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3-Algebras in String Theory

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1. Introduction

In this chapter, we review 3-algebras that appear as fundamental properties of string theory. 3-algebra is a generalization of Lie algebra; it is defined by a tri-linear bracket instead of by a bi-linear bracket, and satisfies fundamental identity, which is a generalization of Jacobi identity [1], [2], [3]. We consider 3-algebras equipped with invariant metrics in order to apply them to physics.

It has been expected that there exists M-theory, which unifies string theories. In M-theory, some structures of 3-algebras were found recently. First, it was found that by using $u(N) \oplus u(N)$ Hermitian 3-algebra, we can describe a low energy effective action of N coincident supermembranes [4], [5], [6], [7], [8], which are fundamental objects in M-theory.

With this as motivation, 3-algebras with invariant metrics were classified [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22]. Lie 3-algebras are defined in real vector spaces and tri-linear brackets of them are totally anti-symmetric in all the three entries. Lie 3-algebras with invariant metrics are classified into $_4$ algebra, and Lorentzian Lie 3-algebras, which have metrics with indefinite signatures. On the other hand, Hermitian 3-algebras are defined in Hermitian vector spaces and their tri-linear brackets are complex linear and anti-symmetric in the first two entries, whereas complex anti-linear in the third entry. Hermitian 3-algebras with invariant metrics are classified into $u(N) \oplus u(M)$ and $sp(2N) \oplus u(1)$ Hermitian 3-algebras.

Moreover, recent studies have indicated that there also exist structures of 3-algebras in the Green-Schwartz supermembrane action, which defines full perturbative dynamics of a supermembrane. It had not been clear whether the total supermembrane action including fermions has structures of 3-algebras, whereas the bosonic part of the action can be described by using a tri-linear bracket, called Nambu bracket [23], [24], which is a generalization of Poisson bracket. If we fix to a light-cone gauge, the total action can be described by using

Poisson bracket, that is, only structures of Lie algebra are left in this gauge [25]. However, it was shown under an approximation that the total action can be described by Nambu bracket if we fix to a semi-light-cone gauge [26]. In this gauge, the eleven dimensional space-time of M-theory is manifest in the supermembrane action, whereas only ten dimensional part is manifest in the light-cone gauge.

The BFSS matrix theory is conjectured to describe an infinite momentum frame (IMF) limit of M-theory [27] and many evidences were found. The action of the BFSS matrix theory can be obtained by replacing Poisson bracket with a finite dimensional Lie algebra's bracket in the supermembrane action in the light-cone gauge. Because of this structure, only variables that represent the ten dimensional part of the eleven-dimensional space-time are manifest in the BFSS matrix theory. Recently, 3-algebra models of M-theory were proposed [26], [28], [29], by replacing Nambu bracket with finite dimensional 3-algebras' brackets in an action that is shown, by using an approximation, to be equivalent to the semi-light-cone supermembrane action. All the variables that represent the eleven dimensional space-time are manifest in these models. It was shown that if the DLCQ limit of the 3-algebra models of M-theory is taken, they reduce to the BFSS matrix theory [26], [28], as they should [30], [31], [32], [33], [34], [35].

2. Definition and classification of metric Hermitian 3-algebra

In this section, we will define and classify the Hermitian 3-algebras equipped with invariant metrics.

2.1. General structure of metric Hermitian 3-algebra

The metric Hermitian 3-algebra is a map $V \times V \times V \rightarrow V$ defined by $(x, y, z) \mapsto [x, y, z]$, where the 3-bracket is complex linear in the first two entries, whereas complex anti-linear in the last entry, equipped with a metric $\langle x, y \rangle$, satisfying the following properties:

the fundamental identity

$$[[x, y, z], v, w] = [[x, v, w], y, z] + [x, [y, v, w], z] - [x, y, [z, w, v]] \quad (\text{id2})$$

the metric invariance

$$\langle [x, v, w], y \rangle - \langle x, [y, w, v] \rangle = 0 \quad (\text{id3})$$

and the anti-symmetry

$$[x, y, z] = -[y, x, z] \quad (\text{id4})$$

for

$$x, y, z, v, w \in V \quad (\text{id5})$$

The Hermitian 3-algebra generates a symmetry, whose generators $D(x, y)$ are defined by

$$D(x, y)z := [z, x; y] \quad (\text{id6})$$

From (≡), one can show that $D(x, y)$ form a Lie algebra,

$$[D(x, y), D(v, w)] = D(D(x, y)v, w) - D(v, D(y, x)w) \quad (\text{id7})$$

There is an one-to-one correspondence between the metric Hermitian 3-algebra and a class of metric complex super Lie algebras [19]. Such a class satisfies the following conditions among complex super Lie algebras $S = S_0 \oplus S_1$, where S_0 and S_1 are even and odd parts, respectively. S_1 is decomposed as $S_1 = V \oplus \bar{V}$, where V is an unitary representation of S_0 ; for $a \in S_0, u, v \in V$,

$$[a, u] \in V \quad (\text{id8})$$

and

$$\langle [a, u], v \rangle + \langle u, [a^*, v] \rangle = 0 \quad (\text{id9})$$

$\bar{v} \in \bar{V}$ is defined by

$$\bar{v} = \langle \cdot, v \rangle \quad (\text{id10})$$

The super Lie bracket satisfies

$$[V, V] = 0, \quad [\bar{V}, \bar{V}] = 0 \quad (\text{id11})$$

From the metric Hermitian 3-algebra, we obtain the class of the metric complex super Lie algebra in the following way. The elements in S_0, V , and $\bar{V}$ are defined by (≡), (≡), and (≡), respectively. The algebra is defined by (≡) and

$$\begin{aligned} [D(x, y), z] &:= D(x, y)z = [z, x; y] \\ [D(x, y), \bar{z}] &:= -\overline{D(y, x)z} = -[z, y; x] \\ [x, \bar{y}] &:= D(x, y) \\ [x, y] &:= 0 \\ [\bar{x}, \bar{y}] &:= 0 \end{aligned} \quad (\text{id12})$$

One can show that this algebra satisfies the super Jacobi identity and $(\square)$ - $(\square)$ as in [19].

Inversely, from the class of the metric complex super Lie algebra, we obtain the metric Hermitian 3-algebra by

$$[x, y; z] : = \alpha [[y, \bar{z}], x] \quad (\text{id13})$$

where α is an arbitrary constant. One can also show that this algebra satisfies $(\square)$ - $(\square)$ for $(\square)$ as in [19].

2.2. Classification of metric Hermitian 3-algebra

The classical Lie super algebras satisfying $(\square)$ - $(\square)$ are $A(m-1, n-1)$ and $C(n+1)$. The even parts of $A(m-1, n-1)$ and $C(n+1)$ are $u(m) \oplus u(n)$ and $sp(2n) \oplus u(1)$, respectively. Because the metric Hermitian 3-algebra one-to-one corresponds to this class of the super Lie algebra, the metric Hermitian 3-algebras are classified into $u(m) \oplus u(n)$ and $sp(2n) \oplus u(1)$ Hermitian 3-algebras.

First, we will construct the $u(m) \oplus u(n)$ Hermitian 3-algebra from $A(m-1, n-1)$, according to the relation in the previous subsection. $A(m-1, n-1)$ is simple and is obtained by dividing $sl(m, n)$ by its ideal. That is, $A(m-1, n-1) = sl(m, n)$ when $m \neq n$ and $A(n-1, n-1) = sl(n, n) / \lambda 1_{2n}$.

