Provisional chapter

An Interpretation of Rosenbrock's Theorem Via Local Rings

A. Amparan, S. Marcaida and I. Zaballa

Additional information is available at the end of the chapter

http://dx.doi.org/10.5772/81051

1. Introduction

Consider a linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{id1}$$

to be identified with the pair of matrices (A, B) where $A \in {}^{n\times n}$, $B \in {}^{n\times m}$ and $\ = \mathbb{R}$ or $\mathbb{C}$ the fields of the real or complex numbers. If state-feedback $u(t) = Fx(t) + v(t)$ is applied to system (▭), Rosenbrock's Theorem on pole assignment (see [1]) characterizes for the closed-loop system

$$\dot{x}(t) = (A + BF)x(t) + Bv(t), \tag{id2}$$

the invariant factors of its state-space matrix $A + BF$. This result can be seen as the solution of an inverse problem; that of finding a non-singular polynomial matrix with prescribed invariant factors and left Wiener–Hopf factorization indices at infinity. To see this we recall that the invariant factors form a complete system of invariants for the finite equivalence of polynomial matrices (this equivalence relation will be revisited in Section ▭) and it will be seen in Section ▭ that any polynomial matrix is left Wiener–Hopf equivalent at infinity to a diagonal matrix $\mathrm{Diag}\left(s^{k_1}, \ \ldots, \ s^{k_m}\right)$, where the non-negative integers $k_1, \ \ldots, \ k_m$ (that can be assumed in non-increasing order) form a complete system of invariants for the left Wiener–Hopf equivalence at infinity. Consider now the transfer function matrix $G(s) = (sI - (A + BF))^{-1}B$ of (▭). This is a rational matrix that can be written as an irreducible matrix fraction description $G(s) = N(s)P(s)^{-1}$, where $N(s)$ and $P(s)$ are right coprime polynomial matrices. In the terminology of [2], $P(s)$ is a polynomial matrix representation of (▭), concept that is closely related to that of polynomial model introduced by Fuhrmann (see for example [3] and the references therein). It turns out that all polynomial matrix representations of a system are right equivalent (see [2], [3]), that is, if $P_1(s)$ and $P_2(s)$ are polynomial matrix representations of the same system there exists a unimodular matrix $U(s)$ such that $P_2(s) = P_1(s)U(s)$. Therefore all polynomial matrix representations of (▭) have the same invariant factors, which are the invariant factors of $sI_n - (A + BF)$ except for some trivial ones. Furthermore, all polynomial matrix representations also have the same left Wiener– Hopf factorization indices at infinity, which are equal to the controllability indices of (▭) and (▭), because the controllability indices are invariant under feedback. With all this in mind it is not hard to see that Rosenbrock's Theorem on pole assignment is equivalent to finding necessary and sufficient conditions for the existence of a non-singular polynomial matrix with prescribed invariant factors and left Wiener–Hopf factorization indices at infinity. This result will be precisely stated in Section ▭ once all the elements that appear are properly defined. In addition, there is a similar result to Rosenbrock's Theorem on pole assignment but involving the infinite structure (see [4]).

Our goal is to generalize both results (the finite and infinite versions of Rosenbrock's Theorem) for rational matrices defined on arbitrary fields via local rings. This will be done in Section ▭ and an extension to arbitrary fields of the concept of Wiener–Hopf equivalence will be needed. This concept is very well established for complex valued rational matrix functions (see for example [5], [6]). Originally it requires a closed contour, γ, that divides the extended complex plane ($\mathbb{C} \cup \{\infty\}$) into two parts: the inner domain (Ω_+) and the region outside γ (Ω_-), which contains the point at infinity. Then two non-singular $m \times m$ complex rational matrices $T_1(s)$ and $T_2(s)$, with no poles and no zeros in γ, are said to be left Wiener–Hopf equivalent with respect to γ if there are $m \times m$ matrices $U_-(s)$ and $U_+(s)$ with no poles and no zeros in $\Omega_- \cup \gamma$ and $\Omega_+ \cup \gamma$, respectively, such that

$$T_2(s) = U_-(s)T_1(s)U_+(s). \tag{id3}$$

It can be seen, then, that any non-singular $m \times m$ complex rational matrix $T(s)$ is left Wiener–Hopf equivalent with respect to γ to a diagonal matrix

$$\mathrm{Diag}\left((s - z_0)^{k_1},\ \ldots,\ (s - z_0)^{k_m}\right) \tag{id4}$$

where z_0 is any complex number in Ω_+ and $k_1 \geq \cdots \geq k_m$ are integers uniquely determined by $T(s)$. They are called the left Wiener–Hopf factorization indices of $T(s)$ with respect to γ (see again [5], [6]). The generalization to arbitrary fields relies on the following idea: We can identify $\Omega_+ \cup \gamma$ and $(\Omega_- \cup \gamma) \setminus \{\infty\}$ with two sets M and M', respectively, of maximal ideals of $\mathbb{C}[s]$. In fact, to each $z_0 \in \mathbb{C}$ we associate the ideal generated by $s - z_0$, which is a maximal ideal of $\mathbb{C}[s]$. Notice that $s - z_0$ is also a prime polynomial of $\mathbb{C}[s]$ but M and M', as defined,

http://dx.doi.org/10.5772/81051

cannot contain the zero ideal, which is prime. Thus we are led to consider the set $\text{Specm}(\mathbb{C}[s])$ of maximal ideals of $\mathbb{C}[s]$. By using this identification we define the left Wiener–Hopf equivalence of rational matrices over an arbitrary field with respect to a subset M of $\text{Specm}([s])$, the set of all maximal ideals of $[s]$. In this study local rings play a fundamental role. They will be introduced in Section ▭. Localization techniques have been used previously in the algebraic theory of linear systems (see, for example, [7]). In Section ▭ the algebraic structure of the rings of proper rational functions with prescribed finite poles is studied (i.e., for a fixed $M \subseteq \text{Specm}([s])$ the ring of proper rational functions $\frac{p(s)}{q(s)}$ with $\gcd(g(s), \pi(s)) = 1$ for all $(\pi(s)) \in M$). It will be shown that if there is an ideal generated by a linear polynomial outside M then the set of proper rational functions with no poles in M is an Euclidean domain and all rational matrices can be classified according to their Smith–McMillan invariants. In this case, two types of invariants live together for any non-singular rational matrix and any set $M \subseteq \text{Specm}([s])$: its Smith–McMillan and left Wiener–Hopf invariants. In Section ▭ we show that a Rosenbrock-like Theorem holds true that completely characterizes the relationship between these two types of invariants.

2. Preliminaries

In the sequel $[s]$ will denote the ring of polynomials with coefficients in an arbitrary field and $\text{Specm}([s])$ the set of all maximal ideals of $[s]$, that is,

$$\text{Specm}([s]) = \{(\pi(s)) : \pi(s) \in [s], \ \text{ irreducible, monic, different from } 1\}. \tag{id5}$$

Let $\pi(s) \in [s]$ be a monic irreducible non-constant polynomial. Let $S = [s] \setminus (\pi(s))$ be the multiplicative subset of $[s]$ whose elements are coprime with $\pi(s)$. We denote by ${}_{\pi}(s)$ the quotient ring of $[s]$ by S; i.e., $S^{-1}[s]$:

$${}_{\pi}(s) = \left\{ \frac{p(s)}{q(s)} : p(s),\ q(s) \in [s], \quad \gcd(q(s), \pi(s)) = 1 \right\}. \tag{id6}$$

This is the localization of $[s]$ at $(\pi(s))$ (see [8]). The units of ${}_{\pi}(s)$ are the rational functions $u(s) = \frac{p(s)}{q(s)}$ such that $\gcd(p(s), \pi(s)) = 1$ and $\gcd(q(s), \pi(s)) = 1$. Consequentially,

$${}_{\pi}(s) = \{u(s)\pi(s)^d : u(s) \text{ is a unit and } d \geq 0\} \cup \{0\}. \tag{id7}$$

For any $M \subseteq \text{Specm}([s])$, let

$$
\begin{aligned}
{}_{M}(s) &= \bigcap_{(\pi(s))\in M} {}_{\pi}(s) \\
&= \left\{ \frac{p(s)}{q(s)} : p(s),\, q(s) \in [s], \quad \gcd\,(q(s),\, \pi(s)) = 1 \;\; \forall \;\; (\pi(s)) \in M \right\}.
\end{aligned}
\tag{id8}
$$

This is a ring whose units are the rational functions $u(s) = \frac{p(s)}{q(s)}$ such that for all ideals $(\pi(s)) \in M$, $\gcd\,(p(s),\, \pi(s)) = 1$ and $\gcd\,(q(s),\, \pi(s)) = 1$. Notice that, in particular, if $M = \text{Specm}([s])$ then ${}_{M}(s) = [s]$ and if $M = \varnothing$ then ${}_{M}(s) = (s)$, the field of rational functions.

Moreover, if $\alpha(s) \in [s]$ is a non-constant polynomial whose prime factorization, $\alpha(s) = k\alpha_1(s)^{d_1} \cdots \alpha_m(s)^{d_m}$, satisfies the condition that $(\alpha_i(s)) \in M$ for all i, we will say that $\alpha(s)$ factorizes in M or $\alpha(s)$ has all its zeros in M. We will consider that the only polynomials that factorize in $M = \varnothing$ are the constants. We say that a non-zero rational function factorizes in M if both its numerator and denominator factorize in M. In this case we will say that the rational function has all its zeros and poles in M. Similarly, we will say that $\frac{p(s)}{q(s)}$ has no poles in M if $p(s) \neq 0$ and $\gcd\,(q(s),\, \pi(s)) = 1$ for all ideals $(\pi(s)) \in M$. And it has no zeros in M if $\gcd\,(p(s),\, \pi(s)) = 1$ for all ideals $(\pi(s)) \in M$. In other words, it is equivalent that $\frac{p(s)}{q(s)}$ has no poles and no zeros in M and that $\frac{p(s)}{q(s)}$ is a unit of ${}_{M}(s)$. So, a non-zero rational function factorizes in M if and only if it is a unit in ${}_{\text{Specm}([s])\setminus M}(s)$.

Let ${}_{M}(s)^{m\times m}$ denote the set of $m \times m$ matrices with elements in ${}_{M}(s)$. A matrix is invertible in ${}_{M}(s)^{m\times m}$ if all its elements are in ${}_{M}(s)$ and its determinant is a unit in ${}_{M}(s)$. We denote by $\text{Gl}_m({}_{M}(s))$ the group of units of ${}_{M}(s)^{m\times m}$.

Remark 1

Let $M_1,\, M_2 \subseteq \text{Specm}([s])$. Notice that

1. If $M_1 \subseteq M_2$ then ${}_{M_1}(s) \supseteq {}_{M_2}(s)$ and $\text{Gl}_m({}_{M_1}(s)) \supseteq \text{Gl}_m({}_{M_2}(s))$.

2. ${}_{M_1 \cup M_2}(s) = {}_{M_1}(s) \cap {}_{M_2}(s)$ and $\text{Gl}_m({}_{M_1\cup M_2}(s)) = \text{Gl}_m({}_{M_1}(s)) \cap \text{Gl}_m({}_{M_2}(s))$.

For any $M \subseteq \text{Specm}([s])$ the ring ${}_{M}(s)$ is a principal ideal domain (see [9]) and its field of fractions is (s). Two matrices $T_1(s),\, T_2(s) \in (s)^{m\times m}$ are equivalent with respect to M if there exist matrices $U(s),\, V(s) \in \text{Gl}_m({}_{M}(s))$ such that $T_2(s) = U(s)T_1(s)V(s)$. Since ${}_{M}(s)$ is a principal ideal domain, for all non-singular $G(s) \in {}_{M}(s)^{m\times m}$ (see [10]) there exist matrices $U(s),\, V(s) \in \text{Gl}_m({}_{M}(s))$ such that

$$
G(s) = U(s)\text{Diag}(\alpha_1(s),\; \ldots,\; \alpha_m(s))V(s) \tag{id10}
$$

with $\alpha_1(s) \mid \cdots \mid \alpha_m(s)$ ("$\mid$" stands for divisibility) monic polynomials factorizing in M, unique up to multiplication by units of ${}_M(s)$. The diagonal matrix is the Smith normal form of $G(s)$ with respect to M and $\alpha_1(s), \ldots, \alpha_m(s)$ are called the invariant factors of $G(s)$ with respect to M. Now we introduce the Smith–McMillan form with respect to M. Assume that $T(s) \in (s)^{m\times m}$ is a non-singular rational matrix. Then $T(s) = \frac{G(s)}{d(s)}$ with $G(s) \in {}_M(s)^{m\times m}$ and $d(s) \in [s]$ monic, factorizing in M. Let $G(s) = U(s)\mathrm{Diag}(\alpha_1(s), \ldots, \alpha_m(s))V(s)$ be the Smith normal form with respect to M of $G(s)$, i.e., $U(s), V(s)$ invertible in ${}_M(s)^{m\times m}$ and $\alpha_1(s) \mid \cdots \mid \alpha_m(s)$ monic polynomials factorizing in M. Then

$$T(s) = U(s)\mathrm{Diag}\left(\frac{{}_1(s)}{\psi_1(s)}, \ldots, \frac{{}_m(s)}{\psi_m(s)}\right)V(s) \qquad \text{(id11)}$$

where $\frac{{}_i(s)}{\psi_i(s)}$ are irreducible rational functions, which are the result of dividing $\alpha_i(s)$ by $d(s)$ and canceling the common factors. They satisfy that ${}_1(s) \mid \cdots \mid {}_m(s)$, $\psi_m(s) \mid \cdots \mid \psi_1(s)$ are monic polynomials factorizing in M. The diagonal matrix in (▭) is the Smith–McMillan form with respect to M. The rational functions $\frac{{}_i(s)}{\psi_i(s)}$, $i = 1, \ldots, m$, are called the invariant rational functions of $T(s)$ with respect to M and constitute a complete system of invariants of the equivalence with respect to M for rational matrices.

In particular, if $M = \mathrm{Specm}([s])$ then ${}_{\mathrm{Specm}([s])}(s) = [s]$, the matrices $U(s), V(s) \in \mathrm{Gl}_m([s])$ are unimodular matrices, (▭) is the global Smith–McMillan form of a rational matrix (see [11] or [1] when $= \mathbb{R}$ or $\mathbb{C}$) and $\frac{{}_i(s)}{\psi_i(s)}$ are the global invariant rational functions of $T(s)$.

From now on rational matrices will be assumed to be non-singular unless the opposite is specified. Given any $M \subseteq \mathrm{Specm}([s])$ we say that an $m \times m$ non-singular rational matrix has no zeros and no poles in M if its global invariant rational functions are units of ${}_M(s)$. If its global invariant rational functions factorize in M, the matrix has its global finite structure localized in M and we say that the matrix has all zeros and poles in M. The former means that $T(s) \in \mathrm{Gl}_m({}_M(s))$ and the latter that $T(s) \in \mathrm{Gl}_m\left({}_{\mathrm{Specm}([s])\setminus M}(s)\right)$ because $\det T(s) = \det U(s) \det V(s) \frac{{}_1(s) \cdots {}_m(s)}{\psi_1(s) \cdots \psi_m(s)}$ and $\det U(s)$, $\det V(s)$ are non-zero constants. The following result clarifies the relationship between the global finite structure of any rational matrix and its local structure with respect to any $M \subseteq \mathrm{Specm}([s])$.

Proposition 2 Let $M \subseteq \mathrm{Specm}([s])$. Let $T(s) \in (s)^{m\times m}$ be non-singular with $\frac{\alpha_1(s)}{\beta_1(s)}, \ldots, \frac{\alpha_m(s)}{\beta_m(s)}$ its global invariant rational functions and let $\frac{{}_1(s)}{\psi_1(s)}, \ldots, \frac{{}_m(s)}{\psi_m(s)}$ be irreducible rational functions such that ${}_1(s) \mid \cdots \mid {}_m(s)$, $\psi_m(s) \mid \cdots \mid \psi_1(s)$ are monic polynomials factorizing in M. The following properties are equivalent:

- There exist $T_L\ (s),\ T_R(s) \in (s)^{m\times m}$ such that the global invariant rational functions of $T_L\ (s)$ are $\frac{{}_1(s)}{\psi_1(s)},\ \ldots,\ \frac{{}_m(s)}{\psi_m(s)}$, $T_R(s) \in \mathrm{Gl}_m({}_M(s))$ and $T(s) = T_L\ (s)T_R(s)$.

- There exist matrices $U_1(s),\ U_2(s)$ invertible in ${}_M(s)^{m\times m}$ such that

$$T(s) = U_1(s)\mathrm{Diag}\left(\frac{{}_1(s)}{\psi_1(s)},\ \ldots,\ \frac{{}_m(s)}{\psi_m(s)}\right)U_2(s), \tag{id15}$$

i.e., $\frac{{}_1(s)}{\psi_1(s)},\ \ldots,\ \frac{{}_m(s)}{\psi_m(s)}$ are the invariant rational functions of $T(s)$ with respect to M.

