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The Lane-Emden-Fowler Equation and Its Generalizations – Lie Symmetry Analysis

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1. Introduction

In the study of stellar structure the Lane-Emden equation (1; 2)

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y^r = 0, \tag{1}$$

where r is a constant, models the thermal behaviour of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. This equation was proposed by Lane (1) (see also (3)) and studied in detail by Emden (2). Fowler (4; 5) considered a generalization of Eq. (1), called Emden-Fowler equation (6), where the last term is replaced by $x^{\nu-1}y^r$.

The Lane-Emden equation (1) also models the equilibria of nonrotating fluids in which internal pressure balances self-gravity. When spherically symmetric solutions of Eq. (1) appeared in (7), they got the attention of astrophysicists. In the latter half of the twentieth century, some interesting applications of the isothermal solution (singular isothermal sphere) and its nonsingular modifications were used in the structures of collisionless systems such as globular clusters and early-type galaxies (8; 9).

The work of Emden (2) also got the attention of physicists outside the field of astrophysics who investigated the generalized polytropic forms of the Lane-Emden equation (1) for specific polytropic indices r. Some singular solutions for $r = 3$ were produced by Fowler (4; 5) and the Emden-Fowler equation in the literature was established, while the works of Thomas (10) and Fermi (11) resulted in the Thomas-Fermi equation, used in atomic theory. Both of these equations, even today, are being investigated by physicists and mathematicians. Other applications of Eq. (1) can be found in the works of Meerson et al (12), Gnutzmann and Ritschel (13), and Bahcall (14; 15).

Many methods, including numerical and perturbation, have been used to solve Eq. (1). The reader is referred to the works of Horedt (16; 17), Bender (18) and Lema (19; 20), Roxbough and Stocken (21), Adomian et al (22), Shawagfeh (23), Burt (24), Wazwaz (25) and Liao (26) for a sample. Exact solutions of Eq. (1) for $r = 0, 1$ and 5 have been obtained (see for example Chandrasekhar (7), Davis (27), Datta (28) and Wrubel (29)). Usually, for $r = 5$, only a one-parameter family of solutions is presented. A more general form of (1), in which the coefficient of y' is considered an arbitrary function of x, was investigated for first integrals by Leach (30).

Many problems in mathematical physics and astrophysics can be formulated by the generalized Lane-Emden equation

$$\frac{d^2y}{dx^2} + \frac{n}{x}\,\frac{dy}{dx} + f(y) = 0, \tag{2}$$

where n is a real constant and $f(y)$ is an arbitrary function of y. For $n = 2$ the approximate analytical solutions to the Eq. (2) were studied by Wazwaz (25) and Dehghan and Shakeri (31).

Another form of $f(y)$ is given by

$$f(y) = (y^2 - C)^{3/2}. \tag{3}$$

Inserting (3) into Eq. (1) gives us the "white-dwarf" equation introduced by Chandrasekhar (7) in his study of the gravitational potential of degenerate white-dwarf stars. In fact, when $C = 0$ this equation reduces to Lane-Emden equation with index $r = 3$.

Another nonlinear form of $f(y)$ is the exponential function

$$f(y) = e^y. \tag{4}$$

Substituting (4) into Eq. (1) results in a model that describes isothermal gas spheres where the temperature remains constant.

Equation (1) with

$$f(y) = e^{-y}$$

gives a model that appears in the theory of thermionic currents when one seeks to determine the density and electric force of an electron gas in the neighbourhood of a hot body in thermal equilibrium was thoroughly investigated by Richardson (32).

Furthermore, the Eq. (1) appears in eight additional cases for the function $f(y)$. The interested reader is referred to Davis (27) for more detail.

The equation

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + e^{\beta y} = 0, \tag{5}$$

where β is a constant, has also been studied by Emden (2). In a recent work (33) an approximate implicit solution has been obtained for Eq. (5) with $\beta = 1$.

