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Kinematic and Dynamic Modelling of Serial Robotic Manipulators Using Dual Number Algebra

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1. Introduction

The kinematic and dynamic modelling of robotic manipulators has, as a specific field of robotics, represented a complex problem. To deal with this, the researchers have based their works on a great variety of mathematical theories (Seiling, 1999). One of these tools is the Dual Algebra, which is a concept originally introduced in 1893 by William Kingdon Clifford (Fisher, 1998; Funda, 1988). A dual number is a compact form that can be used to represent the rigid body motion in the space (Keller, 2000; Pennestrí & Stefanelli, 2007), it has therefore, a natural application in the analysis of spatial mechanisms specifically mechanical manipulators (Bandyopadhyay, 2004, 2006; Bayro-Corrochano & Kähler, 2000, 2001).

Several works related to dual-number in kinematics, dynamics and synthesis of mechanisms have been developed (Cheng, 1994; Fisher, 1998, 1995, 2003) in (Moon & Kota, 2001) is presented a methodology for combining basic building blocks to generate alternative mechanism concepts. The methodology is based on a mathematical framework for carrying out systematic conceptual design of mechanisms using dual vector algebra. The dual vector representation enables separation of kinematic function from mechanism topology, allowing a decomposition of a desired task into subtask, in order to meet either kinematic function or spatial constraints. (Ying et al, 2004) use dual angles as an alternative approach to quantify general spatial human joint motion. Ying proposes that dual Euler angles method provides a way to combine rotational and translational joint motions in Cartesian Coordinate systems, which can avoid the problems caused by the use of the joint coordinate system due to non-orthogonality. Hence the dual angles method is suitable for analyzing the motion characteristics of the ankle joint. The motion at the ankle joint complex involves rotations about and translations along three axes. In the same field of biomechanics (KiatTeu et al, 2006) present a method that provides a convenient assessment of golf-swing effectiveness. The method can also be applied to other sports to examine segmental rotations. In general, this method facilitates the study of human motion with relative ease. The use of a biomechanical model, in conjunction with dual-number coordinate transformation for motion analysis, was shown to provide accurate and reliable results. In particular, the advantage of using the dual Euler angles based on the dual-

number coordinate transformation approach, is that it allows, for a complete 3D motion, to use only six parameters for each anatomical joint. KiatTeu infers that the method has proved to be an effective means to examine high-speed movement in 3D space. It also provides an option in assessing the contributions of the individual segmental rotations in production of the relevant velocity of the end-effector.
(Page et al, 2007) present the location of the instantaneous screw axis (ISA) in order to obtain useful kinematic models of the human body for applications such as prosthesis and orthoses design or even to help in disease diagnosis techniques. Dual vectors are used to represent and operate with kinematic screws with the purpose of locating the instantaneous screw axes which characterize this instantaneous motion. A photogrammetry system based on markers is used to obtain the experimental data from which the kinematic magnitudes are obtained. A comprehensive analysis of the errors in the measurement of kinematic parameters was developed, obtaining explicit expressions for them based on the number of markers and their distribution.

1.1 Dual-number representation in robotics

The dual-number representation has been extended to other fields of mechanics; rigid body mechanics is an area where the dual number formulation has been applied, especially in the kinematics and dynamics of mechanisms.
The homogeneous transformation is a point transformation; in contrast, a line transformation can also naturally be defined in 3D Cartesian space, in which the transformed element is a line in 3D space instead of a point. In robotic kinematics and dynamics, the velocity and acceleration vectors are often the direct targets of analysis. The line transformation will have advantages over the ordinary point transformation, since the combination of the linear and angular quantities can be represented by lines in 3D space. Since a line in 3D space is determined by four independent parameters. (Gu, 1988) presents a procedure that, offers an algorithm which deals with the symbolic analysis for both rotation and translation. In particular, the aforementioned is effective for the direct determination of Jacobian matrices and their derivatives. The dual-number transformation procedure, based on these properties and the principle of transference, can be used for finding Jacobian matrices in robotic kinematics and their derivatives in robotic dynamics and control modeling. A related work was performed in (Pennock & Mattson, 1996) where the forward position problem of two PUMA robots manipulating a planar four bar linkage payload is solved using closed-form solutions for the remaining unknown angular displacements based in orthogonal transformation matrices with dual-number. (Brosky & Shoham, 1998; Sai-Kai, 2000) introduce the generalized Jacobian matrix which consists of the complete dual transformation matrices. The generalized Jacobian matrix relates force and moment at the end-effector to force and moment at the joints for each axe. Furthermore, the generalized Jacobian matrix also relates motion in all directions at the joints to the motion of the end-effector, an essential relation required at the design stage of robot manipulators. An extension of these kinematics and statics schemes into dynamics is possible by applying the dual inertia operator. (Brodsky, 1999) formulated the representation of rigid body dynamic equations introducing the dual inertia operator. Brodsky gives a general expression for the three-dimensional dynamic equation of a rigid body with respect to an arbitrary point. Then the dual Lagrange equation is established by developing derivative rules of a real function with respect to dual variables.

(Bandyopadhyay & Ghosal, 2004) performs a study in order to determinate principal twist of the end-effector of a multi-degree of freedom manipulator, which plays a central role in the analysis, design, motion planning and determination of singularities.
(Yang & Wang, 2010) solve the direct and inverse kinematics of a SCARA robot. They proved that the dual number allows compact formulations considerably facilitating the analysis of robot kinematics. To deal with coordinate transformation in three dimensional Cartesian space, the homogeneous transformation is usually applied. It is defined in the four-dimensional space and its matrix multiplication performs the simultaneous rotation and translation.

2. Mathematical preliminaries

Let us consider a transformation of coordinates between the Cartesian Coordinate system (x,y) and the oblique coordinate system (u,v) given by the equations:

$$x = au + bv\,; \quad y = 0u + av \tag{1}$$

With a, b real numbers. The geometry is depicted in Figure 1.
It is well known that the point (u,v) is localized by the vector $\vec{r} = x\hat{\imath} + y$ from the origin of the Cartesian coordinate. From the transformation (1):

$$\vec{r} = (au + bv)\hat{\imath} + avj \tag{2}$$

The tangent vectors to u and v are:

$$\frac{\partial \vec{r}}{\partial u} = ai\,; \quad \frac{\partial \vec{r}}{\partial v} = bi + aj \tag{3}$$

From the obvious $\hat{e}_u \cdot \hat{e}_v = \frac{b}{\sqrt{b^2 + a^2}}$ it is clear that the coordinates (u, v) are not orthogonal.
The unit vectors of the Cartesian frame can be written in the form of column vectors:

$$i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad j = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4}$$