- $\alpha_i(s) = {}_i(s){}_i^{'}(s)$ and $\beta_i(s) = \psi_i(s)\psi_i^{'}(s)$ with ${}_i^{'}(s),\ \psi_i^{'}(s) \in [s]$ units of ${}_M(s)$, for $i = 1,\ \ldots,\ m$.

Proof.- 1 $\Rightarrow$ 2. Since the global invariant rational functions of $T_L\ (s)$ are $\frac{{}_1(s)}{\psi_1(s)},\ \ldots,\ \frac{{}_m(s)}{\psi_m(s)}$, there exist $W_1(s),\ W_2(s) \in \mathrm{Gl}_m([s])$ such that $T_L\ (s) = W_1(s)\mathrm{Diag}\left(\frac{{}_1(s)}{\psi_1(s)},\ \ldots,\ \frac{{}_m(s)}{\psi_m(s)}\right)W_2(s)$. As ${}_{\mathrm{Specm}([s])}(s) = [s]$, by Remark ▭.1, $W_1(s),\ W_2(s) \in \mathrm{Gl}_m({}_M(s))$. Therefore, putting $U_1(s) = W_1(s)$ and $U_2(s) = W_2(s)T_R(s)$ it follows that $U_1(s)$ and $U_2(s)$ are invertible in ${}_M(s)^{m\times m}$ and $T(s) = U_1(s)\mathrm{Diag}\left(\frac{{}_1(s)}{\psi_1(s)},\ \ldots,\ \frac{{}_m(s)}{\psi_m(s)}\right)U_2(s)$.

2 $\Rightarrow$ 3. There exist unimodular matrices $V_1(s),\ V_2(s) \in [s]^{m\times m}$ such that

$$T(s) = V_1(s)\mathrm{Diag}\left(\frac{\alpha_1(s)}{\beta_1(s)},\ \ldots,\ \frac{\alpha_m(s)}{\beta_m(s)}\right)V_2(s) \tag{id17}$$

with $\frac{\alpha_i(s)}{\beta_i(s)}$ irreducible rational functions such that $\alpha_1(s) \mid \cdots \mid \alpha_m(s)$ and $\beta_m(s) \mid \cdots \mid \beta_1(s)$ are monic polynomials. Write $\frac{\alpha_i(s)}{\beta_i(s)} = \frac{p_i(s)p_i^{'}(s)}{q_i(s)q_i^{'}(s)}$ such that $p_i(s),\ q_i(s)$ factorize in M and $p_i^{'}(s),\ q_i^{'}(s)$ factorize in $\mathrm{Specm}([s]) \setminus M$. Then

$$T(s) = V_1(s)\mathrm{Diag}\left(\frac{p_1(s)}{q_1(s)},\ \ldots,\ \frac{p_m(s)}{q_m(s)}\right)\mathrm{Diag}\left(\frac{p_1^{'}(s)}{q_1^{'}(s)},\ \ldots,\ \frac{p_m^{'}(s)}{q_m^{'}(s)}\right)V_2(s) \tag{id18}$$

with $V_1(s)$ and $\mathrm{Diag}\left(\frac{p_1^{'}(s)}{q_1^{'}(s)},\ \ldots,\ \frac{p_m^{'}(s)}{q_m^{'}(s)}\right)V_2(s)$ invertible in ${}_M(s)^{m\times m}$. Since the Smith–McMillan form with respect to M is unique we get that $\frac{p_i(s)}{q_i(s)} = \frac{{}_i(s)}{\psi_i(s)}$.

3 $\Rightarrow$ 1. Write (▭) as

$$T(s) = V_1(s)\mathrm{Diag}\left(\frac{_1(s)}{\psi_1(s)},\ \ldots,\ \frac{_m(s)}{\psi_m(s)}\right)\mathrm{Diag}\left(\frac{_1^{'}(s)}{\psi_1^{'}(s)},\ \ldots,\ \frac{_m^{'}(s)}{\psi_m^{'}(s)}\right)V_2(s). \quad \text{(id19)}$$

It follows that $T(s) = T_L\ (s)T_R(s)$ with $T_L\ (s) = V_1(s)\mathrm{Diag}\left(\frac{_1(s)}{\psi_1(s)},\ \ldots,\ \frac{_m(s)}{\psi_m(s)}\right)$ and $T_R(s) = \mathrm{Diag}\left(\frac{_1^{'}(s)}{\psi_1^{'}(s)},\ \ldots,\ \frac{_m^{'}(s)}{\psi_m^{'}(s)}\right)V_2(s) \in \mathrm{Gl}_m(_M(s))$.

Corollary 3 Let $T(s) \in (s)^{m\times m}$ be non-singular and $M_1, M_2 \subseteq \mathrm{Specm}([s])$ such that $M_1 \cap M_2 = \emptyset$. If $\frac{_1^i(s)}{\psi_1^i(s)},\ \ldots,\ \frac{_m^i(s)}{\psi_m^i(s)}$ are the invariant rational functions of $T(s)$ with respect to M_i, $i = 1, 2$, then $\frac{_1^1(s)_1^2(s)}{\psi_1^1(s)\psi_1^2(s)},\ \ldots,\ \frac{_m^1(s)_m^2(s)}{\psi_m^1(s)\psi_m^2(s)}$ are the invariant rational functions of $T(s)$ with respect to $M_1 \cup M_2$.

1.08 **Proof**.- Let $\frac{\alpha_1(s)}{\beta_1(s)},\ \ldots,\ \frac{\alpha_m(s)}{\beta_m(s)}$ be the global invariant rational functions of $T(s)$. By Proposition ▭, $\alpha_i(s) = {}_i^1(s)n_i^1(s)$, $\beta_i(s) = \psi_i^1(s)d_i^1(s)$, with $n_i^1(s), d_i^1(s) \in [s]$ units of $_{M_1}(s)$. On the other hand $\alpha_i(s) = {}_i^2(s)n_i^2(s)$, $\beta_i(s) = \psi_i^2(s)d_i^2(s)$, with $n_i^2(s), d_i^2(s) \in [s]$ units of $_{M_2}(s)$. So, ${}_i^1(s)n_i^1(s) = {}_i^2(s)n_i^2(s)$ or equivalently $n_i^1(s) = \frac{_i^2(s)n_i^2(s)}{_i^1(s)}$, $n_i^2(s) = \frac{_i^1(s)n_i^1(s)}{_i^2(s)}$. The polynomials ${}_i^1(s)$, ${}_i^2(s)$ are coprime because ${}_i^1(s)$ factorizes in M_1, ${}_i^2(s)$ factorizes in M_2 and $M_1 \cap M_2 = \emptyset$. In consequence ${}_i^1(s) \mid n_i^2(s)$ and ${}_i^2(s) \mid n_i^1(s)$. Therefore, there exist polynomials $a(s)$, unit of $_{M_2}(s)$, and $a^{'}(s)$, unit of $_{M_1}(s)$, such that $n_i^2(s) = {}_i^1(s)a(s)$, $n_i^1(s) = {}_i^2(s)a^{'}(s)$. Since $\alpha_i(s) = {}_i^1(s)n_i^1(s) = {}_i^1(s)_i^2(s)a^{'}(s)$ and $\alpha_i(s) = {}_i^2(s)n_i^2(s) = {}_i^2(s)_i^1(s)a(s)$. This implies that $a(s) = a^{'}(s)$ unit of $_{M_1}(s) \cap {}_{M_2}(s) = {}_{M_1\cup M_2}(s)$. Following the same ideas we can prove that $\beta_i(s) = \psi_i^1(s)\psi_i^2(s)b(s)$ with $b(s)$ a unit of $_{M_1\cup M_2}(s)$. By Proposition ▭ $\frac{_1^1(s)_1^2(s)}{\psi_1^1(s)\psi_1^2(s)},\ \ldots,\ \frac{_m^1(s)_m^2(s)}{\psi_m^1(s)\psi_m^2(s)}$ are the invariant rational functions of $T(s)$ with respect to $M_1 \cup M_2$.

Corollary 4 Let $M_1, M_2 \subseteq \mathrm{Specm}([s])$. Two non-singular matrices are equivalent with respect to $M_1 \cup M_2$ if and only if they are equivalent with respect to M_1 and with respect to M_2.

Proof.- Notice that by Remark ▭.2 two matrices $T_1(s), T_2(s) \in (s)^{m\times m}$ are equivalent with respect to $M_1 \cup M_2$ if and only if there exist $U_1(s), U_2(s)$ invertible in $_{M_1}(s)^{m\times m} \cap {}_{M_2}(s)^{m\times m}$ such that $T_2(s) = U_1(s)T_1(s)U_2(s)$. Since $U_1(s)$ and $U_2(s)$ are invertible in both $_{M_1}(s)^{m\times m}$ and $_{M_2}(s)^{m\times m}$ then $T_1(s)$ and $T_2(s)$ are equivalent with respect to M_1 and with respect to M_2.

Conversely, if $T_1(s)$ and $T_2(s)$ are equivalent with respect to M_1 and with respect to M_2 then, by the necessity of this result, they are equivalent with respect to $M_1 \setminus (M_1 \cap M_2)$, with respect to $M_2 \setminus (M_1 \cap M_2)$ and with respect to $M_1 \cap M_2$. Let $\frac{{}_1^1(s)}{\psi_1^1(s)}, \ \ldots, \ \frac{{}_m^1(s)}{\psi_m^1(s)}$ be the invariant rational functions of $T_1(s)$ and $T_2(s)$ with respect to $M_1 \setminus (M_1 \cap M_2)$, $\frac{{}_1^2(s)}{\psi_1^2(s)}, \ \ldots, \ \frac{{}_m^2(s)}{\psi_m^2(s)}$ be the invariant rational functions of $T_1(s)$ and $T_2(s)$ with respect to $M_2 \setminus (M_1 \cap M_2)$ and $\frac{{}_1^3(s)}{\psi_1^3(s)}, \ \ldots, \ \frac{{}_m^3(s)}{\psi_m^3(s)}$ be the invariant rational functions of $T_1(s)$ and $T_2(s)$ with respect to $M_1 \cap M_2$. By Corollary ▭ $\frac{{}_1^1(s)}{\psi_1^1(s)} \frac{{}_1^2(s)}{\psi_1^2(s)} \frac{{}_1^3(s)}{\psi_1^3(s)}, \ \ldots, \ \frac{{}_m^1(s)}{\psi_m^1(s)} \frac{{}_m^2(s)}{\psi_m^2(s)} \frac{{}_m^3(s)}{\psi_m^3(s)}$ must be the invariant rational functions of $T_1(s)$ and $T_2(s)$ with respect to $M_1 \cup M_2$. Therefore, $T_1(s)$ and $T_2(s)$ are equivalent with respect to $M_1 \cup M_2$.

Let ${}_{pr}(s)$ be the ring of proper rational functions, that is, rational functions with the degree of the numerator at most the degree of the denominator. The units in this ring are the rational functions whose numerators and denominators have the same degree. They are called biproper rational functions. A matrix $B(s) \in {}_{pr}(s)^{m \times m}$ is said to be biproper if it is a unit in ${}_{pr}(s)^{m \times m}$ or, what is the same, if its determinant is a biproper rational function.

Recall that a rational function $t(s)$ has a pole (zero) at ∞ if $t\left(\frac{1}{s}\right)$ has a pole (zero) at 0. Following this idea, we can define the local ring at ∞ as the set of rational functions, $t(s)$, such that $t\left(\frac{1}{s}\right)$ does not have 0 as a pole, that is, ${}_\infty(s) = \left\{t(s) \in (s) : t\left(\frac{1}{s}\right) \in {}_s(s)\right\}$. If $t(s) = \frac{p(s)}{q(s)}$ with $p(s) = a_t s^t + a_{t+1} s^{t+1} + \cdots + a_p s^p, \ \ a_p \neq 0,$ $q(s) = b_r s^r + b_{r+1} s^{r+1} + \cdots + b_q s^q, \ \ b_q \neq 0,$ $p = d(p(s)), \ \ q = d(q(s))$, where $d(\cdot)$ stands for "degree of", then

$$t\left(\frac{1}{s}\right) = \frac{\frac{a_t}{s^t} + \frac{a_{t+1}}{s^{t+1}} + \cdots + \frac{a_p}{s^p}}{\frac{b_r}{s^r} + \frac{b_{r+1}}{s^{r+1}} + \cdots + \frac{b_q}{s^q}} = \frac{a_t s^{p-t} + a_{t+1} s^{p-t-1} + \cdots + a_p}{b_r s^{q-r} + b_{r+1} s^{q-r-1} + \cdots + b_q} s^{q-p} = \frac{f(s)}{g(s)} s^{q-p}. \tag{id22}$$

As ${}_s(s) = \left\{\frac{f(s)}{g(s)} s^d : f(0) \neq 0, \ g(0) \neq 0 \text{ and } d \geq 0\right\} \cup \{0\}$, then

$${}_\infty(s) = \left\{\frac{p(s)}{q(s)} \in (s) : d(q(s)) \geq d(p(s))\right\}. \tag{id23}$$

Thus, this set is the ring of proper rational functions, ${}_{pr}(s)$.

Two rational matrices $T_1(s), \ T_2(s) \in (s)^{m \times m}$ are equivalent at infinity if there exist biproper matrices $B_1(s), \ B_2(s) \in \mathrm{Gl}_m({}_{pr}(s))$ such that $T_2(s) = B_1(s) T_1(s) B_2(s)$. Given a non-singular rational matrix $T(s) \in (s)^{m \times m}$ (see [11]) there always exist $B_1(s), \ B_2(s) \in \mathrm{Gl}_m({}_{pr}(s))$ such that

$$T(s) = B_1(s)\mathrm{Diag}\left(s^{q_1}, \ \ldots, \ s^{q_m}\right)B_2(s) \tag{id24}$$

where $q_1 \geq \cdots \geq q_m$ are integers. They are called the invariant orders of $T(s)$ at infinity and the rational functions $s^{q_1}, \ \ldots, \ s^{q_m}$ are called the invariant rational functions of $T(s)$ at infinity.

1.05

3. Structure of the ring of proper rational functions with prescribed finite poles

Let $M^{'} \subseteq \mathrm{Specm}([s])$. Any non-zero rational function $t(s)$ can be uniquely written as $t(s) = \frac{n(s)}{d(s)}\frac{n^{'}(s)}{d^{'}(s)}$ where $\frac{n(s)}{d(s)}$ is an irreducible rational function factorizing in $M^{'}$ and $\frac{n^{'}(s)}{d^{'}(s)}$ is a unit of ${}_{M^{'}}(s)$. Define the following function over $(s) \setminus \{0\}$ (see [11], [12]):

$$\begin{array}{cccc} \delta: & (s) \setminus \{0\} & \rightarrow & \mathbb{Z} \\ & t(s) & \mapsto & d\left(d^{'}(s)\right) - d\left(n^{'}(s)\right). \end{array} \tag{id25}$$

This mapping is not a discrete valuation of (s) if $M^{'} \neq \emptyset$: Given two non-zero elements $t_1(s), \ t_2(s) \in (s)$ it is clear that $\delta\left(t_1(s)t_2(s)\right) = \delta\left(t_1(s)\right) + \delta\left(t_2(s)\right)$; but it may not satisfy that $\delta\left(t_1(s) + t_2(s)\right) \geq \min\left(\delta\left(t_1(s)\right), \ \delta\left(t_2(s)\right)\right)$. For example, let $M^{'} = \{(s - a) \in \mathrm{Specm}(\mathbb{R}[s]) : a \notin [-2, \ -1]\}$. Put $t_1(s) = \frac{s + 0.5}{s + 1.5}$ and $t_2(s) = \frac{s + 2.5}{s + 1.5}$. We have that $\delta\left(t_1(s)\right) = d(s + 1.5) - d(1) = 1$, $\delta\left(t_2(s)\right) = d(s + 1.5) - d(1) = 1$ but $\delta\left(t_1(s) + t_2(s)\right) = \delta(2) = 0$.

However, if $M^{'} = \emptyset$ and $t(s) = \frac{n(s)}{d(s)} \in (s)$ where $n(s), \ d(s) \in [s]$, $d(s) \neq 0$, the map

$$\delta_\infty : (s) \rightarrow \mathbb{Z} \cup \{+\infty\} \tag{id26}$$

defined via $\delta_\infty(t(s)) = d(d(s)) - d(n(s))$ if $t(s) \neq 0$ and $\delta_\infty(t(s)) = +\infty$ if $t(s) = 0$ is a discrete valuation of (s).

Consider the subset of (s), ${}_{M^{'}}(s) \cap {}_{pr}(s)$, consisting of all proper rational functions with poles in $\mathrm{Specm}([s]) \setminus M^{'}$, that is, the elements of ${}_{M^{'}}(s) \cap {}_{pr}(s)$ are proper rational functions whose denominators are coprime with all the polynomials $\pi(s)$ such that $(\pi(s)) \in M^{'}$. Notice that $g(s) \in {}_{M^{'}}(s) \cap {}_{pr}(s)$ if and only if $g(s) = n(s)\frac{n^{'}(s)}{d^{'}(s)}$ where:

(a)(b)(c)

- $n(s) \in [s]$ is a polynomial factorizing in $M^{'}$,

- $\frac{n'(s)}{d'(s)}$ is an irreducible rational function and a unit of $_{M'}(s)$,
- $\delta(g(s)) - d(n(s)) \geq 0$ or equivalently $\delta_{\infty}(g(s)) \geq 0$.

