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# Algebraic Theory of Appell Polynomials with Application to General Linear Interpolation Problem

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Additional information is available at the end of the chapter

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## 1. Introduction

In 1880 P. E. Appell ([1]) introduced and widely studied sequences of $n$-degree polynomials

$$A_n(x), \quad n = 0, 1, \ldots \tag{id1}$$

satisfying the differential relation

$$DA_n(x) = nA_{n-1}(x), \quad n = 1, 2, \ldots \tag{id2}$$

Sequences of polynomials, verifying the ($\square$), nowadays called Appell polynomials, have been well studied because of their remarkable applications not only in different branches of mathematics ([2], [3]) but also in theoretical physics and chemistry ([4], [5]). In 1936 an initial bibliography was provided by Davis (p. 25[6]). In 1939 Sheffer ([7]) introduced a new class of polynomials which extends the class of Appell polynomials; he called these polynomials of type zero, but nowadays they are called Sheffer polynomials. Sheffer also noticed the similarities between Appell polynomials and the umbral calculus, introduced in the second half of the 19th century with the work of such mathematicians as Sylvester, Cayley and Blissard (for examples, see [8]). The Sheffer theory is mainly based on formal power series. In 1941 Steffensen ([9]) published a theory on Sheffer polynomials based on formal power series too. However, these theories were not suitable as they did not provide sufficient computational tools. Afterwards Mullin, Roman and Rota ([10], [11], [12]), using operators method, gave a beautiful theory of umbral calculus, including Sheffer polynomials. Recently, Di Bucchianico and Loeb ([13]) summarized and documented more than five hundred old and new findings related to Appell polynomial sequences. In last years attention has centered on finding a

novel representation of Appell polynomials. For instance, Lehemer ([14]) illustrated six different approaches to representing the sequence of Bernoulli polynomials, which is a special case of Appell polynomial sequences. Costabile ([15], [16]) also gave a new form of Bernoulli polynomials, called determinantal form, and later these ideas have been extended to Appell polynomial sequences. In fact, in 2010, Costabile and Longo ([17]) proposed an algebraic and elementary approach to Appell polynomial sequences. At the same time, Yang and Youn ([18]) also gave an algebraic approach, but with different methods. The approach to Appell polynomial sequences via linear algebra is an easily comprehensible mathematical tool, specially for non-specialists; that is very good because many polynomials arise in physics, chemistry and engineering. The present work concerns with these topics and it is organized as follows: in Section ▭ we mention the Appell method ([1]); in Section ▭ we provide the determinantal approach ([17]) and prove the equivalence with other definitions; in Section ▭ classical and non-classical examples are given; in Section ▭, by using elementary tools of linear algebra, general properties of Appell polynomials are provided; in Section ▭ we mention Appell polynomials of second kind ([19], [20]) and, in Section ▭ two classical examples are given; in Section ▭ we provide an application to general linear interpolation problem([21]), giving, in Section ▭, some examples; in Section ▭ the Yang and Youn approach ([18]) is sketched; finally, in Section ▭ conclusions close the work.

## 2. The Appell approach

Let $\{A_n(x)\}_n$ be a sequence of $n$-degree polynomials satisfying the differential relation (▭). Then we have

**Remark 1** There is a one-to-one correspondence of the set of such sequences $\{A_n(x)\}_n$ and the set of numerical sequences $\{\alpha_n\}_n$, $\alpha_0 \neq 0$ given by the explicit representation

$$A_n(x) = \alpha_n + \binom{n}{1}\alpha_{n-1}x + \binom{n}{2}\alpha_{n-2}x^2 + \cdots + \alpha_0 x^n, \quad n = 0, 1, \ldots \tag{id4}$$

Equation (▭), in particular, shows explicitly that for each $n \geq 1$ the polynomial $A_n(x)$ is completely determined by $A_{n-1}(x)$ and by the choice of the constant of integration $\alpha_n$.

**Remark 2** Given the formal power series

$$a(h) = \alpha_0 + \frac{h}{1!}\alpha_1 + \frac{h^2}{2!}\alpha_2 + \cdots + \frac{h^n}{n!}\alpha_n + \cdots, \quad \alpha_0 \neq 0, \tag{id6}$$

with $\alpha_i$ $i = 0, 1, \ldots$ real coefficients, the sequence of polynomials, $A_n(x)$, determined by the power series expansion of the product $a(h)e^{hx}$, i.e.

$$a(h)e^{hx} = A_0(x) + \frac{h}{1!}A_1(x) + \frac{h^2}{2!}A_2(x) + \cdots + \frac{h^n}{n!}A_n(x) + \cdots ,$$  (id7)

satisfies (□).

The function $a(h)$ is said, by Appell, 'generating function' of the sequence $\{A_n(x)\}_n$.

Appell also noticed various examples of sequences of polynomials verifying (□).

He also considered ([1]) an application of these polynomial sequences to linear differential equations, which is out of this context.

## 3. The determinantal approach

Let be $\beta_i \in \mathbb{R}$,    $i = 0, 1, ...,$  with $\beta_0 \neq 0$.

We give the following

**Definition 1** The polynomial sequence defined by

$$
\begin{cases}
A_0(x) = \dfrac{1}{\beta_0}, \\[4mm]
A_n(x) = \dfrac{(-1)^n}{(\beta_0)^{n+1}}
\begin{vmatrix}
1 & x & x^2 & \cdots & \cdots & x^{n-1} & x^n \\
\beta_0 & \beta_1 & \beta_2 & \cdots & \cdots & \beta_{n-1} & \beta_n \\
0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\
0 & 0 & \beta_0 & \cdots & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\
\vdots & & \ddots & & & \vdots & \vdots \\
\vdots & & & \ddots & & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & \beta_0 & \binom{n}{n-1}\beta_1
\end{vmatrix}, & n = 1, 2, ...
\end{cases}
$$  (id9)

is called Appell polynomial sequence for $\beta_i$.

Then we have

**Theorem 1** If $A_n(x)$ is the Appell polynomial sequence for $\beta_i$ the differential relation (□) holds.

Using the properties of linearity we can differentiate the determinant (□), expand the resulting determinant with respect to the first column and recognize the factor $A_{n-1}(x)$ after multiplication of the $i$-th row by $i - 1$,   $i = 2, ..., n$ and $j$-th column by $\frac{1}{j}$,   $j = 1, ..., n$.

**Theorem 2** If $A_n(x)$ is the Appell polynomial sequence for $\beta_i$ we have the equality ($\square$) with

$$\alpha_0 = \frac{1}{\beta_0},$$

$$\alpha_i = \frac{(-1)^i}{(\beta_0)^{i+1}} \begin{vmatrix} \beta_1 & \beta_2 & \cdots & \cdots & \beta_{i-1} & \beta_i \\ \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{i-1}{1}\beta_{i-2} & \binom{i}{1}\beta_{i-1} \\ 0 & \beta_0 & \cdots & \cdots & \binom{i-1}{2}\beta_{i-3} & \binom{i}{2}\beta_{i-2} \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \beta_0 & \binom{i}{i-1}\beta_1 \end{vmatrix} =$$

$$= -\frac{1}{\beta_0}\sum_{k=0}^{i-1}\binom{i}{k}\beta_{i-k}\alpha_k, \qquad i = 1, 2, \ldots, n.$$

(id12)

From ($\square$), by expanding the determinant $A_n(x)$ with respect to the first row, we obtain the ($\square$) with $\alpha_i$ given by ($\square$) and the determinantal form in ($\square$); this is a determinant of an upper Hessenberg matrix of order $i$ ([16]), then setting $\bar{\alpha}_i = (-1)^i(\beta_0)^{i+1}\alpha_i$ for $i = 1, 2, \ldots, n$, we have

$$\bar{\alpha}_i = \sum_{k=0}^{i-1}(-1)^{i-k-1}h_{k+1,i}q_k(i)\bar{\alpha}_k,$$

(id13)

where:

$$h_{l,m} = \begin{cases} \beta_m & \text{for } l = 1, \\ \binom{m}{l-1}\beta_{m-l+1} & \text{for } 1 < l \leq m+1, \quad l, m = 1, 2, \ldots, i, \\ 0 & \text{for } l > m+1, \end{cases}$$

(id14)

$$q_k(i) = \prod_{j=k+2}^{i} h_{j,j-1} = (\beta_0)^{i-k-1}, \quad k = 0, 1, \ldots, i-2,$$

$$q_{i-1}(i) = 1.$$

(id15)

By virtue of the previous setting, ($\square$) implies

$$\bar{\alpha}_i = \sum_{k=0}^{i-2}(-1)^{i-k-1}\binom{i}{k}\beta_{i-k}(\beta_0)^{i-k-1}\bar{\alpha}_k + \binom{i}{i-1}\beta_1\bar{\alpha}_{i-1} =$$

$$= (-1)^i(\beta_0)^{i+1}\left(-\frac{1}{\beta_0}\sum_{k=0}^{i-1}\binom{i}{k}\beta_{i-k}\alpha_k\right),$$

()

and the proof is concluded.