We can describe the oblique frame (u,v) in terms of the tangent vectors ai and bi+aj written as a column vectors:

$$ai = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad ; \quad bi + aj = \begin{pmatrix} b \\ a \end{pmatrix} \tag{5}$$

The column vectors $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$, $\begin{bmatrix} 0 & 1 \end{bmatrix}^T$ can be combined into a single matrix describing the Cartesian Frame:

$$\text{Cartesian Frame: } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

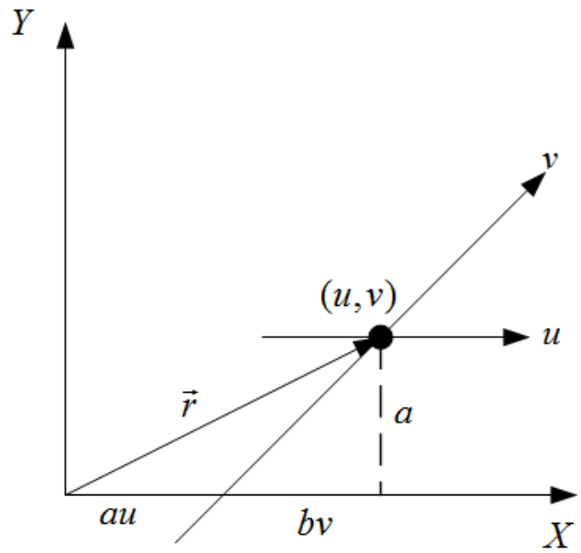

Fig. 1. Oblique plane

The column vectors $\begin{bmatrix} a & 0 \end{bmatrix}^T$ $\begin{bmatrix} b & a \end{bmatrix}^T$ can be combined into a single matrix describing the oblique frame:

$$\text{Oblique Frame: } \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

The matrix $\begin{pmatrix} a & b \\ a & a \end{pmatrix}$ is the transformation matrix that describes the Oblique Frame relative to the Cartesian Frame.

The matrix $\begin{pmatrix} a & b \\ a & a \end{pmatrix}$ can be decomposed as:

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} a + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{6}$$

Where $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$ is the unitary matrix and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \varepsilon$ is a matrix with the following properties:

a. ε is nilpotent:

$$\varepsilon^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

b. ε is a ninety degree rotation operator

$$\varepsilon j = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = i$$

Finally expression (6) is written in the form:

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = a + \varepsilon b \tag{7}$$

Equation (7) is easily extended to 3D:

$$1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \varepsilon = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Study in 1903 called the expression $a + \varepsilon b$ "dual number" because it is constructed from the pair of real numbers (a, b). A dual number is usually denoted in the form:

$$\hat{a} = a + \varepsilon b \tag{8}$$

The algebra of dual numbers has been originally conceived by William Kingdon Clifford. Elementary operation of addition is defined as: $\left(\hat{a} = a + \varepsilon a_0; \quad \hat{b} = b + \varepsilon b_0\right)$

$$\hat{a} + \hat{b} = (a + b) + \varepsilon(a_0 + b_0) \tag{9}$$

Multiplication is defined as:

$$\hat{a}\hat{b} = (a + \varepsilon a_0)(b + \varepsilon b_0) = ab + \varepsilon(ab_0 + a_0 b) \tag{10}$$

Division is defined as:

$$\frac{\hat{a}}{\hat{b}} = \frac{a + \varepsilon a_0}{b + \varepsilon b_0} = \left(\frac{a + \varepsilon a_0}{b + \varepsilon b_0}\right)\left(\frac{b + \varepsilon b_0}{b + \varepsilon b_0}\right) = \frac{a}{b} + \varepsilon \frac{a_0 b - a b_0}{b^2} \tag{11}$$

Division by a pure dual number (εa_0) is no defined. It immediately follows that:

$$\hat{a}^n = (a + \varepsilon a_0)^n = a^n + \varepsilon n a_0 a^{n-1} \tag{12}$$

$$\sqrt{\hat{a}} = \sqrt{a} + \varepsilon \frac{a_0}{a\sqrt{a}} \tag{13}$$

A function F of a dual variable $\hat{x} = x + \varepsilon x_0$ can be represented in the form:

$$F(\hat{x}) = f(x, x_0) + \varepsilon g(x, x_0) \tag{14}$$

Where f and g are real functions of real variables x & x_0. The necessary and sufficient conditions in order that F be analytic are:

$$\frac{\partial f}{\partial x_0} = 0; \quad \frac{\partial f}{\partial x} = \frac{\partial g}{\partial x_0} \tag{15}$$

From these it immediately follows the Taylor Series expansion:

$$f(\hat{x}) = f(x + \varepsilon x_0) = f(x) + \varepsilon x_0 \frac{\partial f(x)}{\partial x} \tag{16}$$

Because $\varepsilon^2 = 0$, $\varepsilon^3 = 0$, $\varepsilon^4 = 0$ and so on, all formal operation of dual number are the same as those of ordinary algebra.

2.1 The dual angle

The dual angle represents the relative displacement and orientation between two lines L_1 and L_2 in the 3D space (Figure 2).

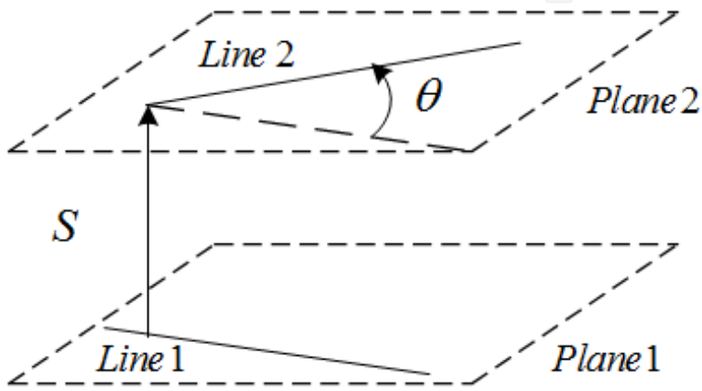

Fig. 2. Geometric representation of a dual angle.

