\int_T \phi_t(f(x_t))d\mu(t),\ g\left(\int_T \phi_t(x_t)d\mu(t)\right)\right] \\ \le &\sup_{m_x\le z\le M_x} F\left[\frac{Mk-z}{M-m}f(m)+\frac{z-km}{M-m}f(M),\ g(z)\right]1_K \end{aligned} \tag{id16}$$

In the dual case (when f is concave) the opposite inequalities hold in (▭) with inf instead of sup .

We prove only the convex case. For convex f the inequality $f(z)\le\frac{M-z}{M-m}f(m)+\frac{z-m}{M-m}f(M)$ holds for every $z\in[m,M]$. Thus, by putting $h_2(z)=\frac{M-z}{M-m}f(m)+\frac{z-m}{M-m}f(M)$ in (▭) we obtain (▭). Numerous applications of the previous theorem can be given (see [15]). Applying Theorem ▭ for the function $F(u,v)=u-\alpha v$ and $k=1$, we obtain the following generalization of Theorem 2.4[15].

Corollary 4 Let $(x_t)_{t\in T}$, m_x, M_x be as in Theorem ▭ and $(\phi_t)_{t\in T}$ be a unital field of positive linear mappings $\phi_t:\ \to\mathcal{B}$. If $f:[m,M]\to\mathbb{R}$ is convex on the interval $[m,M]$, $m<M$, and $g:[m,M]\to\mathbb{R}$, then for any $\alpha\in\mathbb{R}$

$$\int_T \phi_t(f(x_t))d\mu(t)\le\alpha\ g\left(\int_T\phi_t(x_t)d\mu(t)\right)+C1_K \tag{id18}$$

where

$$\begin{aligned} C &= \max_{m_x\le z\le M_x}\left\{\frac{M-z}{M-m}f(m)+\frac{z-m}{M-m}f(M)-\alpha g(z)\right\} \\ &\le \max_{m\le z\le M}\left\{\frac{M-z}{M-m}f(m)+\frac{z-m}{M-m}f(M)-\alpha g(z)\right\} \end{aligned} \tag{}$$

If furthermore αg is strictly convex differentiable, then the constant $C\equiv C(m,M,f,g,\alpha)$ can be written more precisely as

$$C=\frac{M-z_0}{M-m}f(m)+\frac{z_0-m}{M-m}f(M)-\alpha g(z_0) \tag{}$$

where

$$z_0 = \begin{cases} g'^{-1}\left(\frac{f(M)-f(m)}{\alpha(M-m)}\right) & \text{if} \quad \alpha g'(m_x) \le \frac{f(M)-f(m)}{M-m} \le \alpha g'(M_x) \\ m_x & \text{if} \quad \alpha g'(m_x) \ge \frac{f(M)-f(m)}{M-m} \\ M_x & \text{if} \quad \alpha g'(M_x) \le \frac{f(M)-f(m)}{M-m} \end{cases} \qquad ()$$

In the dual case (when f is concave and αg is strictly concave differentiable) the opposite inequalities hold in (▭) with min instead of max with the opposite condition while determining z_0.

3. Inequalities with conditions on spectra

In this section we present Jensens's operator inequality for real valued continuous convex functions with conditions on the spectra of the operators. A discrete version of this result is given in [18]. Also, we obtain generalized converses of Jensen's inequality under the same conditions.

Operator convexity plays an essential role in (▭). In fact, the inequality (▭) will be false if we replace an operator convex function by a general convex function. For example, M.D. Choi in Remark 2.6[2] considered the function $f(t) = t^4$ which is convex but not operator convex. He demonstrated that it is sufficient to put $\dim H = 3$, so we have the matrix case as follows. Let $\Phi : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ be the contraction mapping $\Phi\left((a_{ij})_{1\le i,\, j\le 3}\right) = (a_{ij})_{1\le i,\, j\le 2}$. If $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, then $\Phi(A)^4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neg \le \begin{pmatrix} 9 & 5 \\ 5 & 3 \end{pmatrix} = \Phi\left(A^4\right)$ and no relation between $\Phi(A)^4$ and $\Phi\left(A^4\right)$ under the operator order.

Example 5 It appears that the inequality (▭) will be false if we replace the operator convex function by a general convex function. We give a small example for the matrix cases and $T = \{1, 2\}$. We define mappings $\Phi_1, \Phi_2 : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ by $\Phi_1\left((a_{ij})_{1\le i,\, j\le 3}\right) = \frac{1}{2}(a_{ij})_{1\le i,\, j\le 2}$, $\Phi_2 = \Phi_1$. Then $\Phi_1(I_3) + \Phi_2(I_3) = I_2$.

I)

- If

$$X_1 = 2\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad X_2 = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad ()$$

then

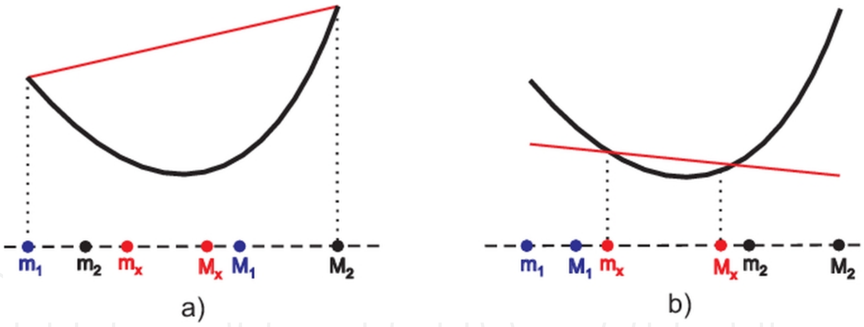


Figure 1. Spectral conditions for a convex function f

$$(\Phi_1(X_1)+\Phi_2(X_2))^4=\begin{pmatrix}16 & 0\\ 0 & 0\end{pmatrix}\neg\leq\begin{pmatrix}80 & 40\\ 40 & 24\end{pmatrix}=\Phi_1\left(X_1^4\right)+\Phi_2\left(X_2^4\right) \qquad ()$$

Given the above, there is no relation between $(\Phi_1(X_1)+\Phi_2(X_2))^4$ and $\Phi_1\left(X_1^4\right)+\Phi_2\left(X_2^4\right)$ under the operator order. We observe that in the above case the following stands $X=\Phi_1(X_1)+\Phi_2(X_2)=\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}$ and $[m_x, M_x]=[0, 2]$, $[m_1, M_1]\subset[-1.60388, 4.49396]$, $[m_2, M_2]=[0, 2]$, i.e.

$$(m_x, M_x)\subset[m_1, M_1]\cup[m_2, M_2] \qquad ()$$

(see Fig. 1.a).

II)

- If

$$X_1=\begin{pmatrix}-14 & 0 & 1\\ 0 & -2 & -1\\ 1 & -1 & -1\end{pmatrix} \text{ and } X_2=\begin{pmatrix}15 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 15\end{pmatrix} \qquad ()$$

then

$$(\Phi_1(X_1)+\Phi_2(X_2))^4=\begin{pmatrix}\frac{1}{16} & 0\\ 0 & 0\end{pmatrix}<\begin{pmatrix}89660 & -247\\ -247 & 51\end{pmatrix}=\Phi_1\left(X_1^4\right)+\Phi_2\left(X_2^4\right) \qquad ()$$

So we have that an inequality of type (▭) now is valid. In the above case the following stands $X=\Phi_1(X_1)+\Phi_2(X_2)=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & 0\end{pmatrix}$ and $[m_x, M_x]=[0, 0.5]$, $[m_1, M_1]\subset[-14.077, -0.328566]$, $[m_2, M_2]=[2, 15]$, i.e.

$$(m_x, M_x) \cap [m_1, M_1] = \varnothing \quad \text{and} \quad (m_x, M_x) \cap [m_2, M_2] = \varnothing \tag{}$$

(see Fig. 1.b).

3.1. Jensen's inequality without operator convexity

It is no coincidence that the inequality (▭) is valid in Example ▭-II). In the following theorem we prove a general result when Jensen's operator inequality (▭) holds for convex functions.

Theorem 6 Let $(x_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. Let m_t and M_t, $m_t \le M_t$, be the bounds of x_t, $t \in T$. Let $(\phi_t)_{t\in T}$ be a unital field of positive linear mappings $\phi_t : \to \mathcal{B}$ from to another unital C^* - algebra $\mathcal{B}$. If

$$(m_x, M_x) \cap [m_t, M_t] = \varnothing, \qquad t \in T \tag{}$$

where m_x and M_x, $m_x \le M_x$, are the bounds of the self-adjoint operator $x = \int_T \phi_t(x_t)\, d\mu(t)$, then

$$f\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \le \int_T \phi_t(f(x_t))\, d\mu(t) \tag{id25}$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval I contains all m_t, M_t.