**Remark 3** We note that (□) and (□) are equivalent to

$$\sum_{k=0}^{i} \binom{i}{k} \beta_{i-k} \alpha_k = \begin{cases} 1 & i = 0 \\ 0 & i > 0 \end{cases} \tag{id17}$$

and that for each sequence of Appell polynomials there exist two sequences of numbers $\alpha_i$ and $\beta_i$ related by (□).

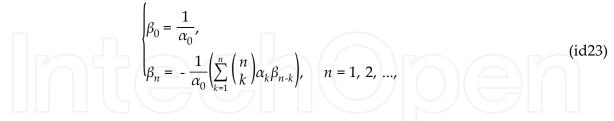**Corollary 1** If $A_n(x)$ is the Appell polynomial sequence for $\beta_i$ we have

$$A_n(x) = \sum_{j=0}^{n} \binom{n}{j} A_{n-j}(0) x^j, \quad n = 0, 1, \dots \tag{id19}$$

Follows from Theorem □ being

$$A_i(0) = \alpha_i, \quad i = 0, 1, \dots, n. \tag{id20}$$

**Remark 4** For computation we can observe that $\alpha_n$ is a $n$-order determinant of a particular upper Hessenberg form and it's known that the algorithm of Gaussian elimination without pivoting for computing the determinant of an upper Hessenberg matrix is stable (p. 27[22]).

**Theorem 3** If $a(h)$ is the function defined in (□) and $A_n(x)$ is the polynomial sequence defined by (□), setting

$$\begin{cases} \beta_0 = \dfrac{1}{\alpha_0}, \\ \beta_n = -\dfrac{1}{\alpha_0} \left( \sum_{k=1}^{n} \binom{n}{k} \alpha_k \beta_{n-k} \right), \quad n = 1, 2, \dots, \end{cases} \tag{id23}$$

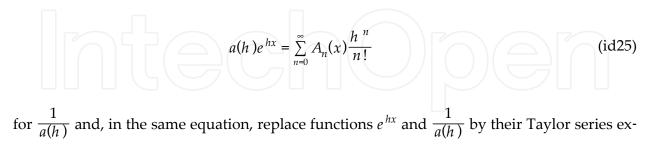we have that $A_n(x)$ satisfies the (□), i.e. $A_n(x)$ is the Appell polynomial sequence for $\beta_i$.

Let be

$$b(h) = \beta_0 + \frac{h}{1!} \beta_1 + \frac{h^2}{2!} \beta_2 + \dots + \frac{h^n}{n!} \beta_n + \dots \tag{id24}$$

with $\beta_n$ as in (□). Then we have $a(h)b(h) = 1$, where the product is intended in the Cauchy sense, i.e.:

$$a(h)b(h) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \alpha_k \beta_{n-k} \frac{h^n}{n!}.$$

()

Let us multiply both hand sides of equation

$$a(h)e^{hx} = \sum_{n=0}^{\infty} A_n(x) \frac{h^n}{n!}$$

(id25)

for $\frac{1}{a(h)}$ and, in the same equation, replace functions $e^{hx}$ and $\frac{1}{a(h)}$ by their Taylor series expansion at the origin; then (⬚) becomes

$$\sum_{n=0}^{\infty} \frac{x^n h^n}{n!} = \sum_{n=0}^{\infty} A_n(x) \frac{h^n}{n!} \sum_{n=0}^{\infty} \frac{h^n}{n!} \beta_n.$$

(id26)

By multiplying the series on the left hand side of (⬚) according to the Cauchy-product rules, previous equality leads to the following system of infinite equations in the unknown $A_n(x)$, $n = 0, 1, \ldots$

$$\left|\begin{array}{l} A_0(x)\beta_0 = 1, \\ A_0(x)\beta_1 + A_1(x)\beta_0 = x, \\ A_0(x)\beta_2 + \binom{2}{1} A_1(x)\beta_1 + A_2(x)\beta_0 = x^2, \\ \vdots \\ A_0(x)\beta_n + \binom{n}{1} A_1(x)\beta_{n-1} + \ldots + A_n(x)\beta_0 = x^n, \\ \vdots \end{array}\right.$$

(id27)

From the first one of (⬚) we obtain the first one of (⬚). Moreover, the special form of the previous system (lower triangular) allows us to work out the unknown $A_n(x)$ operating with the first $n + 1$ equations, only by applying the Cramer rule:

$$A_n(x) = \frac{1}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_0 & 0 & 0 & \cdots & 0 & 1 \\ \beta_1 & \beta_0 & 0 & \cdots & 0 & x \\ \beta_2 & \binom{2}{1}\beta_1 & \beta_0 & \cdots & 0 & x^2 \\ \vdots & & & \ddots & & \vdots \\ \beta_{n-1} & \binom{n-1}{1}\beta_{n-2} & \cdots & \cdots & \beta_0 & x^{n-1} \\ \beta_n & \binom{n}{1}\beta_{n-1} & \cdots & \cdots & \binom{n}{n-1}\beta_1 & x^n \end{vmatrix}. \qquad ()$$

By transposition of the previous, we have

$$A_n(x) = \frac{1}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \\ 1 & x & x^2 & \cdots & x^{n-1} & x^n \end{vmatrix}, \quad n = 1, 2, \ldots, \qquad (id28)$$

that is exactly the second one of ($\square$) after $n$ circular row exchanges: more precisely, the $i$-th row moves to the $(i+1)$-th position for $i = 1, \ldots, n-1$, the $n$-th row goes to the first position.

**Definition 2** The function $a(h)e^{hx}$, as in ($\square$) and ($\square$), is said 'generating function' of the Appell polynomial sequence $A_n(x)$ for $\beta_i$.

Theorems $\square$, $\square$, $\square$ concur to assert the validity of following

**Theorem 4 (Circular)** If $A_n(x)$ is the Appell polynomial sequence for $\beta_i$ we have

$$() \Rightarrow () \Rightarrow () \Rightarrow () \Rightarrow (). \qquad ()$$

**($\square$)$\Rightarrow$ ($\square$):($\square$)$\Rightarrow$ ($\square$):($\square$)$\Rightarrow$ ($\square$):($\square$)$\Rightarrow$ ($\square$):**

•

• Follows from Theorem $\square$.

• Follows from Theorem $\square$, or more simply by direct integration of the differential equation ($\square$).

- Follows ordering the Cauchy product of the developments $a(h)$ and $e^{hx}$ with respect to the powers of $h$ and recognizing polynomials $A_n(x)$, expressed in form (□), as coefficients of $\frac{h^n}{n!}$.

- Follows from Theorem □.

**Remark 5** In virtue of the Theorem □, any of the relations (□), (□), (□), (□) can be assumed as definition of Appell polynomial sequences.

## 4. Examples of Appell polynomial sequences

The following are classical examples of Appell polynomial sequences.

**a)b)c)d)**

- Bernoulli polynomials ([23], [17]):

$$\beta_i = \frac{1}{i+1}, \quad i = 0, 1, ..., \tag{id38}$$

$$a(h) = \frac{h}{e^h - 1}; \tag{id39}$$

- Euler polynomials ([23], [17]):

$$\beta_0 = 1, \quad \beta_i = \frac{1}{2}, \quad i = 1, 2, ..., \tag{id41}$$

$$a(h) = \frac{2}{e^h + 1}; \tag{id42}$$

- Normalized Hermite polynomials ([24], [17]):

$$\beta_i = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-x^2} x^i dx = \begin{cases} 0 & \text{for } i \text{ odd} \\ \dfrac{(i-1)(i-3) \cdot \, \cdots \, \cdot 3 \cdot 1}{2^{\frac{i}{2}}} & \text{for } i \text{ even} \end{cases}, \quad i = 0, 1, ..., \tag{id44}$$

$$a(h) = e^{-\frac{h^2}{4}}; \tag{id45}$$

- Laguerre polynomials ([24], [17]):

$$\beta_i = \int_0^{+\infty} e^{-x} x^i dx = \Gamma(i+1) = i!, \quad i = 0, 1, ..., \tag{id47}$$

$$a(h) = 1 - h; \qquad \text{(id48)}$$

The following are non-classical examples of Appell polynomial sequences.