The dual angle is defined as:

$$\hat{\theta} = \theta + \varepsilon S \tag{17}$$

This concept was introduced by Study in 1903. θ the primary component of $\hat{\theta}$ is the projected angle between L_1 and L_2. S the dual component of $\hat{\theta}$ is the shortest distance between lines L_1 & L_2 (as is obvious S is the length of common perpendicular to plane 1 and plane 2. Table 1 summarizes some properties:

$\hat{c} = \hat{a} + \hat{b}$	$\hat{c} = (a + b) + \varepsilon(a_0 + b_0)$
$\hat{c} = \hat{a}\hat{b}$	$\hat{a}\hat{b} = (a + \varepsilon a_0)(b + \varepsilon b_0)$
$\hat{a}^n$	$\hat{a}^n = a^n + \varepsilon n a_0 a^{n-1}$
$\sqrt{\hat{a}}$	$\sqrt{\hat{a}} = \sqrt{a} + \varepsilon \frac{a_0}{2\sqrt{a}}$
$\hat{a}/\hat{b}$	$\hat{c} = \frac{a}{b} + \varepsilon \frac{a_0 b - a b_0}{b^2}$
$f(\hat{a})$	$f(\hat{a}) = f(a) + \varepsilon a_0 \frac{df(a)}{da}$

Table 1. Basic dual algebra operations

In particular a dual angle is an advantageous tool to represent the coordinates of a rigid body in the space relative to other rigid body, if two planes are parallel and exists a line in each plane, dual angle will be the distance between the planes and the angle produced by the projection of one of the lines onto plane, thus a dual angle is used to describe each one of a robot's joints as a cylindrical one, which means that the entire topology is formulated as a set of dual angles (Fisher, 1995; Cecchini et al, 2004).

2.2 Dual vectors

A dual vector $\hat{V} = \vec{V} + \varepsilon(\vec{r} \times \vec{V})$ is a vector constrained to lie upon a given line L in 3D space. The primary component $\vec{V}$ is called the "resultant vector" and comprises the magnitude and direction of dual vector $\hat{V}$. It is independent of the location frame origin. The dual component $\vec{r} \times \vec{V}$ is called the "moment vector". The vector $\vec{r}$ is the position vector from the frame origin to any point on the line L of dual vector $\vec{V}$. $\vec{r} \times \vec{V}$ is invariant for any choice of point on line L, it depends on the choice of the frame origin. (Figure 3).

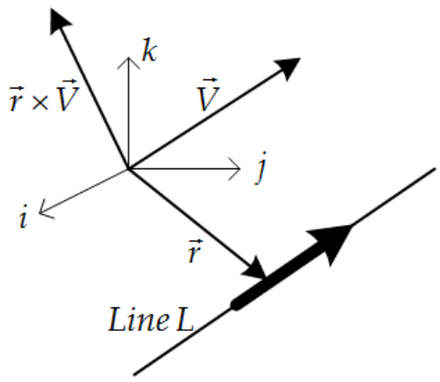

Fig. 3. Dual vector

Among the important dual vectors are:

1. Velocity defined as: $\hat{V} = (\omega + \varepsilon v)\hat{u}$
2. Linear momentum: $\hat{p} = \vec{p} + \varepsilon \vec{r} \times \vec{p}$
3. Force: $\hat{F} = \vec{F} + \varepsilon \vec{r} \times \vec{F}$
4. Angular momentum: $\hat{L} = \vec{L} + \varepsilon \vec{r} \times \vec{L}$

Important dual rotations around and along z, y, x axis are (Figure 4):

$$\hat{R}_{x,\hat{\alpha}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\hat{\alpha} & -\sin\hat{\alpha} \\ 0 & \sin\hat{\alpha} & \cos\hat{\alpha} \end{bmatrix}$$

$$\hat{R}_{y,\hat{\phi}} = \begin{bmatrix} \cos\hat{\phi} & 0 & sen\hat{\phi} \\ 0 & 1 & 0 \\ -sen\hat{\phi} & 0 & \cos\hat{\phi} \end{bmatrix} \qquad \hat{R}_{z,\hat{\theta}} = \begin{bmatrix} \cos\hat{\theta} & -\sin\hat{\theta} & 0 \\ sen\hat{\theta} & \cos\hat{\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

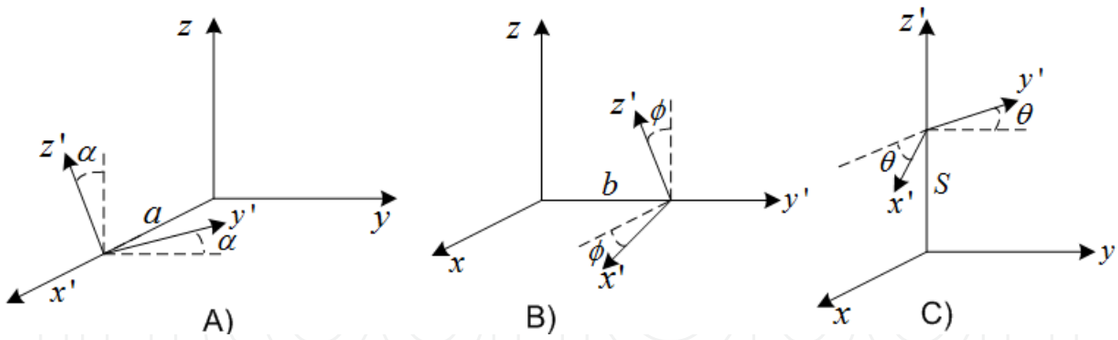

Fig. 4. Dual rotations: A) around and along x, B) around and along y, C) around and along z.

2.3 Algebra of dual vectors and matrices

Let: $\hat{A} = \vec{a} + \varepsilon(\vec{r}_1 \times \vec{a})$ and $\hat{B} = \vec{b} + \varepsilon(\vec{r}_2 \times \vec{b})$:

$$\hat{A} + \hat{B} = \vec{a} + \vec{b} + \varepsilon(\vec{r}_1 \times \vec{a} + \vec{r}_2 \times \vec{b})$$

Definition of dot and cross products are:

$$\hat{A} \cdot \hat{B} = ab\cos\hat{\theta}; \quad \hat{A} \times \hat{B} = ab\sin\hat{\theta}\hat{S}$$

Product of two dual matrices:

Let $\left[\hat{A}\right] = [A] + \varepsilon[A_0]$ & $\left[\hat{B}\right] = [B] + \varepsilon[B_0]$, the definition of their dual product is:

$$\left[\hat{A}\right]\left[\hat{B}\right] = [A][B] + \varepsilon\{[A][B_0] + [A_0][B]\}$$

It is similar with the multiplication rule for dual numbers. The inverse of a square matrix is defined as:

$$\left[\hat{A}\right]\left[\hat{A}\right]^{-1} = [I]$$

$$\left[\hat{A}\right]^{-1} = [A]^{-1} - \varepsilon\left\{[A]^{-1}[A_0][A]^{-1}\right\}$$

3. Denavit – Hartenberg parameters

Mechanisms analysis is facilitated by fixing a coordinate system on each link in a specific manner. With reference to Figure 5, a coordinate frame $\{i+1\}$ is fixed on the distal end of a link i joining joints i and $i+1$ such that:

$\hat{k}_{i+1}$ axis coincident with axis of joint $i+1$

i_{i+1} axis coincident with shortest distance between axes $\hat{k}_i$ & $\hat{k}_{i+1}$

i_{i+1} axis perpendicular to both axes i_{i+1} & $\hat{k}_{i+1}$

The origin of frame $\{i+1\}$ is located at the intersection of axes $\hat{k}_{i+1}$ and i_{i+1}. Frame $\{i\}$ is fixed on the previous link $i-1$. Translation S_i is the distance from point i, the origin of frame $\{i\}$ to the line segment whose length a_i is the shortest distance between joint axes $\hat{k}_i$ and $\hat{k}_{i+1}$. That line segment of shortest distance between join axes intersects axis $\hat{k}_{i+1}$ at point $i+1$, the origin of frame $\{i+1\}$ fixed on the distal end of link i. The projected angle between axes $\hat{k}_i$ and $\hat{k}_{i+1}$ represent the twist α_i of link i. The values α, θ, a, S are the well-known Denavit-Hartenberg parameters.