Real $sl(m, n)$ is defined by

$$\begin{pmatrix} h_1 & c \\ ic^\dagger & h_2 \end{pmatrix} \quad (\text{id15})$$

where h_1 and h_2 are $m \times m$ and $n \times n$ anti-Hermite matrices and c is an $n \times m$ arbitrary complex matrix. Complex $sl(m, n)$ is a complexification of real $sl(m, n)$, given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad (\text{id16})$$

where α, β, γ , and δ are $m \times m, n \times m, m \times n$, and $n \times n$ complex matrices that satisfy

$$\text{tr} \alpha = \text{tr} \delta \quad (\text{id17})$$

Complex $A(m-1, n-1)$ is decomposed as $A(m-1, n-1) = S_0 \oplus V \oplus \bar{V}$, where

$$\begin{aligned} \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} &\in S_0 \\ \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} &\in V \\ \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} &\in \bar{V} \end{aligned} \quad (\text{id18})$$

($\Rightarrow$) is rewritten as $V \rightarrow \bar{V}$ defined by

$$B = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \mapsto B^\dagger = \begin{pmatrix} 0 & 0 \\ \beta^\dagger & 0 \end{pmatrix} \quad (\text{id19})$$

where $B \in V$ and $B^\dagger \in \bar{V}$. ($\Rightarrow$) is rewritten as

$$[X, Y; Z] = \alpha[[Y, Z^\dagger], X] = \alpha \begin{pmatrix} 0 & yz^\dagger x - xz^\dagger y \\ 0 & 0 \end{pmatrix} \quad (\text{id20})$$

for

$$\begin{aligned} X &= \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in V \\ Y &= \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \in V \\ Z &= \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \in V \end{aligned} \quad (\text{id21})$$

As a result, we obtain the $u(m) \oplus u(n)$ Hermitian 3-algebra,

$$[x, y; z] = \alpha(yz^\dagger x - xz^\dagger y) \quad (\text{id22})$$

where x , y , and z are arbitrary $n \times m$ complex matrices. This algebra was originally constructed in [8].

Inversely, from ($\Leftarrow$), we can construct complex $A(m-1, n-1)$. ($\Leftarrow$) is rewritten as

$$D(x, y) = (xy^\dagger, y^\dagger x) \in S_0 \quad (\text{id23})$$

($\Leftarrow$) and ($\Rightarrow$) are rewritten as

$$\begin{aligned}
[(xy^\dagger, y^\dagger x), (x'y'^\dagger, y'^\dagger x')] &= ([xy^\dagger, x'y'^\dagger], [y^\dagger x, y'^\dagger x']) \\
[(xy^\dagger, y^\dagger x), z] &= xy^\dagger z - zy^\dagger x \\
[(xy^\dagger, y^\dagger x), w^\dagger] &= y^\dagger x w^\dagger - w^\dagger x y^\dagger \\
[x, y^\dagger] &= (xy^\dagger, y^\dagger x) \\
[x, y] &= 0 \\
[x^\dagger, y^\dagger] &= 0
\end{aligned} \tag{id24}$$

This algebra is summarized as

$$\left[\begin{pmatrix} xy^\dagger & z \\ w^\dagger & y^\dagger x \end{pmatrix}, \begin{pmatrix} x'y'^\dagger & z' \\ w'^\dagger & y'^\dagger x' \end{pmatrix} \right] \tag{id25}$$

which forms complex $A(m-1, n-1)$.

Next, we will construct the $sp(2n) \oplus u(1)$ Hermitian 3-algebra from $C(n+1)$. Complex $C(n+1)$ is decomposed as $C(n+1) = S_0 \oplus V \oplus \bar{V}$. The elements are given by

$$\begin{aligned}
&\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & -a^T \end{pmatrix} \in S_0 \\
&\begin{pmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & x_2^T & 0 & 0 \\ 0 & -x_1^T & 0 & 0 \end{pmatrix} \in V \\
&\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & y_1 & y_2 \\ y_2^T & 0 & 0 & 0 \\ -y_1^T & 0 & 0 & 0 \end{pmatrix} \in \bar{V}
\end{aligned} \tag{id26}$$

where α is a complex number, a is an arbitrary $n \times n$ complex matrix, b and c are $n \times n$ complex symmetric matrices, and x_1, x_2, y_1 and y_2 are $n \times 1$ complex matrices. ($\Rightarrow$) is rewritten as

$V \rightarrow \bar{V}$ defined by $B \mapsto \bar{B} = U B^* U^{-1}$, where $B \in V, \bar{B} \in \bar{V}$ and

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (\text{id27})$$

Explicitly,

$$B = \begin{pmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & x_2^T & 0 & 0 \\ 0 & -x_1^T & 0 & 0 \end{pmatrix} \mapsto \bar{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x_2^* & -x_1^* \\ -x_1^\dagger & 0 & 0 & 0 \\ -x_2^\dagger & 0 & 0 & 0 \end{pmatrix} \quad (\text{id28})$$

($\Rightarrow$) is rewritten as

$$\begin{aligned} [X, Y; Z] &:= \alpha[[Y, \bar{Z}], X] \\ &= \alpha \left[\begin{pmatrix} 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & 0 \\ 0 & y_2^T & 0 & 0 \\ 0 & -y_1^T & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & z_2^* & -z_1^* \\ -z_1^\dagger & 0 & 0 & 0 \\ -z_2^\dagger & 0 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & x_2^T & 0 & 0 \\ 0 & -x_1^T & 0 & 0 \end{pmatrix} \\ &= \alpha \begin{pmatrix} 0 & 0 & w_1 & w_2 \\ 0 & 0 & 0 & 0 \\ 0 & w_2^T & 0 & 0 \\ 0 & -w_1^T & 0 & 0 \end{pmatrix} \end{aligned} \quad (\text{id29})$$

for

$$\begin{aligned}
X &= \begin{pmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & x_2^T & 0 & 0 \\ 0 & -x_1^T & 0 & 0 \end{pmatrix} \in V \\
Y &= \begin{pmatrix} 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & 0 \\ 0 & y_2^T & 0 & 0 \\ 0 & -y_1^T & 0 & 0 \end{pmatrix} \in V \\
Z &= \begin{pmatrix} 0 & 0 & z_1 & z_2 \\ 0 & 0 & 0 & 0 \\ 0 & z_2^T & 0 & 0 \\ 0 & -z_1^T & 0 & 0 \end{pmatrix} \in V
\end{aligned} \tag{id30}$$

where w_1 and w_2 are given by

$$(w_1, w_2) = -(y_1 z_1^\dagger + y_2 z_2^\dagger)(x_1, x_2) + (x_1 z_1^\dagger + x_2 z_2^\dagger)(y_1, y_2) + (x_2 y_1^T - x_1 y_2^T)(z_2^*, -z_1^*) \tag{id31}$$

As a result, we obtain the $sp(2n) \oplus u(1)$ Hermitian 3-algebra,

$$[x, y; z] = \alpha((y \odot \tilde{z})x + (\tilde{z} \odot x)y - (x \odot y)\tilde{z}) \tag{id32}$$

for $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$, where x_1, x_2, y_1, y_2, z_1 , and z_2 are n-vectors and

$$\begin{aligned}
\tilde{z} &= (z_2^*, -z_1^*) \\
a \odot b &= a_1 \cdot b_2 - a_2 \cdot b_1
\end{aligned} \tag{id33}$$

3. 3-algebra model of M-theory

In this section, we review the fact that the supermembrane action in a semi-light-cone gauge can be described by Nambu bracket, where structures of 3-algebra are manifest. The 3-algebra Models of M-theory are defined based on the semi-light-cone supermembrane action. We also review that the models reduce to the BFSS matrix theory in the DLCQ limit.