If $f : I \to \mathbb{R}$ is concave, then the reverse inequality is valid in (▭).

We prove only the case when f is a convex function. If we denote $m = \inf_{t\in T} \{m_t\}$ and $M = \sup_{t\in T} \{M_t\}$, then $[m, M] \subseteq I$ and $m1_H \le A_t \le M1_H$, $t \in T$. It follows $m1_K \le \int_T \phi_t(x_t)\, d\mu(t) \le M1_K$. Therefore $[m_x, M_x] \subseteq [m, M] \subseteq I$.

a) Let $m_x < M_x$. Since f is convex on $[m_x, M_x]$, then

$$f(z) \le \frac{M_x - z}{M_x - m_x} f(m_x) + \frac{z - m_x}{M_x - m_x} f(M_x), \quad z \in [m_x, M_x] \tag{id26}$$

but since f is convex on $[m_t, M_t]$ and since $(m_x, M_x) \cap [m_t, M_t] = \varnothing$, then

$$f(z) \ge \frac{M_x - z}{M_x - m_x} f(m_x) + \frac{z - m_x}{M_x - m_x} f(M_x), \quad z \in [m_t, M_t], \quad t \in T \tag{id27}$$

Since $m_x 1_K \le \int_T \phi_t(x_t)\, d\mu(t) \le M_x 1_K$, then by using functional calculus, it follows from (▭)

$$f\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \le \frac{M_x 1_K - \int_T \phi_t(x_t)\, d\mu(t)}{M_x - m_x} f(m_x) + \frac{\int_T \phi_t(x_t)\, d\mu(t) - m_x 1_K}{M_x - m_x} f(M_x) \qquad \text{(id28)}$$

On the other hand, since $m_t 1_H \le x_t \le M_t 1_H$, $t \in T$, then by using functional calculus, it follows from (▭)

$$f(x_t) \ge \frac{M_x 1_H - x_t}{M_x - m_x} f(m_x) + \frac{x_t - m_x 1_H}{M_x - m_x} f(M_x), \qquad t \in T \qquad ()$$

Applying a positive linear mapping ϕ_t and summing, we obtain

$$\int_T \phi_t(f(x_t))\, d\mu(t) \ge \frac{M_x 1_K - \int_T \phi_t(x_t)\, d\mu(t)}{M_x - m_x} f(m_x) + \frac{\int_T \phi_t(x_t)\, d\mu(t) - m_x 1_K}{M_x - m_x} f(M_x) \qquad \text{(id29)}$$

since $\int_T \phi_t(1_H)\, d\mu(t) = 1_K$. Combining the two inequalities (▭) and (▭), we have the desired inequality (▭).

b) Let $m_x = M_x$. Since f is convex on $[m, M]$, we have

$$f(z) \ge f(m_x) + l(m_x)(z - m_x) \quad \text{for every } z \in [m, M] \qquad \text{(id30)}$$

where l is the subdifferential of f. Since $m 1_H \le x_t \le M 1_H$, $t \in T$, then by using functional calculus, applying a positive linear mapping ϕ_t and summing, we obtain from (▭)

$$\int_T \phi_t(f(x_t))\, d\mu(t) \ge f(m_x) 1_K + l(m_x)\left(\int_T \phi_t(x_t)\, d\mu(t) - m_x 1_K\right) \qquad \text{(id31)}$$

Since $m_x 1_K = \int_T \phi_t(x_t)\, d\mu(t)$, it follows

$$\int_T \phi_t(f(x_t))\, d\mu(t) \ge f(m_x) 1_K = f\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \qquad \text{(id32)}$$

which is the desired inequality (▭). Putting $\phi_t(y) = a_t y$ for every $y \in$, where $a_t \ge 0$ is a real number, we obtain the following obvious corollary of Theorem ▭.

Corollary 7 Let $(x_t)_{t \in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. Let m_t and M_t, $m_t \le M_t$, be the bounds of x_t, $t \in T$. Let $(a_t)_{t \in T}$ be a continuous field of nonnegative real numbers such that $\int_T a_t\, d\mu(t) = 1$. If

$$(m_x, M_x) \cap [m_t, M_t] = \varnothing, \qquad t \in T \tag{}$$

where m_x and M_x, $m_x \le M_x$, are the bounds of the self-adjoint operator $x = \int_T a_t x_t \, d\mu(t)$, then

$$f\left(\int_T a_t x_t \, d\mu(t)\right) \le \int_T a_t f(x_t) \, d\mu(t) \tag{id34}$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval I contains all m_t, M_t.

3.2. Converses of Jensen's inequality with conditions on spectra

Using the condition on spectra we obtain the following extension of Theorem ▭.

Theorem 8 Let $(x_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. Furthermore, let $(\phi_t)_{t\in T}$ be a field of positive linear mappings $\phi_t : \to \mathcal{B}$ from to another unital C^*-algebra $\mathcal{B}$, such that the function $t \mapsto \phi_t(1_H)$ is integrable with $\int_T \phi_t(1_H) \, d\mu(t) = k 1_K$ for some positive scalar k. Let m_t and M_t, $m_t \le M_t$, be the bounds of x_t, $t \in T$, $m = \inf_{t\in T} \{m_t\}$, $M = \sup_{t\in T} \{M_t\}$, and m_x and M_x, $m_x < M_x$, be the bounds of $x = \int_T \phi_t(x_t) \, d\mu(t)$. If

$$(m_x, M_x) \cap [m_t, M_t] = \varnothing, \qquad t \in T \tag{}$$

and $f : [m, M] \to \mathbb{R}$, $g : [m_x, M_x] \to \mathbb{R}$, $F : U \times V \to \mathbb{R}$ are functions such that $(kf)([m, M]) \subset U$, $g([m_x, M_x]) \subset V$, f is convex, F is bounded and operator monotone in the first variable, then

$$\begin{aligned}
&\inf_{m_x \le z \le M_x} F\left[\frac{M_x k - z}{M_x - m_x} f(m_x) + \frac{z - k m_x}{M_x - m_x} f(M_x), \, g(z)\right] 1_K \\
&\qquad F\left[\int_T \phi_t(f(x_t)) d\mu(t), \, g\left(\int_T \phi_t(x_t) d\mu(t)\right)\right] \\
&\le \sup_{m_x \le z \le M_x} F\left[\frac{Mk - z}{M - m} f(m) + \frac{z - km}{M - m} f(M), \, g(z)\right] 1_K
\end{aligned} \tag{id37}$$

In the dual case (when f is concave) the opposite inequalities hold in (▭) by replacing inf and sup with sup and inf , respectively.

We prove only LHS of (▭). It follows from (▭) (compare it to (▭))

$$\int_T \phi_t(f(x_t))\, d\mu(t) \geq \frac{M_x k 1_K - \int_T \phi_t(x_t)\, d\mu(t)}{M_x - m_x} f(m_x) + \frac{\int_T \phi_t(x_t)\, d\mu(t) - m_x k 1_K}{M_x - m_x} f(M_x) \qquad ()$$

since $\int_T \phi_t(1_H)\, d\mu(t) = k1_K$. By using operator monotonicity of $F(\cdot, v)$ we obtain

$$F\left[\int_T \phi_t(f(x_t)) d\mu(t),\ g\left(\int_T \phi_t(x_t)\, d\mu(t)\right)\right] \geq F\left[\frac{M_x k 1_K - \int_T \phi_t(x_t)\, d\mu(t)}{M_x - m_x} f(m_x) + \frac{\int_T \phi_t(x_t)\, d\mu(t) - m_x k 1_K}{M_x - m_x} f(M_x),\ g\left(\int_T \phi_t(x_t)\, d\mu(t)\right)\right]$$

mxzMx F[Mx k-zMx-mxf(mx)+z-kmxMx-mxf(Mx),g(z)] 1K

$$()$$

Putting $F(u, v) = u - \alpha v$ or $F(u, v) = v^{-1/2} u v^{-1/2}$ in Theorem ▭, we obtain the next corollary.