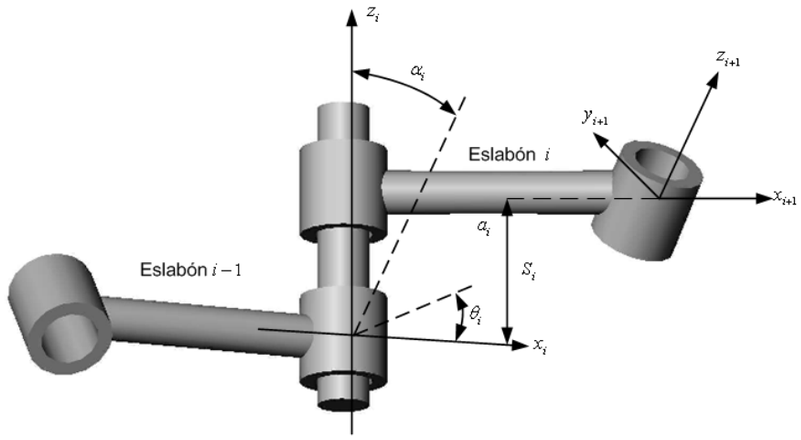

Fig. 5. Denavit and Hartenberg parameters (Pennestrí & Stefanelli, 2007)

A dual matrix rotation that represent the necessary motions of frame $\{i\}$ in terms of an attached $\{i-1\}$ frame is the composition of rotation $R_x(\hat{\alpha}_i)$ and rotation $R_z(\hat{\theta}_i)$, i.e.

$$ {}^{i-1}\hat{M}_i = \left(\hat{R}_{z,\hat{\theta}}\right)\left(\hat{R}_{x,\hat{\alpha}}\right) = \begin{bmatrix} \cos\hat{\theta}_i & -\cos\hat{\alpha}_i \sin\hat{\theta}_i & \sin\hat{\alpha}_i \sin\hat{\theta}_i \\ \sin\hat{\theta}_i & \cos\hat{\alpha}_i \cos\hat{\theta}_i & -\sin\hat{\alpha}_i \cos\hat{\theta}_i \\ 0 & \sin\hat{\alpha}_i & \cos\hat{\alpha}_i \end{bmatrix} \tag{18} $$

So the open chain dual equation is:

$$ {}^{0}\hat{M}_n = \prod_{i=1}^{n} {}^{i-1}\hat{M}_i \tag{19} $$

According with Funda the dual rotation matrix ${}^{0}\hat{M}_n = {}^{0}T_n + \varepsilon\, {}^{0}D_n$

The above expression is useful for modeling prismatic, rotational and cylindrical joints, this represents a main advantage respecting to real numbers, to represent the relative position of a point respecting an inertia frame an alternative is establishing the representation of Denavit and Hartenberg trough the dual angles θ, α:

4. Dual Jacobian matrix

If a point P on a body j is moving with respect to a body I (Fig. 6), the velocity can be expressed in terms of inertial frame R $\left({}^{R}\hat{V}_{j,i}^{P}\right)$.

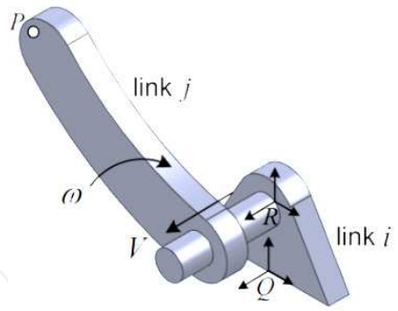

Fig. 6. Dual velocity scheme.

When the dual velocity needs to be represented in terms of frame Q, a rotation from frame R is done:

$$ {}^{Q}\hat{V}_{j,i}^{P} = {}^{Q}T_{p}\,{}^{P}\hat{V}_{j,i}^{P} \tag{20} $$

The relative velocity of a link k with respect to link i $\left(\hat{V}_{k,i}\right)$ in dual form is established as:

$$ \hat{V}_{k,i} = \hat{V}_{j,i} + \hat{V}_{k,j} \tag{21} $$

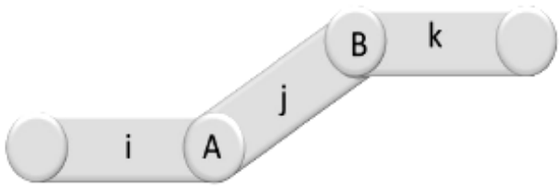

Fig. 7. Relative dual velocity theorem in a kinematic chain.

From dual velocities theorem, the vector of dual velocities in the end of n link in terms of the n frame can be found as:

$$ {}^{0}\hat{V}_{n,0}^{n} = {}^{0}T_{n}\sum_{i=1}^{n} {}^{n}\hat{M}_{i-1}\,{}^{i-1}\hat{V}_{i,i-1}^{i-1} \tag{22} $$

Where ${}^{0}T_{n}$ is the primary component of the dual matrix(19).
The generalized dual Jacobian matrix is obtained by applying the relative velocity theorem in dual form. The differential motions, whether axial or radial, are expressed in a matrix formed by the dual homogenous matrices, in contrast with the conventional Jacobian matrix that is obtained from specific columns of homogeneous transformation matrix (Sai-Kai, 2000).

$$ {}^{0}\hat{V}_{n,0}^{n} = \begin{bmatrix} \omega_x + \varepsilon V_x \\ \omega_y + \varepsilon V_y \\ \omega_z + \varepsilon V_z \end{bmatrix} = \left[{}^{0}T_{n}\right]\left[{}^{n}\hat{M}_{0} \quad {}^{n}\hat{M}_{1} \quad {}^{n}\hat{M}_{2} \ldots \quad {}^{n}\hat{M}_{n-1}\right]\begin{bmatrix} {}^{0}\hat{V}_{1,0}^{0} \\ \hline {}^{1}\hat{V}_{2,1}^{1} \\ \hline \vdots \\ {}^{n-1}\hat{V}_{n,n-1}^{n-1} \end{bmatrix} \tag{23} $$

The block matrix $\left[{}^{n}\hat{M}_0 \quad {}^{n}\hat{M}_1 \quad {}^{n}\hat{M}_2 \ldots \quad {}^{n}\hat{M}_{n-1} \right]$ is called Dual Jacobian matrix (Brodsky).