3.1. Supermembrane and 3-algebra model of M-theory

The fundamental degrees of freedom in M-theory are supermembranes. The action of the covariant supermembrane action in M-theory [36] is given by

$$S_{M2} = \int d^3\sigma \left(\sqrt{-G} + \frac{i}{4} \alpha^{\beta\gamma} \bar{\Psi} \Gamma_{MN} \partial_\alpha \Psi (\Pi_\beta^M \Pi_\gamma^N + \frac{i}{2} \Pi_\beta^M \bar{\Psi} \Gamma^N \partial_\gamma \Psi - \frac{1}{12} \bar{\Psi} \Gamma^M \partial_\beta \Psi \bar{\Psi} \Gamma^N \partial_\gamma \Psi) \right) \quad (\text{id35})$$

where $M, N = 0, \dots, 10$, $\alpha, \beta, \gamma = 0, 1, 2$, $G_{\alpha\beta} = \Pi_\alpha^M \Pi_{\beta M}$ and $\Pi_\alpha^M = \partial_\alpha X^M - \frac{i}{2} \bar{\Psi} \Gamma^M \partial_\alpha \Psi$. Ψ is a $SO(1, 10)$ Majorana fermion.

This action is invariant under dynamical supertransformations,

$$\begin{aligned} \delta \Psi &= \\ \delta X^M &= -i \bar{\Psi} \Gamma^M \end{aligned} \quad (\text{id36})$$

These transformations form the $\mathfrak{su}(1|1)$ supersymmetry algebra in eleven dimensions,

$$[\delta_1, \delta_2] X^M = -2i_1 \Gamma^M_2 \quad (\text{id37})$$

$$[\delta_1, \delta_2] \Psi = 0 \quad (\text{id38})$$

The action is also invariant under the κ -symmetry transformations,

$$\begin{aligned} \delta \Psi &= (1 + \Gamma) \kappa(\sigma) \\ \delta X^M &= i \bar{\Psi} \Gamma^M (1 + \Gamma) \kappa(\sigma) \end{aligned} \quad (\text{id39})$$

where

$$\Gamma = \frac{1}{3! \sqrt{-G}} \alpha^{\beta\gamma} \Pi_\alpha^L \Pi_\beta^M \Pi_\gamma^N \Gamma_{LMN} \quad (\text{id40})$$

If we fix the κ -symmetry ($\Rightarrow$) of the action by taking a semi-light-cone gauge [26] Advantages of a semi-light-cone gauges against a light-cone gauge are shown in [37], [38], [39]

$$\Gamma^{012} \Psi = -\Psi \quad (\text{id42})$$

we obtain a semi-light-cone supermembrane action,

$$S_{M2} = \int d^3\sigma \left(\sqrt{-G} + \frac{i}{4} \alpha^{\beta\gamma} \bar{\Psi} \Gamma_{\mu\nu} \partial_\alpha \Psi \left(\Pi_\beta^\mu \Pi_\gamma^\nu + \frac{i}{2} \Pi_\beta^\mu \bar{\Psi} \Gamma^\nu \partial_\gamma \Psi - \frac{1}{12} \bar{\Psi} \Gamma^\mu \partial_\beta \Psi \bar{\Psi} \Gamma^\nu \partial_\gamma \Psi + \bar{\Psi} \Gamma_{IJ} \partial_\alpha \Psi \partial_\beta X^I \partial_\gamma X^J \right) \right) \quad (\text{id43})$$

where $G_{\alpha\beta} = h_{\alpha\beta} + \Pi_\alpha{}^\mu \Pi_{\beta\mu}$, $\Pi_\alpha{}^\mu = \partial_\alpha X^\mu - \frac{i}{2} \bar{\Psi} \Gamma^\mu \partial_\alpha \Psi$, and $h_{\alpha\beta} = \partial_\alpha X^I \partial_\beta X_I$.

In [26], it is shown under an approximation up to the quadratic order in $\partial_\alpha X^\mu$ and $\partial_\alpha \Psi$ but exactly in X^I , that this action is equivalent to the continuum action of the 3-algebra model of M-theory,

$$S_{cl} = \int d^3\sigma \sqrt{-g} \left(-\frac{1}{12} \{X^I, X^J, X^K\}^2 - \frac{1}{2} (A_{\mu ab} \{\varphi^a, \varphi^b, X^I\})^2 \right. \\ \left. - \frac{1}{3} E^{\mu\nu\lambda} A_{\mu ab} A_{\nu cd} A_{\lambda ef} \{\varphi^a, \varphi^c, \varphi^d\} \{\varphi^b, \varphi^e, \varphi^f\} + \frac{1}{2} \Lambda \right. \\ \left. - \frac{i}{2} \bar{\Psi} \Gamma^\mu A_{\mu ab} \{\varphi^a, \varphi^b, \Psi\} + \frac{i}{4} \bar{\Psi} \Gamma_{IJ} \{X^I, X^J, \Psi\} \right) \quad (\text{id44})$$

where $I, J, K = 3, \dots, 10$ and $\{\varphi^a, \varphi^b, \varphi^c\} = {}^{\alpha\beta\gamma} \partial_\alpha \varphi^a \partial_\beta \varphi^b \partial_\gamma \varphi^c$ is the Nambu-Poisson bracket. An invariant symmetric bilinear form is defined by $\int d^3\sigma \sqrt{-g} \varphi^a \varphi^b$ for complete basis φ^a in three dimensions. Thus, this action is manifestly VPD covariant even when the world-volume metric is flat. X^I is a scalar and Ψ is a $SO(1, 2) \times SO(8)$ Majorana-Weyl fermion satisfying $(\square)$. $E^{\mu\nu\lambda}$ is a Levi-Civita symbol in three dimensions and Λ is a cosmological constant.

The continuum action of 3-algebra model of M-theory $(\square)$ is invariant under 16 dynamical supersymmetry transformations,

$$\delta X^I = i \Gamma^I \Psi \\ \delta A_\mu(\sigma, \sigma') = \frac{i}{2} \Gamma_\mu \Gamma_I (X^I(\sigma) \Psi(\sigma') - X^I(\sigma') \Psi(\sigma)), \\ \delta \Psi = -A_{\mu ab} \{\varphi^a, \varphi^b, X^I\} \Gamma^\mu \Gamma_I - \frac{1}{6} \{X^I, X^J, X^K\} \Gamma_{IJK} \quad (\text{id45})$$

where $\Gamma_{012} = -$. These supersymmetries close into gauge transformations on-shell,

$$[\delta_1, \delta_2] X^I = \Lambda_{cd} \{\varphi^c, \varphi^d, X^I\} \\ [\delta_1, \delta_2] A_{\mu ab} \{\varphi^a, \varphi^b, \quad\} = \Lambda_{ab} \{\varphi^a, \varphi^b, A_{\mu cd} \{\varphi^c, \varphi^d, \quad\}\} \\ \quad - A_{\mu ab} \{\varphi^a, \varphi^b, \Lambda_{cd} \{\varphi^c, \varphi^d, \quad\}\} + 2i_2 \Gamma^\nu{}_1 O_{\mu\nu}^A \\ [\delta_1, \delta_2] \Psi = \Lambda_{cd} \{\varphi^c, \varphi^d, \Psi\} + \left(i_2 \Gamma^\mu{}_1 \Gamma_\mu - \frac{i}{4} {}_2 \Gamma^{KL} {}_1 \Gamma_{KL} \right) O^\Psi \quad (\text{id46})$$

where gauge parameters are given by $\Lambda_{ab} = 2i_2 \Gamma^\mu{}_1 A_{\mu ab} - i_2 \Gamma_{JK} X_a^J X_b^K$. $O_{\mu\nu}^A = 0$ and $O^\Psi = 0$ are equations of motions of $A_{\mu\nu}$ and Ψ , respectively, where

$$\begin{aligned}
 O_{\mu\nu}^A &= A_{\mu ab} \{ \varphi^a, \varphi^b, A_{\nu cd} \{ \varphi^c, \varphi^d, \} \} - A_{\nu ab} \{ \varphi^a, \varphi^b, A_{\mu cd} \{ \varphi^c, \varphi^d, \} \} \\
 &\quad + E_{\mu\nu\lambda} \left(- \{ X^I, A_{ab}^\lambda \{ \varphi^a, \varphi^b, X_I \}, \} + \frac{i}{2} \{ \bar{\Psi}, \Gamma^\lambda \Psi, \} \right) \\
 O^\Psi &= -\Gamma^\mu A_{\mu ab} \{ \varphi^a, \varphi^b, \Psi \} + \frac{1}{2} \Gamma_{IJ} \{ X^I, X^J, \Psi \}
 \end{aligned} \tag{id47}$$

($\Rightarrow$) implies that a commutation relation between the dynamical supersymmetry transformations is

$$\delta_2 \delta_1 - \delta_1 \delta_2 = 0 \tag{id48}$$

up to the equations of motions and the gauge transformations.