Furthermore, more general Emden-type equations were considered in the works (34–38). See also the review paper by Wong (39), which contains more than 140 references on the topic.

The so-called generalized Lane-Emden equation of the first kind

$$x\frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta x^{\nu} y^n = 0, \tag{6}$$

and generalized Lane-Emden equation of the second kind

$$x\frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta x^{\nu} e^{ny} = 0, \tag{7}$$

where α, β, ν and n are constants, have been recently studied in (40; 41). In Goenner (41), the author uncovered symmetries of Eq. (6) to explain integrability of (6) for certain values of the parameters considered in Goenner and Havas (40). Recently, the integrability of the generalized Lane-Emden equations of the first and second kinds has been discussed in Muatjetjeja and Khalique (42).

In this chapter, firstly, a generalized Lane-Emden-Fowler type equation

$$x\frac{d^2y}{dx^2} + n\frac{dy}{dx} + x^{\nu} f(y) = 0, \tag{8}$$

where n and ν are real constants and $f(y)$ is an arbitrary function of y will be studied. We perform the Lie and Noether symmetry analysis of this problem. It should be noted that Eq. (8) for the power function $F(y) = y^r$ is related to the Emden-Fowler equation $y'' + p(X)y^r = 0$ by means of the transformation on the independent variable $X = x^{1-n}$, $n \neq 1$ and $X = \ln x$, $n = 1$.

Secondly, we consider a generalized coupled Lane-Emden system, which occurs in the modelling of several physical phenomena such as pattern formation, population evolution and chemical reactions. We perform Noether symmetry classification of this system and compute the Noether operators corresponding to the standard Lagrangian. In addition the first integrals for the Lane-Emden system will be constructed with respect to Noether operators.

2. Lie point symmetry classification of (8)

We start by determining the equivalence transformations of Eq. (8). We recall (43) that an equivalence transformation

$$\bar{x} = \bar{x}(x, y),\ \bar{y} = \bar{y}(x, y)$$

is a nondegenerate change of variables such that the family of Eqs. (8) remains invariant, i.e., Eq. (8) becomes

$$\bar{x}\frac{d^2\bar{y}}{d\bar{x}^2} + n\frac{d\bar{y}}{d\bar{x}} + \bar{x}^{\nu} \bar{f}(\bar{y}) = 0$$

with $\bar{f}$ depending on $\bar{y}$. Equivalence transformations are essential for simplifying the determining equation and for obtaining disjoint classes.

For Eq. (8) the equivalence transformations are

$$\begin{aligned} \bar{x} &= e^{a_2} x, \\ \bar{y} &= e^{a_3} y + a_1, \\ \bar{f} &= e^{a_3 - (1+\nu)a_2} f, \end{aligned} \tag{9}$$

where a_1, a_2 and a_3 are constants. For details of computations see (44).

If X, given by

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

is an admitted generator of a symmetry group of Eq. (8), then

$$X^{[2]}\left(x\,\frac{d^2y}{dx^2}+n\frac{dy}{dx}+x^{\nu}f(y)\right)\bigg|_{(8)}=0, \tag{10}$$

where $X^{[2]}$ is the second prolongation of X, gives the determining equations for the symmetry. This gives rise to

$$\xi = b(x),$$
$$\eta = c(x)y + d(x),$$

$$x^{\nu-1}(cy+d)f'(y)+\left[x^{\nu-1}(2b'-c)+(\nu-1)x^{\nu-2}b\right]f(y) + \left(\frac{n}{x}c'+c''\right)y+\left(\frac{n}{x}d'+d''\right)=0. \tag{11}$$

If f is an arbitrary function, the above system yields $\xi = 0$, $\eta = 0$, meaning that the principal Lie algebra of Eq. (8) is trivial.