In particular (c) implies that $\frac{n'(s)}{d'(s)} \in {}_{pr}(s)$. The units in $_{M'}(s) \cap {}_{pr}(s)$ are biproper rational functions $\frac{n'(s)}{d'(s)}$, that is $d(n'(s)) = d(d'(s))$, with $n'(s)$, $d'(s)$ factorizing in $\mathrm{Specm}([s]) \setminus M'$. Furthermore, $_{M'}(s) \cap {}_{pr}(s)$ is an integral domain whose field of fractions is (s) provided that $M' \neq \mathrm{Specm}([s])$(see, for example, Prop.5.22[11]). Notice that for $M' = \mathrm{Specm}([s])$, $_{M'}(s) \cap {}_{pr}(s) = [s] \cap {}_{pr}(s) = .$

Assume that there are ideals in $\mathrm{Specm}([s]) \setminus M'$ generated by linear polynomials and let $(s - a)$ be any of them. The elements of $_{M'}(s) \cap {}_{pr}(s)$ can be written as $g(s) = n(s)u(s)\frac{1}{(s-a)^d}$ where $n(s) \in [s]$ factorizes in M', $u(s)$ is a unit in $_{M'}(s) \cap {}_{pr}(s)$ and $d = \delta(g(s)) \geq d(n(s))$. If is algebraically closed, for example $= \mathbb{C}$, and $M' \neq \mathrm{Specm}([s])$ the previous condition is always fulfilled.

The divisibility in $_{M'}(s) \cap {}_{pr}(s)$ is characterized in the following lemma.

Lemma 5 Let $M' \subseteq \mathrm{Specm}([s])$. Let $g_1(s), g_2(s) \in {}_{M'}(s) \cap {}_{pr}(s)$ be such that $g_1(s) = n_1(s)\frac{n_1'(s)}{d_1'(s)}$ and $g_2(s) = n_2(s)\frac{n_2'(s)}{d_2'(s)}$ with $n_1(s), n_2(s) \in [s]$ factorizing in M' and $\frac{n_1'(s)}{d_1'(s)}$, $\frac{n_2'(s)}{d_2'(s)}$ irreducible rational functions, units of $_{M'}(s)$. Then $g_1(s)$ divides $g_2(s)$ in $_{M'}(s) \cap {}_{pr}(s)$ if and only if

$$n_1(s) \mid n_2(s) \text{ in } [s] \tag{id31}$$

$$\delta(g_1(s)) - d(n_1(s)) \leq \delta(g_2(s)) - d(n_2(s)). \tag{id32}$$

Proof.- If $g_1(s) \mid g_2(s)$ then there exists $g(s) = n(s)\frac{n'(s)}{d'(s)} \in {}_{M'}(s) \cap {}_{pr}(s)$, with $n(s) \in [s]$ factorizing in M' and $n'(s), d'(s) \in [s]$ coprime, factorizing in $\mathrm{Specm}([s]) \setminus M'$, such that $g_2(s) = g(s)g_1(s)$. Equivalently, $n_2(s)\frac{n_2'(s)}{d_2'(s)} = n(s)\frac{n'(s)}{d'(s)}n_1(s)\frac{n_1'(s)}{d_1'(s)} = n(s)n_1(s)\frac{n'(s)n_1'(s)}{d'(s)d_1'(s)}$. So $n_2(s) = n(s)n_1(s)$ and $\delta(g_2(s)) - d(n_2(s)) = \delta(g(s)) - d(n(s)) + \delta(g_1(s)) - d(n_1(s))$. Moreover, as $g(s)$ is a proper rational function, $\delta(g(s)) - d(n(s)) \geq 0$ and $\delta(g_2(s)) - d(n_2(s)) \geq \delta(g_1(s)) - d(n_1(s))$.

Conversely, if $n_1(s) \mid n_2(s)$ then there is $n(s) \in [s]$, factorizing in M', such that $n_2(s) = n(s)n_1(s)$. Write $g(s) = n(s)\frac{n'(s)}{d'(s)}$ where $\frac{n'(s)}{d'(s)}$ is an irreducible fraction representation of $\frac{n_2'(s)d_1'(s)}{d_2'(s)n_1'(s)}$, i.e., $\frac{n'(s)}{d'(s)} = \frac{n_2'(s)d_1'(s)}{d_2'(s)n_1'(s)}$ after canceling possible common factors. Thus $\frac{n_2'(s)}{d_2'(s)} = \frac{n'(s)}{d'(s)}\frac{n_1'(s)}{d_1'(s)}$ and

$$\begin{array}{rcl} \delta(g(s)) - d(n(s)) & = & d(d'(s)) - d(n'(s)) - d(n(s)) \\ & = & d(d_2'(s)) + d(n_1'(s)) - d(n_2'(s)) - d(d_1'(s)) - d(n_2(s)) + d(n_1(s)) \\ & = & \delta(g_2(s)) - d(n_2(s)) - (\delta(g_1(s)) - d(n_1(s))) \geq 0. \end{array} \quad \text{(id33)}$$

Then $g(s) \in {}_{M'}(s) \cap {}_{pr}(s)$ and $g_2(s) = g(s)g_1(s)$.

Notice that condition (▭) means that $g_1(s) \mid g_2(s)$ in ${}_{M'}(s)$ and condition (▭) means that $g_1(s) \mid g_2(s)$ in ${}_{pr}(s)$. So, $g_1(s) \mid g_2(s)$ in ${}_{M'}(s) \cap {}_{pr}(s)$ if and only if $g_1(s) \mid g_2(s)$ simultaneously in ${}_{M'}(s)$ and ${}_{pr}(s)$.

Lemma 6 Let $M' \subseteq \mathrm{Specm}([s])$. Let $g_1(s), g_2(s) \in {}_{M'}(s) \cap {}_{pr}(s)$ be such that $g_1(s) = n_1(s)\frac{n_1'(s)}{d_1'(s)}$ and $g_2(s) = n_2(s)\frac{n_2'(s)}{d_2'(s)}$ as in Lemma ▭. If $n_1(s)$ and $n_2(s)$ are coprime in $[s]$ and either $\delta(g_1(s)) = d(n_1(s))$ or $\delta(g_2(s)) = d(n_2(s))$ then $g_1(s)$ and $g_2(s)$ are coprime in ${}_{M'}(s) \cap {}_{pr}(s)$.

Proof.- Suppose that $g_1(s)$ and $g_2(s)$ are not coprime. Then there exists a non-unit $g(s) = n(s)\frac{n'(s)}{d'(s)} \in {}_{M'}(s) \cap {}_{pr}(s)$ such that $g(s) \mid g_1(s)$ and $g(s) \mid g_2(s)$. As $g(s)$ is not a unit, $n(s)$ is not a constant or $\delta(g(s)) > 0$. If $n(s)$ is not a constant then $n(s) \mid n_1(s)$ and $n(s) \mid n_2(s)$ which is impossible because $n_1(s)$ and $n_2(s)$ are coprime. Otherwise, if $n(s)$ is a constant then $\delta(g(s)) > 0$ and we have that $\delta(g(s)) \leq \delta(g_1(s)) - d(n_1(s))$ and $\delta(g(s)) \leq \delta(g_2(s)) - d(n_2(s))$. But this is again impossible.

It follows from this Lemma that if $g_1(s), g_2(s)$ are coprime in both rings ${}_{M'}(s)$ and ${}_{pr}(s)$ then $g_1(s), g_2(s)$ are coprime in ${}_{M'}(s) \cap {}_{pr}(s)$. The following example shows that the converse is not true in general.

Example 7 Suppose that $= \mathbb{R}$ and $M' = \mathrm{Specm}(\mathbb{R}[s]) \setminus \{(s^2+1)\}$. It is not difficult to prove that $g_1(s) = \frac{s^2}{s^2+1}$ and $g_2(s) = \frac{s}{s^2+1}$ are coprime elements in $\mathbb{R}_{M'}(s) \cap \mathbb{R}_{pr}(s)$. Assume that there exists a non-unit $g(s) = n(s)\frac{n'(s)}{d'(s)} \in \mathbb{R}_{M'}(s) \cap \mathbb{R}_{pr}(s)$ such that $g(s) \mid g_1(s)$ and $g(s) \mid g_2(s)$. Then $n(s) \mid s^2$, $n(s) \mid s$ and $\delta(g(s)) - d(n(s)) = 0$. Since $g(s)$ is not a unit, $n(s)$ cannot be a constant. Hence, $n(s) = cs$, $c \neq 0$, and $\delta(g(s)) = 1$, but this is impossible because $d'(s)$ and $n'(s)$ are powers of s^2+1. Therefore $g_1(s)$ and $g_2(s)$ must be coprime. However $n_1(s) = s^2$ and $n_2(s) = s$ are not coprime.

Now, we have the following property when there are ideals in $\mathrm{Specm}([s]) \setminus M'$, $M' \subseteq \mathrm{Specm}([s])$, generated by linear polynomials.

Lemma 8 Let $M' \subseteq \mathrm{Specm}([s])$. Assume that there are ideals in $\mathrm{Specm}([s]) \setminus M'$ generated by linear polynomials and let $(s-a)$ be any of them. Let $g_1(s), g_2(s) \in {}_{M'}(s) \cap {}_{pr}(s)$ be such that $g_1(s) = n_1(s)u_1(s)\frac{1}{(s-a)^{d_1}}$ and $g_2(s) = n_2(s)u_2(s)\frac{1}{(s-a)^{d_2}}$. If $g_1(s)$ and $g_2(s)$ are coprime in ${}_{M'}(s) \cap {}_{pr}(s)$ then $n_1(s)$ and $n_2(s)$ are coprime in $[s]$ and either $d_1 = d(n_1(s))$ or $d_2 = d(n_2(s))$.

Proof.- Suppose that $n_1(s)$ and $n_2(s)$ are not coprime in $[s]$. Then there exists a non-constant $n(s) \in [s]$ such that $n(s) \mid n_1(s)$ and $n(s) \mid n_2(s)$. Let $d = d(n(s))$. Then $g(s) = n(s)\frac{1}{(s-a)^d}$ is not a unit in ${}_{M'}(s) \cap {}_{pr}(s)$ and divides $g_1(s)$ and $g_2(s)$ because $0 = d - d(n(s)) \leq d_1 - d(n_1(s))$ and $0 = d - d(n(s)) \leq d_2 - d(n_2(s))$. This is impossible, so $n_1(s)$ and $n_2(s)$ must be coprime.

Now suppose that $d_1 > d(n_1(s))$ and $d_2 > d(n_2(s))$. Let $d = \min\{d_1 - d(n_1(s)), d_2 - d(n_2(s))\}$. We have that $d > 0$. Thus $g(s) = \frac{1}{(s-a)^d}$ is not a unit in ${}_{M'}(s) \cap {}_{pr}(s)$ and divides $g_1(s)$ and $g_2(s)$ because $d \leq d_1 - d(n_1(s))$ and $d \leq d_2 - d(n_2(s))$. This is again impossible and either $d_1 = d(n_1(s))$ or $d_2 = d(n_2(s))$.

The above lemmas yield a characterization of coprimeness of elements in ${}_{M'}(s) \cap {}_{pr}(s)$ when M' excludes at least one ideal generated by a linear polynomial.

Following the same steps as in p. 11[12] and p. 271[11] we get the following result.

Lemma 9 Let $M' \subseteq \mathrm{Specm}([s])$ and assume that there is at least an ideal in $\mathrm{Specm}([s]) \setminus M'$ generated by a linear polynomial. Then ${}_{M'}(s) \cap {}_{pr}(s)$ is a Euclidean domain.

The following examples show that if all ideals generated by polynomials of degree one are in M', the ring ${}_{M'}(s) \cap {}_{pr}(s)$ may not be a Bezout domain. Thus, it may not be a Euclidean domain. Even more, it may not be a greatest common divisor domain.

Example 10 Let $= \mathbb{R}$ and $M' = \mathrm{Specm}(\mathbb{R}[s]) \setminus \{(s^2+1)\}$. Let $g_1(s) = \frac{s^2}{s^2+1}, g_2(s) = \frac{s}{s^2+1} \in \mathbb{R}_{M'}(s) \cap \mathbb{R}_{pr}(s)$. We have seen, in the previous example, that $g_1(s), g_2(s)$ are coprime. We show now that the Bezout identity is not fulfilled, that is, there are not $a(s), b(s) \in \mathbb{R}_{M'}(s) \cap \mathbb{R}_{pr}(s)$ such that $a(s)g_1(s) + b(s)g_2(s) = u(s)$, with $u(s)$ a unit in $\mathbb{R}_{M'}(s) \cap \mathbb{R}_{pr}(s)$. Elements in $\mathbb{R}_{M'}(s) \cap \mathbb{R}_{pr}(s)$ are of the form $\frac{n(s)}{(s^2+1)^d}$ with $n(s)$ relatively prime with s^2+1 and $2d \geq d(n(s))$ and the units in $\mathbb{R}_{M'}(s) \cap \mathbb{R}_{pr}(s)$ are non-zero constants. We will see that there are not elements $a(s) = \frac{n(s)}{(s^2+1)^d}$, $b(s) = \frac{n'(s)}{(s^2+1)^{d'}}$ with $n(s)$ and $n'(s)$ coprime with s^2+1, $2d \geq d(n(s))$ and $2d' \geq d(n'(s))$ such that $a(s)g_1(s) + b(s)g_2(s) = c$, with c non-zero constant. Assume that $\frac{n(s)}{(s^2+1)^d}\frac{s^2}{s^2+1} + \frac{n'(s)}{(s^2+1)^{d'}}\frac{s}{s^2+1} = c$. We conclude that $c(s^2+1)^{d+1}$ or $c(s^2+1)^{d'+1}$ is a multiple of s, which is impossible.

Example 11 Let $= \mathbb{R}$ and $M^{'} = \text{Specm}(\mathbb{R}[s]) \setminus \{(s^2+1)\}$. A fraction $g(s) = \frac{n(s)}{(s^2+1)^d} \in \mathbb{R}_{M^{'}}(s) \cap \mathbb{R}_{pr}(s)$ if and only if $2d - d(n(s)) \geq 0$. Let $g_1(s) = \frac{s^2}{(s^2+1)^3}$, $g_2(s) = \frac{s(s+1)}{(s^2+1)^4} \in \mathbb{R}_{M^{'}}(s) \cap \mathbb{R}_{pr}(s)$. By Lemma ▭:

- $g(s) \mid g_1(s) \Leftrightarrow n(s) \mid s^2$ and $0 \leq 2d - d(n(s)) \leq 6 - 2 = 4$
- $g(s) \mid g_2(s) \Leftrightarrow n(s) \mid s(s+1)$ and $0 \leq 2d - d(n(s)) \leq 8 - 2 = 6$.

If $n(s) \mid s^2$ and $n(s) \mid s(s+1)$ then $n(s) = c$ or $n(s) = cs$ with c a non-zero constant. Then $g(s) \mid g_1(s)$ and $g(s) \mid g_2(s)$ if and only if $n(s) = c$ and $d \leq 2$ or $n(s) = cs$ and $2d \leq 5$. So, the list of common divisors of $g_1(s)$ and $g_2(s)$ is:

$$\left\{c, \frac{c}{s^2+1}, \frac{c}{(s^2+1)^2}, \frac{cs}{s^2+1}, \frac{cs}{(s^2+1)^2} : c \in , c \neq 0\right\}. \tag{id42}$$

If there would be a greatest common divisor, say $\frac{n(s)}{(s^2+1)^d}$, then $n(s) = cs$ because $n(s)$ must be a multiple of c and cs. Thus such a greatest common divisor should be either $\frac{cs}{s^2+1}$ or $\frac{cs}{(s^2+1)^2}$, but $\frac{c}{(s^2+1)^2}$ does not divide neither of them because

$$4 = \delta\left(\frac{c}{(s^2+1)^2}\right) - d(c) > max\left\{\delta\left(\frac{cs}{s^2+1}\right) - d(cs), \delta\left(\frac{cs}{(s^2+1)^2}\right) - d(cs)\right\} = 3. \tag{id43}$$

Thus, $g_1(s)$ and $g_2(s)$ do not have greatest common divisor.

3.1. Smith–McMillan form

A matrix $U(s)$ is invertible in $_{M^{'}}(s)^{m\times m} \cap {}_{pr}(s)^{m\times m}$ if $U(s) \in {}_{M^{'}}(s)^{m\times m} \cap {}_{pr}(s)^{m\times m}$ and its determinant is a unit in both rings, $_{M^{'}}(s)$ and $_{pr}(s)$, i.e., $U(s) \in \text{Gl}_m\left({}_{M^{'}}(s) \cap {}_{pr}(s)\right)$ if and only if $U(s) \in \text{Gl}_m\left({}_{M^{'}}(s)\right) \cap \text{Gl}_m\left({}_{pr}(s)\right)$.