This action is invariant under a translation,

$$\delta X^I(\sigma) = \eta^I, \quad \delta A^\mu(\sigma, \sigma') = \eta^\mu(\sigma) - \eta^\mu(\sigma') \tag{id49}$$

where η^I are constants.

The action is also invariant under 16 kinematical supersymmetry transformations

$$\delta \Psi = \epsilon \tag{id50}$$

and the other fields are not transformed. ϵ is a constant and satisfy $\Gamma_{012} \epsilon = \epsilon$ and should come from sixteen components of thirty-two $= 1$ supersymmetry parameters in eleven dimensions, corresponding to eigen values ± 1 of Γ_{012} , respectively. This $= 1$ supersymmetry consists of remaining 16 target-space supersymmetries and transmuted 16 κ -symmetries in the semi-light-cone gauge [26], [25], [40].

A commutation relation between the kinematical supersymmetry transformations is given by

$$\delta_2 \delta_1 - \delta_1 \delta_2 = 0 \tag{id51}$$

A commutator of dynamical supersymmetry transformations and kinematical ones acts as

$$\begin{aligned}
 (\delta_2 \delta_1 - \delta_1 \delta_2) X^I(\sigma) &= i_1 \Gamma^I \epsilon_2 \equiv \eta_0^I \\
 (\delta_2 \delta_1 - \delta_1 \delta_2) A^\mu(\sigma, \sigma') &= \frac{i}{2} \Gamma^\mu \Gamma_I (X^I(\sigma) - X^I(\sigma'))_2 \equiv \eta_0^\mu(\sigma) - \eta_0^\mu(\sigma')
 \end{aligned} \tag{id52}$$

where the commutator that acts on the other fields vanishes. Thus, the commutation relation is given by

$$\delta_2 \delta_1 - \delta_1 \delta_2 = \delta_\eta \quad (\text{id53})$$

where δ_η is a translation.

If we change a basis of the supersymmetry transformations as

$$\begin{aligned} \delta' &= \delta + \delta \\ \delta' &= i(\delta - \delta) \end{aligned} \quad (\text{id54})$$

we obtain

$$\begin{aligned} \delta'_2 \delta'_1 - \delta'_1 \delta'_2 &= \delta_\eta \\ \delta'_2 \delta'_1 - \delta'_1 \delta'_2 &= \delta_\eta \\ \delta'_2 \delta'_1 - \delta'_1 \delta'_2 &= 0 \end{aligned} \quad (\text{id55})$$

These thirty-two supersymmetry transformations are summarised as $\Delta = (\delta', \delta')$ and $(\Rightarrow)$ implies the $= 1$ supersymmetry algebra in eleven dimensions,

$$\Delta_2 \Delta_1 - \Delta_1 \Delta_2 = \delta_\eta \quad (\text{id56})$$

3.2. Lie 3-algebra models of M-theory

In this and next subsection, we perform the second quantization on the continuum action of the 3-algebra model of M-theory: By replacing the Nambu-Poisson bracket in the action $(\Rightarrow)$ with brackets of finite-dimensional 3-algebras, Lie and Hermitian 3-algebras, we obtain the Lie and Hermitian 3-algebra models of M-theory [26], [28], respectively. In this section, we review the Lie 3-algebra model.

If we replace the Nambu-Poisson bracket in the action $(\Rightarrow)$ with a completely antisymmetric real 3-algebra's bracket [21], [22],

$$\begin{aligned} \int d^3 \sigma \sqrt{-g} &\rightarrow \langle \quad \rangle \\ \{\varphi^a, \varphi^b, \varphi^c\} &\rightarrow [T^a, T^b, T^c] \end{aligned} \quad (\text{id58})$$

we obtain the Lie 3-algebra model of M-theory [26], [28],

$$\begin{aligned}
 S_0 = & \int d^3x \left\{ -\frac{1}{12} [X^I, X^J, X^K]^2 - \frac{1}{2} (A_{\mu ab} [T^a, T^b, X^I])^2 \right. \\
 & - \frac{1}{3} E^{\mu\nu\lambda} A_{\mu ab} A_{\nu cd} A_{\lambda ef} [T^a, T^c, T^d] [T^b, T^e, T^f] \\
 & \left. - \frac{i}{2} \bar{\Psi} \Gamma^\mu A_{\mu ab} [T^a, T^b, \Psi] + \frac{i}{4} \bar{\Psi} \Gamma_{IJ} [X^I, X^J, \Psi] \right\}
 \end{aligned} \quad (\text{id59})$$

We have deleted the cosmological constant Λ , which corresponds to an operator ordering ambiguity, as usual as in the case of other matrix models [27], [41].

This model can be obtained formally by a dimensional reduction of the $d=8$ BLG model [4], [5], [6],

$$\begin{aligned}
 S_{=8BLG} = & \int d^3x \left\{ -\frac{1}{12} [X^I, X^J, X^K]^2 - \frac{1}{2} (D_\mu X^I)^2 - E^{\mu\nu\lambda} \left(\frac{1}{2} A_{\mu ab} \partial_\nu A_{\lambda cd} T^a [T^b, T^c, T^d] \right. \right. \\
 & \left. \left. + \frac{1}{3} A_{\mu ab} A_{\nu cd} A_{\lambda ef} [T^a, T^c, T^d] [T^b, T^e, T^f] \right) \right. \\
 & \left. + \frac{i}{2} \bar{\Psi} \Gamma^\mu D_\mu \Psi + \frac{i}{4} \bar{\Psi} \Gamma_{IJ} [X^I, X^J, \Psi] \right\}
 \end{aligned} \quad (\text{id60})$$

The formal relations between the Lie (Hermitian) 3-algebra models of M-theory and the $d=8$ ($d=6$) BLG models are analogous to the relation among the $d=4$ super Yang-Mills in four dimensions, the BFSS matrix theory [27], and the IIB matrix model [41]. They are completely different theories although they are related to each others by dimensional reductions. In the same way, the 3-algebra models of M-theory and the BLG models are completely different theories.

The fields in the action (59) are spanned by the Lie 3-algebra T^a as $X^I = X_a^I T^a$, $\Psi = \Psi_a T^a$ and $A^\mu = A_{ab}^\mu T^a \otimes T^b$, where $I = 3, \dots, 10$ and $\mu = 0, 1, 2$. $\langle \rangle$ represents a metric for the 3-algebra. Ψ is a Majorana spinor of $SO(1,10)$ that satisfies $\Gamma_{012} \Psi = \Psi$. $E^{\mu\nu\lambda}$ is a Levi-Civita symbol in three-dimensions.