**e)f)g)h)i)**

- Generalized Bernoulli polynomials

  ○ with Jacobi weight ([17]):

$$\beta_i = \int_0^1 (1 - x)^\alpha x^\beta x^i dx = \frac{\Gamma(\alpha + 1)\Gamma(\beta + i + 1)}{\Gamma(\alpha + \beta + i + 2)}, \qquad \alpha, \beta > -1, \quad i = 0, 1, \ldots, \qquad \text{(id51)}$$
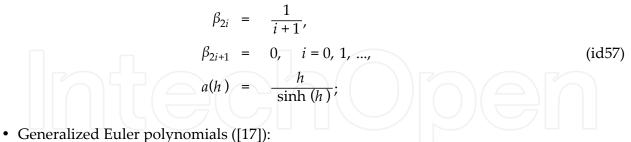
$$a(h) = \frac{1}{\int_0^1 (1 - x)^\alpha x^\beta e^{hx} dx}; \qquad \text{(id52)}$$

  ○ of order $k$ ([11]):

$$\beta_i = \left(\frac{1}{i + 1}\right)^k, \quad k \text{ integer}, \quad i = 0, 1, \ldots, \qquad \text{(id54)}$$

$$a(h) = \left(\frac{h}{e^h - 1}\right)^k; \qquad \text{(id55)}$$

- Central Bernoulli polynomials ([25]):

$$\begin{aligned}
\beta_{2i} &= \frac{1}{i + 1}, \\
\beta_{2i+1} &= 0, \quad i = 0, 1, \ldots, \\
a(h) &= \frac{h}{\sinh(h)};
\end{aligned} \qquad \text{(id57)}$$

- Generalized Euler polynomials ([17]):

$$\begin{aligned}
\beta_0 &= 1, \\
\beta_i &= \frac{w_1}{w_1 + w_2}, \quad w_1, w_2 > 0, \quad i = 1, 2, \ldots, \\
a(h) &= \frac{w_1 + w_2}{w_1 e^h + w_2};
\end{aligned} \qquad \text{(id59)}$$

- Generalized Hermite polynomials ([17]):

$$\beta_i = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-|x|^\alpha}x^i dx$$

$$= \begin{cases} 0 & \text{for } i \text{ odd} \\ \dfrac{2}{\alpha\sqrt{\pi}}\Gamma\!\left(\dfrac{i+1}{\alpha}\right) & \text{for } i \text{ even} \end{cases} \quad \begin{array}{l} i = 0,\,1,\,...,\\ \alpha > 0, \end{array} \tag{id61}$$

$$a(h) = \frac{\sqrt{\pi}}{\int_{-\infty}^{\infty}e^{-|x|^\alpha}e^{hx}dx};$$

- Generalized Laguerre polynomials ([17]):

$$\beta_i = \int_0^{+\infty}e^{-\alpha x}x^i dx$$

$$= \frac{\Gamma(i+1)}{\alpha^{i+1}} = \frac{i!}{\alpha^{i+1}}, \quad \alpha > 0, \quad i = 0,\,1,\,..., \tag{id63}$$

$$a(h) = \alpha - h.$$

# 5. General properties of Appell polynomials

By elementary tools of linear algebra we can prove the general properties of Appell polynomials.

Let $A_n(x)$, $n = 0,\,1,\,...,$ be a polynomial sequence and $\beta_i \in \mathbb{R}$, $\quad i = 0,\,1,\,...,$ with $\beta_0 \neq 0$.

**Theorem 5 (Recurrence)** $A_n(x)$ is the Appell polynomial sequence for $\beta_i$ if and only if

$$A_n(x) = \frac{1}{\beta_0}\left(x^n - \sum_{k=0}^{n-1}\binom{n}{k}\beta_{n-k}A_k(x)\right), \quad n = 1,\,2,\,... \tag{id65}$$

Follows observing that the following holds:

$$A_n(x) = \frac{(-1)^n}{(\beta_0)^{n+1}}\begin{vmatrix} 1 & x & x^2 & \cdots & \cdots & x^{n-1} & x^n \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix} = \tag{id66}$$

$$= \frac{1}{\beta_0}\left(x^n - \sum_{k=0}^{n-1}\binom{n}{k}\beta_{n-k}A_k(x)\right), \quad n = 1, 2, \dots \tag{id67}$$

In fact, if $A_n(x)$ is the Appell polynomial sequence for $\beta_i$, from ($\square$), we can observe that $A_n(x)$ is a determinant of an upper Hessenberg matrix of order $n+1$ ([16]) and, proceeding as in Theorem $\square$, we can obtain the ($\square$).

**Corollary 2** If $A_n(x)$ is the Appell polynomial sequence for $\beta_i$ then

$$x^n = \sum_{k=0}^{n}\binom{n}{k}\beta_{n-k}A_k(x), \quad n = 0, 1, \dots \tag{id69}$$

Follows from ($\square$).

**Corollary 3** Let $_n$ be the space of polynomials of degree $\leq n$ and $\{A_n(x)\}_n$ be an Appell polynomial sequence, then $\{A_n(x)\}_n$ is a basis for $_n$.

If we have

$$P_n(x) = \sum_{k=0}^{n} a_{n,k}x^k, \quad a_{n,k} \in \mathbb{R}, \tag{id71}$$

then, by Corollary $\square$, we get

$$P_n(x) = \sum_{k=0}^{n} a_{n,k}\sum_{j=0}^{k}\binom{k}{j}\beta_{k-j}A_j(x) = \sum_{k=0}^{n} c_{n,k}A_k(x), \tag{}$$

where

$$c_{n,k} = \sum_{j=0}^{n-k}\binom{k+j}{k}a_{k+j}\beta_j. \tag{id72}$$

**Remark 6** An alternative recurrence relation can be determined from ($\square$) after differentiation with respect to $h$ ([18], [26]).

Let be $\beta_i, \ \gamma_i \in \mathbb{R}, \ i = 0, 1, \dots,$ with $\beta_0, \ \gamma_0 \neq 0$.

Let us consider the Appell polynomial sequences $A_n(x)$ and $B_n(x), \ n = 0, 1, \dots,$ for $\beta_i$ and $\gamma_i$, respectively, and indicate with $(AB)_n(x)$ the polynomial that is obtained replacing in $A_n(x)$ the powers $x^0, x^1, \dots, x^n$, respectively, with the polynomials $B_0(x), B_1(x), \dots, B_n(x)$. Then we have

**Theorem 6** The sequences

*i) ii)*

- $\lambda A_n(x) + \mu B_n(x), \quad \lambda, \mu \in \mathbb{R},$

- $(AB)_n(x)$

are sequences of Appell polynomials again.

*i) ii)*

- Follows from the property of linearity of determinant.

- Expanding the determinant $(AB)_n(x)$ with respect to the first row we obtain

$$
\begin{aligned}
(AB)_n(x) &= \frac{(-1)^n}{(\beta_0)^{n+1}} \sum_{j=0}^{n} (-1)^j (\beta_0)^j \binom{n}{j} \bar{\alpha}_{n-j} B_j(x) = \\
&= \sum_{j=0}^{n} \frac{(-1)^{n-j}}{(\beta_0)^{n-j+1}} \binom{n}{j} \bar{\alpha}_{n-j} B_j(x),
\end{aligned}
\tag{id79}
$$

where

$$
\bar{\alpha}_0 = 1,
$$

$$
\bar{\alpha}_i = \begin{vmatrix}
\beta_1 & \beta_2 & \cdots & \cdots & \beta_{i-1} & \beta_i \\
\beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{i-1}{1}\beta_{i-2} & \binom{i}{1}\beta_{i-1} \\
0 & \beta_0 & \cdots & \cdots & \binom{i-1}{2}\beta_{i-3} & \binom{i}{2}\beta_{i-2} \\
\vdots & & \ddots & & \vdots & \vdots \\
\vdots & & \ddots & & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & \beta_0 & \binom{i}{i-1}\beta_1
\end{vmatrix}, \quad i = 1, 2, \ldots, n.
\tag{}
$$

We observe that

$$
A_i(0) = \frac{(-1)^i}{(\beta_0)^{i+1}} \bar{\alpha}_i, \quad i = 1, 2, \ldots, n
\tag{}
$$

and hence (□) becomes

$$
(AB)_n(x) = \sum_{j=0}^{n} \binom{n}{j} A_{n-j}(0) B_j(x).
\tag{id80}
$$

Differentiating both hand sides of (□) and since $B_j(x)$ is a sequence of Appell polynomials, we deduce

$$((AB)_n(x))' = n(AB)_{n-1}(x). \tag{id81}$$

Let us, now, introduce the Appell vector.