Two matrices $G_1(s), G_2(s) \in {}_{M^{'}}(s)^{m\times m} \cap {}_{pr}(s)^{m\times m}$ are equivalent in $_{M^{'}}(s) \cap {}_{pr}(s)$ if there exist $U_1(s), U_2(s)$ invertible in $_{M^{'}}(s)^{m\times m} \cap {}_{pr}(s)^{m\times m}$ such that

$$G_2(s) = U_1(s)G_1(s)U_2(s). \tag{id45}$$

If there are ideals in $\text{Specm}([s]) \setminus M^{'}$ generated by linear polynomials then $_{M^{'}}(s) \cap {}_{pr}(s)$ is an Euclidean ring and any matrix with elements in $_{M^{'}}(s) \cap {}_{pr}(s)$ admits a Smith normal form (see [10], [11] or [12]). Bearing in mind the characterization of divisibility in ${}_{M'}(s) \cap {}_{pr}(s)$ given in Lemma ▭ we have

Theorem 12 (Smith normal form in ${}_{M'}(s) \cap {}_{pr}(s)$) Let $M' \subseteq \text{Specm}([s])$. Assume that there are ideals in $\text{Specm}([s]) \setminus M'$ generated by linear polynomials and let $(s-a)$ be one of them. Let $G(s) \in {}_{M'}(s)^{m\times m} \cap {}_{pr}(s)^{m\times m}$ be non-singular. Then there exist $U_1(s), U_2(s)$ invertible in ${}_{M'}(s)^{m\times m} \cap {}_{pr}(s)^{m\times m}$ such that

$$G(s) = U_1(s)\text{Diag}\left(n_1(s)\frac{1}{(s-a)^{d_1}}, \ \ldots, \ n_m(s)\frac{1}{(s-a)^{d_m}}\right)U_2(s) \qquad \text{(id47)}$$

with $n_1(s) \mid \cdots \mid n_m(s)$ monic polynomials factorizing in M' and $d_1, \ \ldots, d_m$ integers such that $0 \le d_1 - d(n_1(s)) \le \cdots \le d_m - d(n_m(s))$.

Under the hypothesis of the last theorem $n_1(s)\frac{1}{(s-a)^{d_1}}, \ \ldots, n_m(s)\frac{1}{(s-a)^{d_m}}$ form a complete system of invariants for the equivalence in ${}_{M'}(s) \cap {}_{pr}(s)$ and are called the invariant rational functions of $G(s)$ in ${}_{M'}(s) \cap {}_{pr}(s)$. Notice that $0 \le d_1 \le \cdots \le d_m$ because $n_i(s)$ divides $n_{i+1}(s)$.

Recall that the field of fractions of ${}_{M'}(s) \cap {}_{pr}(s)$ is (s) when $M' \ne \text{Specm}([s])$. Thus we can talk about equivalence of matrix rational functions. Two rational matrices $T_1(s), T_2(s) \in (s)^{m\times m}$ are equivalent in ${}_{M'}(s) \cap {}_{pr}(s)$ if there are $U_1(s), U_2(s)$ invertible in ${}_{M'}(s)^{m\times m} \cap {}_{pr}(s)^{m\times m}$ such that

$$T_2(s) = U_1(s)T_1(s)U_2(s). \qquad \text{(id48)}$$

When all ideals generated by linear polynomials are not in M', each rational matrix admits a reduction to Smith–McMillan form with respect to ${}_{M'}(s) \cap {}_{pr}(s)$.

Theorem 13 (Smith–McMillan form in ${}_{M'}(s) \cap {}_{pr}(s)$) Let $M' \subseteq \text{Specm}([s])$. Assume that there are ideals in $\text{Specm}([s]) \setminus M'$ generated by linear polynomials and let $(s-a)$ be any of them. Let $T(s) \in (s)^{m\times m}$ be a non-singular matrix. Then there exist $U_1(s), U_2(s)$ invertible in ${}_{M'}(s)^{m\times m} \cap {}_{pr}(s)^{m\times m}$ such that

$$T(s) = U_1(s)\text{Diag}\left(\frac{\frac{{}_1(s)}{(s-a)^{n_1}}}{\frac{\psi_1(s)}{(s-a)^{d_1}}}, \ \ldots, \ \frac{\frac{{}_m(s)}{(s-a)^{n_m}}}{\frac{\psi_m(s)}{(s-a)^{d_m}}}\right)U_2(s) \qquad \text{(id50)}$$

with $\frac{{}_i(s)}{(s-a)^{n_i}}, \frac{\psi_i(s)}{(s-a)^{d_i}} \in {}_{M'}(s) \cap {}_{pr}(s)$ coprime for all i such that ${}_i(s)$, $\psi_i(s)$ are monic polynomials factorizing in M', $\frac{{}_i(s)}{(s-a)^{n_i}}$ divides $\frac{{}_{i+1}(s)}{(s-a)^{n_{i+1}}}$ for $i = 1, \ldots, m-1$ while $\frac{\psi_i(s)}{(s-a)^{d_i}}$ divides $\frac{\psi_{i-1}(s)}{(s-a)^{d_{i-1}}}$ for $i = 2, \ldots, m$.

The elements $\frac{\frac{{}_i(s)}{(s-a)^{n_i}}}{\frac{\psi_i(s)}{(s-a)^{d_i}}}$ of the diagonal matrix, satisfying the conditions of the previous theorem, constitute a complete system of invariant for the equivalence in ${}_{M'}(s) \cap {}_{pr}(s)$ of rational matrices. However, this system of invariants is not minimal. A smaller one can be obtained by substituting each pair of positive integers (n_i, d_i) by its difference $l_i = n_i - d_i$.

Theorem 14 Under the conditions of Theorem ▭, $\frac{{}_i(s)}{\psi_i(s)} \frac{1}{(s-a)^{l_i}}$ with ${}_i(s)$, $\psi_i(s)$ monic and coprime polynomials factorizing in M', ${}_i(s) \mid {}_{i+1}(s)$ while $\psi_i(s) \mid \psi_{i-1}(s)$ and $l_1, \ldots, l_m$ integers such that $l_1 + d(\psi_1(s)) - d({}_1(s)) \leq \cdots \leq l_m + d(\psi_m(s)) - d({}_m(s))$ also constitute a complete system of invariants for the equivalence in ${}_{M'}(s) \cap {}_{pr}(s)$.

Proof.- We only have to show that from the system $\frac{{}_i(s)}{\psi_i(s)} \frac{1}{(s-a)^{l_i}}$, $i = 1, \ldots, m$, satisfying the conditions of Theorem ▭, the system $\frac{\frac{{}_i(s)}{(s-a)^{n_i}}}{\frac{\psi_i(s)}{(s-a)^{d_i}}}$, $i = 1, \ldots, n$, can be constructed satisfying the conditions of Theorem ▭.

Suppose that ${}_i(s)$, $\psi_i(s)$ are monic and coprime polynomials factorizing in M' such that ${}_i(s) \mid {}_{i+1}(s)$ and $\psi_i(s) \mid \psi_{i-1}(s)$. And suppose also that $l_1, \ldots, l_m$ are integers such that $l_1 + d(\psi_1(s)) - d({}_1(s)) \leq \cdots \leq l_m + d(\psi_m(s)) - d({}_m(s))$. If $l_i + d(\psi_i(s)) - d({}_i(s)) \leq 0$ for all i, we define non-negative integers $n_i = d({}_i(s))$ and $d_i = d({}_i(s)) - l_i$ for $i = 1, \ldots, m$. If $l_i + d(\psi_i(s)) - d({}_i(s)) > 0$ for all i, we define $n_i = l_i + d(\psi_i(s))$ and $d_i = d(\psi_i(s))$. Otherwise there is an index $k \in \{2, \ldots, m\}$ such that

$$l_{k-1} + d(\psi_{k-1}(s)) - d({}_{k-1}(s)) \leq 0 < l_k + d(\psi_k(s)) - d({}_k(s)). \qquad \text{(id52)}$$

Define now the non-negative integers n_i, d_i as follows:

$$n_i = \begin{cases} d({}_i(s)) & \text{if } i < k \\ l_i + d(\psi_i(s)) & \text{if } i \geq k \end{cases} \qquad d_i = \begin{cases} d({}_i(s)) - l_i & \text{if } i < k \\ d(\psi_i(s)) & \text{if } i \geq k \end{cases} \qquad \text{(id53)}$$

Notice that $l_i = n_i - d_i$. Moreover,

$$n_i - d(\,_i(s)) = \begin{cases} 0 & \text{if } i < k \\ l_i + d(\psi_i(s)) - d(\,_i(s)) & \text{if } i \geq k \end{cases} \tag{id54}$$

$$d_i - d(\psi_i(s)) = \begin{cases} -l_i - d(\psi_i(s)) + d(\,_i(s)) & \text{if } i < k \\ 0 & \text{if } i \geq k \end{cases} \tag{id55}$$

and using (▭), (▭)

$$n_1 - d(\,_1(s)) = \cdots = n_{k-1} - d(\,_{k-1}(s)) = 0 < n_k - d(\,_k(s)) \leq \cdots \leq n_m - d(\,_m(s)) \tag{id56}$$

$$d_1 - d(\psi_1(s)) \geq \cdots \geq d_{k-1} - d(\psi_{k-1}(s)) \geq 0 = d_k - d(\psi_k(s)) = \cdots = d_m - d(\psi_m(s)). \tag{id57}$$

In any case $\frac{\,_i(s)}{(s-a)^{n_i}}$ and $\frac{\psi_i(s)}{(s-a)^{d_i}}$ are elements of $_{M'}(s) \cap \,_{pr}(s)$. Now, on the one hand $_i(s)$, $\psi_i(s)$ are coprime and $n_i - d(\,_i(s)) = 0$ or $d_i - d(\psi_i(s)) = 0$. This means (Lemma ▭) that $\frac{\,_i(s)}{(s-a)^{n_i}}, \frac{\psi_i(s)}{(s-a)^{d_i}}$ are coprime for all i. On the other hand $_i(s) \mid \,_{i+1}(s)$ and $0 \leq n_i - d(\,_i(s)) \leq n_{i+1} - d(\,_{i+1}(s))$. Then (Lemma ▭) $\frac{\,_i(s)}{(s-a)^{n_i}}$ divides $\frac{\,_{i+1}(s)}{(s-a)^{n_{i+1}}}$. Similarly, since $\psi_i(s) \mid \psi_{i-1}(s)$ and $0 \leq d_i - d(\psi_i(s)) \leq d_{i-1} - d(\psi_{i-1}(s))$, it follows that $\frac{\psi_i(s)}{(s-a)^{d_i}}$ divides $\frac{\psi_{i-1}(s)}{(s-a)^{d_{i-1}}}$.

We call $\frac{\,_i(s)}{\psi_i(s)} \frac{1}{(s-a)^{l_i}}$, $i = 1, \ldots, m$, the invariant rational functions of $T(s)$ in $_{M'}(s) \cap \,_{pr}(s)$.

There is a particular case worth considering: If $M' = \emptyset$ then $_{\emptyset}(s) \cap \,_{pr}(s) = \,_{pr}(s)$ and $(s) \in \text{Specm}([s]) \setminus M' = \text{Specm}([s])$. In this case, we obtain the invariant rational functions of $T(s)$ at infinity (recall (▭)).

4. Wiener–Hopf equivalence

The left Wiener–Hopf equivalence of rational matrices with respect to a closed contour in the complex plane has been extensively studied ([5] or [6]). Now we present the generalization to arbitrary fields ([13]).

Definition 15 Let M and M' be subsets of $\text{Specm}([s])$ such that $M \cup M' = \text{Specm}([s])$. Let $T_1(s), T_2(s) \in (s)^{m\times m}$ be two non-singular rational matrices with no zeros and no poles in $M \cap M'$. The matrices $T_1(s), T_2(s)$ are said to be left Wiener–Hopf equivalent with respect to (M, M') if there exist both $U_1(s)$ invertible in $_{M'}(s)^{m\times m} \cap \,_{pr}(s)^{m\times m}$ and $U_2(s)$ invertible in $_M(s)^{m\times m}$ such that

$$T_2(s) = U_1(s)T_1(s)U_2(s). \quad \text{(id59)}$$

This is, in fact, an equivalence relation as it is easily seen. It would be an equivalence relation even if no condition about the union and intersection of M and M' were imposed. It will be seen later on that these conditions are natural assumptions for the existence of unique diagonal representatives in each class.

The right Wiener–Hopf equivalence with respect to (M, M') is defined in a similar manner: There are invertible matrices $U_1(s)$ in $_{M'}(s)^{m\times m} \cap {}_{pr}(s)^{m\times m}$ and $U_2(s)$ in $_{M}(s)^{m\times m}$ such that

$$T_2(s) = U_2(s)T_1(s)U_1(s). \quad \text{(id60)}$$

In the following only the left Wiener–Hopf equivalence will be considered, but, by transposition, all results hold for the right Wiener–Hopf equivalence as well.

The aim of this section is to obtain a complete system of invariants for the Wiener–Hopf equivalence with respect to (M, M') of rational matrices, and to obtain, if possible, a canonical form.

There is a particular case that is worth-considering: If $M = \mathrm{Specm}([s])$ and $M' = \emptyset$, the invertible matrices in $_{\emptyset}(s)^{m\times m} \cap {}_{pr}(s)^{m\times m}$ are the biproper matrices and the invertible matrices in $_{\mathrm{Specm}([s])}(s)^{m\times m}$ are the unimodular matrices. In this case, the left Wiener–Hopf equivalence with respect to $(M, M') = (\mathrm{Specm}([s]), \emptyset)$ is the so-called left Wiener–Hopf equivalence at infinity (see [14]). It is known that any non-singular rational matrix is left Wiener–Hopf equivalent at infinity to a diagonal matrix $\mathrm{Diag}(s^{g_1}, \ldots, s^{g_m})$ where $g_1, \ldots, g_m$ are integers, that is, for any non-singular $T(s) \in (s)^{m\times m}$ there exist both a biproper matrix $B(s) \in \mathrm{Gl}_m({}_{pr}(s))$ and a unimodular matrix $U(s) \in \mathrm{Gl}_m([s])$ such that

$$T(s) = B(s)\mathrm{Diag}(s^{g_1}, \ldots, s^{g_m})U(s) \quad \text{(id61)}$$

where $g_1 \geq \cdots \geq g_m$ are integers uniquely determined by $T(s)$. They are called the left Wiener–Hopf factorization indices at infinity and form a complete system of invariants for the left Wiener–Hopf equivalence at infinity. These are the basic objects that will produce the complete system of invariants for the left Wiener–Hopf equivalence with respect to (M, M').

For polynomial matrices, their left Wiener–Hopf factorization indices at infinity are the column degrees of any right equivalent (by a unimodular matrix) column proper matrix. Namely, a polynomial matrix is column proper if it can be written as $P_c\mathrm{Diag}(s^{g_1}, \ldots, s^{g_m}) + L(s)$ with $P_c \in {}^{m\times m}$ non-singular, $g_1, \ldots, g_m$ non-negative integers and $L(s)$ a polynomial matrix such that the degree of the ith column of $L(s)$ smaller than g_i, $1 \leq i \leq m$. Let $P(s) \in [s]^{m\times m}$ be non-singular polynomial. There exists a unimodular matrix

$V(s) \in [s]^{m\times m}$ such that $P(s)V(s)$ is column proper. The column degrees of $P(s)V(s)$ are uniquely determined by $P(s)$, although $V(s)$ is not (see [14], p. 388[15], [16]). Since $P(s)V(s)$ is column proper, it can be written as $P(s)V(s) = P_c D(s) + L\ (s)$ with P_c non-singular, $D(s) = \mathrm{Diag}\left(s^{g_1},\ \ldots,\ s^{g_m}\right)$ and the degree of the ith column of $L\ (s)$ smaller than g_i, $1 \le i \le m$. Then $P(s)V(s) = \left(P_c + L\ (s)D(s)^{-1}\right)D(s)$. Put $B(s) = P_c + L\ (s)D(s)^{-1}$. Since P_c is non-singular and $L\ (s)D(s)^{-1}$ is a strictly proper matrix, $B(s)$ is biproper, and $P(s) = B(s)D(s)U(s)$ where $U(s) = V(s)^{-1}$.

The left Wiener–Hopf factorization indices at infinity can be used to associate a sequence of integers with every non-singular rational matrix and every $M \subseteq \mathrm{Specm}([s])$. This is done as follows: If $T(s) \in (s)^{m\times m}$ then it can always be written as $T(s) = T_L\ (s)T_R(s)$ such that the global invariant rational functions of $T_L\ (s)$ factorize in M and $T_R(s) \in \mathrm{Gl}_m\left(_M(s)\right)$ or, equivalently, the global invariant rational functions of $T_R(s)$ factorize in $\mathrm{Specm}([s]) \setminus M$ (see Proposition ▭). There may be many factorizations of this type, but it turns out (see Proposition 3.2[4] for the polynomial case) that the left factors in all of them are right equivalent. This means that if $T(s) = T_{L\ 1}(s)T_{R1}(s) = T_{L\ 2}(s)T_{R2}(s)$ with the global invariant rational functions of $T_{L\ 1}(s)$ and $T_{L\ 2}(s)$ factorizing in M and the global invariant rational functions of $T_{R1}(s)$ and $T_{R2}(s)$ factorizing in $\mathrm{Specm}([s]) \setminus M$ then there is a unimodular matrix $U(s)$ such that $T_{L\ 1}(s) = T_{L\ 2}(s)U(s)$. In particular, $T_{L\ 1}(s)$ and $T_{L\ 2}(s)$ have the same left Wiener–Hopf factorization indices at infinity. Thus the following definition makes sense:

Definition 16 Let $T(s) \in (s)^{m\times m}$ be a non-singular rational matrix and $M \subseteq \mathrm{Specm}([s])$. Let $T_L\ (s),\ T_R(s) \in (s)^{m\times m}$ such that

i)ii)iii)

- $T(s) = T_L\ (s)T_R(s)$,
- the global invariant rational functions of $T_L\ (s)$ factorize in M, and
- the global invariant rational functions of $T_R(s)$ factorize in $\mathrm{Specm}([s]) \setminus M$.