Corollary 9 Let $(x_t)_{t\in T}$, m_t, M_t, m_x, M_x, m, M, $(\phi_t)_{t\in T}$ be as in Theorem ▭ and $f : [m, M] \to \mathbb{R}$, $g : [m_x, M_x] \to \mathbb{R}$ be continuous functions. If

$$(m_x, M_x) \cap [m_t, M_t] = \emptyset, \qquad t \in T \qquad ()$$

and f is convex, then for any $\alpha \in \mathbb{R}$

$$\begin{aligned} &\min_{m_x \leq z \leq M_x} \left\{\frac{M_x k - z}{M_x - m_x} f(m_x) + \frac{z - k m_x}{M_x - m_x} f(M_x) - g(z)\right\} 1_K + \alpha g\left(\int_T \phi_t(x_t) d\mu(t)\right) \\ &\leq \int_T \phi_t(f(x_t)) d\mu(t) \\ &\leq \alpha g\left(\int_T \phi_t(x_t) d\mu(t)\right) + \max_{m_x \leq z \leq M_x} \left\{\frac{Mk - z}{M - m} f(m) + \frac{z - km}{M - m} f(M) - g(z)\right\} 1_K \end{aligned} \qquad \text{(id39)}$$

If additionally $g > 0$ on $[m_x, M_x]$, then

$$\begin{aligned} &\min_{m_x \leq z \leq M_x} \left\{\frac{\frac{M_x k - z}{M_x - m_x} f(m_x) + \frac{z - k m_x}{M_x - m_x} f(M_x)}{g(z)}\right\} g\left(\int_T \phi_t(x_t) d\mu(t)\right) \\ &\leq \int_T \phi_t(f(x_t)) d\mu(t) \leq \max_{m_x \leq z \leq M_x} \left\{\frac{\frac{Mk - z}{M - m} f(m) + \frac{z - km}{M - m} f(M)}{g(z)}\right\} g\left(\int_T \phi_t(x_t) d\mu(t)\right) \end{aligned} \qquad \text{(id40)}$$

In the dual case (when f is concave) the opposite inequalities hold in (▭) by replacing min and max with max and min, respectively. If additionally $g > 0$ on $[m_x, M_x]$, then the opposite inequalities also hold in (▭) by replacing min and max with max and min, respectively.

4. Refined Jensen's inequality

In this section we present a refinement of Jensen's inequality for real valued continuous convex functions given in Theorem ▭. A discrete version of this result is given in [19].

To obtain our result we need the following two lemmas.

Lemma 10 Let f be a convex function on an interval I, $m, M \in I$ and $p_1, p_2 \in [0, 1]$ such that $p_1 + p_2 = 1$. Then

$$\min\{p_1, p_2\}\left[f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right] \leq p_1 f(m) + p_2 f(M) - f(p_1 m + p_2 M) \qquad \text{(id42)}$$

These results follows from Theorem 1, p. 717[20].

Lemma 11 Let x be a bounded self-adjoint elements in a unital C^*-algebra of operators on some Hilbert space H. If the spectrum of x is in $[m, M]$, for some scalars $m < M$, then

$$\begin{array}{lcl} f(x) & \leq & \dfrac{M1_H - x}{M - m} f(m) + \dfrac{x - m1_H}{M - m} f(M) - \delta_f \tilde{x} \\ \text{(resp.} \quad f(x) & \geq & \dfrac{M1_H - x}{M - m} f(m) + \dfrac{x - m1_H}{M - m} f(M) + \delta_f \tilde{x} \quad) \end{array} \qquad \text{(id44)}$$

holds for every continuous convex (resp. concave) function $f : [m, M] \to \mathbb{R}$, where

$$\begin{array}{c} \delta_f = f(m) + f(M) - 2f\left(\dfrac{m+M}{2}\right) \quad \left(\text{resp. } \delta_f = 2f\left(\dfrac{m+M}{2}\right) - f(m) - f(M)\right) \\ \text{and} \quad \tilde{x} = \dfrac{1}{2}1_H - \dfrac{1}{M - m}\left|x - \dfrac{m+M}{2}1_H\right| \end{array} \qquad ()$$

We prove only the convex case. It follows from (▭) that

$$\begin{array}{lcl} f(p_1 m + p_2 M) & \leq & p_1 f(m) + p_2 f(M) \\ & - & \min\{p_1, p_2\}\left(f(m) + f(M) - 2f\left(\dfrac{m+M}{2}\right)\right) \end{array} \qquad \text{(id45)}$$

for every $p_1, p_2 \in [0, 1]$ such that $p_1 + p_2 = 1$. For any $z \in [m, M]$ we can write

$$f(z) = f\left(\frac{M - z}{M - m}m + \frac{z - m}{M - m}M\right) \tag*{()}$$

Then by using (▭) for $p_1 = \frac{M - z}{M - m}$ and $p_2 = \frac{z - m}{M - m}$ we obtain

$$\begin{aligned} f(z) &\leq \frac{M - z}{M - m}f(m) + \frac{z - m}{M - m}f(M) \\ &- \left(\frac{1}{2} - \frac{1}{M - m}\left|z - \frac{m + M}{2}\right|\right)\left(f(m) + f(M) - 2f\left(\frac{m + M}{2}\right)\right) \end{aligned} \tag*{(id46)}$$

since

$$\min\left\{\frac{M - z}{M - m}, \frac{z - m}{M - m}\right\} = \frac{1}{2} - \frac{1}{M - m}\left|z - \frac{m + M}{2}\right| \tag*{()}$$

Finally we use the continuous functional calculus for a self-adjoint operator x: $f, g \in (I)$, $Sp(x) \subseteq I$ and $f \leq g$ on I implies $f(x) \leq g(x)$; and $h(z) = |z|$ implies $h(x) = |x|$. Then by using (▭) we obtain the desired inequality (▭).

Theorem 12 Let $(x_t)_{t \in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. Let m_t and M_t, $m_t \leq M_t$, be the bounds of x_t, $t \in T$. Let $(\phi_t)_{t \in T}$ be a unital field of positive linear mappings $\phi_t : \to \mathcal{B}$ from to another unital C^* - algebra $\mathcal{B}$. Let

$$(m_x, M_x) \cap [m_t, M_t] = \emptyset, \quad t \in T, \qquad \text{and} \qquad m < M \tag*{()}$$

where m_x and M_x, $m_x \leq M_x$, be the bounds of the operator $x = \int_T \phi_t(x_t)\, d\mu(t)$ and

$$m = \sup\{M_t : M_t \leq m_x,\ t \in T\},\ M = \inf\{m_t : m_t \geq M_x,\ t \in T\} \tag*{()}$$

If $f : I \to \mathbb{R}$ is a continuous convex (resp. concave) function provided that the interval I contains all m_t, M_t, then

$$f\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \leq \int_T \phi_t(f(x_t))\, d\mu(t) - \delta_f \tilde{x} \leq \int_T \phi_t(f(x_t))\, d\mu(t) \tag*{(id48)}$$

(resp.

$$f\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \geq \int_T \phi_t(f(x_t))\, d\mu(t) - \delta_f \tilde{x} \geq \int_T \phi_t(f(x_t))\, d\mu(t)\ ) \tag*{(id49)}$$

holds, where

$$\begin{array}{rrcl} & \delta_f \equiv \delta_f(\bar{m}, \bar{M}) & = & f(\bar{m}) + f(\bar{M}) - 2f\left(\frac{\bar{m}+\bar{M}}{2}\right) \\ (\text{resp.} & \delta_f \equiv \delta_f(\bar{m}, \bar{M}) & = & 2f\left(\frac{\bar{m}+\bar{M}}{2}\right) - f(\bar{m}) - f(\bar{M})\Big) \\ & \tilde{x} \equiv \tilde{x}_x(\bar{m}, \bar{M}) & = & \frac{1}{2}1_K - \frac{1}{\bar{M}-\bar{m}}\left|x - \frac{\bar{m}+\bar{M}}{2}1_K\right| \end{array} \tag{id50}$$

and $\bar{m} \in [m, m_A], \bar{M} \in [M_A, M], \bar{m} < \bar{M}$, are arbitrary numbers.

We prove only the convex case. Since $x = \int_T \phi_t(x_t)\, d\mu(t) \in \mathcal{B}$ is the self-adjoint elements such that $\bar{m}1_K \le m_x 1_K \le \int_T \phi_t(x_t)\, d\mu(t) \le M_x 1_K \le \bar{M}1_K$ and f is convex on $[\bar{m}, \bar{M}] \subseteq I$, then by Lemma ▭ we obtain

$$f\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \le \frac{\bar{M}1_K - \int_T \phi_t(x_t)\, d\mu(t)}{\bar{M}-\bar{m}} f(\bar{m}) + \frac{\int_T \phi_t(x_t)\, d\mu(t) - \bar{m}1_K}{\bar{M}-\bar{m}} f(\bar{M}) - \delta_f \tilde{x} \tag{id51}$$

where δ_f and $\tilde{x}$ are defined by (▭).