The function f depends upon y only. Thus Eq. (11) only holds if its coefficients identically vanish or they are proportional to a function $\alpha = \alpha(x)$, i.e.,

$$c = r\alpha,\ \ d = q\alpha,\ \ 2b' - c + (\nu-1)x^{-1}b = p\alpha,$$
$$c''x^{-\nu+1} + nx^{-\nu}c' = h\alpha,\ \ d''x^{-\nu+1} + nx^{-\nu}d' = g\alpha, \tag{12}$$

where r, q, p, h and g are constants. Thus Eq. (11) becomes

$$(ru+q)F'(u) + pF(u) + hu + g = 0, \tag{13}$$

which is our classifying relation. This relation is invariant under the equivalence transformations (9) if

$$\bar{r} = r,\ \ \bar{q} = (ra_1+q)e^{-a_3},\ \ \bar{p} = p,$$
$$\bar{h} = he^{(\nu+1)a_2},\ \ \bar{g} = e^{-a_3+(\nu+1)a_2}(ha_1+g). \tag{14}$$

The relations in (14) are used to find the non-equivalent forms of f and this leads to the following eight cases.

Case 1. $n \neq (1-\nu)/2$, $f(y)$ arbitrary but not of the form contained in Cases 3, 4, 5 and 6.

No Lie point symmetry exits in this case.

Case 2. $n = (1-\nu)/2$, $f(y)$ arbitrary but not of the form contained in Cases 4, 5 and 6.

We obtain one Lie point symmetry

$$X = x^{(1-\nu)/2}\frac{\partial}{\partial x} \tag{15}$$

for the corresponding Eq. (8).

Case 3. $f(y)$ is linear in y.

This case is well known and the corresponding Eq. (8) has $sl(3, \Re)$ symmetry algebra. (See, for example, (45)).

Case 4. $f(y) = K - \delta y^2/2$, where $\delta = \pm 1$ and K is a constant.

Here we have six subcases:

4.1. $n = 2\nu + 3$, $K = 0$. The corresponding Eq. (8) admits a single Lie point symmetry

$$X_1 = x\frac{\partial}{\partial x} - (\nu+1)y\frac{\partial}{\partial y}. \tag{16}$$

Note that this is subsumed in Case 5.1 below.

4.2. $n = 12\nu + 13$, $K = 0$. Here the corresponding Eq. (8) admits the same symmetry as in Case 4.1.

4.3. $n = (\nu+4)/3$, $K = 0$. In this subcase the corresponding Eq. (8) admits a two-dimensional symmetry Lie algebra which is spanned by the operators (16) and

$$X_2 = x^{(2-\nu)/3}\frac{\partial}{\partial x} - \frac{\nu+1}{3}x^{-(\nu+1)/3}y\frac{\partial}{\partial y}.$$

Note that this is contained in Case 5.2 below.

4.4. $n = 7\nu + 8$, $K = 0$. The corresponding Eq. (8) admits the symmetry operator (16) and in addition, the symmetry operator

$$X_2 = x^{\nu+2}\frac{\partial}{\partial x} - \left[3(\nu+1)x^{\nu+1}y + \frac{24(\nu+1)^3}{\delta}\right]\frac{\partial}{\partial y}.$$

4.5. $n = (7\nu+13)/6$, $K = 0$. In this subcase the corresponding Eq. (8) admits two Lie point symmetries, namely, the symmetry given by (16) and the symmetry

$$X_2 = x^{(5-\nu)/6}\frac{\partial}{\partial x} - \left[\frac{2}{3}(\nu+1)x^{-(\nu+1)/6}y - \frac{(\nu+1)^3}{9\delta}x^{-7(\nu+1)/6}\right]\frac{\partial}{\partial y}.$$

4.6. $n = (1-\nu)/2$, $K = 0$. The corresponding Eq. (8) admits two Lie point symmetries and they are (15) and (16).

Case 5. $f(y) = -\delta_1/\sigma - y\delta_2/(\sigma+1) + Ky^{-\sigma}$, where $\delta_1, \delta_2 = 0, \pm 1$, $\sigma \neq -1, 0$ and K is a constant.