**Definition 3** If $A_n(x)$ is the Appell polynomial sequence for $\beta_i$ the vector of functions $\overline{A}_n(x) = [A_0(x), ..., A_n(x)]^T$ is called Appell vector for $\beta_i$.

Then we have

**Theorem 7 (Matrix form)** Let $\overline{A}_n(x)$ be a vector of polynomial functions. Then $\overline{A}_n(x)$ is the Appell vector for $\beta_i$ if and only if, putting

$$(M)_{i,j} = \begin{cases} \binom{i}{j}\beta_{i-j} & i \geq j \\ 0 & otherwise \end{cases}, \qquad i, j = 0, ..., n, \tag{id84}$$

and $X(x) = [1, x, ..., x^n]^T$ the following relation holds
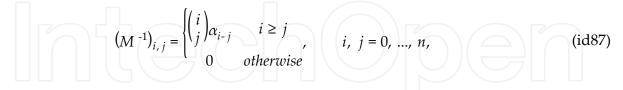
$$X(x) = M\overline{A}_n(x) \tag{id85}$$

or, equivalently,

$$\overline{A}_n(x) = (M^{-1})X(x), \tag{id86}$$

being $M^{-1}$ the inverse matrix of $M$.

If $\overline{A}_n(x)$ is the Appell vector for $\beta_i$ the result easily follows from Corollary □.

Vice versa, observing that the matrix $M$ defined by (□) is invertible, setting

$$(M^{-1})_{i,j} = \begin{cases} \binom{i}{j}\alpha_{i-j} & i \geq j \\ 0 & otherwise \end{cases}, \qquad i, j = 0, ..., n, \tag{id87}$$

we have the (□) and therefore the (□) and, being the coefficients $\alpha_k$ and $\beta_k$ related by (□), we have that $A_n(x)$ is the Appell polynomial sequence for $\beta_i$.

**Theorem 8 (Connection constants)** Let $\overline{A}_n(x)$ and $\overline{B}_n(x)$ be the Appell vectors for $\beta_i$ and $\gamma_i$, respectively. Then

$$\overline{A}_n(x) = C\overline{B}_n(x), \tag{id89}$$

where

$$(C)_{i,j} = \begin{cases} \binom{i}{j} c_{i-j} & i \geq j \\ 0 & otherwise \end{cases}, \qquad i, \ j = 0, \ ..., \ n. \tag{id90}$$

with

$$c_n = \sum_{k=0}^{n} \binom{n}{k} \alpha_{n-k} \gamma_k. \tag{id91}$$

From Theorem ▯ we have

$$X(x) = M \overline{A}_n(x) \tag{}$$

with $M$ as in (▯) or, equivalently,

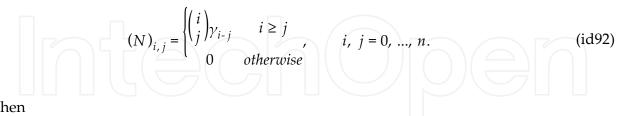$$\overline{A}_n(x) = (M^{-1}) X(x), \tag{}$$

with $M^{-1}$ as in (▯).

Always from Theorem ▯ we get

$$X(x) = N \overline{B}_n(x) \tag{}$$

with

$$(N)_{i,j} = \begin{cases} \binom{i}{j} \gamma_{i-j} & i \geq j \\ 0 & otherwise \end{cases}, \qquad i, \ j = 0, \ ..., \ n. \tag{id92}$$

Then

$$\overline{A}_n(x) = M^{-1} N \overline{B}_n(x), \tag{}$$

from which, setting $C = M^{-1}N$, we have the thesis.

**Theorem 9 (Inverse relations)** Let $A_n(x)$ be the Appell polynomial sequence for $\beta_i$ then the following are inverse relations:

$$
\begin{cases}
y_n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} \beta_{n-k} x_k \\[2mm]
x_n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} A_{n-k}(0) y_k.
\end{cases}
\tag{id94}
$$

Let us remember that

$$
A_k(0) = \alpha_k,
\tag{}
$$

where the coefficients $\alpha_k$ and $\beta_k$ are related by ($\square$).

Moreover, setting $\bar{y}_n = [y_0, \ldots, y_n]^T$ and $\bar{x}_n = [x_0, \ldots, x_n]^T$, from ($\square$) we have

$$
\begin{cases}
\bar{y}_n = M_1 \bar{x}_n \\
\bar{x}_n = M_2 \bar{y}_n
\end{cases}
\tag{}
$$

with

$$
(M_1)_{i,j} = \begin{cases} \binom{i}{j} \beta_{i-j} & i \geq j \\[2mm] 0 & otherwise \end{cases}, \qquad i, j = 0, \ldots, n,
\tag{id95}
$$

$$
(M_2)_{i,j} = \begin{cases} \binom{i}{j} \alpha_{i-j} & i \geq j \\[2mm] 0 & otherwise \end{cases}, \qquad i, j = 0, \ldots, n,
\tag{id96}
$$

and, from ($\square$) we get

$$
M_1 M_2 = I_{n+1},
\tag{}
$$

i.e. ($\square$) are inverse relations.

**Theorem 10 (Inverse relation between two Appell polynomial sequences)** Let $\overline{A}_n(x)$ and $\overline{B}_n(x)$ be the Appell vectors for $\beta_i$ and $\gamma_i$, respectively. Then the following are inverse relations:

$$
\begin{cases}
\overline{A}_n(x) = C \overline{B}_n(x) \\
\overline{B}_n(x) = \widetilde{C} \overline{A}_n(x)
\end{cases}
\tag{id98}
$$

with

$$(C)_{i,j} = \begin{cases} \binom{i}{j} c_{i-j} & i \geq j \\ 0 & \text{otherwise} \end{cases}, \qquad (\widetilde{C})_{i,j} = \begin{cases} \binom{i}{j} \tilde{c}_{i-j} & i \geq j \\ 0 & \text{otherwise} \end{cases}, \qquad i, j = 0, \ldots, n, \tag{id99}$$

$$c_n = \sum_{k=0}^{n} \binom{n}{k} A_{n-k}(0) \gamma_k, \qquad \tilde{c}_n = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(0) \beta_k. \tag{id100}$$

Follows from Theorem □, after observing that

$$\sum_{k=0}^{n} \binom{n}{k} c_{n-k} \tilde{c}_k = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases} \tag{id101}$$

and therefore

$$C\widetilde{C} = I_{n+1}. \tag{}$$

**Theorem 11 (Binomial identity)** If $A_n(x)$ is the Appell polynomial sequence for $\beta_i$ we have

$$A_n(x + y) = \sum_{i=0}^{n} \binom{n}{i} A_i(x) y^{n-i}, \qquad n = 0, 1, \ldots \tag{id103}$$

Starting by the Definition □ and using the identity

$$(x + y)^i = \sum_{k=0}^{i} \binom{i}{k} y^k x^{i-k}, \tag{id104}$$

we infer

$$A_n(x + y) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & (x+y)^1 & \cdots & (x+y)^{n-1} & (x+y)^n \\ \beta_0 & \beta_1 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & \beta_0 & \beta_1 \binom{n}{n-1} \end{vmatrix} = \tag{}$$

$$= \sum_{i=0}^{n} y^{i} \frac{(-1)^{n-i}}{(\beta_0)^{n-i+1}} \begin{vmatrix} \binom{i}{i} & \binom{i+1}{i}x^1 & \binom{i+2}{i}x^2 & \cdots & \binom{n-1}{i}x^{n-i-1} & \binom{n}{i}x^{n-i} \\ \beta_0 & \beta_1\binom{i+1}{i} & \beta_2\binom{i+2}{i} & \cdots & \beta_{n-i-1}\binom{n-1}{i} & \beta_{n-i}\binom{n}{i} \\ 0 & \beta_0 & \beta_1\binom{i+2}{i+1} & \cdots & \beta_{n-i-2}\binom{n-1}{i+1} & \beta_{n-i-1}\binom{n}{i+1} \\ \vdots & & \beta_0 & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & \beta_0 & \beta_1\binom{n}{n-1} \end{vmatrix}. \qquad ()$$