But since f is convex on $[m_t, M_t]$ and $(m_x, M_x) \cap [m_t, M_t] = \emptyset$ implies $(\bar{m}, \bar{M}) \cap [m_t, M_t] = \emptyset$, then

$$f(x_t) \ge \frac{\bar{M}1_H - x_t}{\bar{M}-\bar{m}} f(\bar{m}) + \frac{x_t - \bar{m}1_H}{\bar{M}-\bar{m}} f(\bar{M}), \quad t \in T \tag{}$$

Applying a positive linear mapping ϕ_t, integrating and adding $-\delta_f \tilde{x}$, we obtain

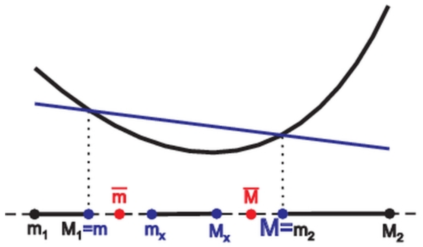
$$\int_T \phi_t(f(x_t))\, d\mu(t) - \delta_f \tilde{x} \ge \frac{\bar{M}1_K - \int_T \phi_t(x_t)\, d\mu(t)}{\bar{M}-\bar{m}} f(\bar{m}) + \frac{\int_T \phi_t(x_t)\, d\mu(t) - \bar{m}1_K}{\bar{M}-\bar{m}} f(\bar{M}) - \delta_f \tilde{x} \tag{id52}$$

since $\int_T \phi_t(1_H)\, d\mu(t) = 1_K$. Combining the two inequalities (▭) and (▭), we have LHS of (▭). Since $\delta_f \ge 0$ and $\tilde{x} \ge 0$, then we have RHS of (▭).

If $m < M$ and $m_x = M_x$, then the inequality (▭) holds, but $\delta_f(m_x, M_x)\, \tilde{x}(m_x, M_x)$ is not defined (see Example ▭ I) and II)).

Example 13 We give examples for the matrix cases and $T = \{1, 2\}$. Then we have refined inequalities given in Fig. 2.

We put $f(t) = t^4$ which is convex but not operator convex in (▭). Also, we define mappings $\Phi_1, \Phi_2 : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ as follows: $\Phi_1((a_{ij})_{1\le i,\, j\le 3}) = \frac{1}{2}(a_{ij})_{1\le i,\, j\le 2}$, $\Phi_2 = \Phi_1$ (then $\Phi_1(I_3) + \Phi_2(I_3) = I_2$).



$$f(\phi_1(x_1))+f(\phi_2(x_2)) \le \phi_1(f(x_1))+\phi_2(f(x_2))-\delta_f\tilde{x},$$

where

$$\delta_f = f(\bar{m})+f(\bar{M})-2f\left((\bar{M}+\bar{m})/2\right),$$

$$\tilde{x} = \frac{1}{2}1_K - \frac{1}{\bar{M}-\bar{m}}\left|\phi_1(x_1)+\phi_2(x_2)-\frac{\bar{M}+\bar{m}}{2}1_K\right|$$

Figure 2. Refinement for two operators and a convex function f

I) First, we observe an example when $\delta_f \widetilde{X}$ is equal to the difference RHS and LHS of Jensen's inequality. If $X_1 = -3I_3$ and $X_2 = 2I_3$, then $X = \Phi_1(X_1) + \Phi_2(X_2) = -0.5I_2$, so $m = -3$, $M = 2$. We also put $\bar{m} = -3$ and $\bar{M} = 2$. We obtain

$$(\Phi_1(X_1) + \Phi_2(X_2))^4 = 0.0625I_2 < 48.5I_2 = \Phi_1\left(X_1^4\right) + \Phi_2\left(X_2^4\right) \qquad ()$$

and its improvement

$$(\Phi_1(X_1) + \Phi_2(X_2))^4 = 0.0625I_2 = \Phi_1\left(X_1^4\right) + \Phi_2\left(X_2^4\right) - 48.4375I_2 \qquad ()$$

since $\delta_f = 96.875$, $\widetilde{X} = 0.5I_2$. We remark that in this case $m_x = M_x = -1/2$ and $\widetilde{X}(m_x, M_x)$ is not defined.

II) Next, we observe an example when $\delta_f \widetilde{X}$ is not equal to the difference RHS and LHS of Jensen's inequality and $m_x = M_x$. If

$$X_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad \text{then } X = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } m = -1, \ M = 2 \qquad ()$$

In this case $\tilde{x}(m_x, M_x)$ is not defined, since $m_x = M_x = 1/2$. We have

$$(\Phi_1(X_1) + \Phi_2(X_2))^4 = \frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} < \begin{pmatrix} \frac{17}{2} & 0 \\ 0 & \frac{97}{2} \end{pmatrix} = \Phi_1\left(X_1^4\right) + \Phi_2\left(X_2^4\right) \qquad ()$$

and putting $\bar{m} = -1$, $\bar{M} = 2$ we obtain $\delta_f = 135/8$, $\widetilde{X} = I_2/2$ which give the following improvement

$$(\Phi_1(X_1) + \Phi_2(X_2))^4 = \frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} < \frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 641 \end{pmatrix} = \Phi_1\left(X_1^4\right) + \Phi_2\left(X_2^4\right) - \frac{135}{16}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad ()$$

III) Next, we observe an example with matrices that are not special. If

$$X_1 = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & -1 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 3 \end{pmatrix}, \quad \text{then} \quad X = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad ()$$

so $m_1 = -4.8662$, $M_1 = -0.3446$, $m_2 = 1.3446$, $M_2 = 5.8662$, $m = -0.3446$, $M = 1.3446$ and we put $\overline{m} = m$, $\overline{M} = M$ (rounded to four decimal places). We have

$$(\Phi_1(X_1) + \Phi_2(X_2))^4 = \frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} < \begin{pmatrix} \frac{1283}{2} & -255 \\ -255 & \frac{237}{2} \end{pmatrix} = \Phi_1\left(X_1^4\right) + \Phi_2\left(X_2^4\right) \qquad ()$$

and its improvement

$$\begin{aligned} (\Phi_1(X_1) + \Phi_2(X_2))^4 &= \frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ < \begin{pmatrix} 639.9213 & -255 \\ -255 & 117.8559 \end{pmatrix} &= \Phi_1\left(X_1^4\right) + \Phi_2\left(X_2^4\right) - \begin{pmatrix} 1.5787 & 0 \\ 0 & 0.6441 \end{pmatrix} \end{aligned} \qquad ()$$

(rounded to four decimal places), since $\delta_f = 3.1574$, $\widetilde{X} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2040 \end{pmatrix}$. But, if we put $\overline{m} = m_x = 0$, $\overline{M} = M_x = 0.5$, then $\widetilde{X} = \mathbf{0}$, so we do not have an improvement of Jensen's inequality. Also, if we put $\overline{m} = 0$, $\overline{M} = 1$, then $\widetilde{X} = 0.5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\delta_f = 7/8$ and $\delta_f \widetilde{X} = 0.4375\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which is worse than the above improvement.

Putting $\Phi_t(y) = a_t y$ for every $y \in$, where $a_t \geq 0$ is a real number, we obtain the following obvious corollary of Theorem ▭.

Corollary 14 Let $(x_t)_{t \in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ. Let m_t and M_t, $m_t \leq M_t$, be the bounds of x_t, $t \in T$. Let $(a_t)_{t \in T}$ be a continuous field of nonnegative real numbers such that $\int_T a_t \, d\mu(t) = 1$. Let

$$(m_x, M_x) \cap [m_t, M_t] = \emptyset, \quad t \in T, \qquad \text{and} \qquad m < M \qquad ()$$

where m_x and M_x, $m_x \leq M_x$, are the bounds of the operator $x = \int_T \phi_t(x_t) \, d\mu(t)$ and

$$m = \sup\{M_t \colon M_t \leq m_x, \ t \in T\}, \ M = \inf\{m_t \colon m_t \geq M_x, \ t \in T\} \qquad ()$$

If $f: I \to \mathbb{R}$ is a continuous convex (resp. concave) function provided that the interval I contains all m_t, M_t, then

$$\begin{array}{rcl} f\left(\int_T a_t x_t \, d\mu(t)\right) & \leq & \int_T a_t f(x_t) \, d\mu(t) - \delta_f \, \tilde{\tilde{x}} \leq \int_T a_t f(x_t) \, d\mu(t) \\ \left(\text{resp.} \quad f\left(\int_T a_t x_t \, d\mu(t)\right)\right. & \geq & \left. \int_T a_t f(x_t) \, d\mu(t) + \delta_f \, \tilde{\tilde{x}} \geq \int_T a_t f(x_t) \, d\mu(t) \right) \end{array} \quad ()$$

holds, where δ_f is defined by (▭), $\tilde{\tilde{x}} = \frac{1}{2} 1_H - \frac{1}{M - \bar{m}} \left| \int_T a_t x_t \, d\mu(t) - \frac{\bar{m} + \bar{M}}{2} 1_H \right|$ and $\bar{m} \in [m, m_A]$, $\bar{M} \in [M_A, M]$, $\bar{m} < \bar{M}$, are arbitrary numbers.