Three subcases arise:

5.1. $n = \dfrac{\sigma - 2\nu - 1}{\sigma + 1}$, $\sigma \neq 3$, $\delta_1, \delta_2 = 0$. In this subcase we have one Lie point symmetry generator

$$X_1 = x\frac{\partial}{\partial x} + \frac{\nu+1}{\sigma+1}y\frac{\partial}{\partial y} \tag{17}$$

admitted by the corresponding Eq. (8).

5.2. $n = \dfrac{\sigma - \nu - 2}{\sigma - 1}$, $\sigma \neq 3$, $\delta_1, \delta_2 = 0$. Here the corresponding Eq. (8) admits a two-dimensional symmetry Lie algebra spanned by the operators (17) and

$$X_2 = x^{\frac{\sigma+\nu}{\sigma-1}} \frac{\partial}{\partial x} + \frac{\nu+1}{\sigma-1} x^{\frac{\nu+1}{\sigma-1}} y \frac{\partial}{\partial y}. \tag{18}$$

5.3. $n = \dfrac{1-\nu}{2}$, (This subcase corresponds to $\sigma = 3$), $\delta_1, \delta_2 = 0$. The corresponding Eq. (8) in this case admits three Lie point symmetry generators and these are given by (17), (18) with $\sigma = 3$ and (15).

Case 6. $f(y) = Ke^{-\delta_1 y} + \delta_2 y + \delta_3$, where $\delta_1 = \pm 1$, $\delta_2, \delta_3 = 0, \pm 1$ and K is a constant.

We have three subcases.

6.1. For all values of $n \neq 1, (1-\nu)/2$, $\delta_2, \delta_3 = 0$ one Lie point symmetry generator

$$X_1 = x\frac{\partial}{\partial x} + \frac{\nu+1}{\delta_1}\frac{\partial}{\partial y} \tag{19}$$

is admitted by the corresponding Eq. (8).

6.2. $n = 1$, $\delta_2, \delta_3 = 0$. In this subcase the corresponding Eq. (8) admits the Lie point symmetry (19) and in addition the Lie point symmetry

$$X_2 = x \ln x \frac{\partial}{\partial x} + \frac{1}{\delta_1}[2 + (\nu+1)\ln x]\frac{\partial}{\partial y}.$$

6.3. $n = (1-\nu)/2$, $\delta_2, \delta_3 = 0$. The corresponding Eq. (8) admits two Lie point symmetries. These symmetries are given by (15) and (19).

Case 7. $f(y) = -\delta_1 \ln y - \delta_2 y + K$, where $\delta_1, \delta_2 = 0, \pm 1$ and K is a constant.

This reduces to Case 2.

Case 8. $f(y) = -\delta_1 y \ln y + Ky + \delta_2$, where $\delta_1, \delta_2 = 0, \pm 1$ and K is a constant.

This also reduces to Case 2.

2.1 Integration of (8) for different fs

The main purpose for calculating symmetries is to use them to solve or reduce the order of differential equations. Here we use the symmetries calculated above to integrate Eq. (8) for three functions f. Other cases can be dealt in a similar manner. We recall that for any two-dimensional Lie algebra with symmetries G_1 and G_2 satisfying the Lie bracket relationship $[G_1, G_2] = \lambda G_1$, for some constant λ, the usual reduction of order is through the normal subgroup G_1 (46). We first consider Case 4.4. The corresponding Eq. (8) admits the two symmetries

$$X_1 = x\frac{\partial}{\partial x} - (\nu+1)y\frac{\partial}{\partial y},\ X_2 = x^{\nu+2}\frac{\partial}{\partial x} - \left[3(\nu+1)x^{\nu+1}y + \frac{24(\nu+1)^3}{\delta}\right]\frac{\partial}{\partial y}.$$