We divide, now, each $j$ - th column, $j = 2, ..., n - i + 1$, for $\binom{i+j-1}{i}$ and multiply each $h$ - th row, $h = 3, ..., n - i + 1$, for $\binom{i+h-2}{i}$. Thus we finally obtain

$$A_n(x + y) =$$

$$= \sum_{i=0}^{n} \frac{\binom{i+1}{i} \cdots \binom{n}{i}}{\binom{i+1}{i} \cdots \binom{n-1}{i}} y^{i} \frac{(-1)^{n-i}}{(\beta_0)^{n-i+1}}$$

$$\begin{vmatrix} 1 & x^1 & x^2 & \cdots & x^{n-i-1} & x^{n-i} \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-i-1} & \beta_{n-i} \\ 0 & \beta_0 & \beta_1\binom{2}{1} & \cdots & \beta_{n-i-2}\binom{n-i-1}{1} & \beta_{n-i-1}\binom{n-i}{1} \\ \vdots & & \beta_0 & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & \beta_0 & \beta_1\binom{n-i}{n-i-1} \end{vmatrix} = \qquad ()$$

$$= \sum_{i=0}^{n} \binom{n}{i} A_{n-i}(x) y^{i} = \sum_{i=0}^{n} \binom{n}{i} A_{i}(x) y^{n-i}.$$

**Theorem 12 (Generalized Appell identity)** Let $A_n(x)$ and $B_n(x)$ be the Appell polynomial sequences for $\beta_i$ and $\gamma_i$, respectively. Then, if $C_n(x)$ is the Appell polynomial sequence for $\delta_i$ with

$$\begin{cases} \delta_0 = \dfrac{1}{C_0(0)}, \\ \delta_i = -\dfrac{1}{C_0(0)} \sum_{k=1}^{i} \binom{i}{k} \delta_{i-k} C_k(0), \quad i = 1, ..., \end{cases} \qquad \text{(id106)}$$

and

$$C_i(0) = \sum_{j=0}^{i} \binom{i}{j} B_{i-j}(0) A_j(0),$$  (id107)

where $A_i(0)$ and $B_i(0)$ are related to $\beta_i$ and $\gamma_i$, respectively, by relations similar to ($\square$), we have

$$C_n(y+z) = \sum_{k=0}^{n} \binom{n}{k} A_k(y) B_{n-k}(z).$$  (id108)

Starting from ($\square$) we have

$$C_n(y+z) = \sum_{k=0}^{n} \binom{n}{k} C_{n-k}(0)(y+z)^k.$$  (id109)

Then, applying ($\square$) and the well-known classical binomial identity, after some calculation, we obtain the thesis.

**Theorem 13 (Combinatorial identities)** Let $A_n(x)$ and $B_n(x)$ be the Appell polynomial sequences for $\beta_i$ and $\gamma_i$, respectively. Then the following relations holds:

$$\sum_{k=0}^{n} \binom{n}{k} A_k(x) B_{n-k}(-x) = \sum_{k=0}^{n} \binom{n}{k} A_k(0) B_{n-k}(0),$$  (id111)

$$\sum_{k=0}^{n} \binom{n}{k} A_k(x) B_{n-k}(z) = \sum_{k=0}^{n} \binom{n}{k} A_k(x+z) B_{n-k}(0).$$  (id112)

If $C_n(x)$ is the Appell polynomial sequence for $\delta_i$ defined as in ($\square$), from the generalized Appell identity, we have

$$\sum_{k=0}^{n} \binom{n}{k} A_k(x) B_{n-k}(-x) = C_n(0) = \sum_{k=0}^{n} \binom{n}{k} A_k(0) B_{n-k}(0)$$  ()

and

$$\sum_{k=0}^{n} \binom{n}{k} A_k(x) B_{n-k}(z) = C_n(x+z) = \sum_{k=0}^{n} \binom{n}{k} A_k(x+z) B_{n-k}(0).$$  ()

**Theorem 14 (Forward difference)** If $A_n(x)$ is the Appell polynomial sequence for $\beta_i$ we have

$$\Delta A_n(x) \equiv A_n(x+1) - A_n(x) = \sum_{i=0}^{n-1} \binom{n}{i} A_i(x), \quad n = 0, 1, \ldots \tag{id114}$$

The desired result follows from ($\square$) with $y = 1$.

**Theorem 15 (Multiplication Theorem)** Let $\overline{A}_n(x)$ be the Appell vector for $\beta_i$.

The following identities hold:

$$\overline{A}_n(mx) = B(x)\overline{A}_n(x) \qquad n = 0, 1, \ldots, \qquad m = 1, 2, \ldots, \tag{id116}$$

$$\overline{A}_n(mx) = M^{-1}DX(x) \qquad n = 0, 1, \ldots, \qquad m = 1, 2, \ldots, \tag{id117}$$

where

$$(B(x))_{i,j} = \begin{cases} \binom{i}{j}(m-1)^{i-j}x^{i-j} & i \geq j \\ 0 & otherwise \end{cases}, \qquad i, j = 0, \ldots, n, \tag{id118}$$

$D = diag[1, m, \ldots, m^n]$ and $M^{-1}$ defined as in ($\square$).

The ($\square$) follows from ($\square$) setting $y = x(m-1)$. In fact we get

$$A_n(mx) = \sum_{i=0}^{n} \binom{n}{i} A_i(x)(m-1)^{n-i}x^{n-i}. \tag{id119}$$

The ($\square$) follows from Theorem $\square$. In fact we get

$$\overline{A}_n(mx) = M^{-1}X(mx) = M^{-1}DX(x), \tag{id120}$$

and

$$A_n(mx) = \sum_{i=0}^{n} \binom{n}{i} \alpha_{n-i} m^i x^i. \tag{id121}$$

**Theorem 16 (Differential equation)** If $A_n(x)$ is the Appell polynomial sequence for $\beta_i$ then $A_n(x)$ satisfies the linear differential equation:

$$\frac{\beta_n}{n!} y^{(n)}(x) + \frac{\beta_{n-1}}{(n-1)!} y^{(n-1)}(x) + \ldots + \frac{\beta_2}{2!} y^{(2)}(x) + \beta_1 y^{(1)}(x) + \beta_0 y(x) = x^n \tag{id123}$$

From Theorem $\square$ we have

$$A_{n+1}(x) = \frac{1}{\beta_0}\left(x^{n+1} - \sum_{k=0}^{n}\binom{n+1}{k+1}\beta_{k+1}A_{n-k}(x)\right). \tag{id124}$$

From Theorem ▭ we find that

$$A'_{n+1}(x) = (n+1)A_n(x), \quad \text{and} \quad A_{n-k}(x) = \frac{A_n^{(k)}(x)}{n(n-1)...(n-k+1)}, \tag{id125}$$

and replacing $A_{n-k}(x)$ in the (▭) we obtain

$$A_{n+1}(x) = \frac{1}{\beta_0}\left(x^{n+1} - (n+1)\sum_{k=0}^{n}\beta_{k+1}\frac{A_n^{(k)}(x)}{(k+1)!}\right). \tag{id126}$$

Differentiating both hand sides of the last one and replacing $A'_{n+1}(x)$ with $(n+1)A_n(x)$, after some calculation we obtain the thesis.

**Remark 7** An alternative differential equation for Appell polynomial sequences can be determined by the recurrence relation referred to in Remark ▭ ([18], [26]).

## 6. Appell polynomial sequences of second kind

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ and $\Delta$ be the finite difference operator ([23]), i.e.:

$$\Delta[f](x) = f(x+1) - f(x), \tag{id128}$$

we define the finite difference operator of order $i$, with $i \in \mathbb{N}$, as

$$\Delta^i[f](x) = \Delta(\Delta^{i-1}[f](x)) = \sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}f(x+j), \tag{id129}$$

meaning $\Delta^0 = I$ and $\Delta^1 = \Delta$, where $I$ is the identity operator.