5. Extension Jensen's inequality

In this section we present an extension of Jensen's operator inequality for n - tuples of self-adjoint operators, unital n - tuples of positive linear mappings and real valued continuous convex functions with conditions on the spectra of the operators.

In a discrete version of Theorem ▭ we prove that Jensen's operator inequality holds for every continuous convex function and for every n - tuple of self-adjoint operators $(A_1, \ldots, A_n)$, for every n - tuple of positive linear mappings $(\Phi_1, \ldots, \Phi_n)$ in the case when the interval with bounds of the operator $A = \sum_{i=1}^{n} \Phi_i(A_i)$ has no intersection points with the interval with bounds of the operator A_i for each $i = 1, \ldots, n$, i.e. when $(m_A, M_A) \cap [m_i, M_i] = \emptyset$ for $i = 1, \ldots, n$, where m_A and M_A, $m_A \leq M_A$, are the bounds of A, and m_i and M_i, $m_i \leq M_i$, are the bounds of A_i, $i = 1, \ldots, n$. It is interesting to consider the case when $(m_A, M_A) \cap [m_i, M_i] = \emptyset$ is valid for several $i \in \{1, \ldots, n\}$, but not for all $i = 1, \ldots, n$. We study it in the following theorem (see [21]).

Theorem 15 Let $(A_1, \ldots, A_n)$ be an n - tuple of self-adjoint operators $A_i \in B(H)$ with the bounds m_i and M_i, $m_i \leq M_i$, $i = 1, \ldots, n$. Let $(\Phi_1, \ldots, \Phi_n)$ be an n - tuple of positive linear mappings $\Phi_i : B(H) \to B(K)$, such that $\sum_{i=1}^{n} \Phi_i(1_H) = 1_K$. For $1 \leq n_1 < n$, we denote $m = \min\{m_1, \ldots, m_{n_1}\}$, $M = \max\{M_1, \ldots, M_{n_1}\}$ and $\sum_{i=1}^{n_1} \Phi_i(1_H) = \alpha \, 1_K$, $\sum_{i=n_1+1}^{n} \Phi_i(1_H) = \beta \, 1_K$, where $\alpha, \beta > 0$, $\alpha + \beta = 1$. If

$$(m, M) \cap [m_i, M_i] = \emptyset, \qquad i = n_1 + 1, \ldots, n \quad ()$$

and one of two equalities

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i) = \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(A_i) = \sum_{i=1}^{n} \Phi_i(A_i) \quad ()$$

is valid, then

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le \sum_{i=1}^{n}\Phi_i(f(A_i)) \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) \qquad \text{(id57)}$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval I contains all $m_i, M_i, i = 1, \ldots, n$. If $f : I \to \mathbb{R}$ is concave, then the reverse inequality is valid in (▭).

We prove only the case when f is a convex function. Let us denote

$$A = \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i), \qquad B = \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i), \qquad C = \sum_{i=1}^{n}\Phi_i(A_i) \qquad ()$$

It is easy to verify that $A = B$ or $B = C$ or $A = C$ implies $A = B = C$.

a) Let $m < M$. Since f is convex on $[m, M]$ and $[m_i, M_i] \subseteq [m, M]$ for $i = 1, \ldots, n_1$, then

$$f(z) \le \frac{M - z}{M - m} f(m) + \frac{z - m}{M - m} f(M), \quad z \in [m_i, M_i] \text{ for } i = 1, \ldots, n_1 \qquad \text{(id58)}$$

but since f is convex on all $[m_i, M_i]$ and $(m, M) \cap [m_i, M_i] = \emptyset$ for $i = n_1 + 1, \ldots, n$, then

$$f(z) \ge \frac{M - z}{M - m} f(m) + \frac{z - m}{M - m} f(M), \quad z \in [m_i, M_i] \text{ for } i = n_1 + 1, \ldots, n \qquad \text{(id59)}$$

Since $m_i 1_H \le A_i \le M_i 1_H$, $i = 1, \ldots, n_1$, it follows from (▭)

$$f(A_i) \le \frac{M 1_H - A_i}{M - m} f(m) + \frac{A_i - m 1_H}{M - m} f(M), \qquad i = 1, \ldots, n_1 \qquad ()$$

Applying a positive linear mapping Φ_i and summing, we obtain

$$\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le \frac{M\alpha 1_K - \sum_{i=1}^{n_1}\Phi_i(A_i)}{M - m} f(m) + \frac{\sum_{i=1}^{n_1}\Phi_i(A_i) - m\alpha 1_K}{M - m} f(M) \qquad ()$$

since $\sum_{i=1}^{n_1}\Phi_i(1_H) = \alpha 1_K$. It follows

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le \frac{M 1_K - A}{M - m} f(m) + \frac{A - m 1_K}{M - m} f(M) \qquad \text{(id60)}$$

Similarly to (▭) in the case $m_i 1_H \le A_i \le M_i 1_H$, $i = n_1 + 1, \ldots, n$, it follows from (▭)

$$\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) \geq \frac{M1_K - B}{M - m} f(m) + \frac{B - m1_K}{M - m} f(M) \tag{id61}$$

Combining (▭) and (▭) and taking into account that $A = B$, we obtain

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \leq \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) \tag{id62}$$

It follows

$$\begin{aligned}
\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) &= \sum_{i=1}^{n_1}\Phi_i(f(A_i)) + \frac{\beta}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) && \text{(by } \alpha + \beta = 1\text{)}\\
&\leq \sum_{i=1}^{n_1}\Phi_i(f(A_i)) + \sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) && \text{(by ())}\\
&= \sum_{i=1}^{n}\Phi_i(f(A_i)) && \text{(id63)}\\
&\leq \frac{\alpha}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) + \sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) && \text{(by ())}\\
&= \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) && \text{(by } \alpha + \beta = 1\text{)}
\end{aligned}$$

which gives the desired double inequality (▭).

b) Let $m = M$. Since $[m_i, M_i] \subseteq [m, M]$ for $i = 1, \ldots, n_1$, then $A_i = m1_H$ and $f(A_i) = f(m)1_H$ for $i = 1, \ldots, n_1$. It follows

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i) = m1_K \qquad \text{and} \qquad \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) = f(m)1_K \tag{id64}$$

On the other hand, since f is convex on I, we have

$$f(z) \geq f(m) + l(m)(z - m) \quad \text{for every } z \in I \tag{id65}$$

where l is the subdifferential of f. Replacing z by A_i for $i = n_1 + 1, \ldots, n$, applying Φ_i and summing, we obtain from (▭) and (▭)

$$\begin{aligned}
\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) &\geq f(m)1_K + l(m)\left(\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i) - m1_K\right)\\
&= f(m)1_K = \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i))
\end{aligned} \tag{}$$

So (▭) holds again. The remaining part of the proof is the same as in the case a).

Remark 16 We obtain the equivalent inequality to the one in Theorem ▭ in the case when $\sum_{i=1}^{n}\Phi_i(1_H)=\gamma\, 1_K$, for some positive scalar γ. If $\alpha+\beta=\gamma$ and one of two equalities

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i)=\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i)=\frac{1}{\gamma}\sum_{i=1}^{n}\Phi_i(A_i) \qquad ()$$

is valid, then

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i))\le\frac{1}{\gamma}\sum_{i=1}^{n}\Phi_i(f(A_i))\le\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) \qquad ()$$

holds for every continuous convex function f.

Remark 17 Let the assumptions of Theorem ▭ be valid.

1. We observe that the following inequality

$$f\left(\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i)\right)\le\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) \qquad (\mathrm{id}68)$$

holds for every continuous convex function $f: I\to\mathbb{R}$.