Since $[X_1, X_2] = (\nu+1)X_2$, we may use X_2 to reduce the corresponding Eq. (8) to quadratures. The invariants of X_2 are found from

$$\frac{dx}{x^{\nu+2}} = \frac{dy}{-[3(\nu+1)x^{\nu+1}y + 24(\nu+1)^3/\delta]} = \frac{dy'}{-[3(\nu+1)^2x^\nu y + (4\nu+5)x^{\nu+1}y']}$$

and are

$$t = x^{3\nu+3}y + \frac{12}{\delta}(\nu+1)^2x^{2\nu+2},\ s = x^{4\nu+5}y' + 3(\nu+1)x^{4\nu+4}y + \frac{24}{\delta}(\nu+1)^3x^{3\nu+3}.$$

This leads to the first-order equation

$$\frac{ds}{dt} = \frac{\delta t^2}{2s}$$

which can be immediately integrated to give

$$s^2 = \frac{\delta}{3}t^3 + C_1,$$

where C_1 is an arbitrary constant of integration. Reverting to the x and y variables we obtain a first-order differential equation whose solution can be written as

$$y = x^{-3(\nu+1)}t - \frac{12}{\delta}(\nu+1)^2x^{-(\nu+1)},$$

where t is given by

$$\int \frac{dt}{\pm\sqrt{C_1 + \delta t^3/3}} = -\frac{1}{\nu+1}x^{-(\nu+1)} + C_2,$$

in which C_1 and C_2 are integration constants. Hence we have quadrature of Eq. (8) for given f.

We now consider Case 5.2. The two symmetries admitted by the corresponding Eq. (8) are

$$X_1 = x\frac{\partial}{\partial x} + \frac{\nu+1}{\sigma+1}y\frac{\partial}{\partial y},\ X_2 = x^{\frac{\sigma+\nu}{\sigma-1}}\frac{\partial}{\partial x} + \frac{\nu+1}{\sigma-1}x^{\frac{\nu+1}{\sigma-1}}y\frac{\partial}{\partial y}$$

with $[X_1, X_2] = \dfrac{(\nu+1)}{\sigma-1}X_2$. Following the above procedure we find that the solution of the corresponding Eq. (8) is

$$y = tx^{(\nu+1)/(\sigma-1)},$$

where t is defined by

$$\int \frac{dt}{\pm\sqrt{C_1 + 2Kt^{1-\sigma}/(1-\sigma)}} = \frac{1-\sigma}{1+\nu}x^{(1+\nu)/(1-\sigma)} + C_2,$$

in which C_1 and C_2 are arbitrary constants of integration.

Finally for Case 6.2 the corresponding Eq. (8) admits the two symmetries

$$X_1 = x\frac{\partial}{\partial x} + \frac{\nu+1}{\delta_1}\frac{\partial}{\partial y}, \ X_2 = x\ln x\frac{\partial}{\partial x} + \frac{1}{\delta_1}[2+(\nu+1)\ln x]\frac{\partial}{\partial y}$$

with $[X_1, X_2] = X_1$. In this case the solution of the corresponding Eq. (8) is

$$y = \frac{1}{\delta_1}\ln\left(\frac{x^{\nu+1}}{t}\right),$$

where t is given by

$$\int \frac{dt}{\pm\sqrt{2\delta_1 K t^3 + t^2[(\nu+1)^2 - 2\delta_1 C_1]}} = \ln x + C_2,$$

in which C_1 and C_2 are arbitrary integration constants.

3. Noether classification and integration of (8) for different fs

In this section we perform a Nother point symmetry classification of Eq. (8) with respect to the standard Lagrangian. We then obtain first integrals of the various cases, which admit Noether point symmetries and reduce the corresponding equations to quadratures.