Then the left Wiener–Hopf factorization indices of $T(s)$ with respect to M are defined to be the left Wiener–Hopf factorization indices of $T_L\ (s)$ at infinity.

In the particular case that $M = \mathrm{Specm}([s])$, we can put $T_L\ (s) = T(s)$ and $T_R(s) = I_m$. Therefore, the left Wiener–Hopf factorization indices of $T(s)$ with respect to $\mathrm{Specm}([s])$ are the left Wiener–Hopf factorization indices of $T(s)$ at infinity.

We prove now that the left Wiener–Hopf equivalence with respect to $(M,\ M')$ can be characterized through the left Wiener–Hopf factorization indices with respect to M.

Theorem 17 Let $M, M' \subseteq \mathrm{Specm}([s])$ be such that $M \cup M' = \mathrm{Specm}([s])$. Let $T_1(s)$, $T_2(s) \in (s)^{m\times m}$ be two non-singular rational matrices with no zeros and no poles in $M \cap M'$.

The matrices $T_1(s)$ and $T_2(s)$ are left Wiener–Hopf equivalent with respect to (M, M') if and only if $T_1(s)$ and $T_2(s)$ have the same left Wiener–Hopf factorization indices with respect to M.

Proof.- By Proposition ▭ we can write $T_1(s) = T_{L\ 1}(s)T_{R1}(s)$, $T_2(s) = T_{L\ 2}(s)T_{R2}(s)$ with the global invariant rational functions of $T_{L\ 1}(s)$ and of $T_{L\ 2}(s)$ factorizing in $M \setminus M'$ (recall that $T_1(s)$ and $T_2(s)$ have no zeros and no poles in $M \cap M'$) and the global invariant rational functions of $T_{R1}(s)$ and of $T_{R2}(s)$ factorizing in $M' \setminus M$.

Assume that $T_1(s)$, $T_2(s)$ have the same left Wiener–Hopf factorization indices with respect to M. By definition, $T_1(s)$ and $T_2(s)$ have the same left Wiener–Hopf factorization indices with respect to M if $T_{L\ 1}(s)$ and $T_{L\ 2}(s)$ have the same left Wiener–Hopf factorization indices at infinity. This means that there exist matrices $B(s) \in \mathrm{Gl}_m(_{pr}(s))$ and $U(s) \in \mathrm{Gl}_m([s])$ such that $T_{L\ 2}(s) = B(s)T_{L\ 1}(s)U(s)$. We have that $T_2(s) = T_{L\ 2}(s)T_{R2}(s) = B(s)T_{L\ 1}(s)U(s)T_{R2}(s) = B(s)T_1(s)\left(T_{R1}(s)^{-1}U(s)T_{R2}(s)\right)$. We aim to prove that $B(s) = T_{L\ 2}(s)U(s)^{-1}T_{L\ 1}(s)^{-1}$ is invertible in $_{M'}(s)^{m\times m}$ and $T_{R1}(s)^{-1}U(s)T_{R2}(s) \in Gl_m(_M(s))$. Since the global invariant rational functions of $T_{L\ 2}(s)$ and $T_{L\ 1}(s)$ factorize in $M \setminus M'$, $T_{L\ 2}(s), T_{L\ 1}(s) \in {}_{M'}(s)^{m\times m}$ and $B(s) \in {}_{M'}(s)^{m\times m}$. Moreover, $\det B(s)$ is a unit in $_{M'}(s)^{m\times m}$ as desired. Now, $T_{R1}(s)^{-1}U(s)T_{R2}(s) \in Gl_m(_M(s))$ because $T_{R1}(s), T_{R2}(s) \in {}_M(s)^{m\times m}$ and $\det T_{R1}(s)$ and $\det T_{R2}(s)$ factorize in $M' \setminus M$. Therefore $T_1(s)$ and $T_2(s)$ are left Wiener–Hopf equivalent with respect to (M, M').

Conversely, let $U_1(s) \in \mathrm{Gl}_m(_{M'}(s)) \cap \mathrm{Gl}_m(_{pr}(s))$ and $U_2(s) \in \mathrm{Gl}_m(_M(s))$ such that $T_1(s) = U_1(s)T_2(s)U_2(s)$. Hence, $T_1(s) = T_{L\ 1}(s)T_{R1}(s) = U_1(s)T_{L\ 2}(s)T_{R2}(s)U_2(s)$. Put $\overline{T}_{L\ 2}(s) = U_1(s)T_{L\ 2}(s)$ and $\overline{T}_{R2}(s) = T_{R2}(s)U_2(s)$. Therefore,

(i)(ii)(iii)

- $T_1(s) = T_{L\ 1}(s)T_{R1}(s) = \overline{T}_{L\ 2}(s)\overline{T}_{R2}(s)$,
- the global invariant rational functions of $T_{L\ 1}(s)$ and of $\overline{T}_{L\ 2}(s)$ factorize in M, and
- the global invariant rational functions of $T_{R1}(s)$ and of $\overline{T}_{R2}(s)$ factorize in $\mathrm{Specm}([s]) \setminus M$.

Then $T_{L\ 1}(s)$ and $\overline{T}_{L\ 2}(s)$ are right equivalent (see the remark previous to Definition ▭). So, there exists $U(s) \in \mathrm{Gl}_m([s])$ such that $T_{L\ 1}(s) = \overline{T}_{L\ 2}(s)U(s)$. Thus, $T_{L\ 1}(s) = U_1(s)T_{L\ 2}(s)U(s)$. Since $U_1(s)$ is biproper and $U(s)$ is unimodular $T_{L\ 1}(s)$, $T_{L\ 2}(s)$ have the same left Wiener–Hopf factorization indices at infinity. Consequentially, $T_1(s)$ and $T_2(s)$ have the same left Wiener–Hopf factorization indices with respect to M.

In conclusion, for non-singular rational matrices with no zeros and no poles in $M \cap M'$ the left Wiener–Hopf factorization indices with respect to M form a complete system of invariants for the left Wiener–Hopf equivalence with respect to (M, M') with $M \cup M' = \text{Specm}([s])$.

A straightforward consequence of the above theorem is the following Corollary

Corollary 18 Let $M, M' \subseteq \text{Specm}([s])$ be such that $M \cup M' = \text{Specm}([s])$. Let $T_1(s)$, $T_2(s) \in (s)^{m\times m}$ be non-singular with no zeros and no poles in $M \cap M'$. Then $T_1(s)$ and $T_2(s)$ are left Wiener–Hopf equivalent with respect to (M, M') if and only if for any factorizations $T_1(s) = T_{L\ 1}(s)T_{R1}(s)$ and $T_2(s) = T_{L\ 2}(s)T_{R2}(s)$ satisfying the conditions (i)–(iii) of Definition ▭, $T_{L\ 1}(s)$ and $T_{L\ 2}(s)$ are left Wiener–Hopf equivalent at infinity.

Next we deal with the problem of factorizing or reducing a rational matrix to diagonal form by Wiener–Hopf equivalence. It will be shown that if there exists in M an ideal generated by a monic irreducible polynomial of degree equal to 1 which is not in M', then any non-singular rational matrix, with no zeros and no poles in $M \cap M'$ admits a factorization with respect to (M, M'). Afterwards, some examples will be given in which these conditions on M and M' are removed and factorization fails to exist.

Theorem 19 Let $M, M' \subseteq \text{Specm}([s])$ be such that $M \cup M' = \text{Specm}([s])$. Assume that there are ideals in $M \setminus M'$ generated by linear polynomials. Let $(s - a)$ be any of them and $T(s) \in (s)^{m\times m}$ a non-singular matrix with no zeros and no poles in $M \cap M'$. There exist both $U_1(s)$ invertible in ${}_{M'}(s)^{m\times m} \cap {}_{pr}(s)^{m\times m}$ and $U_2(s)$ invertible in ${}_{M}(s)^{m\times m}$ such that

$$T(s) = U_1(s)\text{Diag}\left((s-a)^{k_1}, \ \ldots, \ (s-a)^{k_m}\right)U_2(s), \tag{id72}$$

where $k_1 \geq \cdots \geq k_m$ are integers uniquely determined by $T(s)$. Moreover, they are the left Wiener–Hopf factorization indices of $T(s)$ with respect to M.

Proof.- The matrix $T(s)$ can be written (see Proposition ▭) as $T(s) = T_L(s)T_R(s)$ with the global invariant rational functions of $T_L(s)$ factorizing in $M \setminus M'$ and the global invariant rational functions of $T_R(s)$ factorizing in $\text{Specm}([s]) \setminus M = M' \setminus M$. As $k_1, \ \ldots, \ k_m$ are the left Wiener–Hopf factorization indices of $T_L(s)$ at infinity, there exist matrices $U(s) \in \text{Gl}_m([s])$ and $B(s) \in \text{Gl}_m({}_{pr}(s))$ such that $T_L(s) = B(s)D_1(s)U(s)$ with $D_1(s) = \text{Diag}\left(s^{k_1}, \ \ldots, \ s^{k_m}\right)$. Put $D(s) = \text{Diag}\left((s-a)^{k_1}, \ \ldots, \ (s-a)^{k_m}\right)$ and $U_1(s) = B(s)\text{Diag}\left(\frac{s^{k_1}}{(s-a)^{k_1}}, \ \ldots, \ \frac{s^{k_m}}{(s-a)^{k_m}}\right)$. Then $T_L(s) = U_1(s)D(s)U(s)$. If $U_2(s) = U(s)T_R(s)$ then this matrix is invertible in ${}_{M}(s)^{m\times m}$ and $T(s) = U_1(s)\text{Diag}\left((s-a)^{k_1}, \ \ldots, \ (s-a)^{k_m}\right)U_2(s)$. We only have to prove that $U_1(s)$ is invertible in ${}_{M'}(s)^{m\times m} \cap {}_{pr}(s)^{m\times m}$. It is clear that $U_1(s)$ is in ${}_{pr}(s)^{m\times m}$ and biproper. Moreover, the global invariant rational functions of $T_L(s)\ U_1(s) = T_L(s)(D(s)U(s))^{-1}$ factorize in $M \setminus M'$. Therefore, $U_1(s)$ is invertible in ${}_{M'}(s)^{m\times m}$.

We prove now the uniqueness of the factorization. Assume that $T(s)$ also factorizes as

$$T(s) = \widetilde{U}_1(s)\mathrm{Diag}\left((s-a)^{\tilde{k}_1},\ \dots,\ (s-a)^{\tilde{k}_m}\right)\widetilde{U}_2(s), \tag{id73}$$

with $\tilde{k}_1 \geq \cdots \geq \tilde{k}_m$ integers. Then,

$$\mathrm{Diag}\left((s-a)^{\tilde{k}_1},\ \dots,\ (s-a)^{\tilde{k}_m}\right) = \widetilde{U}_1(s)^{-1}U_1(s)\mathrm{Diag}\left((s-a)^{k_1},\ \dots,\ (s-a)^{k_m}\right)U_2(s)\widetilde{U}_2(s)^{-1}. \tag{id74}$$

The diagonal matrices have no zeros and no poles in $M \cap M'$ (because $(s-a) \in M \setminus M'$) and they are left Wiener–Hopf equivalent with respect to (M, M'). By Theorem ▭, they have the same left Wiener–Hopf factorization indices with respect to M. Thus, $\tilde{k}_i = k_i$ for all $i = 1, \dots, m$.

Following [5] we could call left Wiener–Hopf factorization indices with respect to (M, M') the exponents $k_1 \geq \cdots \geq k_m$ appearing in the diagonal matrix of Theorem ▭. They are, actually, the left Wiener–Hopf factorization indices with respect to M.

Several examples follow that exhibit some remarkable features about the results that have been proved so far. The first two examples show that if no assumption is made on the intersection and/or union of M and M' then existence and/or uniqueness of diagonal factorization may fail to exist.

Example 20

If $P(s)$ is a polynomial matrix with zeros in $M \cap M'$ then the existence of invertible matrices $U_1(s) \in \mathrm{Gl}_m({}_{M'}(s)) \cap \mathrm{Gl}_m({}_{pr}(s))$ and $U_2(s) \in \mathrm{Gl}_m({}_{M}(s))$ such that $P(s) = U_1(s)\mathrm{Diag}\left((s-a)^{k_1},\ \dots,\ (s-a)^{k_m}\right)U_2(s)$ with $(s-a) \in M \setminus M'$ may fail. In fact, suppose that $M = \{(s), (s+1)\}$, $M' = \mathrm{Specm}[s] \setminus \{(s)\}$. Therefore, $M \cap M' = \{(s+1)\}$ and $(s) \in M \setminus M'$. Consider $p_1(s) = s+1$. Assume that $s+1 = u_1(s)s^k u_2(s)$ with $u_1(s)$ a unit in ${}_{M'}(s) \cap {}_{pr}(s)$ and $u_2(s)$ a unit in ${}_{M}(s)$. Thus, $u_1(s) = c$ a nonzero constant and $u_2(s) = \frac{1}{c}\frac{s+1}{s^k}$ which is not a unit in ${}_{M}(s)$.

Example 21

If $M \cup M' \neq \mathrm{Specm}[s]$ then the factorization indices with respect to (M, M') may be not unique. Suppose that $(\beta(s)) \notin M \cup M'$, $(\pi(s)) \in M \setminus M'$ with $d(\pi(s)) = 1$ and $p(s) = u_1(s)\pi(s)^k u_2(s)$, with $u_1(s)$ a unit in ${}_{M'}(s) \cap {}_{pr}(s)$ and $u_2(s)$ a unit in ${}_{M}(s)$. Then $p(s)$ can also be factorized as $p(s) = \tilde{u}_1(s)\pi(s)^{k-d(\beta(s))}\tilde{u}_2(s)$ with $\tilde{u}_1(s) = u_1(s)\frac{\pi(s)^{d(\beta(s))}}{\beta(s)}$ a unit in ${}_{M'}(s) \cap {}_{pr}(s)$ and $\tilde{u}_2(s) = \beta(s)u_2(s)$ a unit in ${}_{M}(s)$.

The following example shows that if all ideals generated by polynomials of degree equal to one are in $M' \setminus M$ then a factorization as in Theorem ▭ may not exist.

Example 22 Suppose that $= \mathbb{R}$. Consider $M = \{(s^2+1)\} \subseteq \text{Specm}(\mathbb{R}[s])$ and $M' = \text{Specm}(\mathbb{R}[s]) \setminus \{(s^2+1)\}$. Let

$$P(s) = \begin{bmatrix} s & 0 \\ -s^2 & (s^2+1)^2 \end{bmatrix}. \tag{id78}$$

Notice that $P(s)$ has no zeros and no poles in $M \cap M' = \varnothing$. We will see that it is not possible to find invertible matrices $U_1(s) \in \mathbb{R}_{M'}(s)^{2\times 2} \cap \mathbb{R}_{pr}(s)^{2\times 2}$ and $U_2(s) \in \mathbb{R}_M(s)^{2\times 2}$ such that

$$U_1(s)P(s)U_2(s) = \text{Diag}\left((p(s)/q(s))^{c_1},\ (p(s)/q(s))^{c_2}\right). \tag{id79}$$

We can write $\frac{p(s)}{q(s)} = u(s)(s^2+1)^a$ with $u(s)$ a unit in $\mathbb{R}_M(s)$ and $a \in \mathbb{Z}$. Therefore,

$$\text{Diag}\left((p(s)/q(s))^{c_1},\ (p(s)/q(s))^{c_2}\right) = \text{Diag}\left((s^2+1)^{ac_1},\ (s^2+1)^{ac_2}\right)\text{Diag}\left(u(s)^{c_1},\ u(s)^{c_2}\right). \tag{id80}$$

$\text{Diag}\left(u(s)^{c_1},\ u(s)^{c_2}\right)$ is invertible in $\mathbb{R}_M(s)^{2\times 2}$ and $P(s)$ is also left Wiener–Hopf equivalent with respect to (M, M') to the diagonal matrix $\text{Diag}\left((s^2+1)^{ac_1},\ (s^2+1)^{ac_2}\right)$.