Indeed, by the assumptions of Theorem ▭ we have

$$m\alpha 1_H\le\sum_{i=1}^{n_1}\Phi_i(f(A_i))\le M\alpha 1_H \quad \text{and} \quad \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i)=\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i) \qquad ()$$

which implies

$$m1_H\le\sum_{i=n_1+1}^{n}\frac{1}{\beta}\Phi_i(f(A_i))\le M\,1_H \qquad ()$$

Also $(m, M)\cap[m_i, M_i]=\emptyset$ for $i=n_1+1, \dots, n$ and $\sum_{i=n_1+1}^{n}\frac{1}{\beta}\Phi_i(1_H)=1_K$ hold. So we can apply Theorem ▭ on operators $A_{n_1+1}, \dots, A_n$ and mappings $\frac{1}{\beta}\Phi_i$ and obtain the desired inequality.

2. We denote by m_C and M_C the bounds of $C=\sum_{i=1}^{n}\Phi_i(A_i)$. If $(m_C, M_C)\cap[m_i, M_i]=\emptyset$, $i=1, \dots, n_1$ or f is an operator convex function on $[m, M]$, then the double inequality (▭) can be extended from the left side if we use Jensen's operator inequality (see Theorem 2.1[16])

$$
\begin{aligned}
f\big(\sum_{i=1}^{n}\Phi_i(A_i)\big) &= f\Big(\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i)\Big) \\
&\le \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le \sum_{i=1}^{n}\Phi_i(f(A_i)) \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i))
\end{aligned}
\qquad ()
$$

Example 18 If neither assumptions $(m_C, M_C)\cap[m_i, M_i]=\emptyset$, $i=1, \ldots, n_1$, nor f is operator convex in Remark ▭ - 2. is satisfied and if $1<n_1<n$, then (▭) can not be extended by Jensen's operator inequality, since it is not valid. Indeed, for $n_1=2$ we define mappings $\Phi_1, \Phi_2 : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ by $\Phi_1\big((a_{ij})_{1\le i,\, j\le 3}\big)=\frac{\alpha}{2}(a_{ij})_{1\le i,\, j\le 2}$, $\Phi_2=\Phi_1$. Then $\Phi_1(I_3)+\Phi_2(I_3)=\alpha I_2$. If

$$
A_1 = 2\begin{pmatrix}1 & 0 & 1\\ 0 & 0 & 1\\ 1 & 1 & 1\end{pmatrix} \quad \text{and} \quad A_2 = 2\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \qquad ()
$$

then

$$
\Big(\frac{1}{\alpha}\Phi_1(A_1)+\frac{1}{\alpha}\Phi_2(A_2)\Big)^4 = \frac{1}{\alpha^4}\begin{pmatrix}16 & 0\\ 0 & 0\end{pmatrix} \neg \le \frac{1}{\alpha}\begin{pmatrix}80 & 40\\ 40 & 24\end{pmatrix} = \frac{1}{\alpha}\Phi_1(A_1^4)+\frac{1}{\alpha}\Phi_2(A_2^4) \qquad ()
$$

for every $\alpha\in(0, 1)$. We observe that $f(t)=t^4$ is not operator convex and $(m_C, M_C)\cap[m_i, M_i]\ne\emptyset$, since $C=A=\frac{1}{\alpha}\Phi_1(A_1)+\frac{1}{\alpha}\Phi_2(A_2)=\frac{1}{\alpha}\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}$, $[m_C, M_C]=[0, 2/\alpha]$, $[m_1, M_1]\subset[-1.60388, 4.49396]$ and $[m_2, M_2]=[0, 2]$.

With respect to Remark ▭, we obtain the following obvious corollary of Theorem ▭.

Corollary 19 Let $(A_1, \ldots, A_n)$ be an n - tuple of self-adjoint operators $A_i\in B(H)$ with the bounds m_i and M_i, $m_i\le M_i$, $i=1, \ldots, n$. For some $1\le n_1<n$, we denote $m=\min\{m_1, \ldots, m_{n_1}\}$, $M=\max\{M_1, \ldots, M_{n_1}\}$. Let $(p_1, \ldots, p_n)$ be an n - tuple of non-negative numbers, such that $0<\sum_{i=1}^{n_1}p_i = {}_1 < \; = \sum_{i=1}^{n}p_i$. If

$$
(m, M)\cap[m_i, M_i]=\emptyset, \qquad i=n_1+1, \ldots, n \qquad ()
$$

and one of two equalities

$$
\frac{1}{{}_1}\sum_{i=1}^{n_1}p_iA_i = \frac{1}{}\sum_{i=1}^{n}p_iA_i = \frac{1}{-{}_1}\sum_{i=n_1+1}^{n}p_iA_i \qquad ()
$$

is valid, then

$$\frac{1}{{}_{1}}\sum_{i=1}^{n_1} p_i f(A_i) \leq \frac{1}{}\sum_{i=1}^{n} p_i f(A_i) \leq \frac{1}{-{}_{1}}\sum_{i=n_1+1}^{n} p_i f(A_i) \qquad \text{(id71)}$$

holds for every continuous convex function $f: I \to \mathbb{R}$ provided that the interval I contains all m_i, M_i, $i = 1, \ldots, n$.

If $f: I \to \mathbb{R}$ is concave, then the reverse inequality is valid in (▭).

As a special case of Corollary ▭ we can obtain a discrete version of Corollary ▭ as follows.

Corollary 20 (Discrete version of Corollary ▭) Let $(A_1, \ldots, A_n)$ be an n - tuple of self-adjoint operators $A_i \in B(H)$ with the bounds m_i and M_i, $m_i \leq M_i$, $i = 1, \ldots, n$. Let $(\alpha_1, \ldots, \alpha_n)$ be an n - tuple of nonnegative real numbers such that $\sum_{i=1}^{n} \alpha_i = 1$. If

$$(m_A, M_A) \cap [m_i, M_i] = \varnothing, \qquad i = 1, \ldots, n \qquad \text{(id73)}$$

where m_A and M_A, $m_A \leq M_A$, are the bounds of $A = \sum_{i=1}^{n} \alpha_i A_i$, then

$$f\Big(\sum_{i=1}^{n} \alpha_i A_i\Big) \leq \sum_{i=1}^{n} \alpha_i f(A_i) \qquad \text{(id74)}$$

holds for every continuous convex function $f: I \to \mathbb{R}$ provided that the interval I contains all m_i, M_i.

We prove only the convex case. We define $(n+1)$ - tuple of operators $(B_1, \ldots, B_{n+1})$, $B_i \in B(H)$, by $B_1 = A = \sum_{i=1}^{n} \alpha_i A_i$ and $B_i = A_{i-1}$, $i = 2, \ldots, n+1$. Then $m_{B_1} = m_A$, $M_{B_1} = M_A$ are the bounds of B_1 and $m_{B_i} = m_{i-1}$, $M_{B_i} = M_{i-1}$ are the ones of B_i, $i = 2, \ldots, n+1$. Also, we define $(n+1)$ - tuple of non-negative numbers $(p_1, \ldots, p_{n+1})$ by $p_1 = 1$ and $p_i = \alpha_{i-1}$, $i = 2, \ldots, n+1$. Then $\sum_{i=1}^{n+1} p_i = 2$ and by using (▭) we have

$$(m_{B_1}, M_{B_1}) \cap [m_{B_i}, M_{B_i}] = \varnothing, \qquad i = 2, \ldots, n+1 \qquad \text{(id75)}$$

Since

$$\sum_{i=1}^{n+1} p_i B_i = B_1 + \sum_{i=2}^{n+1} p_i B_i = \sum_{i=1}^{n} \alpha_i A_i + \sum_{i=1}^{n} \alpha_i A_i = 2B_1 \qquad \text{()}$$

then

$$p_1 B_1 = \frac{1}{2}\sum_{i=1}^{n+1} p_i B_i = \sum_{i=2}^{n+1} p_i B_i \qquad \text{(id76)}$$

Taking into account (▭) and (▭), we can apply Corollary ▭ for $n_1 = 1$ and B_i, p_i as above, and we get

$$p_1 f(B_1) \le \frac{1}{2}\sum_{i=1}^{n+1} p_i f(B_i) \le \sum_{i=2}^{n+1} p_i f(B_i) \qquad ()$$

which gives the desired inequality (▭).

6. Extension of the refined Jensen's inequality

There is an extensive literature devoted to Jensen's inequality concerning different refinements and extensive results, see, for example [22], [23], [24], [25], [26], [27], [28], [29].

In this section we present an extension of the refined Jensen's inequality obtained in Section ▭ and a refinement of the same inequality obtained in Section ▭.