It can easily be verified that the standard Lagrangian of Eq. (8) is

$$L = \frac{1}{2}x^n y'^2 - x^{n+\nu-1}\int f(y)dy. \tag{20}$$

The determining equation (see (47)) for the Noether point symmetries corresponding to L in (20) is

$$X^{[1]}(L) + LD(\xi) = D(B), \tag{21}$$

where X given by

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{22}$$

is the generator of Noether symmetry and $B(x,y)$ is the gauge term and D is the total differentiation operator defined by (48)

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots. \tag{23}$$

The solution of Eq. (21) results in

$$\xi = a(x),$$

$$\eta = \frac{1}{2}[a' - nx^{-1}a]y + b(x), \tag{24}$$

$$B = \frac{1}{4}x^n\left[a'' - n\left(\frac{a}{x}\right)'\right]y^2 + b'x^n y + c(x), \tag{25}$$

$$
\begin{aligned}
&[-(n+\nu-1)x^{n+\nu-2}a - a'x^{n+\nu-1}]\int f(y)dy + [-\frac{1}{2}x^{n+\nu-1}a'y \\
&\quad +\frac{1}{2}nx^{n+\nu-2}ay - x^{n+\nu-1}b]f(y) = \frac{1}{4}a'''x^n y^2 + \frac{1}{2}nx^{n-2}a'y^2 \\
&-\frac{1}{2}nx^{n-3}ay^2 - \frac{1}{4}n^2x^{n-1}\left(\frac{a}{x}\right)' y^2 + b''x^n y + b'nx^{n-1}y + c'(x).
\end{aligned} \tag{26}
$$

The analysis of Eq. (26) leads to the following eight cases:

Case 1. $n \neq \dfrac{1-\nu}{2}$, $f(y)$ arbitrary but not of the form contained in cases 3, 4, 5 and 6.

We find that $\xi = 0$, $\eta = 0$, $B =$ constant and we conclude that there is no Noether point symmetry.

Noether point symmetries exist in the following cases.

Case 2. $n = \dfrac{1-\nu}{2}$, $f(y)$ arbitrary.

We obtain $\xi = x^{\frac{1-\nu}{2}}$, $\eta = 0$ and $B =$ constant. Therefore we have a single Noether symmetry generator $X = x^{\frac{1-\nu}{2}}\partial/\partial x$. For this case the integration is trivial even without a Noether symmetry. The Noetherian first integral (47) is

$$I = \frac{1}{2}\,x^{1-\nu}\,y'^2 + \int f(y)dy$$

from which, setting $I = C$, one gets quadrature.

Case 3. $f(y)$ is linear in y.

We have five Noether point symmetries associated with the standard Lagrangian for the corresponding differential equation (8) and $sl(3,\Re)$ symmetry algebra. This case is well-known, see, e.g., (45).

Case 4. $f = \alpha y^2 + \beta y + \gamma$, $\alpha \neq 0$

There are four subcases. They are as follows:

4.1. If $n = 2\nu + 3$, $\beta = 0$ and $\gamma = 0$, we obtain $\xi = x$, $\eta = -(\nu+1)y$ and $B =$ constant. This is contained in Case 5.1 below.

4.2. If $n = 2\nu+3$, $\nu \neq -1$, $\beta^2 = 4\alpha\gamma$, we get $\xi = x$, $\eta = -(\nu+1)(y+\beta/2\alpha)$ and $B = \dfrac{\beta\gamma}{6\alpha}\,x^{3\nu+3}$. We have

$$X = x\frac{\partial}{\partial x} - (\nu+1)(y+\beta/2\alpha)\frac{\partial}{\partial y}.$$

In this case the Noetherian first integral (47) is

$$
\begin{aligned}
I = -\frac{1}{2}x^{2\nu+4}y'^2 - \frac{1}{3}\alpha x^{3\nu+3}y^3 - \frac{1}{2}\beta x^{3\nu+3}y^2 - \gamma x^{3\nu+3}y - (\nu+1)x^{2\nu+3}yy' \\
-(\nu+1)\frac{\beta}{2\alpha}x^{2\nu+3}y' - \frac{\beta\gamma}{6\alpha}x^{3\nu+3}.
\end{aligned}
$$