Assume that there exist invertible matrices $U_1(s) \in \mathbb{R}_{M'}(s)^{2\times 2} \cap \mathbb{R}_{pr}(s)^{2\times 2}$ and $U_2(s) \in \mathbb{R}_M(s)^{2\times 2}$ such that $U_1(s)P(s)U_2(s) = \text{Diag}\left((s^2+1)^{d_1},\ (s^2+1)^{d_2}\right)$, with $d_1 \geq d_2$ integers. Notice first that $\det U_1(s)$ is a nonzero constant and since $\det P(s) = s(s^2+1)^2$ and $\det U_2(s)$ is a rational function with numerator and denominator relatively prime with s^2+1, it follows that $cs(s^2+1)^2 \det U_2(s) = (s^2+1)^{d_1+d_2}$. Thus, $d_1 + d_2 = 2$. Let

$$U_1(s)^{-1} = \begin{bmatrix} b_{11}(s) & b_{12}(s) \\ b_{21}(s) & b_{22}(s) \end{bmatrix}, \quad U_2(s) = \begin{bmatrix} u_{11}(s) & u_{12}(s) \\ u_{21}(s) & u_{22}(s) \end{bmatrix}. \tag{id81}$$

From $P(s)U_2(s) = U_1(s)^{-1}\text{Diag}\left((s^2+1)^{d_1},\ (s^2+1)^{d_2}\right)$ we get

$$su_{11}(s) = b_{11}(s)(s^2+1)^{d_1}, \tag{id82}$$

$$-s^2u_{11}(s) + (s^2+1)^2u_{21}(s) = b_{21}(s)(s^2+1)^{d_1}, \tag{id83}$$

$$su_{12}(s) = b_{12}(s)(s^2+1)^{d_2}, \tag{id84}$$

$$-s^2u_{12}(s)+(s^2+1)^2u_{22}(s)=b_{22}(s)(s^2+1)^{d_2}. \qquad \text{(id85)}$$

As $u_{11}(s)\in\mathbb{R}_M(s)$ and $b_{11}(s)\in\mathbb{R}_{M'}(s)\cap\mathbb{R}_{pr}(s)$, we can write $u_{11}(s)=\frac{f_1(s)}{g_1(s)}$ and $b_{11}(s)=\frac{h_1(s)}{(s^2+1)^{q_1}}$ with $f_1(s),\ g_1(s),\ h_1(s)\in\mathbb{R}[s]$, $\gcd\left(g_1(s),\ s^2+1\right)=1$ and $d(h_1(s))\le 2q_1$. Therefore, by (▭), $s\frac{f_1(s)}{g_1(s)}=\frac{h_1(s)}{(s^2+1)^{q_1}}(s^2+1)^{d_1}$. Hence, $u_{11}(s)=f_1(s)$ or $u_{11}(s)=\frac{f_1(s)}{s}$. In the same way and using (▭), $u_{12}(s)=f_2(s)$ or $u_{12}(s)=\frac{f_2(s)}{s}$ with $f_2(s)$ a polynomial. Moreover, by (▭), d_2 must be non-negative. Hence, $d_1\ge d_2\ge 0$. Using now (▭) and (▭) and bearing in mind again that $u_{21}(s),\ u_{22}(s)\in\mathbb{R}_M(s)$ and $b_{21}(s),\ b_{22}(s)\in\mathbb{R}_{M'}(s)\cap\mathbb{R}_{pr}(s)$, we conclude that $u_{21}(s)$ and $u_{22}(s)$ are polynomials.

We can distinguish two cases: $d_1=2$, $d_2=0$ and $d_1=d_2=1$. If $d_1=2$ and $d_2=0$, by (▭), $b_{12}(s)$ is a polynomial and since $b_{12}(s)$ is proper, it is constant: $b_{12}(s)=c_1$. Thus $u_{12}(s)=\frac{c_1}{s}$. By (▭), $b_{22}(s)=-c_1s+(s^2+1)^2u_{22}(s)$. Since $u_{22}(s)$ is polynomial and $b_{22}(s)$ is proper, $b_{22}(s)$ is also constant and then $u_{22}(s)=0$ and $c_1=0$. Consequentially, $b_{22}(s)=0$, and $b_{12}(s)=0$. This is impossible because $U_1(s)$ is invertible.

If $d_1=d_2=1$ then , using (▭),

$$\begin{aligned} b_{21}(s) &= \frac{-s^2u_{11}(s)+(s^2+1)^2u_{21}(s)}{s^2+1}=\frac{-s^2\frac{b_{11}(s)}{s}(s^2+1)+(s^2+1)^2u_{21}(s)}{s^2+1}\\ &= -sb_{11}(s)+(s^2+1)u_{21}(s)=-s\frac{h_1(s)}{(s^2+1)^{q_1}}+(s^2+1)u_{21}(s)\\ &= \frac{-sh_1(s)+(s^2+1)^{q_1+1}u_{21}(s)}{(s^2+1)^{q_1}}. \end{aligned} \qquad \text{(id86)}$$

Notice that $d(-sh_1(s))\le 1+2q_1$ and $d\left((s^2+1)^{q_1+1}u_{21}(s)\right)=2(q_1+1)+d(u_{21}(s))\ge 2q_1+2$ unless $u_{21}(s)=0$. Hence, if $u_{21}(s)\ne 0$, $d\left(-sh_1(s)+(s^2+1)^{q_1+1}u_{21}(s)\right)\ge 2q_1+2$ which is greater than $d\left((s^2+1)^{q_1}\right)=2q_1$. This cannot happen because $b_{21}(s)$ is proper. Thus, $u_{21}(s)=0$. In the same way and reasoning with (▭) we get that $u_{22}(s)$ is also zero. This is again impossible because $U_2(s)$ is invertible. Therefore no left Wiener–Hopf factorization of $P(s)$ with respect to (M,M') exits.

We end this section with an example where the left Wiener–Hopf factorization indices of the matrix polynomial in the previous example are computed. Then an ideal generated by a polynomial of degree 1 is added to M and the Wiener–Hopf factorization indices of the same matrix are obtained in two different cases.

Example 23 Let $= \mathbb{R}$ and $M = \{(s^2+1)\}$. Consider the matrix

$$P(s) = \begin{bmatrix} s & 0 \\ -s^2 & (s^2+1)^2 \end{bmatrix}, \tag{id88}$$

which has a zero at 0. It can be written as $P(s) = P_1(s)P_2(s)$ with

$$P_1(s) = \begin{bmatrix} 1 & 0 \\ -s & (s^2+1)^2 \end{bmatrix}, \quad P_2(s) = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}, \tag{id89}$$

where the global invariant factors of $P_1(s)$ are powers of s^2+1 and the global invariant factors of $P_2(s)$ are relatively prime with s^2+1. Moreover, the left Wiener–Hopf factorization indices of $P_1(s)$ at infinity are 3, 1 (add the first column multiplied by s^3+2s to the second column; the result is a column proper matrix with column degrees 1 and 3). Therefore, the left Wiener–Hopf factorization indices of $P(s)$ with respect to M are 3, 1.

Consider now $\widetilde{M} = \{(s^2+1), (s)\}$ and $\widetilde{M}' = \mathrm{Specm}(\mathbb{R}[s]) \setminus \widetilde{M}$. There is a unimodular matrix $U(s) = \begin{bmatrix} 1 & s^2+2 \\ 0 & 1 \end{bmatrix}$, invertible in $\mathbb{R}_{\widetilde{M}}(s)^{2\times 2}$, such that $P(s)U(s) = \begin{bmatrix} s & s^3+2s \\ -s^2 & 1 \end{bmatrix}$ is column proper with column degrees 3 and 2. We can write

$$P(s)U(s) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} s^2 & 0 \\ 0 & s^3 \end{bmatrix} + \begin{bmatrix} s & 2s \\ 0 & 1 \end{bmatrix} = B(s)\begin{bmatrix} s^2 & 0 \\ 0 & s^3 \end{bmatrix}, \tag{id90}$$

where $B(s)$ is the following biproper matrix

$$B(s) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} s & 2s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s^{-2} & 0 \\ 0 & s^{-3} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{s^2+2}{s^2} \\ -1 & \frac{1}{s^3} \end{bmatrix}. \tag{id91}$$

Moreover, the denominators of its entries are powers of s and $\det B(s) = \frac{(s^2+1)^2}{s^4}$. Therefore, $B(s)$ is invertible in $\mathbb{R}_{\widetilde{M}'}(s)^{2\times 2} \cap \mathbb{R}_{pr}(s)^{2\times 2}$. Since $B(s)^{-1}P(s)U(s) = \mathrm{Diag}(s^2, s^3)$, the left Wiener–Hopf factorization indices of $P(s)$ with respect to $\widetilde{M}$ are 3, 2.

If $\widetilde{M} = \{(s^2+1), (s-1)\}$, for example, a similar procedure shows that $P(s)$ has 3, 1 as left Wiener–Hopf factorization indices with respect to $\widetilde{M}$; the same indices as with respect to M. The reason is that s - 1 is not a divisor of det $P(s)$ and so $P(s) = P_1(s)P_2(s)$ with $P_1(s)$ and $P_2(s)$ as in (▭) and $P_1(s)$ factorizing in $\widetilde{M}$.

Remark 24 It must be noticed that a procedure has been given to compute, at least theoretically, the left Wiener–Hopf factorization indices of any rational matrix with respect to any subset M of Specm($[s]$). In fact, given a rational matrix $T(s)$ and M, write $T(s) = T_L(s)T_R(s)$ with the global invariant rational functions of $T_L(s)$ factorizing in M, and the global invariant rational functions of $T_R(s)$ factorizing in Specm($[s]$) \ M (for example, using the global Smith–McMillan form of $T(s)$). We need to compute the left Wiener–Hopf factorization indices at infinity of the rational matrix $T_L(s)$. The idea is as follows: Let $d(s)$ be the monic least common denominator of all the elements of $T_L(s)$. The matrix $T_L(s)$ can be written as $T_L(s) = \frac{P(s)}{d(s)}$, with $P(s)$ polynomial. The left Wiener–Hopf factorization indices of $P(s)$ at infinity are the column degrees of any column proper matrix right equivalent to $P(s)$. If $k_1, \ldots, k_m$ are the left Wiener–Hopf factorization indices at infinity of $P(s)$ then $k_1 + d, \ldots, k_m + d$ are the left Wiener–Hopf factorization indices of $T_L(s)$, where $d = d(d(s))$ (see [4]). Free and commercial software exists that compute such column degrees.

5. Rosenbrock's Theorem via local rings

As said in the Introduction, Rosenbrock's Theorem ([1]) on pole assignment by state feedback provides, in its polynomial formulation, a complete characterization of the relationship between the invariant factors and the left Wiener–Hopf factorization indices at infinity of any non-singular matrix polynomial. The precise statement of this result is the following theorem:

Theorem 25 Let $g_1 \geq \cdots \geq g_m$ and $\alpha_1(s) \mid \cdots \mid \alpha_m(s)$ be non-negative integers and monic polynomials, respectively. Then there exists a non-singular matrix $P(s) \in [s]^{m\times m}$ with $\alpha_1(s), \ldots, \alpha_m(s)$ as invariant factors and $g_1, \ldots, g_m$ as left Wiener–Hopf factorization indices at infinity if and only if the following relation holds:

$$(g_1, \ldots, g_m) \prec (d(\alpha_m(s)), \ldots, d(\alpha_1(s))). \qquad \text{(id94)}$$

Symbol $\prec$ appearing in (▭) is the majorization symbol (see [17]) and it is defined as follows: If $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_m)$ are two finite sequences of real numbers and $a_{[1]} \geq \cdots \geq a_{[m]}$ and $b_{[1]} \geq \cdots \geq b_{[m]}$ are the given sequences arranged in non-increasing order then $(a_1, \ldots, a_m) \prec (b_1, \ldots, b_m)$ if

$$\sum_{i=1}^{j} a_{[i]} \leq \sum_{i=1}^{j} b_{[i]}, \quad 1 \leq j \leq m - 1 \qquad \text{(id95)}$$

with equality for $j = m$.

The above Theorem ▭ can be extended to cover rational matrix functions. Any rational matrix $T(s)$ can be written as $\frac{N(s)}{d(s)}$ where $d(s)$ is the monic least common denominator of all the elements of $T(s)$ and $N(s)$ is polynomial. It turns out that the invariant rational functions of $T(s)$ are the invariant factors of $N(s)$ divided by $d(s)$ after canceling common factors. We also have the following characterization of the left Wiener– Hopf factorization indices at infinity of $T(s)$: these are those of $N(s)$ plus the degree of $d(s)$ (see [4]). Bearing all this in mind one can easily prove (see [4])

Theorem 26 Let $g_1 \geq \cdots \geq g_m$ be integers and $\frac{\alpha_1(s)}{\beta_1(s)}, \ldots, \frac{\alpha_m(s)}{\beta_m(s)}$ irreducible rational functions, where $\alpha_i(s), \beta_i(s) \in [s]$ are monic such that $\alpha_1(s) \mid \cdots \mid \alpha_m(s)$ while $\beta_m(s) \mid \cdots \mid \beta_1(s)$. Then there exists a non-singular rational matrix $T(s) \in (s)^{m \times m}$ with $g_1, \ldots, g_m$ as left Wiener–Hopf factorization indices at infinity and $\frac{\alpha_1(s)}{\beta_1(s)}, \ldots, \frac{\alpha_m(s)}{\beta_m(s)}$ as global invariant rational functions if and only if

$$(g_1, \ldots, g_m) \prec \left(d(\alpha_m(s)) - d(\beta_m(s)), \ldots, d(\alpha_1(s)) - d(\beta_1(s))\right). \qquad \text{(id97)}$$

Recall that for $M \subseteq \mathrm{Specm}([s])$ any rational matrix $T(s)$ can be factorized into two matrices (see Proposition ▭) such that the global invariant rational functions and the left Wiener–Hopf factorization indices at infinity of the left factor of $T(s)$ give the invariant rational functions and the left Wiener–Hopf factorization indices of $T(s)$ with respect to M. Using Theorem ▭ on the left factor of $T(s)$ we get:

Theorem 27 Let $M \subseteq \mathrm{Specm}([s])$. Let $k_1 \geq \cdots \geq k_m$ be integers and $\frac{{}_1(s)}{\psi_1(s)}, \ldots, \frac{{}_m(s)}{\psi_m(s)}$ be irreducible rational functions such that ${}_1(s) \mid \cdots \mid {}_m(s), \psi_m(s) \mid \cdots \mid \psi_1(s)$ are monic polynomials factorizing in M. Then there exists a non-singular matrix $T(s) \in (s)^{m \times m}$ with $\frac{{}_1(s)}{\psi_1(s)}, \ldots, \frac{{}_m(s)}{\psi_m(s)}$ as invariant rational functions with respect to M and $k_1, \ldots, k_m$ as left Wiener–Hopf factorization indices with respect to M if and only if

$$(k_1, \ldots, k_m) \prec \left(d({}_m(s)) - d(\psi_m(s)), \ldots, d({}_1(s)) - d(\psi_1(s))\right). \qquad \text{(id99)}$$

Theorem ▭ relates the left Wiener–Hopf factorization indices with respect to M and the finite structure inside M. Our last result will relate the left Wiener–Hopf factorization indices with respect to M and the structure outside M, including that at infinity. The next Theorem is an extension of Rosenbrock's Theorem to the point at infinity, which was proved in [4]:

Theorem 28 Let $g_1 \geq \cdots \geq g_m$ and $q_1 \geq \cdots \geq q_m$ be integers. Then there exists a non-singular matrix $T(s) \in (s)^{m \times m}$ with $g_1, \ldots, g_m$ as left Wiener–Hopf factorization indices at infinity and $s^{q_1}, \ldots, s^{q_m}$ as invariant rational functions at infinity if and only if

http://dx.doi.org/10.5772/81051

$$(g_1, \ \ldots, g_m) \prec (q_1, \ \ldots, q_m). \qquad \text{(id101)}$$

Notice that Theorem ▭ can be obtained from Theorem ▭ when $M = \mathrm{Specm}([s])$. In the same way, taking into account that the equivalence at infinity is a particular case of the equivalence in $_{M'}(s) \cap {}_{pr}(s)$ when $M' = \varnothing$, we can give a more general result than that of Theorem ▭. Specifically, necessary and sufficient conditions can be provided for the existence of a non-singular rational matrix with prescribed left Wiener–Hopf factorization indices with respect to M and invariant rational functions in $_{M'}(s) \cap {}_{pr}(s)$.

Theorem 29 Let $M, M' \subseteq \mathrm{Specm}([s])$ be such that $M \cup M' = \mathrm{Specm}([s])$. Assume that there are ideals in $M \setminus M'$ generated by linear polynomials and let $(s - a)$ be any of them. Let $k_1 \geq \cdots \geq k_m$ be integers, $\frac{{}_1(s)}{\psi_1(s)}, \ \ldots, \frac{{}_m(s)}{\psi_m(s)}$ irreducible rational functions such that ${}_1(s) \mid \cdots \mid {}_m(s)$, $\psi_m(s) \mid \cdots \mid \psi_1(s)$ are monic polynomials factorizing in $M' \setminus M$ and $l_1, \ \ldots, l_m$ integers such that $l_1 + d(\psi_1(s)) - d({}_1(s)) \leq \cdots \leq l_m + d(\psi_m(s)) - d({}_m(s))$. Then there exists a non-singular matrix $T(s) \in (s)^{m \times m}$ with no zeros and no poles in $M \cap M'$ with $k_1, \ \ldots, k_m$ as left Wiener–Hopf factorization indices with respect to M and $\frac{{}_1(s)}{\psi_1(s)} \frac{1}{(s-a)^{l_1}}, \ \ldots, \frac{{}_m(s)}{\psi_m(s)} \frac{1}{(s-a)^{l_m}}$ as invariant rational functions in $_{M'}(s) \cap {}_{pr}(s)$ if and only if the following condition holds:

$$(k_1, \ \ldots, k_m) \prec (-l_1, \ \ldots, -l_m). \qquad \text{(id103)}$$

The proof of this theorem will be given along the following two subsections. We will use several auxiliary results that will be stated and proved when needed.