Theorem 21 Let $(A_1, \ldots, A_n)$ be an n - tuple of self-adjoint operators $A_i \in B(H)$ with the bounds m_i and M_i, $m_i \le M_i$, $i = 1, \ldots, n$. Let $(\Phi_1, \ldots, \Phi_n)$ be an n - tuple of positive linear mappings $\Phi_i : B(H) \to B(K)$, such that $\sum_{i=1}^{n_1} \Phi_i(1_H) = \alpha\, 1_K$, $\sum_{i=n_1+1}^{n} \Phi_i(1_H) = \beta\, 1_K$, where $1 \le n_1 < n$, $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Let $m_L = \min\{m_1, \ldots, m_{n_1}\}$, $M_R = \max\{M_1, \ldots, M_{n_1}\}$ and

$$\begin{aligned} m &= \max\{M_i : M_i \le m_L,\ i \in \{n_1+1, \ldots, n\}\} \\ M &= \min\{m_i : m_i \ge M_R,\ i \in \{n_1+1, \ldots, n\}\} \end{aligned} \qquad ()$$

If

$$(m_L, M_R) \cap [m_i, M_i] = \emptyset, \quad i = n_1+1, \ldots, n, \quad \text{and} \quad m < M \qquad ()$$

and one of two equalities

$$\frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(A_i) = \sum_{i=1}^{n} \Phi_i(A_i) = \frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i(A_i) \qquad ()$$

is valid, then

$$\begin{aligned} \frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) &\le \frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \beta\delta_f \widetilde{A} \le \sum_{i=1}^{n} \Phi_i(f(A_i)) \\ &\le \frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \alpha\delta_f \widetilde{A} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) \end{aligned} \qquad \text{(id78)}$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval I contains all m_i, M_i, $i = 1, \ldots, n$, where

$$\delta_f \equiv \delta_f(\bar{m}, \bar{M}) = f(\bar{m}) + f(\bar{M}) - 2f\left(\frac{\bar{m}+\bar{M}}{2}\right)$$
$$\widetilde{A} \equiv \widetilde{A}_{A,\Phi,n_1,\alpha}(\bar{m}, \bar{M}) = \frac{1}{2}1_K - \frac{1}{\alpha(\bar{M}-\bar{m})}\sum_{i=1}^{n_1}\Phi_i\left(\left|A_i - \frac{\bar{m}+\bar{M}}{2}1_H\right|\right) \tag{id79}$$

and $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, are arbitrary numbers. If $f : I \to \mathbb{R}$ is concave, then the reverse inequality is valid in (▭).

We prove only the convex case. Let us denote

$$A = \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i), \qquad B = \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i), \qquad C = \sum_{i=1}^{n}\Phi_i(A_i) \tag{}$$

It is easy to verify that $A = B$ or $B = C$ or $A = C$ implies $A = B = C$.

Since f is convex on $[\bar{m}, \bar{M}]$ and $(A_i) \subseteq [m_i, M_i] \subseteq [\bar{m}, \bar{M}]$ for $i = 1, \ldots, n_1$, it follows from Lemma ▭ that

$$f(A_i) \le \frac{\bar{M}1_H - A_i}{\bar{M}-\bar{m}}f(\bar{m}) + \frac{A_i - \bar{m}1_H}{\bar{M}-\bar{m}}f(\bar{M}) - \delta_f\widetilde{A}_i, \qquad i = 1, \ldots, n_1 \tag{}$$

holds, where $\delta_f = f(\bar{m}) + f(\bar{M}) - 2f\left(\frac{\bar{m}+\bar{M}}{2}\right)$ and $\widetilde{A}_i = \frac{1}{2}1_H - \frac{1}{\bar{M}-\bar{m}}\left|A_i - \frac{\bar{m}+\bar{M}}{2}1_H\right|$. Applying a positive linear mapping Φ_i and summing, we obtain

$$\begin{aligned}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le\ & \frac{\bar{M}\alpha 1_K - \sum_{i=1}^{n_1}\Phi_i(A_i)}{\bar{M}-\bar{m}}f(\bar{m}) + \frac{\sum_{i=1}^{n_1}\Phi_i(A_i) - \bar{m}\alpha 1_K}{\bar{M}-\bar{m}}f(\bar{M}) \\ & - \delta_f\left(\frac{\alpha}{2}1_K - \frac{1}{\bar{M}-\bar{m}}\sum_{i=1}^{n_1}\Phi_i\left(\left|A_i - \frac{\bar{m}+\bar{M}}{2}1_H\right|\right)\right)\end{aligned} \tag{}$$

since $\sum_{i=1}^{n_1}\Phi_i(1_H) = \alpha 1_K$. It follows that

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le \frac{\bar{M}1_K - A}{\bar{M}-\bar{m}}f(\bar{m}) + \frac{A - \bar{m}1_K}{\bar{M}-\bar{m}}f(\bar{M}) - \delta_f\widetilde{A} \tag{id80}$$

where $\widetilde{A} = \frac{1}{2}1_K - \frac{1}{\alpha(\bar{M}-\bar{m})}\sum_{i=1}^{n_1}\Phi_i\left(\left|A_i - \frac{\bar{m}+\bar{M}}{2}1_H\right|\right)$.

Additionally, since f is convex on all $[m_i, M_i]$ and $(\bar{m}, \bar{M}) \cap [m_i, M_i] = \emptyset$, $i = n_1 + 1, \ldots, n$, then

$$f(A_i) \geq \frac{\bar{M}1_H - A_i}{\bar{M} - \bar{m}} f(\bar{m}) + \frac{A_i - \bar{m}1_H}{\bar{M} - \bar{m}} f(\bar{M}), \qquad i = n_1 + 1, \ldots, n \tag{id81}$$

It follows

$$\frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \delta_f \widetilde{A} \geq \frac{\bar{M}1_K - B}{\bar{M} - \bar{m}} f(\bar{m}) + \frac{B - \bar{m}1_K}{\bar{M} - \bar{m}} f(\bar{M}) - \delta_f \widetilde{A} \tag{id82}$$

Combining (▭) and (▭) and taking into account that $A = B$, we obtain

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \delta_f \widetilde{A} \tag{id83}$$

Next, we obtain

$$\begin{aligned} & \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \\ = \quad & \sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \frac{\beta}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \quad (\text{by } \alpha + \beta = 1) \\ \leq \quad & \sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \beta\delta_f \widetilde{A} \qquad (\text{by } ()) \\ \leq \quad & \frac{\alpha}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \alpha\delta_f \widetilde{A} + \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \beta\delta_f \widetilde{A} \qquad (\text{by } ()) \\ = \quad & \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \delta_f \widetilde{A} \quad (\text{by } \alpha + \beta = 1) \end{aligned} \tag{}$$

which gives the following double inequality

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \leq \sum_{i=1}^{n} \Phi_i(f(A_i)) - \beta\delta_f \widetilde{A} \leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \delta_f \widetilde{A} \tag{id84}$$

Adding $\beta\delta_f \widetilde{A}$ in the above inequalities, we get

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \beta\delta_f \widetilde{A} \leq \sum_{i=1}^{n} \Phi_i(f(A_i)) \leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \alpha\delta_f \widetilde{A} \tag{id85}$$

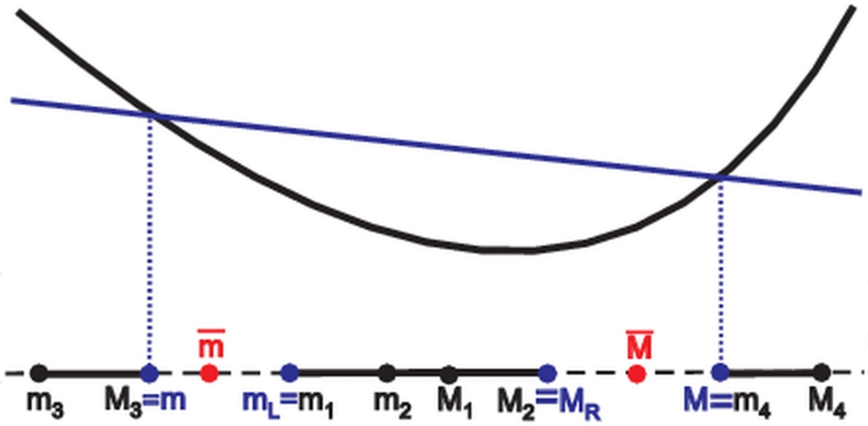


Figure 3. An example a convex function and the bounds of four operators

Now, we remark that $\delta_f \geq 0$ and $\widetilde{A} \geq 0$. (Indeed, since f is convex, then $f((\overline{m}+\overline{M})/2) \leq (f(\overline{m})+f(\overline{M}))/2$, which implies that $\delta_f \geq 0$. Also, since

$$(A_i) \subseteq [\overline{m}, \overline{M}] \quad \Rightarrow \quad \left| A_i - \frac{\overline{M}+\overline{m}}{2} 1_H \right| \leq \frac{\overline{M}-\overline{m}}{2} 1_H, \qquad i = 1, \ldots, n_1 \qquad ()$$

then

$$\sum_{i=1}^{n_1} \Phi_i\left(\left| A_i - \frac{\overline{M}+\overline{m}}{2} 1_H \right| \right) \leq \frac{\overline{M}-\overline{m}}{2} \alpha 1_K \qquad ()$$

which gives

$$0 \leq \frac{1}{2} 1_K - \frac{1}{\alpha(\overline{M}-\overline{m})} \sum_{i=1}^{n_1} \Phi_i\left(\left| A_i - \frac{\overline{M}+\overline{m}}{2} 1_H \right| \right) = \widetilde{A} \Big) \qquad ()$$