Finite dimensional Lie 3-algebras with an invariant metric is classified into four-dimensional Euclidean e_4 algebra and the Lie 3-algebras with indefinite metrics in [9], [10], [11], [21], [22]. We do not choose e_4 algebra because its degrees of freedom are just four. We need an algebra with arbitrary dimensions N , which is taken to infinity to define M-theory. Here we choose the most simple indefinite metric Lie 3-algebra, so called the Lorentzian Lie 3-algebra associated with $u(N)$ Lie algebra,

$$\begin{aligned}
 [T^{-1}, T^a, T^b] &= 0 \\
 [T^0, T^i, T^j] &= [T^i, T^j] = f^{ij}{}_k T^k \\
 [T^i, T^j, T^k] &= f^{ijk} T^{-1}
 \end{aligned} \quad (\text{id61})$$

where $a = -1, 0, i$ ($i = 1, \dots, N^2$). T^i are generators of $u(N)$. A metric is defined by a symmetric bilinear form,

$$\begin{aligned} \langle T^{-1}, T^0 \rangle &= -1 \\ \langle T^i, T^j \rangle &= h^{ij} \end{aligned} \quad (\text{id62})$$

and the other components are 0. The action is decomposed as

$$\begin{aligned} S = \text{Tr} \bigg(& -\frac{1}{4}(x_0^K)^2 [x^I, x^J]^2 + \frac{1}{2}(x_0^I [x_I, x^J])^2 - \frac{1}{2}(x_0^I b_\mu + [a_\mu, x^I])^2 - \frac{1}{2} E^{\mu\nu\lambda} b_\mu [a_\nu, a_\lambda] \\ & + i\bar{\psi}_0 \Gamma^\mu b_\mu \psi - \frac{i}{2} \bar{\psi} \Gamma^\mu [a_\mu, \psi] + \frac{i}{2} x_0^I \bar{\psi} \Gamma_{IJ} [x^J, \psi] - \frac{i}{2} \bar{\psi}_0 \Gamma_{IJ} [x^I, x^J] \psi \bigg) \end{aligned} \quad (\text{id63})$$

where we have renamed $X_0^I \rightarrow x_0^I$, $X_i^I T^i \rightarrow x^I$, $\Psi_0 \rightarrow \psi_0$, $\Psi_i T^i \rightarrow \psi$, $2A_{\mu 0i} T^i \rightarrow a_\mu$, and $A_{\mu ij} [T^i, T^j] \rightarrow b_\mu$. a_μ correspond to the target coordinate matrices X^μ , whereas b_μ are auxiliary fields.

In this action, T^{-1} mode; X_{-1}^I , Ψ_{-1} or A_{-1a}^μ does not appear, that is they are unphysical modes. Therefore, the indefinite part of the metric ($\ominus$) does not exist in the action and the Lie 3-algebra model of M-theory is ghost-free like a model in [42]. This action can be obtained by a dimensional reduction of the three-dimensional = 8 BLG model [4], [5], [6] with the same 3-algebra. The BLG model possesses a ghost mode because of its kinetic terms with indefinite signature. On the other hand, the Lie 3-algebra model of M-theory does not possess a kinetic term because it is defined as a zero-dimensional field theory like the IIB matrix model [41].

This action is invariant under the translation

$$\delta x^I = \eta^I, \quad \delta a^\mu = \eta^\mu \quad (\text{id64})$$

where η^I and η^μ belong to $u(1)$. This implies that eigen values of x^I and a^μ represent an eleven-dimensional space-time.

The action is also invariant under 16 kinematical supersymmetry transformations

$$\delta \psi = \dot{} \quad (\text{id65})$$

and the other fields are not transformed. $\dot{}$ belong to $u(1)$ and satisfy $\Gamma_{012} \dot{} = \dot{}$ and should come from sixteen components of thirty-two = 1 supersymmetry parameters in eleven dimensions, corresponding to eigen values ± 1 of Γ_{012} , respectively, as in the previous subsection.

A commutation relation between the kinematical supersymmetry transformations is given by

$$\delta_2 \delta_1 - \delta_1 \delta_2 = 0 \quad (\text{id66})$$

The action is invariant under 16 dynamical supersymmetry transformations,

$$\begin{aligned} \delta X^I &= i\Gamma^I \Psi \\ \delta A_{\mu ab}[T^a, T^b, \quad] &= i\Gamma_\mu \Gamma_I [X^I, \Psi, \quad] \\ \delta \Psi &= -A_{\mu ab}[T^a, T^b, X^I] \Gamma^\mu \Gamma_I - \frac{1}{6} [X^I, X^J, X^K] \Gamma_{IJK} \end{aligned} \quad (\text{id67})$$

where $\Gamma_{012} = -$. These supersymmetries close into gauge transformations on-shell,

$$\begin{aligned} [\delta_1, \delta_2] X^I &= \Lambda_{cd} [T^c, T^d, X^I] \\ [\delta_1, \delta_2] A_{\mu ab}[T^a, T^b, \quad] &= \Lambda_{ab} [T^a, T^b, A_{\mu cd} [T^c, T^d, \quad]] \\ &\quad - A_{\mu ab} [T^a, T^b, \Lambda_{cd} [T^c, T^d, \quad]] + 2i_2 \Gamma^\nu_1 O_{\mu\nu}^A \\ [\delta_1, \delta_2] \Psi &= \Lambda_{cd} [T^c, T^d, \Psi] + \left(i_2 \Gamma^\mu_1 \Gamma_\mu - \frac{i}{4} 2 \Gamma^{KL}_1 \Gamma_{KL} \right) O^\Psi \end{aligned} \quad (\text{id68})$$

where gauge parameters are given by $\Lambda_{ab} = 2i_2 \Gamma^\mu_1 A_{\mu ab} - i_2 \Gamma_{JK1} X^J_a X^K_b$. $O_{\mu\nu}^A = 0$ and $O^\Psi = 0$ are equations of motions of $A_{\mu\nu}$ and Ψ , respectively, where

$$\begin{aligned} O_{\mu\nu}^A &= A_{\mu ab} [T^a, T^b, A_{\nu cd} [T^c, T^d, \quad]] - A_{\nu ab} [T^a, T^b, A_{\mu cd} [T^c, T^d, \quad]] \\ &\quad + E_{\mu\nu\lambda} \left(-[X^I, A_{ab}^\lambda [T^a, T^b, X_I], \quad] + \frac{i}{2} [\bar{\Psi}, \Gamma^\lambda \Psi, \quad] \right) \\ O^\Psi &= -\Gamma^\mu A_{\mu ab} [T^a, T^b, \Psi] + \frac{1}{2} \Gamma_{IJ} [X^I, X^J, \Psi] \end{aligned} \quad (\text{id69})$$

($\Rightarrow$) implies that a commutation relation between the dynamical supersymmetry transformations is

$$\delta_2 \delta_1 - \delta_1 \delta_2 = 0 \quad (\text{id70})$$

up to the equations of motions and the gauge transformations.

The 16 dynamical supersymmetry transformations ($\Rightarrow$) are decomposed as

$$\begin{aligned}
\delta x^I &= i\Gamma^I \psi \\
\delta x_0^I &= i\Gamma^I \psi_0 \\
\delta x_{-1}^I &= i\Gamma^I \psi_{-1} \\
\delta \psi &= -(b_\mu x_0^I + [a_\mu, x^I])\Gamma^\mu \Gamma_I - \frac{1}{2}x_0^I [x^J, x^K]\Gamma_{JK} \\
\delta \psi_0 &= 0 \\
\delta \psi_{-1} &= -\text{Tr}(b_\mu x^I)\Gamma^\mu \Gamma_I - \frac{1}{6}\text{Tr}([x^I, x^J]x^K)\Gamma_{JK} \\
\delta a_\mu &= i\Gamma_\mu \Gamma_I (x_0^I \psi - \psi_0 x^I) \\
\delta b_\mu &= i\Gamma_\mu \Gamma_I [x^I, \psi] \\
\delta A_{\mu-1i} &= i\Gamma_\mu \Gamma_I \frac{1}{2}(x_{-1}^I \psi_i - \psi_{-1} x_i^I) \\
\delta A_{\mu-10} &= i\Gamma_\mu \Gamma_I \frac{1}{2}(x_{-1}^I \psi_0 - \psi_{-1} x_0^I)
\end{aligned} \tag{id71}$$

and thus a commutator of dynamical supersymmetry transformations and kinematical ones acts as

$$\begin{aligned}
(\delta_2 \delta_1 - \delta_1 \delta_2)x^I &= i_1 \Gamma^I{}_2 \equiv \eta^I \\
(\delta_2 \delta_1 - \delta_1 \delta_2)a^\mu &= i_1 \Gamma^\mu \Gamma_I x_0^I{}_2 \equiv \eta^\mu \\
(\delta_2 \delta_1 - \delta_1 \delta_2)A_{-1i}^\mu T^i &= \frac{1}{2}i_1 \Gamma^\mu \Gamma_I x_{-1}^I{}_2
\end{aligned} \tag{id72}$$

where the commutator that acts on the other fields vanishes. Thus, the commutation relation for physical modes is given by

$$\delta_2 \delta_1 - \delta_1 \delta_2 = \delta_\eta \tag{id73}$$

where δ_η is a translation.