5.1. Necessity

We can give the following result for rational matrices using a similar result given in Lemma 4.2 in [18] for matrix polynomials.

Lemma 30 Let $M, M' \subseteq \mathrm{Specm}([s])$ be such that $M \cup M' = \mathrm{Specm}([s])$. Let $T(s) \in (s)^{m \times m}$ be a non-singular matrix with no zeros and no poles in $M \cap M'$ with $g_1 \geq \cdots \geq g_m$ as left Wiener–Hopf factorization indices at infinity and $k_1 \geq \cdots \geq k_m$ as left Wiener–Hopf factorization indices with respect to M. If $\frac{{}_1(s)}{\psi_1(s)}, \ \ldots, \frac{{}_m(s)}{\psi_m(s)}$ are the invariant rational functions of $T(s)$ with respect to M' then

$$(g_1 - k_1, \ \ldots, g_m - k_m) \prec \left(d({}_m(s)) - d(\psi_m(s)), \ \ldots, d({}_1(s)) - d(\psi_1(s))\right). \qquad \text{(id106)}$$

It must be pointed out that $(g_1 - k_1, \ \ldots, g_m - k_m)$ may be an unordered m-tuple.

Proof.- By Proposition ▭ there exist unimodular matrices $U(s),\ V(s) \in [s]^{m\times m}$ such that

$$T(s) = U(s)\mathrm{Diag}\left(\frac{\alpha_1(s)}{\beta_1(s)},\ \ldots,\ \frac{\alpha_m(s)}{\beta_m(s)}\right)\mathrm{Diag}\left(\frac{{}_1(s)}{\psi_1(s)},\ \ldots,\ \frac{{}_m(s)}{\psi_m(s)}\right)V(s) \qquad \text{(id107)}$$

with $\alpha_i(s) \mid \alpha_{i+1}(s),\ \beta_i(s) \mid \beta_{i-1}(s),\ {}_i(s) \mid {}_{i+1}(s),\ \psi_i(s) \mid \psi_{i-1}(s),\ \alpha_i(s),\ \beta_i(s)$ units in ${}_{M'\setminus M}(s)$ and ${}_i(s),\ \psi_i(s)$ factorizing in $M' \setminus M$ because $T(s)$ has no poles and no zeros in $M \cap M'$. Therefore $T(s) = T_L(s)T_R(s)$, where $T_L(s) = U(s)\mathrm{Diag}\left(\frac{\alpha_1(s)}{\beta_1(s)},\ \ldots,\ \frac{\alpha_m(s)}{\beta_m(s)}\right)$ has $k_1,\ \ldots,\ k_m$ as left Wiener–Hopf factorization indices at infinity and $T_R(s) = \mathrm{Diag}\left(\frac{{}_1(s)}{\psi_1(s)},\ \ldots,\ \frac{{}_m(s)}{\psi_m(s)}\right)V(s)$ has $\frac{{}_1(s)}{\psi_1(s)},\ \ldots,\ \frac{{}_m(s)}{\psi_m(s)}$ as global invariant rational functions. Let $d(s) = \beta_1(s)\psi_1(s)$. Hence,

$$d(s)T(s) = U(s)\mathrm{Diag}(\bar{\alpha}_1(s),\ \ldots,\ \bar{\alpha}_m(s))\mathrm{Diag}(\bar{}_1(s),\ \ldots,\ \bar{}_m(s))V(s) \qquad \text{(id108)}$$

with $\bar{\alpha}_i(s) = \frac{\alpha_i(s)}{\beta_i(s)}\beta_1(s)$ units in ${}_{M'\setminus M}(s)$ and $\bar{}_i(s) = \frac{{}_i(s)}{\psi_i(s)}\psi_1(s)$ factorizing in $M' \setminus M$. Put $P(s) = d(s)T(s)$. Its left Wiener–Hopf factorization indices at infinity are $g_1 + d(d(s)),\ \ldots,\ g_m + d(d(s))$ Lemma 2.3[4]. The matrix $P_1(s) = U(s)\mathrm{Diag}(\bar{\alpha}_1(s),\ \ldots,\ \bar{\alpha}_m(s)) = \beta_1(s)T_L(s)$ has $k_1 + d(\beta_1(s)),\ \ldots,\ k_m + d(\beta_1(s))$ as left Wiener–Hopf factorization indices at infinity. Now if $P_2(s) = \mathrm{Diag}(\bar{}_1(s),\ \ldots,\ \bar{}_m(s))V(s) = \psi_1(s)T_R(s)$ then its invariant factors are $\bar{}_1(s),\ \ldots,\ \bar{}_m(s)$, $P(s) = P_1(s)P_2(s)$ and, by Lemma 4.2[18],

$$(g_1 + d(d(s)) - k_1 - d(\beta_1(s)),\ \ldots,\ g_m + d(d(s)) - k_m - d(\beta_1(s))) \prec (d(\bar{}_m(s)),\ \ldots,\ d(\bar{}_1(s))). \qquad \text{(id109)}$$

Therefore, (▭) follows.

5.1.1. Proof of Theorem : Necessity

If $\frac{{}_1(s)}{\psi_1(s)}\frac{1}{(s-a)^{l_1}},\ \ldots,\ \frac{{}_m(s)}{\psi_m(s)}\frac{1}{(s-a)^{l_m}}$ are the invariant rational functions of $T(s)$ in ${}_{M'}(s) \cap {}_{pr}(s)$ then there exist matrices $U_1(s),\ U_2(s)$ invertible in ${}_{M'}(s)^{m\times m} \cap {}_{pr}(s)^{m\times m}$ such that

$$T(s) = U_1(s)\mathrm{Diag}\left(\frac{{}_1(s)}{\psi_1(s)}\frac{1}{(s-a)^{l_1}},\ \ldots,\ \frac{{}_m(s)}{\psi_m(s)}\frac{1}{(s-a)^{l_m}}\right)U_2(s). \qquad \text{(id111)}$$

We analyze first the finite structure of $T(s)$ with respect to M'. If $D_1(s) = \mathrm{Diag}((s-a)^{-l_1},\ \ldots,\ (s-a)^{-l_m}) \in {}_{M'}(s)^{m\times m}$, we can write $T(s)$ as follows:

$$T(s) = U_1(s)\mathrm{Diag}\left(\frac{{}_1(s)}{\psi_1(s)},\ \ldots,\ \frac{{}_m(s)}{\psi_m(s)}\right)D_1(s)U_2(s), \tag{id112}$$

with $U_1(s)$ and $D_1(s)U_2(s)$ invertible matrices in ${}_{M'}(s)^{m\times m}$. Thus $\frac{{}_1(s)}{\psi_1(s)},\ \ldots,\ \frac{{}_m(s)}{\psi_m(s)}$ are the invariant rational functions of $T(s)$ with respect to M'. Let $g_1 \geq \cdots \geq g_m$ be the left Wiener–Hopf factorization indices of $T(s)$ at infinity. By Lemma ▭ we have

$$(g_1 - k_1,\ \ldots,\ g_m - k_m) \prec \left(d({}_m(s)) - d(\psi_m(s)),\ \ldots,\ d({}_1(s)) - d(\psi_1(s))\right). \tag{id113}$$

As far as the structure of $T(s)$ at infinity is concerned, let

$$D_2(s) = \mathrm{Diag}\left(\frac{{}_1(s)}{\psi_1(s)}\frac{s^{l_1+d(\psi_1(s))-d({}_1(s))}}{(s-a)^{l_1}},\ \ldots,\ \frac{{}_m(s)}{\psi_m(s)}\frac{s^{l_m+d(\psi_m(s))-d({}_m(s))}}{(s-a)^{l_m}}\right). \tag{id114}$$

Then $D_2(s) \in Gl({}_{pr}(s))$ and

$$T(s) = U_1(s)\mathrm{Diag}\left(s^{-l_1-d(\psi_1(s))+d({}_1(s))},\ \ldots,\ s^{-l_m-d(\psi_m(s))+d({}_m(s))}\right)D_2(s)U_2(s) \tag{id115}$$

where $U_1(s) \in {}_{pr}(s)^{m\times m}$ and $D_2(s)U_2(s) \in {}_{pr}(s)^{m\times m}$ are biproper matrices. Therefore $s^{-l_1-d(\psi_1(s))+d({}_1(s))},\ \ldots,\ s^{-l_m-d(\psi_m(s))+d({}_m(s))}$ are the invariant rational functions of $T(s)$ at infinity. By Theorem ▭

$$(g_1,\ \ldots,\ g_m) \prec \left(-l_1 - d(\psi_1(s)) + d({}_1(s)),\ \ldots,\ -l_m - d(\psi_m(s)) + d({}_m(s))\right). \tag{id116}$$

Let $\sigma \in \Sigma_m$ (the symmetric group of order m) be a permutation such that $g_{\sigma(1)} - k_{\sigma(1)} \geq \cdots \geq g_{\sigma(m)} - k_{\sigma(m)}$ and define $c_i = g_{\sigma(i)} - k_{\sigma(i)}$, $i = 1,\ \ldots,\ m$. Using (▭) and (▭) we obtain

$$\begin{aligned}
\sum_{j=1}^{r} k_j + \sum_{j=1}^{r}\left(d({}_j(s)) - d(\psi_j(s))\right) &\leq \sum_{j=1}^{r} k_j + \sum_{j=m-r+1}^{m} c_j \\
&\leq \sum_{j=1}^{r} k_j + \sum_{j=1}^{r}(g_j - k_j) = \sum_{j=1}^{r} g_j \\
&\leq \sum_{j=1}^{r} -l_j + \sum_{j=1}^{r}\left(d({}_j(s)) - d(\psi_j(s))\right)
\end{aligned} \tag{id117}$$

for $r = 1,\ \ldots,\ m-1$. When $r = m$ the previous inequalities are all equalities and condition (▭) is satisfied.

Remark 31 It has been seen in the above proof that if a matrix has $\frac{{}_1(s)}{\psi_1(s)}\frac{1}{(s-a)^{l_1}}, \ldots, \frac{{}_m(s)}{\psi_m(s)}\frac{1}{(s-a)^{l_m}}$ as invariant rational functions in ${}_{M'}(s) \cap {}_{pr}(s)$ then $\frac{{}_1(s)}{\psi_1(s)}, \ldots, \frac{{}_m(s)}{\psi_m(s)}$ are its invariant rational functions with respect to M' and $s^{-l_1-d(\psi_1(s))+d({}_1(s))}, \ldots, s^{-l_m-d(\psi_m(s))+d({}_m(s))}$ are its invariant rational functions at infinity.

5.2. Sufficiency

Let $a, b \in$ be arbitrary elements such that $ab \neq 1$. Consider the changes of indeterminate

$$f(s) = a + \frac{1}{s-b}, \quad \tilde{f}(s) = b + \frac{1}{s-a} \tag{id120}$$

and notice that $f(\tilde{f}(s)) = \tilde{f}(f(s)) = s$. For $\alpha(s) \in [s]$, let $[s] \setminus (\alpha(s))$ denote the multiplicative subset of $[s]$ whose elements are coprime with $\alpha(s)$. For $a, b \in$ as above define

$$\begin{array}{rcl} t_{a,b}: \quad [s] & \to & [s] \setminus (s-b) \\ \pi(s) & \mapsto & (s-b)^{d(\pi(s))}\pi\left(a + \frac{1}{s-b}\right) = (s-b)^{d(\pi(s))}\pi(f(s)) \ . \end{array} \tag{id121}$$

In words, if $\pi(s) = p_d(s-a)^d + p_{d-1}(s-a)^{d-1} + \cdots + p_1(s-a) + p_0$ $(p_d \neq 0)$ then

$$t_{a,b}(\pi(s)) = p_0(s-b)^d + p_1(s-b)^{d-1} + \cdots + p_{d-1}(s-b) + p_d. \tag{id122}$$

In general $d(t_{a,b}(\pi(s))) \leq d(\pi(s))$ with equality if and only if $\pi(s) \in [s] \setminus (s-a)$. This shows that the restriction $h_{a,b} : [s] \setminus (s-a) \to [s] \setminus (s-b)$ of $t_{a,b}$ to $[s] \setminus (s-a)$ is a bijection. In addition $h_{a,b}^{-1}$ is the restriction of $t_{b,a}$ to $[s] \setminus (s-b)$; i.e.,

$$\begin{array}{rcl} h_{a,b}^{-1}: \quad [s] \setminus (s-b) & \to & [s] \setminus (s-a) \\ \alpha(s) & \mapsto & (s-a)^{d(\alpha(s))}\alpha\left(b + \frac{1}{s-a}\right) = (s-a)^{d(\alpha(s))}\alpha(\tilde{f}(s)) \end{array} \tag{id123}$$

or $h_{a,b}^{-1} = h_{b,a}$.

In what follows we will think of a, b as given elements of and the subindices of $t_{a,b}$, $h_{a,b}$ and $h_{a,b}^{-1}$ will be removed. The following are properties of h (and h^{-1}) that can be easily proved.

Lemma 32 Let $\pi_1(s), \pi_2(s) \in [s] \setminus (s-a)$. The following properties hold:

- $h(\pi_1(s)\pi_2(s)) = h(\pi_1(s))h(\pi_2(s))$.

http://dx.doi.org/10.5772/81051

- If $\pi_1(s) \mid \pi_2(s)$ then $h(\pi_1(s)) \mid h(\pi_2(s))$.
- If $\pi_1(s)$ is an irreducible polynomial then $h(\pi_1(s))$ is an irreducible polynomial.
- If $\pi_1(s),\ \pi_2(s)$ are coprime polynomials then $h(\pi_1(s)), h(\pi_2(s))$ are coprime polynomials.

As a consequence the map

$$\begin{array}{rccc} H: & \mathrm{Specm}([s]) \setminus \{(s-a)\} & \to & \mathrm{Specm}([s]) \setminus \{(s-b)\} \\ & (\pi(s)) & \mapsto & \left(\frac{1}{p_0} h(\pi(s))\right) \end{array} \tag{id129}$$

with $p_0 = \pi(a)$, is a bijection whose inverse is

$$\begin{array}{rccc} H^{-1}: & \mathrm{Specm}([s]) \setminus \{(s-b)\} & \to & \mathrm{Specm}([s]) \setminus \{(s-a)\} \\ & (\alpha(s)) & \mapsto & \left(\frac{1}{a_0} h^{-1}(\alpha(s))\right) \end{array} \tag{id130}$$

where $a_0 = \alpha(b)$. In particular, if $M' \subseteq \mathrm{Specm}([s]) \setminus \{(s-a)\}$ and $\widetilde{M} = \mathrm{Specm}([s]) \setminus (M' \cup \{(s-a)\})$ (i.e. the complementary subset of M' in $\mathrm{Specm}([s]) \setminus \{(s-a)\}$) then

$$H(\widetilde{M}) = \mathrm{Specm}([s]) \setminus (H(M') \cup \{(s-b)\}). \tag{id131}$$

In what follows and for notational simplicity we will assume $b = 0$.

Lemma 33 Let $M' \subseteq \mathrm{Specm}([s]) \setminus \{(s-a)\}$ where $a \in$ is an arbitrary element of .

- If $\pi(s) \in [s]$ factorizes in M' then $h(\pi(s))$ factorizes in $H(M')$.
- If $\pi(s) \in [s]$ is a unit of $_{M'}(s)$ then $t(\pi(s))$ is a unit of $_{H(M')}(s)$.

Proof.- 1. Let $\pi(s) = c\pi_1(s)^{g_1} \cdots \pi_m(s)^{g_m}$ with $c \neq 0$ constant, $(\pi_i(s)) \in M'$ and $g_i \geq 1$. Then $h(\pi(s)) = c(h(\pi_1(s)))^{g_1} \cdots (h(\pi_m(s)))^{g_m}$. By Lemma ▭ $h(\pi_i(s))$ is an irreducible polynomial (that may not be monic). If c_i is the leading coefficient of $h(\pi_i(s))$ then $\frac{1}{c_i} h(\pi_i(s))$ is monic, irreducible and $\left(\frac{1}{c_i} h(\pi_i(s))\right) \in H(M')$. Hence $h(\pi(s))$ factorizes in $H(M')$.