Consequently, the following inequalities

$$\begin{aligned} &\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \leq \frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \beta \delta_f \widetilde{A} \\ &\frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \alpha \delta_f \widetilde{A} \leq \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) \end{aligned} \qquad ()$$

hold, which with (▭) proves the desired series inequalities (▭). 1.05

Example 22 We observe the matrix case of Theorem ▭ for $f(t) = t^4$, which is the convex function but not operator convex, $n = 4$, $n_1 = 2$ and the bounds of matrices as in Fig. 3.

We show an example such that

$$\begin{aligned}\frac{1}{\alpha}\left(\Phi_1\left(A_1^4\right)+\Phi_2\left(A_2^4\right)\right) &< \frac{1}{\alpha}\left(\Phi_1\left(A_1^4\right)+\Phi_2\left(A_2^4\right)\right)+\beta\delta_f\widetilde{A} \\ &< \Phi_1\left(A_1^4\right)+\Phi_2\left(A_2^4\right)+\Phi_3\left(A_3^4\right)+\Phi_4\left(A_4^4\right) \\ &< \frac{1}{\beta}\left(\Phi_3\left(A_3^4\right)+\Phi_4\left(A_4^4\right)\right)-\alpha\delta_f\widetilde{A} < \frac{1}{\beta}\left(\Phi_3\left(A_3^4\right)+\Phi_4\left(A_4^4\right)\right)\end{aligned} \tag{id88}$$

holds, where $\delta_f = \overline{M}^4 + \overline{m}^4 - (\overline{M} + \overline{m})^4 8$ and

$$\widetilde{A} = \frac{1}{2}I_2 - \frac{1}{\alpha(\overline{M} - \overline{m})}\left(\Phi_1\left(\left| \ A_1 - \frac{\overline{M}+\overline{m}}{2}I_h \right| \ \right) + \Phi_2\left(\left| \ A_2 - \frac{\overline{M}+\overline{m}}{2}I_3 \right| \ \right)\right) \tag{}$$

We define mappings $\Phi_i : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ as follows: $\Phi_i\left((a_{jk})_{1\le j,k\le 3}\right) = \frac{1}{4}\left(a_{jk}\right)_{1\le j,k\le 2}$, $i = 1, \ \ldots, 4$. Then $\sum_{i=1}^{4}\Phi_i\left(I_3\right) = I_2$ and $\alpha = \beta = \frac{1}{2}$.

Let

$$A_1 = 2\begin{pmatrix} 2 & 9/8 & 1 \\ 9/8 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}, \ A_2 = 3\begin{pmatrix} 2 & 9/8 & 0 \\ 9/8 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \ A_3 = -3\begin{pmatrix} 4 & 1/2 & 1 \\ 1/2 & 4 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \ A_4 = 12\begin{pmatrix} 5/3 & 1/2 & 0 \\ 1/2 & 3/2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{}$$

Then $m_1 = 1.28607$, $M_1 = 7.70771$, $m_2 = 0.53777$, $M_2 = 5.46221$, $m_3 = -14.15050$, $M_3 = -4.71071$, $m_4 = 12.91724$, $M_4 = 36.$, so $m_L = m_2$, $M_R = M_1$, $m = M_3$ and $M = m_4$ (rounded to five decimal places). Also,

$$\frac{1}{\alpha}\left(\Phi_1(A_1)+\Phi_2(A_2)\right) = \frac{1}{\beta}\left(\Phi_3(A_3)+\Phi_4(A_4)\right) = \begin{pmatrix} 4 & 9/4 \\ 9/4 & 3 \end{pmatrix} \tag{}$$

and

$$\begin{aligned} A_f \equiv \frac{1}{\alpha}\left(\Phi_1\left(A_1^4\right)+\Phi_2\left(A_2^4\right)\right) &= \begin{pmatrix} 989.00391 & 663.46875 \\ 663.46875 & 526.12891 \end{pmatrix} \\ C_f \equiv \Phi_1\left(A_1^4\right)+\Phi_2\left(A_2^4\right)+\Phi_3\left(A_3^4\right)+\Phi_4\left(A_4^4\right) &= \begin{pmatrix} 68093.14258 & 48477.98437 \\ 48477.98437 & 51335.39258 \end{pmatrix} \\ B_f \equiv \frac{1}{\beta}\left(\Phi_3\left(A_3^4\right)+\Phi_4\left(A_4^4\right)\right) &= \begin{pmatrix} 135197.28125 & 96292.5 \\ 96292.5 & 102144.65625 \end{pmatrix} \end{aligned} \tag{}$$

Then

$$A_f < C_f < B_f \tag{id89}$$

holds (which is consistent with (▭)).

We will choose three pairs of numbers $(\bar{m}, \bar{M})$, $\bar{m} \in [-4.71071, 0.53777]$, $\bar{M} \in [7.70771, 12.91724]$ as follows

i) $\bar{m} = m_L = 0.53777$, $\bar{M} = M_R = 7.70771$, then

$$\tilde{\Delta}_1 = \beta\delta_f \tilde{A} = 0.5 \cdot 2951.69249 \cdot \begin{pmatrix} 0.15678 & 0.09030 \\ 0.09030 & 0.15943 \end{pmatrix} = \begin{pmatrix} 231.38908 & 133.26139 \\ 133.26139 & 235.29515 \end{pmatrix}$$

ii) $\bar{m} = m = -4.71071$, $\bar{M} = M = 12.91724$, then

$$\tilde{\Delta}_2 = \beta\delta_f \tilde{A} = 0.5 \cdot 27766.07963 \cdot \begin{pmatrix} 0.36022 & 0.03573 \\ 0.03573 & 0.36155 \end{pmatrix} = \begin{pmatrix} 5000.89860 & 496.04498 \\ 496.04498 & 5019.50711 \end{pmatrix}$$

iii) $\bar{m} = -1$, $\bar{M} = 10$, then

$$\tilde{\Delta}_3 = \beta\delta_f \tilde{A} = 0.5 \cdot 9180.875 \cdot \begin{pmatrix} 0.28203 & 0.08975 \\ 0.08975 & 0.27557 \end{pmatrix} = \begin{pmatrix} 1294.66 & 411.999 \\ 411.999 & 1265. \end{pmatrix}$$

New, we obtain the following improvement of (▭) (see (▭))

Table 1.

Using Theorem ▭ we get the following result.

Corollary 23 Let the assumptions of Theorem ▭ hold. Then

$$\frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) \le \frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \gamma_1\delta_f \tilde{A} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) \qquad \text{(id91)}$$

and

$$\frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) \le \frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \gamma_2\delta_f \tilde{A} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) \qquad \text{(id92)}$$

holds for every γ_1, γ_2 in the close interval joining α and β, where δ_f and $\tilde{A}$ are defined by (▭).

Adding $\alpha\delta_f \tilde{A}$ in (▭) and noticing $\delta_f \tilde{A} \ge 0$, we obtain

$$\frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) \le \frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \alpha\delta_f \tilde{A} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) \qquad \text{(id93)}$$

Taking into account the above inequality and the left hand side of (▭) we obtain (▭).

Similarly, subtracting $\beta\delta_f \widetilde{A}$ in (▭) we obtain (▭).

Remark 24 We can obtain extensions of inequalities which are given in Remark ▭ and ▭. Also, we can obtain a special case of Theorem ▭ with the convex combination of operators A_i putting $\Phi_i(B) = \alpha_i B$, for $i = 1, \ldots, n$, similarly as in Corollary ▭. Finally, applying this result, we can give another proof of Corollary ▭. The interested reader can see the details in [30].

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