5. Dynamic analysis: Dual force

One of the most important features of dual number formulation is the capability of generalization for a great variety of robot topologies, without modifying the main program, this is an advantage when compared to typical homogenous matrices wherein is required to specify in dynamical model whether a joint is rotational or prismatic.
In dual algebra, if a force and a momentum act with respect a coordinate system, they can be represented in an expression called dual force:

$$\hat{F} = \vec{F} + \varepsilon\vec{\tau} \tag{24}$$

A clear example would be a screwdriver where is necessary to apply a force axially and around to screw.

Fig. 8. Example of dual force

If a dual force is applied on a point "B" different to the origin point "A", the effect on the point "B" will be determined by a coordinate transformation. Then a dual force applied on "A" in terms of the frame "B" is given by:

$${}^{B}\hat{F}_A = {}^{B}T_A \, {}^{A}\vec{F}_A + \varepsilon \, {}^{B}T_A \, {}^{A}\vec{\tau}_A \tag{25}$$

5.1 Dual momentum

Dual momentum concept is introduced due to the acceleration is a dual pseudo-vector, that means that it can not be established as a dual vector.

$${}^{B}\hat{H}_A = \int_A \left({}^{B}\vec{P}_p + \varepsilon \, {}^{B}\vec{H}_p \right) \tag{26}$$

The terms ${}^{B}\vec{P}_p$ & ${}^{B}\vec{H}_p$ are the linear and angular momentum of a particle "p" on a body "A" respectively, in terms of frame "B".

$$ {}^{B}P_A = m_A\left[{}^{B}V_A^B\right] - \left[{}^{B}S_A^B\right]\left[{}^{B}\omega_A^B\right] $$

$$ {}^{B}H_A = \left[{}^{B}S_A^B\right]\left[{}^{B}V_A^B\right] + \left[{}^{B}J_A^B\right]\left[{}^{B}\omega_A^B\right] $$

5.2 Dual inertial force

According with (Pennock & Meehan, 2000) the dual inertial forces on a rigid body are the derivative of the dual momentum:

$$ {}^{B}\hat{f}_A = \frac{d}{dt}\left({}^{B}\hat{H}_A\right) \tag{27} $$

$$ {}^{B}\hat{f}_A = \begin{bmatrix} {}^{B}\dot{\hat{H}}_{Ai} \\ {}^{B}\dot{\hat{H}}_{Aj} \\ {}^{B}\dot{\hat{H}}_{Ak} \end{bmatrix} + \begin{bmatrix} 0 & -{}^{B}\hat{V}_{Ak}^B & {}^{B}\hat{V}_{Aj}^B \\ {}^{B}\hat{V}_{Ak}^B & 0 & -{}^{B}\hat{V}_{Ai}^B \\ -{}^{B}\hat{V}_{Aj}^B & {}^{B}\hat{V}_{Ai}^B & 0 \end{bmatrix} \begin{bmatrix} {}^{B}\hat{H}_{Ai} \\ {}^{B}\hat{H}_{Aj} \\ {}^{B}\hat{H}_{Ak} \end{bmatrix} $$

$$ \begin{bmatrix} {}^{B}\hat{f}_{Ai} \\ {}^{B}\hat{f}_{Aj} \\ {}^{B}\hat{f}_{Ak} \end{bmatrix} = \begin{bmatrix} {}^{B}\dot{\hat{H}}_{Ai} - {}^{B}\hat{V}_{Ak}^B\,{}^{B}\hat{H}_{Aj} + {}^{B}\hat{V}_{Aj}^B\,{}^{B}\hat{H}_{Ak} \\ {}^{B}\dot{\hat{H}}_{Aj} - {}^{B}\hat{V}_{Ai}^B\,{}^{B}\hat{H}_{Ak} + {}^{B}\hat{V}_{Ak}^B\,{}^{B}\hat{H}_{Ai} \\ {}^{B}\dot{\hat{H}}_{Ak} - {}^{B}\hat{V}_{Aj}^B\,{}^{B}\hat{H}_{Ai} + {}^{B}\hat{V}_{Ai}^B\,{}^{B}\hat{H}_{Aj} \end{bmatrix} $$

5.3 Dynamic equilibrium

Extending the D'Alembert principle to dual numbers for dynamic equilibrium

$$ {}^{B}\hat{M}_A\,{}^{A}\hat{F}_A - {}^{B}F_B = {}^{B}\hat{f}_A \tag{28} $$

6. Example: Robot with cylindrical joints

The robot shown in the Figure 9 is a clear example where the dual numbers can be employed:

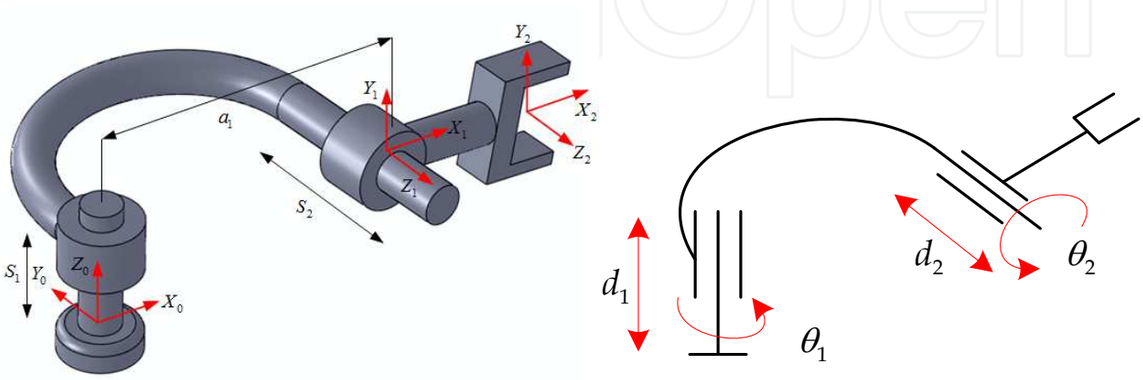

Fig. 9. Assignment of reference systems and Denavit-Hartenberg parameters.

i	θ_i	S_i	a_i	α_i
1	θ_1	d_1	l_1	90°
2	θ_2	d_2	l_2	0°

Table 2. Denavith and Hartemberg parameters of 2C robotic arm.