2. If $\pi(s) \in [s]$ is a unit of $_{M'}(s)$ then it can be written as $\pi(s) = (s-a)^g \pi_1(s)$ where $g \geq 0$ and $\pi_1(s)$ is a unit of $_{M' \cup \{(s-a)\}}(s)$. Therefore $\pi_1(s)$ factorizes in $\mathrm{Specm}([s]) \setminus (M' \cup \{(s-a)\})$. Since $t(\pi(s)) = h(\pi_1(s))$, it factorizes in (recall that we are assuming $b = 0$) $H(\mathrm{Specm}([s]) \setminus (M' \cup \{(s-a)\}) = \mathrm{Specm}([s]) \setminus (H(M') \cup \{(s)\})$. So, $t(\pi(s))$ is a unit of $_{H(M')}(s)$.

Lemma 34 Let $a \in$ be an arbitrary element. Then

- If $M' \subseteq \text{Specm}([s]) \setminus \{(s-a)\}$ and $U(s) \in \text{Gl}_m\big(_{M'}(s)\big)$ then $U(f(s)) \in \text{Gl}_m\big(_{H(M')}(s)\big)$.
- If $U(s) \in \text{Gl}_m\big(_{s-a}(s)\big)$ then $U(f(s)) \in \text{Gl}_m\big(_{pr}(s)\big)$.
- If $U(s) \in \text{Gl}_m\big(_{pr}(s)\big)$ then $U(f(s)) \in \text{Gl}_m\big(_{s}(s)\big)$.
- If $(s-a) \in M' \subseteq \text{Specm}([s])$ and $U(s) \in \text{Gl}_m\big(_{M'}(s)\big)$ then the matrix $U(f(s)) \in \text{Gl}_m\big(_{H(M' \setminus \{(s-a)\})}(s)\big) \cap \text{Gl}_m\big(_{pr}(s)\big)$

Proof.- Let $\frac{p(s)}{q(s)}$ with $p(s),\ q(s) \in [s]$.

$$\frac{p(f(s))}{q(f(s))} = \frac{s^{d(p(s))}p(f(s))}{s^{d(q(s))}q(f(s))}s^{d(q(s))-d(p(s))} = \frac{t(p(s))}{t(q(s))}s^{d(q(s))-d(p(s))}. \tag{id140}$$

1. Assume that $U(s) \in \text{Gl}_m\big(_{M'}(s)\big)$ and let $\frac{p(s)}{q(s)}$ be any element of $U(s)$. Therefore $q(s)$ is a unit of $_{M'}(s)$ and, by Lemma ▭.2, $t(q(s))$ is a unit of $_{H(M')}(s)$. Moreover, s is also a unit of $_{H(M')}(s)$. Hence, $\frac{p(f(s))}{q(f(s))} \in {}_{H(M')}(s)$. Furthermore, if $\det U(s) = \frac{\tilde{p}(s)}{\tilde{q}(s)}$, it is a unit of $_{M'}(s)$ and $\det U(f(s)) = \frac{\tilde{p}(f(s))}{\tilde{q}(f(s))}$ is a unit of $_{H(M')}(s)$.

2. If $\frac{p(s)}{q(s)}$ is any element of $U(s) \in \text{Gl}_m\big(_{s-a}(s)\big)$ then $q(s) \in [s] \setminus (s-a)$ and so $d(h(q(s))) = d(q(s))$. Since $s-a$ may divide $p(s)$ we have that $d(t(p(s))) \le d(p(s))$. Hence, $d(h(q(s))) - d(q(s)) \ge d(t(p(s)) - d(p(s))$ and $\frac{p(f(s))}{q(f(s))} = \frac{t(p(s))}{h(q(s))}s^{d(q(s))-d(p(s))} \in {}_{pr}(s)$. Moreover if $\det U(s) = \frac{\tilde{p}(s)}{\tilde{q}(s)}$ then $\tilde{p}(s),\ \tilde{q}(s) \in [s] \setminus (s-a)$, $d(h(\tilde{p}(s))) = d(\tilde{p}(s))$ and $d(h(\tilde{q}(s))) = d(\tilde{q}(s))$. Thus, $\det U(f(s)) = \frac{h(\tilde{p}(s))}{h(\tilde{q}(s))}s^{d(\tilde{q}(s))-d(\tilde{p}(s))}$ is a biproper rational function, i.e., a unit of $_{pr}(s)$.

3. If $U(s) \in \text{Gl}_m\big(_{pr}(s)\big)$ and $\frac{p(s)}{q(s)}$ is any element of $U(s)$ then $d(q(s)) \ge d(p(s))$. Since $\frac{p(f(s))}{q(f(s))} = \frac{t(p(s))}{t(q(s))}s^{d(q(s))-d(p(s))}$ and $t(p(s)),\ t(q(s)) \in [s] \setminus (s)$ we obtain that $U(f(s)) \in {}_{s}(s)^{m\times m}$. In addition, if $\det U(s) = \frac{\tilde{p}(s)}{\tilde{q}(s)}$, which is a unit of $_{pr}(s)$, then $d(\tilde{q}(s)) = d(\tilde{p}(s))$ and since $t(\tilde{p}(s)),\ t(\tilde{q}(s)) \in [s] \setminus (s)$ we conclude that $\det U(f(s)) = \frac{t(\tilde{p}(s))}{t(\tilde{q}(s))}$ is a unit of $_{s}(s)$.

4. It is a consequence of 1., 2. and Remark ▭.2.

Proposition 35 Let $M \subseteq \text{Specm}([s])$ and $(s-a) \in M$. If $T(s) \in (s)^{m\times m}$ is non-singular with $\frac{n_i(s)}{d_i(s)} = (s-a)^{g_i}\frac{_i(s)}{\psi_i(s)}$ $\big(_i(s),\ \psi_i(s) \in [s] \setminus (s-a)\big)$ as invariant rational functions with respect to M then $T(f(s))^T \in (s)^{m\times m}$ is a non-singular matrix with $\frac{1}{c_i}\frac{h(_i(s))}{h(\psi_i(s))}s^{-g_i+d(\psi_i(s))-d(_i(s))}$ as invariant rational functions in $_{H(M \setminus \{(s-a)\})}(s)^{m\times m} \cap {}_{pr}(s)^{m\times m}$ where $c_i = \frac{_i(a)}{\psi_i(a)}$.

Proof.- Since $(s - a)^{g_i} \frac{{}_i(s)}{\psi(s)}$ are the invariant rational functions of $T(s)$ with respect to M, there are $U_1(s),\ U_2(s) \in Gl_m({}_M(s))$ such that

$$T(s) = U_1(s)\mathrm{Diag}\left((s - a)^{g_1} \frac{{}_1(s)}{\psi_1(s)},\ \ldots,\ (s - a)^{g_m} \frac{{}_m(s)}{\psi_m(s)}\right) U_2(s). \tag{id142}$$

Notice that $(f(s) - a)^{g_i} \frac{{}_i(f(s))}{\psi_i(f(s))} = \frac{h({}_i(s))}{h(\psi_i(s))} s^{-g_i + d(\psi_i(s)) - d({}_i(s))}$. Let $c_i = \frac{{}_i(a)}{\psi_i(a)}$, which is a non-zero constant, and put $D = \mathrm{Diag}(c_1,\ \ldots,\ c_m)$. Hence,

$$T(f(s))^T = U_2(f(s))^T DL\ (s) U_1(f(s))^T \tag{id143}$$

with

$$L\ (s) = \mathrm{Diag}\left(\frac{1}{c_1} \frac{h({}_1(s))}{h(\psi_1(s))} s^{-g_1 + d(\psi_1(s)) - d({}_1(s))},\ \ldots,\ \frac{1}{c_m} \frac{h({}_m(s))}{h(\psi_m(s))} s^{-g_m + d(\psi_m(s)) - d({}_m(s))}\right). \tag{id144}$$

By 4 of Lemma ▭ matrices $U_1(f(s))^T,\ U_2(f(s))^T \in \mathrm{Gl}_m({}_{H(M \setminus \{(s-a)\})}(s)) \cap \mathrm{Gl}_m({}_{pr}(s))$ and the Proposition follows.

Proposition 36 Let $M, M' \subseteq \mathrm{Specm}([s])$ such that $M \cup M' = \mathrm{Specm}([s])$. Assume that there are ideals in $M \setminus M'$ generated by linear polynomials and let $(s - a)$ be any of them. If $T(s) \in (s)^{m\times m}$ is a non-singular rational matrix with no poles and no zeros in $M \cap M'$ and $k_1,\ \ldots,\ k_m$ as left Wiener–Hopf factorization indices with respect to M then $T(f(s))^T \in (s)^{m\times m}$ is a non-singular rational matrix with no poles and no zeros in $H(M \cap M')$ and $-k_m,\ \ldots,\ -k_1$ as left Wiener–Hopf factorization indices with respect to $H(M') \cup \{(s)\}$.

Proof.- By Theorem ▭ there are matrices $U_1(s)$ invertible in ${}_M(s)^{m\times m} \cap {}_{pr}(s)^{m\times m}$ and $U_2(s)$ invertible in ${}_M(s)^{m\times m}$ such that $T(s) = U_1(s)\mathrm{Diag}((s - a)^{k_1},\ \ldots,\ (s - a)^{k_m})U_2(s)$. By Lemma ▭ $U_2(f(s))^T$ is invertible in ${}_{H(M \setminus \{(s-a)\})}(s)^{m\times m} \cap {}_{pr}(s)^{m\times m}$ and $U_1(f(s))^T$ is invertible in ${}_{H(M')}(s)^{m\times m} \cap {}_s(s)^{m\times m} = {}_{H(M') \cup \{(s)\}}(s)^{m\times m}$. Moreover, $H(M \setminus \{(s - a)\}) \cup H(M') \cup \{(s)\} = \mathrm{Specm}([s])$ and $H(M \setminus \{(s - a)\}) \cap (H(M') \cup \{(s)\}) = H(M \cap M')$. Thus, $T(f(s))^T = U_2(f(s))^T \mathrm{Diag}(s^{-k_1},\ \ldots,\ s^{-k_m}) U_1(f(s))^T$ has no poles and no zeros in $H(M \cap M')$ and $-k_m,\ \ldots,\ -k_1$ are its left Wiener–Hopf factorization indices with respect to $H(M') \cup \{(s)\}$.

5.2.1. Proof of Theorem : Sufficiency

Let $k_1 \geq \cdots \geq k_m$ be integers, $\frac{{}_1(s)}{\psi_1(s)}, \ldots, \frac{{}_m(s)}{\psi_m(s)}$ irreducible rational functions such that ${}_1(s) \mid \cdots \mid {}_m(s)$, $\psi_m(s) \mid \cdots \mid \psi_1(s)$ are monic polynomials factorizing in $M' \setminus M$ and $l_1, \ldots, l_m$ integers such that $l_1 + d(\psi_1(s)) - d({}_1(s)) \leq \cdots \leq l_m + d(\psi_m(s)) - d({}_m(s))$ and satisfying (▭).

Since ${}_i(s)$ and $\psi_i(s)$ are coprime polynomials that factorize in $M' \setminus M$ and $(s-a) \in M \setminus M'$, by Lemmas ▭ and ▭, $\frac{h({}_1(s))}{h(\psi_1(s))} s^{l_1 + d(\psi_1(s)) - d({}_1(s))}, \ldots, \frac{h({}_m(s))}{h(\psi_m(s))} s^{l_m + d(\psi_m(s)) - d({}_m(s))}$ are irreducible rational functions with numerators and denominators polynomials factorizing in $H(M') \cup \{(s)\}$ (actually, in $H(M' \setminus M) \cup \{(s)\}$) and such that each numerator divides the next one and each denominator divides the previous one.

By (▭) and Theorem ▭ there is a matrix $G(s) \in (s)^{m \times m}$ with $-k_m, \ldots, -k_1$ as left Wiener–Hopf factorization indices with respect to $H(M') \cup \{(s)\}$ and $\frac{1}{c_1} \frac{h({}_1(s))}{h(\psi_1(s))} s^{l_1 + d(\psi_1(s)) - d({}_1(s))}, \ldots, \frac{1}{c_m} \frac{h({}_m(s))}{h(\psi_m(s))} s^{l_m + d(\psi_m(s)) - d({}_m(s))}$ as invariant rational functions with respect to $H(M') \cup \{(s)\}$ where $c_i = \frac{{}_i(a)}{\psi_i(a)}$, $i = 1, \ldots, m$. Notice that $G(s)$ has no zeros and poles in $H(M \cap M')$ because the numerator and denominator of each rational function $\frac{h({}_i(s))}{h(\psi_i(s))} s^{l_i + d(\psi_i(s)) - d({}_i(s))}$ factorizes in $H(M' \setminus M) \cup \{(s)\}$ and so it is a unit of ${}_{H(M \cap M')}(s)$.

Put $\hat{M} = H(M') \cup \{(s)\}$ and $\hat{M}' = H(M \setminus \{(s-a)\})$. As remarked in the proof of Proposition ▭, $\hat{M} \cup \hat{M}' = \text{Specm}([s])$ and $\hat{M} \cap \hat{M}' = H(M \cap M')$. Now $(s) \in \hat{M}$ so that we can apply Proposition ▭ to $G(s)$ with the change of indeterminate $\tilde{f}(s) = \frac{1}{s-a}$. Thus the invariant rational functions of $G(\tilde{f}(s))^T$ in ${}_{M'}(s) \cap {}_{pr}(s)$ are $\frac{{}_1(s)}{\psi_1(s)} \frac{1}{(s-a)^{l_1}}, \ldots, \frac{{}_m(s)}{\psi_m(s)} \frac{1}{(s-a)^{l_m}}$.

On the other hand $\hat{M}' = H(M \setminus \{(s-a)\}) \subseteq \text{Specm}([s]) \setminus \{(s)\}$ and so $(s) \in \hat{M} \setminus \hat{M}'$. Then we can apply Proposition ▭ to $G(s)$ with $\tilde{f}(s) = \frac{1}{s-a}$ so that $G(\tilde{f}(s))^T$ is a non-singular matrix with no poles and no zeros in $H^{-1}(\hat{M} \cap \hat{M}') = H^{-1}(H(M \cap M')) = M \cap M'$ and $k_1, \ldots, k_m$ as left Wiener–Hopf factorization indices with respect to $H^{-1}(\hat{M}') \cup \{(s-a)\} = (M \setminus \{(s-a)\}) \cup \{(s-a)\} = M$. The theorem follows by letting $T(s) = G(\tilde{f}(s))^T$.

Remark 37 Notice that when $M' = \varnothing$ and $M = \text{Specm}([s])$ in Theorem ▭ we obtain Theorem ▭ ($q_i = -l_i$).

http://dx.doi.org/10.5772/81051

Author details

A. Amparan[1], S. Marcaida[1] and I. Zaballa[1]

1 Universidad del País Vasco/Euskal Herriko Unibertsitatea UPV/EHU,, Spain

References

[1] Amparan, A., Marcaida, S. & Zaballa, I. (2004). Wiener–hopf factorization indices and infinite structure of rational matrices, *SIAM J. Control Optim.* Vol. 42(No. 6): 2130–2144.

[2] Amparan, A., Marcaida, S. & Zaballa, I. (2006). On the existence of linear systems with prescribed invariants for system similarity, *Linear Algebra and its Applications* Vol. 413: 510–533.

[3] Amparan, A., Marcaida, S. & Zaballa, I. (2007). Local realizations and local polynomial matrix representations of systems, *Linear Algebra and its Applications* Vol. 425: 757–775.

[4] Amparan, A., Marcaida, S. & Zaballa, I. (2009). Local wiener–hopf factorization and indices over arbitrary fields, *Linear Algebra and its Applications* Vol. 430: 1700–1722.

[5] Atiyah, M. F. & MacDonald, I. G. (1969). *Introduction to commutative algebra*, Addison-Wesley.

[6] Clancey, K. & Gohberg, I. (1981). *Factorization of matrix functions and singular integral operators*, Birkhäuser Verlag, Basel, Boston, Stuttgart.

[7] Cullen, D. J. (1986). Local system equivalence, *Mathematical Systems Theory* 19: 67–78.

[8] Fuhrmann, P. & Helmke, U. (2001). On the parametrization of conditioned invariant subspaces and observer theory, *Linear Algebra and its Applications* 332–334: 265–353.

[9] Fuhrmann, P. & Willems, J. C. (1979). Factorization indices at infinity for rational matrix functions, *Integral Equations Operator Theory* 2/3: 287–301.

[10] Gohberg, I., Kaashoek, M. A. & van Schagen F. (1995). *Partially specified matrices and operators: classification, completion, applications*, Birkhäuser Verlag, Basel.

[11] Hardy, G. H., Littlewood, J. E. & G., P. (1967). *Inequalities*, Cambridge Univ. Press, Cambridge.

[12] Kailath, T. (1980). *Linear systems*, Prentice Hall, New Jersey.

[13] Newman, M. (1972). *Integral matrices*, Academic Press, New York and London.

[14] Rosenbrock, H. H. (1970). *State-space and multivariable theory*, Thomas Nelson and Sons, London.

[15] Vardulakis, A. I. G. (1991). *Linear multivariable control*, John Wiley and Sons, New York.

[16] Vidyasagar, M. (1985). *Control system synthesis. A factorization approach*, The MIT Press, New York.

[17] Wolovich, W. A. (1974). *Linear multivariable systems*, Springer-Verlag, New York.

[18] Zaballa, I. (1997). Controllability and hermite indices of matrix pairs, *Int. J. Control* 68(1): 61–86.