($\Rightarrow$), ($\Leftarrow$), and ($\Leftrightarrow$) imply the $\mathfrak{su}(1,1)$ supersymmetry algebra in eleven dimensions as in the previous subsection.

3.3. Hermitian 3-algebra model of M-theory

In this subsection, we study the Hermitian 3-algebra models of M-theory [26]. Especially, we study mostly the model with the $u(N) \oplus u(N)$ Hermitian 3-algebra ($\Leftarrow$).

The continuum action ($\Leftarrow$) can be rewritten by using the triality of $SO(8)$ and the $SU(4) \times U(1)$ decomposition [8], [43], [44] as

$$\begin{aligned}
 S_{cl} = & \int d^3\sigma \sqrt{-g} \left(-V - A_{\mu ba} \{Z^A, T^a, T^b\} A_{dc}^\mu \{Z_A, T^c, T^d\} \right. \\
 & + \frac{1}{3} E^{\mu\nu\lambda} A_{\mu ba} A_{\nu dc} A_{\lambda fe} \{T^a, T^c, T^d\} \{T^b, T^f, T^e\} \\
 & + i\bar{\psi}^A \Gamma^\mu A_{\mu ba} \{\psi_A, T^a, T^b\} + \frac{i}{2} E_{ABCD} \bar{\psi}^A \{Z^C, Z^D, \psi^B\} - \frac{i}{2} E^{ABCD} Z_D \{\bar{\psi}_A, \psi_B, Z_C\} \\
 & \left. - i\bar{\psi}^A \{\psi_A, Z^B, Z_B\} + 2i\bar{\psi}^A \{\psi_B, Z^B, Z_A\} \right)
 \end{aligned} \tag{id75}$$

where fields with a raised A index transform in the 4 of $SU(4)$, whereas those with lowered one transform in the $\bar{4}$. $A_{\mu ba}$ ($\mu = 0, 1, 2$) is an anti-Hermitian gauge field, Z^A and Z_A are a complex scalar field and its complex conjugate, respectively. ψ_A is a fermion field that satisfies

$$\Gamma^{012} \psi_A = -\psi_A \tag{id76}$$

and ψ^A is its complex conjugate. $E^{\mu\nu\lambda}$ and E^{ABCD} are Levi-Civita symbols in three dimensions and four dimensions, respectively. The potential terms are given by

$$\begin{aligned}
 V &= \frac{2}{3} \gamma_B^{CD} \gamma_{CD}^B \\
 \gamma_B^{CD} &= \{Z^C, Z^D, Z_B\} - \frac{1}{2} \delta_B^C \{Z^E, Z^D, Z_E\} + \frac{1}{2} \delta_B^D \{Z^E, Z^C, Z_E\}
 \end{aligned} \tag{id77}$$

If we replace the Nambu-Poisson bracket with a Hermitian 3-algebra's bracket [19], [20],

$$\begin{aligned}
 \int d^3\sigma \sqrt{-g} &\rightarrow \langle \quad \rangle \\
 \{\varphi^a, \varphi^b, \varphi^c\} &\rightarrow [T^a, T^b; \bar{T}^c]
 \end{aligned} \tag{id78}$$

we obtain the Hermitian 3-algebra model of M-theory [26],

$$\begin{aligned}
 S = & \langle -V - A_{\mu ba} [Z^A, T^a; \bar{T}^b] \overline{A_{dc}^\mu [Z_A, T^c; \bar{T}^d]} + \frac{1}{3} E^{\mu\nu\lambda} A_{\mu ba} A_{\nu dc} A_{\lambda fe} [T^a, T^c; \bar{T}^d] \overline{[T^b, T^f; \bar{T}^e]} \\
 & + i\bar{\psi}^A \Gamma^\mu A_{\mu ba} [\psi_A, T^a; \bar{T}^b] + \frac{i}{2} E_{ABCD} \bar{\psi}^A [Z^C, Z^D; \bar{\psi}^B] - \frac{i}{2} E^{ABCD} \bar{Z}_D [\bar{\psi}_A, \psi_B; \bar{Z}_C] \\
 & - i\bar{\psi}^A [\psi_A, Z^B; \bar{Z}_B] + 2i\bar{\psi}^A [\psi_B, Z^B; \bar{Z}_A] \rangle
 \end{aligned} \tag{id79}$$

where the cosmological constant has been deleted for the same reason as before. The potential terms are given by

$$\begin{aligned}
V &= \frac{2}{3} \gamma_B^{CD} \bar{\gamma}_{CD}^B \\
\gamma_B^{CD} &= [Z^C, Z^D; \bar{Z}_B] - \frac{1}{2} \delta_B^C [Z^E, Z^D; \bar{Z}_E] + \frac{1}{2} \delta_B^D [Z^E, Z^C; \bar{Z}_E]
\end{aligned} \tag{id80}$$

This matrix model can be obtained formally by a dimensional reduction of the $= 6$ BLG action [8], which is equivalent to ABJ(M) action [7], [45]. The authors of [46], [47], [48], [49] studied matrix models that can be obtained by a dimensional reduction of the ABJM and ABJ gauge theories on S^3 . They showed that the models reproduce the original gauge theories on S^3 in planar limits.,

$$\begin{aligned}
S_{=6BLG} &= \int d^3x < -V - D_\mu Z^A \overline{D^\mu Z_A} + E^{\mu\nu\lambda} \left(\frac{1}{2} A_{\mu\bar{c}b} \partial_\nu A_{\lambda\bar{d}a} \bar{T}^{\bar{d}} [T^a, T^b; \bar{T}^{\bar{c}}] \right. \\
&\quad \left. + \frac{1}{3} A_{\mu\bar{b}a} A_{\nu\bar{d}c} A_{\lambda\bar{f}e} [T^a, T^c; \bar{T}^{\bar{d}}] \overline{[T^b, T^f; \bar{T}^{\bar{e}}]} \right) \\
&\quad - i\bar{\Psi}^A \Gamma^\mu D_\mu \psi_A + \frac{i}{2} E_{ABCD} \bar{\Psi}^A [Z^C, Z^D; \psi^B] - \frac{i}{2} E^{ABCD} \bar{Z}_D [\bar{\psi}_A, \psi_B; \bar{Z}_C] \\
&\quad \left. - i\bar{\Psi}^A [\psi_A, Z^B; \bar{Z}_B] + 2i\bar{\Psi}^A [\psi_B, Z^B; \bar{Z}_A] >
\end{aligned} \tag{id82}$$