Thus the reduced equation is

$$\frac{1}{2}x^{2\nu+4}y'^2 + \frac{1}{3}\alpha x^{3\nu+3}y^3 + \frac{1}{2}\beta x^{3\nu+3}y^2 + \gamma x^{3\nu+3}y + (\nu+1)x^{2\nu+3}yy' + (\nu+1)\frac{\beta}{2\alpha}x^{2\nu+3}y' + \frac{\beta\gamma}{6\alpha}x^{3\nu+3} = C, \tag{27}$$

where C is an arbitrary constant. We now solve Eq. (27). For this purpose we use an invariant of X (see (49)) as the new dependent variable. This invariant is obtained by solving the Lagrange's system associated with X, viz.,

$$\frac{dx}{x} = \frac{dy}{-(\nu+1)(y+\beta/2\alpha)},$$

and is

$$u = x^{\nu+1}y + \frac{\beta}{2\alpha}x^{\nu+1}.$$

In terms of u Eq. (27) becomes

$$C = \frac{1}{2}(\nu+1)^2u^2 - \frac{1}{2}x^2u'^2 - \frac{1}{3}\alpha u^3,$$

which is a first-order variables separable ordinary differential equation. Separating the variables we obtain

$$\frac{du}{\pm\sqrt{(\nu+1)^2u^2 - (2/3)\alpha u^3 - 2C}} = \frac{dx}{x}.$$

Hence we have quadrature or double reduction of our Eq. (8) for the given f.

4.3. If $n = (\nu+4)/3$, $n \neq (1-\nu)/2, -1$, $\beta = 0$ and $\gamma = 0$, we find $\xi = x^{(2-\nu)/3}$, $\eta = -\frac{\nu+1}{3}x^{-(\nu+1)/3}y$ and $B = \frac{(\nu+1)^2}{18}y^2 + k$, k a constant. This is subsumed in Case 5.2 below.

4.4. If $n = (1-\nu)/2$, $n \neq (\nu+4)/3$, β and γ are arbitrary, we obtain $\xi = x^{\frac{1-\nu}{2}}$, $\eta = 0$. This reduces to Case 2.

Case 5. $f = \alpha y^r, \alpha \neq 0, r \neq 0, 1$.
Here we have two subcases.

5.1. If $n = \frac{r+2\nu+1}{r-1}$, we obtain $\xi = x$, $\eta = \frac{\nu+1}{1-r}y$ and $B =$ constant. The solution of Eq. (8) for the above n and f is given by

$$y = ux^{\frac{\nu+1}{1-r}}, \tag{28}$$

where u satisfies

$$\int \frac{du}{\pm\sqrt{(\nu+1)^2(1-r)^{-2}u^2 - 2\alpha(1+r)^{-1}u^{1+r} - 2C_1}} = \ln xC_2, \tag{29}$$

in which, C_1 and C_2 are arbitrary constants of integration.

We note that when $r = 5$ and $\nu = 1$, we get $n = 2$. This gives us the Lane-Emden equation $y'' + (2/x)y' + y^5 = 0$. Its general solution is given by Eq. (29) and we recover the solution given in (50). Only a one-parameter family of solutions is known in the other literature, namely,

$y = [3a/(x^2+3a^2)]^{1/2}$, $a =$ constant (see, e.g., (27) or (51)). Here we have determined a two-parameter family of solutions. Another almost unknown exact solution of $y'' + (2/x)y' + y^5 = 0$, which is worth mentioning here, is given by

$$xy^2 = \left[1 + 3\cot^2\left(\frac{1}{2}\ln\frac{x}{c}\right)\right]^{-1}, \tag{30}$$

where c is an arbitrary constant.