Let the sequence of falling factorial defined by

$$\begin{cases} (x)_0 = 1, \\ (x)_n = x(x-1)(x-2)\cdots(x-n+1), \quad n = 1, 2, ..., \end{cases} \tag{id130}$$

we give the following

**Definition 4** Let $i \in \mathbb{R}$, $i = 0, 1, ...,$ with $0 \neq 0$. The polynomial sequence

$$\left\{\begin{array}{l} {}_0(x) = \dfrac{1}{0}, \\[2em] {}_n(x) = \dfrac{(-1)^n}{(0)^{n+1}} \begin{vmatrix} 1 & (x)_1 & (x)_2 & \cdots & \cdots & (x)_{n-1} & (x)_n \\ 0 & 1 & 2 & \cdots & \cdots & n-1 & n \\ 0 & 0 & \binom{2}{1}1 & \cdots & \cdots & \binom{n-1}{1}_{n-2} & \binom{n}{1}_{n-1} \\ 0 & 0 & 0 & \cdots & \cdots & \binom{n-1}{2}_{n-3} & \binom{n}{2}_{n-2} \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \binom{n}{n-1}1 \end{vmatrix}, \quad n = 1, 2, \ldots \end{array}\right.$$

$$\text{(id132)}$$

is called Appell polynomial sequence of second kind.

Then, we have

**Theorem 17** For Appell polynomial sequences of second kind we get

$$\Delta {}_n(x) = n {}_{n-1}(x) \quad n = 1, 2, \ldots \tag{id134}$$

By the well-known relation ([23])

$$\Delta(x)_n = n(x)_{n-1}, \qquad n = 1, 2, \ldots, \tag{id135}$$

applying the operator $\Delta$ to the definition ($\square$) and using the properties of linearity of $\Delta$ we have

$$\Delta {}_n(x) = \dfrac{(-1)^n}{(0)^{n+1}} \begin{vmatrix} \Delta 1 & \Delta(x)_1 & \Delta(x)_2 & \cdots & \cdots & \Delta(x)_{n-1} & \Delta(x)_n \\ 0 & 1 & 2 & \cdots & \cdots & n-1 & n \\ 0 & 0 & \binom{2}{1}_1 & \cdots & \cdots & \binom{n-1}{1}_{n-2} & \binom{n}{1}_{n-1} \\ 0 & 0 & 0 & \cdots & \cdots & \binom{n-1}{2}_{n-3} & \binom{n}{2}_{n-2} \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \binom{n}{n-1}_1 \end{vmatrix}, \quad n = 1, 2, \ldots \tag{id136}$$

We can expand the determinant in (□) with respect to the first column and, after multiplying the $i$-th row by $i - 1$, $i = 2, ..., n$ and the $j$-th column by $\frac{1}{j}$, $j = 1, ..., n$, we can recognize the factor $_{n-1}(x)$.

We can observe that the structure of the determinant in (□) is similar to that one of the determinant in (□). In virtue of this it is possible to obtain a dual theory of Appell polynomials of first kind, in the sense that similar properties can be proven ([20]).

For example, the generating function is

$$H(x, h) = a(h)(1 + h)^x,$$
(id137)

where $a(h)$ is an invertible formal series of power.

# 7. Examples of Appell polynomial sequences of second kind

The following are classical examples of Appell polynomial sequences of second kind.

**a)b)**

- Bernoulli polynomials of second kind ([23], [20]):

$$_i = \frac{(-1)^i}{i+1} i!, \quad i = 0, 1, ...,$$
(id139)

$$H(x, h) = \frac{h(1 + h)^x}{ln(1 + h)};$$
(id140)

- Boole polynomials ([23], [20]):

$$_i = \begin{vmatrix} 1, & i = 0 \\ \frac{1}{2}, & i = 1 \\ 0, & i = 2, ... \end{vmatrix}$$
(id142)

$$H(x, h) = \frac{2(1 + h)^x}{2 + h}.$$
(id143)

## 8. An application to general linear interpolation problem

Let $X$ be the linear space of real functions defined in the interval $[0, 1]$ continuous and with continuous derivatives of all necessary orders. Let $L$ be a linear functional on $X$ such that $L(1) \neq 0$. If in $(\Box)$ and respectively in $(\Box)$ we set

$$\beta_i = L(x^i), \qquad _i = L((x)_i), \quad i = 0, 1, ..., \tag{id144}$$

$A_n(x)$ and $_n(x)$ will be said Appell polynomial sequences of first or of second kind related to the functional $L$ and denoted by $A_{L,n}(x)$ and $_{L,n}(x)$, respectively.

**Remark 8** The generating function of the sequence $A_{L,n}(x)$ is

$$G(x, h) = \frac{e^{xh}}{L_x(e^{xh})}, \tag{id146}$$

and for $_{L,n}(x)$ is

$$H(x, h) = \frac{(1 + h)^x}{L_x((1 + h)^x)}, \tag{id147}$$

where $L_x$ means that the functional $L$ is applied to the argument as a function of $x$.

For $A_{L,n}(x)$ if $G(x, h) = a(h)e^{xh}$ with $\frac{1}{a(h)} = \sum_{i=0}^{\infty} \beta_i \frac{h^i}{i!}$ we have

$$G(x, t) = \frac{e^{xh}}{\frac{1}{a(h)}} = \frac{e^{xh}}{\sum_{i=0}^{\infty} \beta_i \frac{h^i}{i!}} = \frac{e^{xh}}{\sum_{i=0}^{\infty} L(x^i)\frac{h^i}{i!}} = \frac{e^{xh}}{L\left(\sum_{i=0}^{\infty} x^i \frac{h^i}{i!}\right)} = \frac{e^{xh}}{L_x(e^{xh})}. \tag{}$$

For $_{L,n}(x)$, the proof similarly follows. Then, we have

**Theorem 18** Let $\omega_i \in \mathbb{R}$, $i = 0, ..., n$, the polynomials

$$P_n(x) = \sum_{i=0}^{n} \frac{\omega_i}{i!} A_{L,i}(x), \tag{id149}$$

$$P_n^*(x) = \sum_{i=0}^{n} \frac{\omega_i}{i!} {}_{L,i}(x) \tag{id150}$$

are the unique polynomials of degree less than or equal to $n$, such that

$$L\left(P_n^{(i)}\right) = i! \, \omega_i, \quad i = 0, ..., n, \tag{id151}$$

$$L\left(\Delta^i P_n^*\right) = i! \, \omega_i, \quad i = 0, ..., n. \tag{id152}$$

The proof follows observing that, by the hypothesis on functional $L$ there exists a unique polynomial of degree $\leq n$ verifying ($\square$) and , respectively, ($\square$); moreover from the properties of $A_{L,i}(x)$ and $_{L,i}(x)$, we have

$$L\left(A_{L,i}^{(j)}(x)\right) = i(i-1)...(i-j+1)L\left(A_{L,i-j}(x)\right) = j! \binom{i}{j}\delta_{ij}, \tag{id153}$$

$$L\left(\Delta^i{}_{L,i}(x)\right) = i(i-1)...(i-j+1)L\left(_{L,i-j}(x)\right) = j! \binom{i}{j}\delta_{ij}, \tag{id154}$$

where $\delta_{ij}$ is the Kronecker symbol.

From ($\square$) and ($\square$) it is easy to prove that the polynomials ($\square$) and ($\square$) verify ($\square$) and ($\square$), respectively.

**Remark 9** For every linear functional $L$ on $X$, $\{A_{L,i}(x)\}$, $\{_{L,i}(x)\}$, $i = 0, ..., n$, are basis for $_n$ and, $\forall P_n(x) \in {}_n$, we have

$$P_n(x) = \sum_{i=0}^{n} \frac{L\left(P_n^{(i)}\right)}{i!} A_{L,i}(x), \tag{id156}$$

$$P_n(x) = \sum_{i=0}^{n} \frac{L\left(\Delta^i P_n\right)}{i!} {}_{L,i}(x). \tag{id157}$$

Let us consider a function $f \in X$. Then we have the following

**Theorem 19** The polynomials

$$P_{L,n}[f](x) = \sum_{i=0}^{n} \frac{L\left(f^{(i)}\right)}{i!} A_{L,i}(x), \tag{id159}$$

$$P_{L,n}^*[f](x) = \sum_{i=0}^{n} \frac{L\left(\Delta^i f\right)}{i!} {}_{L,i}(x) \tag{id160}$$

are the unique polynomial of degree $\leq n$ such that

$$L\left(P_{L,n}[f]^{(i)}\right) = L\left(f^{(i)}\right), \quad i = 0, ..., n, \tag{}$$

$$L\left(\Delta^i P^*_{L,n}[f]\right) = L\left(\Delta^i f\right), \quad i = 0, \ldots, n. \tag{}$$

Setting $\omega_i = \dfrac{L\left(f^{(i)}\right)}{i!}$, and respectively, $\omega_i = \dfrac{L\left(\Delta^i f\right)}{i!}$, $i = 0, \ldots, n$, the result follows from Theorem □.