The Hermitian 3-algebra models of M-theory are classified into the models with $u(m) \oplus u(n)$ Hermitian 3-algebra $(=)$ and $sp(2n) \oplus u(1)$ Hermitian 3-algebra $(=)$. In the following, we study the $u(N) \oplus u(N)$ Hermitian 3-algebra model. By substituting the $u(N) \oplus u(N)$ Hermitian 3-algebra $(=)$ to the action $(=)$, we obtain

$$\begin{aligned}
S &= \text{Tr} \left(-\frac{(2\pi)^2}{k^2} V - (Z^A A_\mu^R - A_\mu^L Z^A) (Z^A A^{R\mu} - A^{L\mu} Z^A)^\dagger - \frac{k}{2\pi} \frac{i}{3} E^{\mu\nu\lambda} (A_\mu^R A_\nu^R A_\lambda^R - A_\mu^L A_\nu^L A_\lambda^L) \right. \\
&\quad \left. - \bar{\Psi}^A \Gamma^\mu (\psi_A A_\mu^R - A_\mu^L \psi_A) + \frac{2\pi}{k} (i E_{ABCD} \bar{\Psi}^A Z^C \psi^\dagger B Z^D - i E^{ABCD} Z_D^\dagger \bar{\psi}^\dagger A Z_C^\dagger \psi_B \right. \\
&\quad \left. - i\bar{\Psi}^A \psi_A Z_B^\dagger Z^B + i\bar{\Psi}^A Z^B Z_B^\dagger \psi_A + 2i\bar{\Psi}^A \psi_B Z_A^\dagger Z^B - 2i\bar{\Psi}^A Z^B Z_A^\dagger \psi_B) \right)
\end{aligned} \tag{id83}$$

where $A_\mu^R \equiv -\frac{k}{2\pi} i A_{\mu\bar{b}a} T^{\dagger\bar{b}} T^a$ and $A_\mu^L \equiv -\frac{k}{2\pi} i A_{\mu\bar{b}a} T^a T^{\dagger\bar{b}}$ are $N \times N$ Hermitian matrices. In the algebra, we have set $\alpha = \frac{2\pi}{k}$, where k is an integer representing the Chern-Simons level. We choose $k = 1$ in order to obtain 16 dynamical supersymmetries. V is given by

$$\begin{aligned}
V &= +\frac{1}{3} Z_A^\dagger Z^A Z_B^\dagger Z^B Z_C^\dagger Z^C + \frac{1}{3} Z^A Z_A^\dagger Z^B Z_B^\dagger Z^C Z_C^\dagger + \frac{4}{3} Z_A^\dagger Z^B Z_C^\dagger Z^A Z_B^\dagger Z^C \\
&\quad - Z_A^\dagger Z^A Z_B^\dagger Z^C Z_C^\dagger Z^B - Z^A Z_A^\dagger Z^B Z_C^\dagger Z^C Z_B^\dagger
\end{aligned} \tag{id84}$$

By redefining fields as

$$\begin{aligned} Z^A &\rightarrow \left(\frac{k}{2\pi}\right)^{\frac{1}{3}} Z^A \\ A^\mu &\rightarrow \left(\frac{2\pi}{k}\right)^{\frac{1}{3}} A^\mu \\ \psi^A &\rightarrow \left(\frac{k}{2\pi}\right)^{\frac{1}{6}} \psi^A \end{aligned} \quad (\text{id85})$$

we obtain an action that is independent of Chern-Simons level:

$$\begin{aligned} S = & \text{Tr} \left(-V - (Z^A A_\mu^R - A_\mu^L Z^A)(Z^A A^{R\mu} - A^{L\mu} Z^A)^\dagger - \frac{i}{3} E^{\mu\nu\lambda} (A_\mu^R A_\nu^R A_\lambda^R - A_\mu^L A_\nu^L A_\lambda^L) \right. \\ & - \bar{\psi}^A \Gamma^\mu (\psi_A A_\mu^R - A_\mu^L \psi_A) + i E_{ABCD} \bar{\psi}^A Z^C \psi^{+B} Z^D - i E^{ABCD} Z_D^\dagger \bar{\psi}^+ A Z_C^\dagger \psi_B \\ & \left. - i \bar{\psi}^A \psi_A Z_B^\dagger Z^B + i \bar{\psi}^A Z^B Z_B^\dagger \psi_A + 2i \bar{\psi}^A \psi_B Z_A^\dagger Z^B - 2i \bar{\psi}^A Z^B Z_A^\dagger \psi_B \right) \end{aligned} \quad (\text{id86})$$

as opposed to three-dimensional Chern-Simons actions.

If we rewrite the gauge fields in the action as $A_\mu^L = A_\mu + b_\mu$ and $A_\mu^R = A_\mu - b_\mu$, we obtain

$$\begin{aligned} S = & \text{Tr} \left(-V + ([A_\mu, Z^A] + \{b_\mu, Z^A\})([A^\mu, Z_A] - \{b^\mu, Z_A\}) + i E^{\mu\nu\lambda} \left(\frac{2}{3} b_\mu b_\nu b_\lambda + 2A_\mu A_\nu b_\lambda \right) \right. \\ & + \bar{\psi}^A \Gamma^\mu ([A_\mu, \psi_A] + \{b_\mu, \psi_A\}) + i E_{ABCD} \bar{\psi}^A Z^C \psi^{+B} Z^D - i E^{ABCD} Z_D^\dagger \bar{\psi}^+ A Z_C^\dagger \psi_B \\ & \left. - i \bar{\psi}^A \psi_A Z_B^\dagger Z^B + i \bar{\psi}^A Z^B Z_B^\dagger \psi_A + 2i \bar{\psi}^A \psi_B Z_A^\dagger Z^B - 2i \bar{\psi}^A Z^B Z_A^\dagger \psi_B \right) \end{aligned} \quad (\text{id87})$$

where $[,]$ and $\{ , \}$ are the ordinary commutator and anticommutator, respectively. The $u(1)$ parts of A^μ decouple because A^μ appear only in commutators in the action. b^μ can be regarded as auxiliary fields, and thus A^μ correspond to matrices X^μ that represents three space-time coordinates in M-theory. Among $N \times N$ arbitrary complex matrices Z^A , we need to identify matrices X^I ($I = 3, \dots, 10$) representing the other space coordinates in M-theory, because the model possesses not $SO(8)$ but $SU(4) \times U(1)$ symmetry. Our identification is

$$\begin{aligned} Z^A &= iX^{A+2} - X^{A+6}, \\ X^I &= \hat{X}^I - ix^I 1 \end{aligned} \quad (\text{id88})$$

where $\hat{X}^I$ and x^I are $su(N)$ Hermitian matrices and real scalars, respectively. This is analogous to the identification when we compactify ABJM action, which describes N M2 branes, and obtain the action of N D2 branes [50], [7], [51]. We will see that this identification works also in our case. We should note that while the $su(N)$ part is Hermitian, the $u(1)$ part is anti-Hermitian. That is, an eigen-value distribution of X^μ, Z^A , and not X^I determine the space-

time in the Hermitian model. In order to define light-cone coordinates, we need to perform Wick rotation: $a^0 \rightarrow -ia^0$. After the Wick rotation, we obtain

$$A^0 = \hat{A}^0 - ia^0 1 \quad (\text{id89})$$

where $\hat{A}^0$ is a $su(N)$ Hermitian matrix.

3.4. DLCQ Limit of 3-algebra model of M-theory

It was shown that M-theory in a DLCQ limit reduces to the BFSS matrix theory with matrices of finite size [30], [31], [32], [33], [34], [35]. This fact is a strong criterion for a model of M-theory. In [26], [28], it was shown that the Lie and Hermitian 3-algebra models of M-theory reduce to the BFSS matrix theory with matrices of finite size in the DLCQ limit. In this subsection, we show an outline of the mechanism.