From Table 2, the dual angles $\hat{\theta}$ & $\hat{\alpha}$ are constructed as:

$$\hat{\theta}_1 = \theta_1 + \varepsilon d_1 \qquad \hat{\theta}_2 = \theta_2 + \varepsilon d_2 \qquad \hat{\alpha}_1 = \alpha_1 + \varepsilon a_1 \qquad \hat{\alpha}_2 = \alpha_2 + \varepsilon a_2$$

It is observed that different topologies can be solved from the assigned values to $\theta_1, d_1, \theta_2, d_2$; for example if θ_1 is 0 the robot will change the original topology CC to PC then nine different robots can be solved from the same aforementioned equations.

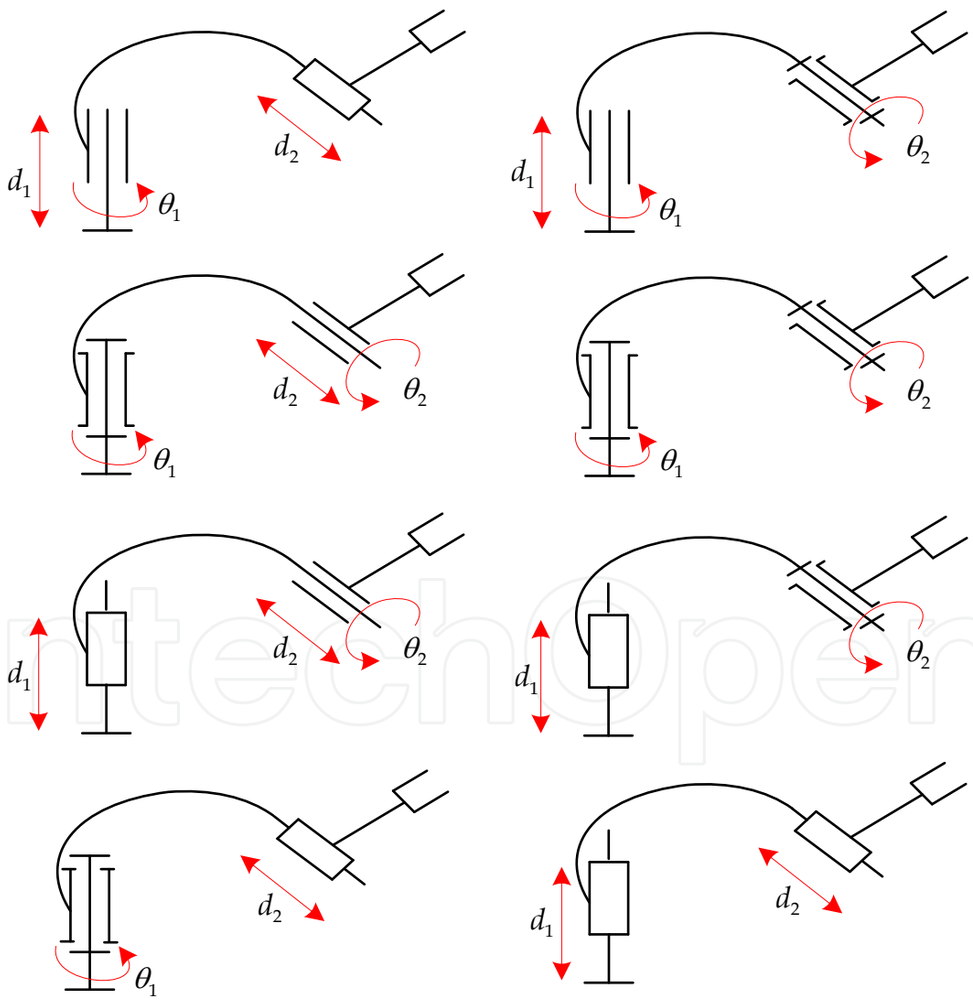

Fig. 10. Possible topologies for different values of θ_1 & θ_2.

$$ {}^{0}\hat{M}_2 = \begin{bmatrix} \cos\hat{\theta}_1 & -\cos\hat{\alpha}_1\sin\hat{\theta}_1 & \sin\hat{\alpha}_1\sin\hat{\theta}_1 \\ \sin\hat{\theta}_1 & \cos\hat{\alpha}_1\cos\hat{\theta}_1 & -\sin\hat{\alpha}_1\cos\hat{\theta}_1 \\ 0 & \sin\hat{\alpha}_1 & \cos\hat{\alpha}_1 \end{bmatrix} \begin{bmatrix} \cos\hat{\theta}_2 & -\cos\hat{\alpha}_2\sin\hat{\theta}_2 & \sin\hat{\alpha}_2\sin\hat{\theta}_2 \\ \sin\hat{\theta}_2 & \cos\hat{\alpha}_2\cos\hat{\theta}_2 & -\sin\hat{\alpha}_2\cos\hat{\theta}_2 \\ 0 & \sin\hat{\alpha}_2 & \cos\hat{\alpha}_2 \end{bmatrix} $$

For the inverse solution:

$$ {}^{0}\hat{M}_2 = \begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix} $$

$$ +\varepsilon \begin{bmatrix} -p_z n_y + p_y n_z & -p_z o_y + p_y o_z & -p_z a_y + p_y a_z \\ p_z n_x - p_x n_z & p_z o_x - p_x o_z & p_z a_x - p_x a_z \\ -p_y n_x + p_x n_y & -p_y o_x + p_x o_y & -p_y a_x + p_x a_y \end{bmatrix} $$

$$ {}^{0}\hat{M}_2 = \begin{bmatrix} c_1 c_2 & -c_1 s_2 & s_1 \\ s_1 c_2 & -s_1 s_2 & -c_1 \\ s_2 & c_2 & 0 \end{bmatrix} $$

$$ +\varepsilon \begin{bmatrix} 100.5\,s_1 s_2 - d_1 s_1 c_2 - d_2 c_1 s_2 & 70 s_1 + 100.5 s_1 c_2 - d_2 c_1 c_2 + d_1 s_1 s_2 & c_1(d_1 + 70 s_2) \\ 100.5\,c_1 s_2 + d_1 c_1 c_2 - d_2 s_1 s_2 & 70 c_1 - 100.5 c_1 c_2 - d_1 c_1 s_2 - d_2 s_1 c_2 & s_1(d_1 + 70 s_2) \\ d_2 c_2 & -d_2 s_2 & -70 c_2 - 100.5 \end{bmatrix} $$

Dividing elements (2,1) and (1,1) in both matrices:

$$ \theta_1 = tg^{-1}\left(\frac{n_y}{n_x}\right) \; ; \; \theta_2 = tg^{-1}\left(\frac{\pm\sqrt{1-o_z^2}}{o_z^2}\right) $$