**Definition 5** The polynomials (□) and (□) are called Appell interpolation polynomial for $f$ of first and of second kind, respectively.

Now it is interesting to consider the estimation of the remainders

$$R_{L,n}[f](x) = f(x) - P_{L,n}[f](x), \quad \forall\, x \in [0, 1], \tag{id162}$$

$$R^*_{L,n}[f](x) = f(x) - P^*_{L,n}[f](x), \quad \forall\, x \in [0, 1]. \tag{id163}$$

**Remark 10** For any $f \in \,_n$

$$R_{L,n}[f](x) = 0, \quad R_{L,n}[x^{n+1}] \neq 0, \quad \forall\, x \in [0, 1], \tag{id165}$$

$$R^*_{L,n}[f](x) = 0, \quad R^*_{L,n}[(x)_{n+1}] \neq 0, \quad \forall\, x \in [0, 1], \tag{id166}$$

i. e. the polynomial operators (□) and (□) are exact on $_n$.

For a fixed $x$ we may consider the remainder $R_{L,n}[f](x)$ and $R^*_{L,n}[f](x)$ as linear functionals which act on $f$ and annihilate all elements of $_n$. From Peano's Theorem (p. 69[27]) if a linear functional has this property, then it must also have a simple representation in terms of $f^{(n+1)}$. Therefore we have

**Theorem 20** Let $f \in C^{n+1}[a, b]$, the following relations hold

$$R_{L,n}(f, x) = \frac{1}{n!}\int_0^1 K_n(x, t) f^{(n+1)}(t)\,dt, \quad \forall\, x \in [0, 1], \tag{id168}$$

$$R^*_{L,n}(f, x) = \frac{1}{n!}\int_0^1 K^*_n(x, t) f^{(n+1)}(t)\,dt, \quad \forall\, x \in [0, 1], \tag{id169}$$

where

$$K_n(x, t) = R_{L,n}[(x - t)^n_+] = (x - t)^n_+ - \sum_{i=0}^n \binom{n}{i} L\left((x - t)^{n-i}_+\right) A_{L,i}(x), \tag{id170}$$

$$K_n^*(x, t) = R_{L,n}^*\big[(x - t)_+^n\big] = (x - t)_+^n - \sum_{i=0}^{n} \frac{L\left(\Delta^i(x - t)_+^n\right)}{i!} \, L_{,i}(x). \tag{id171}$$

After some calculation, the results follow by Remark ☐ and Peano's Theorem.

**Remark 11 (Bounds)** If $f^{(n+1)} \in \mathcal{L}^p[0, 1]$ and $K_n(x, t)$, $K_n^*(x, t) \in \mathcal{L}^q[0, 1]$ with $\frac{1}{p} + \frac{1}{q} = 1$ then we apply the Hölder's inequality so that

$$|R_{L,n}[f](x)| \le \frac{1}{n!}\Big(\int_0^1 |K_n(x, t)|^q dt\Big)^{\frac{1}{q}}\Big(\int_0^1 |f^{(n+1)}(t)|^p dt\Big)^{\frac{1}{p}}, \tag{}$$

$$|R_{L,n}^*[f](x)| \le \frac{1}{n!}\Big(\int_0^1 |K_n^*(x, t)|^q dt\Big)^{\frac{1}{q}}\Big(\int_0^1 |f^{(n+1)}(t)|^p dt\Big)^{\frac{1}{p}}. \tag{}$$

The two most important cases are $p = q = 2$ and $q = 1$, $p = \infty$ :

**i)ii)**

• for $p = q = 2$ we have the estimates

$$|R_{L,n}[f](x)| \le \sigma_n |||f|||, \quad |R_{L,n}^*[f](x)| \le \sigma_n^* |||f|||, \tag{id174}$$

where

$$(\sigma_n)^2 = \Big(\frac{1}{n!}\Big)^2 \int_0^1 (K_n(x, t))^2 dt, \quad (\sigma_n^*)^2 = \Big(\frac{1}{n!}\Big)^2 \int_0^1 (K_n^*(x, t))^2 dt, \tag{id175}$$

and

$$|||f|||^2 = \int_0^1 (f^{(n+1)}(t))^2 dt; \tag{id176}$$

• for $q = 1$, $p = \infty$ we have that

$$|R_{L,n}[f](x)| \le \frac{1}{n!} M_{n+1} \int_0^1 |K_n(x, t)| dt, \quad |R_{L,n}^*[f](x)| \le \frac{1}{n!} M_{n+1} \int_0^1 |K_n^*(x, t)| dt, \tag{id178}$$

where

$$M_{n+1} = \sup_{a \le x \le b} |f^{(n+1)}(x)|. \tag{id179}$$

A further polynomial operator can be determined as follows:

for any fixed $z \in [0, 1]$ we consider the polynomial

$$\overline{P}_{L\ ,n}[f](x) \equiv f(z) + P_{L\ ,n}[f](x) - P_{L\ ,n}[f](z) = f(z) + \sum_{i=1}^{n} \frac{L\left(f^{(i)}\right)}{i!}(A_{L\ ,i}(x) - A_{L\ ,i}(z)), \quad \text{(id180)}$$

and, respectively,

$$\overline{P}^{*}_{L\ ,n}[f](x) \equiv f(z) + P^{*}_{L\ ,n}[f](x) - P^{*}_{L\ ,n}[f](z) = f(z) + \sum_{i=1}^{n} \frac{L\left(\Delta^{i}f\right)}{i!}(_{L\ ,i}(x) - _{L\ ,i}(z)). \quad \text{(id181)}$$

Then we have the following

**Theorem 21** The polynomials $\overline{P}_{L\ ,n}[f](x)$, $\overline{P}^{*}_{L\ ,n}[f](x)$ are approximating polynomials of degree $n$ for $f(x)$, i.e.:

$$\forall\ x \in [0,\ 1], \qquad f(x) = \overline{P}_{L\ ,n}[f](x) + \overline{R}_{L\ ,n}[f](x), \qquad \text{(id183)}$$

$$f(x) = \overline{P}^{*}_{L\ ,n}[f](x) + \overline{R}^{*}_{L\ ,n}[f](x), \qquad \text{(id184)}$$

where

$$\overline{R}_{L\ ,n}[f](x) = R_{L\ ,n}[f](x) - R_{L\ ,n}[f](z), \qquad \text{(id185)}$$

$$\overline{R}^{*}_{L\ ,n}[f](x) = R^{*}_{L\ ,n}[f](x) - R^{*}_{L\ ,n}[f](z), \qquad \text{(id186)}$$

with

$$\overline{R}_{L\ ,n}[x^{i}] = 0, \quad i = 0,\ ..,\ n, \quad \overline{R}_{L\ ,n}[x^{n+1}] \neq 0, \qquad \text{(id187)}$$

$$\overline{R}^{*}_{L\ ,n}[(x)_{i}] = 0, \quad i = 0,\ ..,\ n, \quad \overline{R}^{*}_{L\ ,n}[(x)_{n+1}] \neq 0. \qquad \text{(id188)}$$

$\forall\ x \in [0,\ 1]$ and for any fixed $z \in [0,\ 1]$, from ($\square$), we have

$$f(x) - f(z) = P_{L\ ,n}[f](x) - P_{L\ ,n}[f](z) + R_{L\ ,n}[f](x) - R_{L\ ,n}[f](z), \qquad ()$$

from which we get ($\square$) and ($\square$). The exactness of the polynomial $\overline{P}_{L\ ,n}[f](x)$ follows from the exactness of the polynomial $P_{L\ ,n}[f](x)$.

Proceeding in the same manner we can prove the result for the polynomial $\overline{P}^{*}_{L\ ,n}[f](x)$.