DLCQ limit of M-theory consists of a light-cone compactification, $x^- \approx x^- + 2\pi R$, where $x^\pm = \frac{1}{\sqrt{2}}(x^{10} \pm x^0)$, and Lorentz boost in x^{10} direction with an infinite momentum. After appropriate scalings of fields [26], [28], we define light-cone coordinate matrices as

$$\begin{aligned} X^0 &= \frac{1}{\sqrt{2}}(X^+ - X^-) \\ X^{10} &= \frac{1}{\sqrt{2}}(X^+ + X^-) \end{aligned} \quad (\text{id91})$$

We integrate out b^μ by using their equations of motion.

A matrix compactification [52] on a circle with a radius R imposes the following conditions on X^- and the other matrices Y :

$$\begin{aligned} X^- - (2\pi R)1 &= U^\dagger X^- U \\ Y &= U^\dagger Y U \end{aligned} \quad (\text{id92})$$

where U is a unitary matrix. In order to obtain a solution to (□), we need to take $N \rightarrow \infty$ and consider matrices of infinite size [52]. A solution to (□) is given by $X^- = \bar{X}^- + \tilde{X}^-$, $Y = \tilde{Y}$ and

$$U = \begin{pmatrix} \ddots & & & & \\ & \ddots & & & \\ & & 0 & 1 & 0 \\ & & & 0 & 1 \\ & & & & 0 & 1 \\ & 0 & & & & 0 & \ddots \\ & & & & & & \ddots \end{pmatrix} \otimes 1_{n \times n} \in U(N) \quad (\text{id93})$$

Backgrounds $\bar{X}^-$ are

$$\bar{X}^- = -T^3 \bar{x}_0 T^0 - (2\pi R) \text{diag}(\dots, s-1, s, s+1, \dots) \otimes 1_{n \times n} \quad (\text{id94})$$

in the Lie 3-algebra case, whereas

$$\bar{X}^- = -i(T^3 \bar{x}^-)1 - i(2\pi R) \text{diag}(\dots, s-1, s, s+1, \dots) \otimes 1_{n \times n} \quad (\text{id95})$$

in the Hermitian 3-algebra case. A fluctuation $\tilde{x}$ that represents $u(N)$ parts of $\tilde{X}^-$ and $\tilde{Y}$ is

$$\begin{pmatrix} \ddots & & & & & & \\ \ddots & \tilde{x}(0) & \tilde{x}(1) & \tilde{x}(2) & & & \ddots \\ \ddots & \tilde{x}(-1) & \tilde{x}(0) & \tilde{x}(1) & \tilde{x}(2) & & \\ & \tilde{x}(-2) & \tilde{x}(-1) & \tilde{x}(0) & \tilde{x}(1) & \tilde{x}(2) & \\ & & \tilde{x}(-2) & \tilde{x}(-1) & \tilde{x}(0) & \tilde{x}(1) & \tilde{x}(2) \\ & & & \tilde{x}(-2) & \tilde{x}(-1) & \tilde{x}(0) & \tilde{x}(1) & \ddots \\ & & & & \tilde{x}(-2) & \tilde{x}(-1) & \tilde{x}(0) & \ddots \\ & \ddots & & & & \ddots & \ddots & \ddots \end{pmatrix} \quad (\text{id96})$$

Each $\tilde{x}(s)$ is a $n \times n$ matrix, where s is an integer. That is, the (s, t) -th block is given by $\tilde{x}_{s,t} = \tilde{x}(s-t)$.

We make a Fourier transformation,

$$\tilde{x}(s) = \frac{1}{2\pi\tilde{R}} \int_0^{2\pi\tilde{R}} d\tau x(\tau) e^{is\frac{\tau}{\tilde{R}}} \quad (\text{id97})$$

where $x(\tau)$ is a $n \times n$ matrix in one-dimension and $R\tilde{R} = 2\pi$. From (□)-(□), the following identities hold:

$$\begin{aligned} \sum_t \tilde{x}_{s,t} \tilde{x}'_{t,u} &= \frac{1}{2\pi\tilde{R}} \int_0^{2\pi\tilde{R}} d\tau x(\tau) x'(\tau) e^{i(s-u)\frac{\tau}{\tilde{R}}} \\ \text{tr} \left(\sum_{s,t} \tilde{x}_{s,t} \tilde{x}'_{t,s} \right) &= V \frac{1}{2\pi\tilde{R}} \int_0^{2\pi\tilde{R}} d\tau \text{tr}(x(\tau) x'(\tau)) \\ [\tilde{x}^-, \tilde{x}]_{s,t} &= \frac{1}{2\pi\tilde{R}} \int_0^{2\pi\tilde{R}} d\tau \partial_\tau x(\tau) e^{i(s-t)\frac{\tau}{\tilde{R}}} \end{aligned} \quad (\text{id98})$$

where tr is a trace over $n \times n$ matrices and $V = \sum_s 1$.

Next, we boost the system in x^{10} direction:

$$\begin{aligned}\widetilde{X}^+ &= \frac{1}{T} \widetilde{X}^+ \\ \widetilde{X}^- &= T \widetilde{X}^-\end{aligned}\tag{id99}$$

The DLCQ limit is achieved when $T \rightarrow \infty$, where the "novel Higgs mechanism" [50] is realized. In $T \rightarrow \infty$, the actions of the 3-algebra models of M-theory reduce to that of the BFSS matrix theory [27] with matrices of finite size,

$$S = \frac{1}{g^2} \int_{-\infty}^{\infty} d\tau \text{tr} \left(\frac{1}{2} (D_0 x^P)^2 - \frac{1}{4} [x^P, x^Q]^2 + \frac{1}{2} \bar{\psi} \Gamma^0 D_0 \psi - \frac{i}{2} \bar{\psi} \Gamma^P [x_P, \psi] \right) \tag{id100}$$

where $P, Q = 1, 2, \dots, 9$.

3.5. Supersymmetric deformation of Lie 3-algebra model of M-theory

A supersymmetric deformation of the Lie 3-algebra Model of M-theory was studied in [53] (see also [54], [55], [56]). If we add mass terms and a flux term,

$$S_m = \left\langle -\frac{1}{2} \mu^2 (X^I)^2 - \frac{i}{2} \mu \bar{\Psi} \Gamma_{3456} \Psi + H_{IJKL} [X^I, X^J, X^K] X^L \right\rangle \tag{id102}$$

such that

$$H_{IJKL} = \begin{cases} -\frac{\mu}{6} \epsilon_{IJKL} & (I, J, K, L = 3, 4, 5, 6 \text{ or } 7, 8, 9, 10) \\ 0 & (\text{otherwise}) \end{cases} \tag{id103}$$

to the action (□), the total action $S_0 + S_m$ is invariant under dynamical 16 supersymmetries,

$$\begin{aligned}\delta X^I &= i \Gamma^I \Psi \\ \delta A_{\mu ab} [T^a, T^b, \quad] &= i \Gamma_\mu \Gamma_I [X^I, \Psi, \quad] \\ \delta \Psi &= -\frac{1}{6} [X^I, X^J, X^K] \Gamma_{IJK} - A_{\mu ab} [T^a, T^b, X^I] \Gamma^\mu \Gamma_I + \mu \Gamma_{3456} X^I \Gamma_I\end{aligned}\tag{id104}$$

From this action, we obtain various interesting solutions, including fuzzy sphere solutions [53].

4. Conclusion

The metric Hermitian 3-algebra corresponds to a class of the super Lie algebra. By using this relation, the metric Hermitian 3-algebras are classified into $u(m) \oplus u(n)$ and $sp(2n) \oplus u(1)$ Hermitian 3-algebras.

The Lie and Hermitian 3-algebra models of M-theory are obtained by second quantizations of the supermembrane action in a semi-light-cone gauge. The Lie 3-algebra model possesses manifest $= 1$ supersymmetry in eleven dimensions. In the DLCQ limit, both the models reduce to the BFSS matrix theory with matrices of finite size as they should.

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