The velocities in the cylindrical joints are given by:

$$ {}^{0}\hat{V}_{1,0}^{0} = \begin{bmatrix} 0 \\ 0 \\ \omega_1 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ 0 \\ v_1 \end{bmatrix} \qquad {}^{1}\hat{V}_{2,1}^{1} = \begin{bmatrix} 0 \\ 0 \\ \omega_2 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ 0 \\ v_2 \end{bmatrix} $$

Computing the velocities on the end-effector:

$$ {}^{0}\hat{V}_{2,0}^{2} = {}^{0}T_2 \left\{ {}^{2}\hat{M}_0 \, {}^{0}\hat{V}_{1,0}^{0} + {}^{2}\hat{M}_1 \, {}^{1}\hat{V}_{2,1}^{1} \right\} $$

The above expression can be rewritten in terms of dual Jacobian matrix.

$$ {}^{0}\hat{V}_{2,0}^{2} = \begin{bmatrix} \omega_x + \varepsilon V_x \\ \omega_y + \varepsilon V_y \\ \omega_z + \varepsilon V_z \end{bmatrix} = \begin{bmatrix} {}^{0}T_2 \end{bmatrix} \begin{bmatrix} {}^{2}\hat{M}_0 & | & {}^{2}\hat{M}_1 \end{bmatrix} \begin{bmatrix} {}^{0}\hat{V}_{1,0}^{0} \\ \hline {}^{1}\hat{V}_{2,1}^{1} \end{bmatrix} $$

$$ {}^{0}\hat{V}_{2,0}^{2} = \begin{bmatrix} c_1c_2 & -c_1s_2 & s_1 \\ s_1c_2 & -s_1s_2 & -c_1 \\ s_2 & c_2 & 0 \end{bmatrix} \begin{bmatrix} {}^{0}\hat{M}_{2}^{T} & {}^{1}\hat{M}_{2}^{T} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega_1 + \varepsilon v_1 \\ 0 \\ 0 \\ \omega_2 + \varepsilon v_2 \end{bmatrix} $$

Dual velocities:

$$ {}^{1}\hat{V}_{1,0}^{1} = {}^{1}\hat{M}_{0}\,{}^{0}\hat{V}_{1,0}^{0} = \begin{bmatrix} 0 \\ \sin\hat{\alpha}_1(\omega_1 + \varepsilon v_1) \\ \cos\hat{\alpha}_1(\omega_1 + \varepsilon v_1) \end{bmatrix} \qquad {}^{2}\hat{V}_{2,0}^{2} = {}^{2}\hat{M}_{1}\,{}^{1}\hat{V}_{2,0}^{1} = \begin{bmatrix} 0 \\ \sin\hat{\alpha}_2(\omega_2 + \varepsilon v_2) \\ \cos\hat{\alpha}_2(\omega_2 + \varepsilon v_2) \end{bmatrix} $$

6.1 Dynamic analysis

Once obtained the velocities, the next step for solving the dynamic equations is, according with Fisher, to compute the dual momentum, being for each link:

$$ {}^{1}\hat{H}_0 = m_1\left[{}^{1}\vec{V}_{1,0}^{1}\right] - \left[{}^{1}S_1\right]\left[{}^{1}\vec{\omega}_{1,0}^{1}\right] + \varepsilon\left\{\left[{}^{1}S_1\right]\left[{}^{1}\vec{V}_{1,0}^{1}\right] + \left[{}^{1}I_1\right]\left[{}^{1}\vec{\omega}_{1,0}^{1}\right]\right\} $$

$$ {}^{2}\hat{H}_1 = m_2\left[{}^{2}\vec{V}_{2,0}^{2}\right] - \left[{}^{2}S_2\right]\left[{}^{2}\vec{\omega}_{2,0}^{2}\right] + \varepsilon\left\{\left[{}^{2}S_2\right]\left[{}^{2}\vec{V}_{2,0}^{2}\right] + \left[{}^{2}I_2\right]\left[{}^{2}\vec{\omega}_{2,0}^{2}\right]\right\} $$

Derivating the above expressions:

$$ {}^{1}\dot{\hat{H}}_0 = m_1\left[\frac{d}{dt}\left({}^{1}\vec{V}_{1,0}^{1}\right)\right] - \left[{}^{1}S_1\right]\left[\frac{d}{dt}\left({}^{1}\vec{\omega}_{1,0}^{1}\right)\right] $$

$$ + \varepsilon\left\{\left[{}^{1}S_1\right]\left[\frac{d}{dt}\left({}^{1}\vec{V}_{1,0}^{1}\right)\right] + \left[{}^{1}I_1\right]\left[\frac{d}{dt}\left({}^{1}\vec{\omega}_{1,0}^{1}\right)\right]\right\} $$

$$ \begin{bmatrix} {}^{1}f_{0i} + \varepsilon\,{}^{1}t_{0i} \\ {}^{1}f_{0j} + \varepsilon\,{}^{1}t_{0j} \\ {}^{1}f_{0k} + \varepsilon\,{}^{1}t_{0k} \end{bmatrix} = \begin{bmatrix} {}^{1}\dot{P}_{0i} - {}^{1}\omega_{1,0k}^{1}\,{}^{1}P_{0j} + {}^{1}\omega_{1,0j}^{1}\,{}^{1}P_{0k} \\ {}^{1}\dot{P}_{0j} - {}^{1}\omega_{1,0i}^{1}\,{}^{1}P_{0k} + {}^{1}\omega_{1,0k}^{1}\,{}^{1}P_{0i} \\ {}^{1}\dot{P}_{0k} - {}^{1}\omega_{1,0j}^{1}\,{}^{1}P_{0i} + {}^{1}\omega_{1,0i}^{1}\,{}^{1}P_{0j} \end{bmatrix} $$

$$ + \varepsilon\begin{bmatrix} {}^{1}\dot{H}_{0i} - {}^{1}\omega_{1,0k}^{1}\,{}^{1}H_{0j} - {}^{1}V_{1,0k}^{1}\,{}^{1}P_{0j} + {}^{1}\omega_{1,0j}^{1}\,{}^{1}H_{0k} + {}^{1}V_{1,0j}^{1}\,{}^{1}P_{0k} \\ {}^{1}\dot{H}_{0j} - {}^{1}\omega_{1,0i}^{1}\,{}^{1}H_{0k} - {}^{1}V_{1,0i}^{1}\,{}^{1}P_{0k} + {}^{1}\omega_{1,0k}^{1}\,{}^{1}H_{0i} + {}^{1}V_{1,0k}^{1}\,{}^{1}P_{0i} \\ {}^{1}\dot{H}_{0k} - {}^{1}\omega_{1,0j}^{1}\,{}^{1}H_{0i} - {}^{1}V_{1,0j}^{1}\,{}^{1}P_{0i} + {}^{1}\omega_{1,0i}^{1}\,{}^{1}H_{0j} + {}^{1}V_{1,0i}^{1}\,{}^{1}P_{0j} \end{bmatrix} $$