**Remark 12** The polynomials $\overline{P}_{L\ ,n}[f](x)$, $\overline{P}^{*}_{L\ ,n}[f](x)$ satisfy the interpolation conditions

$$\overline{P}_{L,n}[f](z) = f(z), \qquad L\left(\overline{P}^{(i)}_{L,n}[f]\right) = L\left(f^{(i)}\right), \quad i = 1, \dots, n, \tag{id190}$$

$$\overline{P}^{*}_{L,n}[f](z) = f(z), \qquad L\left(\Delta^{i}\overline{P}^{*}_{L,n}[f]\right) = L\left(\Delta^{i}f\right), \quad i = 1, \dots, n. \tag{id191}$$

## 9. Examples of Appell interpolation polynomials

**a)b)c)**

- Taylor interpolation and classical interpolation on equidistant points:

  Assuming

$$L(f) = f(x_0), \qquad x_0 \in [0, 1], \tag{id193}$$

  the polynomials $P_{L,n}[f](x)$ and $P^{*}_{L,n}[f](x)$ are, respectively, the Taylor interpolation polynomial and the classical interpolation polynomial on equidistant points;

- Bernoulli interpolation of first and of second kind:

  - Bernoulli interpolation of first kind ([21], [15]):

    Assuming

$$L(f) = \int_0^1 f(x)dx, \tag{id196}$$

    the interpolation polynomials $P_{L,n}[f](x)$ and $\overline{P}_{L,n}[f](x)$ become

$$P_{L,n}[f](x) = \int_0^1 f(x)dx + \sum_{i=1}^{n} \frac{f^{(i-1)}(1) - f^{(i-1)}(0)}{i!} B_i(x), \tag{id197}$$

$$\overline{P}_{L,n}[f](x) = f(0) + \sum_{i=1}^{n} \frac{f^{(i-1)}(1) - f^{(i-1)}(0)}{i!} (B_i(x) - B_i(0)), \tag{id198}$$

    where $B_i(x)$ are the classical Bernoulli polynomials ([17], [23]);

  - Bernoulli interpolation of second kind ([20]):

    Assuming

$$L(f) = \left[D\Delta^{-1}f\right]_{x=0}, \tag{id200}$$

where $\Delta^{-1}$ denote the indefinite summation operator and is defined as the linear operator inverse of the finite difference operator $\Delta$, the interpolation polynomials $P_{L,n}^*[f](x)$ and $\bar{P}_{L,n}^*[f](x)$ become

$$P_{L,n}^*[f](x) = \left[\Delta^{-1}Df\right]_{x=0} + \sum_{i=0}^{n-1} f'(i)\mathcal{B}_{n,i}^{II}(x), \tag{id201}$$

$$\bar{P}_{L,n}^*[f](x) = f(0) + \sum_{i=0}^{n-1} f'(i)\left(\mathcal{B}_{n,i}^{II}(x) - \mathcal{B}_{n,i}^{II}(0)\right), \tag{id202}$$

where

$$\mathcal{B}_{n,i}^{II}(x) = \sum_{j=i}^{n-1} \binom{j}{i} \frac{(-1)^{j-i}}{(j+1)!} B_{j+1}^{II}(x), \tag{id203}$$

and $B_j^{II}(x)$ are the Bernoulli polynomials of second kind ([20]);

- Euler and Boole interpolation:

  ○ Euler interpolation ([21]):

  Assuming

$$L(f) = \frac{f(0) + f(1)}{2}, \tag{id206}$$

the interpolation polynomials $P_{L,n}[f](x)$ and $\bar{P}_{L,n}[f](x)$ become

$$P_{L,n}[f](x) = \frac{f(0) + f(1)}{2} + \sum_{i=1}^{n} \frac{f^{(i)}(0) + f^{(i)}(1)}{2i!} E_i(x), \tag{id207}$$

$$\bar{P}_{L,n}[f](x) = f(0) + \sum_{i=1}^{n} \frac{f^{(i)}(0) + f^{(i)}(1)}{2i!} \left(E_i(x) - E_i(0)\right); \tag{id208}$$

  ○ Boole interpolation ([20]):

  Assuming

$$L(f) = [Mf]_{x=0}, \tag{id210}$$

where $Mf$ is defined by

$$Mf(x) = \frac{f(x) + f(x+1)}{2}, \tag{id211}$$

the interpolation polynomials $P_{L,n}^*[f](x)$ and $\overline{P}_{L,n}^*[f](x)$ become

$$P_{L,n}^*[f](x) = \frac{f(0) + f(1)}{2}\mathcal{E}_{n,0}^{II}(x) + \sum_{i=1}^{n}\frac{f(i) + f(i+1)}{2}\mathcal{E}_{n,i}^{II}(x), \tag{id212}$$

$$\overline{P}_{L,n}^*[f](x) = f(0) + \sum_{i=1}^{n}\frac{f(i) + f(i+1)}{2}\left(\mathcal{E}_{n,i}^{II}(x) - \mathcal{E}_{n,i}^{II}(0)\right), \tag{id213}$$

where

$$\mathcal{E}_{n,i}^{II}(x) = \sum_{j=i}^{n}\binom{j}{i}\frac{(-1)^{j-i}}{j!}E_j^{II}(x), \tag{id214}$$

and $E_j^{II}(x)$ are the Boole polynomials ([20]).

# 10. The algebraic approach of Yang and Youn

Yang and Youn ([18]) also proposed an algebraic approach to Appell polynomial sequences but with different methods. In fact, they referred the Appell sequence, $s_n(x)$, to an invertible analytic function g(t):

$$s_n(x) = \left[\frac{d^n}{dt}\left(\frac{1}{g(t)}e^{xt}\right)\right]_{t=0}, \tag{id215}$$

and called Appell vector for $g(t)$ the vector

$$\overline{S}_n(x) = [s_0(x), ..., s_n(x)]^T. \tag{id216}$$

Then, they proved that

$$\overline{S}_n(x) = P_n\left[\frac{1}{g(t)}\right]_{t=0}W_n[e^{xt}]_{t=0} = W_n\left[\frac{1}{g(t)}e^{xt}\right]_{t=0}, \tag{id217}$$

being $W_n[f(t)] = [f(t), f'(t), ..., f^{(n)}(t)]^T$ and $P_n[f(t)]$ the generalized Pascal functional matrix of $f(t)$ ([28]) defined by

$$\left(P_n[f(t)]\right)_{i,j} = \begin{cases} \binom{i}{j} f^{(i-j)}(t) & i \geq j \\ 0 & otherwise \end{cases}, \qquad i, j = 0, ..., n. \tag{id218}$$

Expressing the (□) in matrix form we have

$$\bar{S}_n(x) = SX(x), \tag{id219}$$

with

$$S = \begin{bmatrix} s_{00} & 0 & 0 & \cdots & 0 \\ s_{10} & s_{11} & 0 & \cdots & 0 \\ s_{20} & s_{21} & s_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n0} & s_{n1} & s_{n2} & \cdots & s_{nn} \end{bmatrix}, \quad X(x) = [1, x, ..., x^n]^T, \tag{id220}$$

where

$$s_{i,j} = \binom{i}{j}\left[\left(\frac{1}{g(t)}\right)^{(i-j)}\right]_{t=0}, \qquad i = 0, ..., n, \quad j = 0, ..., i. \tag{id221}$$

It is easy to see that the matrix $S$ coincides with the matrix $M^{-1}$ introduced in Section □, Theorem □.

## 11. Conclusions

We have presented an elementary algebraic approach to the theory of Appell polynomials. Given a sequence of real numbers $\beta_i$, $i = 0, 1, ...,$ $\beta_0 \neq 0$, a polynomial sequence on determinantal form, called of Appell, has been built. The equivalence of this approach with others existing was proven and, almost always using elementary tools of linear algebra, most important properties od Appell polynomials were proven too. A dual theory referred to the finite difference operator $\Delta$ has been proposed. This theory has provided a class of polynomials called Appell polynomials of second kind. Finally, given a linear functional $L$, with $L(1) \neq 0$, and defined

$$L(x^i) = \beta_i, \quad (L((x)_i) =_i ), \tag{id222}$$

the linear interpolation problem

$$L\left(P_n^{(i)}\right) = i\,!\,\omega_i, \qquad \left(L\left(\Delta^i P_n\right) = i\,!\,\omega_i\right), \quad P_n \in {}_n, \quad \omega_i \in \mathbb{R}, \tag{id223}$$

has been considered and its solution has been expressed by the basis of Appell polynomials related to the functional $L$ by ($\square$). This problem can be extended to appropriate real functions, providing a new approximating polynomial, the remainder of which can be estimated too. This theory is susceptible of extension to the more general class of Sheffer polynomials and to the bi-dimensional case.

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