$$\begin{bmatrix} {}^2f_{1i}+\varepsilon\,{}^2t_{1i} \\ {}^2f_{1j}+\varepsilon\,{}^2t_{1j} \\ {}^2f_{1k}+\varepsilon\,{}^2t_{1k} \end{bmatrix}$$

$$=\begin{bmatrix} {}^2\dot{P}_{1i}-{}^2\omega^2_{2,0k}\,{}^2P_{1j}+{}^2\omega^2_{2,0j}\,{}^2P_{1k} \\ {}^2\dot{P}_{1j}-{}^2\omega^2_{2,0i}\,{}^2P_{1k}+{}^2\omega^2_{2,0k}\,{}^2P_{1i} \\ {}^2\dot{P}_{1k}-{}^2\omega^2_{2,0j}\,{}^2P_{1i}+{}^2\omega^2_{2,0i}\,{}^2P_{1j} \end{bmatrix}$$

$$+\varepsilon\begin{bmatrix} {}^2\dot{H}_{1i}-{}^2\omega^2_{2,0k}\,{}^2H_{1j}-{}^2V^2_{2,0k}\,{}^2P_{1j}+{}^2\omega^2_{2,0j}\,{}^2H_{1k}+{}^2V^2_{2,0j}\,{}^2P_{1k} \\ {}^2\dot{H}_{1j}-{}^2\omega^2_{2,0i}\,{}^2H_{1k}-{}^2V^2_{2,0i}\,{}^2P_{1k}+{}^2\omega^2_{2,0k}\,{}^2H_{1i}+{}^2V^2_{2,0k}\,{}^2P_{1i} \\ {}^2\dot{H}_{1k}-{}^2\omega^2_{2,0j}\,{}^2H_{1i}-{}^2V^2_{2,0j}\,{}^2P_{1i}+{}^2\omega^2_{2,0i}\,{}^2H_{1j}+{}^2V^2_{2,0i}\,{}^2P_{1j} \end{bmatrix}$$

From dynamic equilibrium:

$${}^BM_A\,{}^A\vec{F}_A-{}^B\vec{F}_B-{}^B\vec{f}_A+\varepsilon\left({}^BM_A\,{}^A\vec{T}_A+{}^BD_A\,{}^A\vec{F}_A-{}^B\vec{T}_B-{}^B\vec{t}_A\right)=0$$

$${}^BM_A\,{}^A\vec{F}_A-{}^B\vec{F}_B-{}^B\vec{f}_A=0$$

$${}^BM_A\,{}^A\vec{T}_A+{}^BD_A\,{}^A\vec{F}_A-{}^B\vec{T}_B-{}^B\vec{t}_A=0$$

$${}^2\vec{F}_2=\left[{}^2M_1\right]^{-1}\left\{{}^2\vec{f}_1\right\}$$

$${}^1\vec{F}_1=\left[{}^1M_0\right]^{-1}\left\{{}^2\vec{F}_2+{}^1\vec{f}_0\right\}$$

$${}^2\vec{T}_2=\left[{}^2M_1\right]^{-1}\left\{{}^2\vec{t}_1-{}^2D_1\,{}^2\vec{F}_2\right\}$$

$${}^1\vec{T}_1=\left[{}^1M_0\right]^{-1}\left\{{}^2\vec{T}_2+{}^1\vec{t}_0-{}^1D_0\,{}^1\vec{F}_1\right\}$$

7. Conclusions

The presented method, based on dual-number representation, has demonstrated be a powerful tool for solving a great variety of problems, that imply motions simultaneity off rotation and translation of rigid bodies in the space; the aforementioned, allows establishing dual rotation matrices. Robotics is a field wherein dual numbers have been employed to describe the motion of a rigid body, in particular of serial robotic arms. The methodology proposed is useful for robotic arms with cylindrical, prismatic and rotational joints. Once established the dual angles $\hat{\theta}$ and $\hat{\alpha}$, if the dual part of $\hat{\theta}$ is zero, the mechanism has only revolute joints, otherwise if the primary part of $\hat{\theta}$ is zero, only exist prismatic joints. So the developed methodology can be generalized to different topologies, which is a great advantage that allows that only one program solves a great variety of topologies.

The dynamic model is treated by using the dual momentum, wherein the inertial forces are computed by means of a set of linear equations, thus a $6 \times n$ vector of forces is calculated, and in consequence one obtains a complete description of the robotic manipulator. An appropriate way of dual numbers programming will yield a suitable software alternative to simulate and analyze different serial robotic manipulators topologies.

8. Acknowledgments

The authors gratefully acknowledge the support of CONACYT, IPN and ICyTDF for research projects and scholarships. They also would like to thank the anonymous reviewers for their valuable comments and suggestions.

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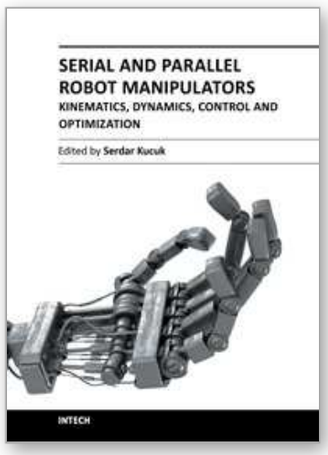

Serial and Parallel Robot Manipulators - Kinematics, Dynamics, Control and Optimization

Edited by Dr. Serdar Kucuk

ISBN 978-953-51-0437-7
Hard cover, 458 pages
Publisher InTech
Published online 30, March, 2012
Published in print edition March, 2012

The robotics is an important part of modern engineering and is related to a group of branches such as electric & electronics, computer, mathematics and mechanism design. The interest in robotics has been steadily increasing during the last decades. This concern has directly impacted the development of the novel theoretical research areas and products. This new book provides information about fundamental topics of serial and parallel manipulators such as kinematics & dynamics modeling, optimization, control algorithms and design strategies. I would like to thank all authors who have contributed the book chapters with their valuable novel ideas and current developments.

How to reference

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R. Tapia Herrera, Samuel M. Alcantara, Jesus A. Meda C. and Alejandro S. Velazquez (2012). Kinematic and Dynamic Modelling of Serial Robotic Manipulators Using Dual Number Algebra, Serial and Parallel Robot Manipulators - Kinematics, Dynamics, Control and Optimization, Dr. Serdar Kucuk (Ed.), ISBN: 978-953-51-0437-7, InTech, Available from: http://www.intechopen.com/books/serial-and-parallel-robot-manipulators-kinematics-dynamics-control-and-optimization/kinematic-and-dynamic-modelling-of-serial-robotic-manipulators-using-dual-number-algebra

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