5.2. If $n = \dfrac{r+\nu+2}{r+1}$, with $r \neq -1$, we have $\xi = x^{\frac{r-\nu}{r+1}}$, $\eta = -\left(\dfrac{\nu+1}{r+1}\right)x^{-\frac{\nu+1}{r+1}}\,y$ and $B = \dfrac{(\nu+1)^2}{2(r+1)^2}\,y^2 + k$, where k is a constant.

In this case the solution of the corresponding Eq. (8) is

$$y = ux^{-\frac{\nu+1}{r+1}}, \tag{31}$$

where u is given by

$$\int \frac{du}{\pm\sqrt{C_1 - 2\alpha(r+1)^{-1}u^{r+1}}} = \left(\frac{r+1}{\nu+1}\right)x^{\frac{\nu+1}{r+1}} + C_2, \tag{32}$$

in which, C_1 and C_2 are arbitrary constants.

5.3. If $n = \dfrac{1-\nu}{2}$, we obtain $\xi = x^{\frac{1-\nu}{2}}$, $\eta = 0$ and $B =$ constant. This reduces to Case 2.

Case 6. $f = \alpha\exp(\beta y) + \gamma y + \delta$, $\alpha \neq 0, \beta \neq 0$.
Here again we have two subcases.

6.1. If $n = \dfrac{1-\nu}{2}$, we obtain $\xi = x^{\frac{1-\nu}{2}}$, $\eta = 0$ and $B = k$, k a constant. This reduces to Case 2.

6.2. If $n = 1$, $\nu \neq -1$, $\gamma = 0$ and $\delta = 0$, we deduce that $\xi = x$, $\eta = -(\nu+1)/\beta$ and $B = k$, k a constant.

The solution of the corresponding Eq. (8) for this case is

$$y = \frac{\nu+1}{\beta}\ln\left(\frac{u}{x}\right), \tag{33}$$

where u is defined by

$$\int \frac{du}{\pm u\sqrt{1 - 2\alpha\beta(\nu+1)^{-2}u^{\nu+1} + 2C_1\beta^2(\nu+1)^{-2}}} = \ln xC_2, \tag{34}$$

in which, C_1 and C_2 are integration constants.

Case 7. $f = \alpha\ln y + \gamma y + \delta$, $\alpha \neq 0$.
If $n = \dfrac{1-\nu}{2}$, we obtain $\xi = x^{\frac{1-\nu}{2}}$, $\eta = 0$ and $B = k$, k a constant. This reduces to Case 2.

Case 8. $f = \alpha y\ln y + \gamma y + \delta$, $\alpha \neq 0$.
If $n = \dfrac{1-\nu}{2}$, we obtain $\xi = x^{\frac{1-\nu}{2}}$, $\eta = 0$ and $B = k$, k a constant. This reduces to Case 2.

4. Systems of Lane-Emden-Fowler equations

The modelling of several physical phenomena such as pattern formation, population evolution, chemical reactions, and so on (see, for example (52)), gives rise to the systems of Lane-Emden equations, and have attracted much attention in recent years. Several authors have proved existence and uniqueness results for the Lane-Emden systems (53; 54) and other related systems (see, for example (55–57) and references therein). Here we consider the following generalized coupled Lane-Emden system (58)

$$\frac{d^2u}{dt^2} + \frac{n}{t}\frac{du}{dt} + f(v) = 0, \tag{35}$$

$$\frac{d^2v}{dt^2} + \frac{n}{t}\frac{dv}{dt} + g(u) = 0, \tag{36}$$

where n is real constant and $f(v)$ and $g(u)$ are arbitrary functions of v and u, respectively. Note that system (35)-(36) is a natural extension of the well-known Lane-Emden equation. We will classify the Noether operators and construct first integrals for this coupled Lane-Emden system.

It can readily be verified that the natural Lagrangian of system (35)-(36) is

$$L = t^n \dot{u}\dot{v} - t^n \int f(v)dv - t^n \int g(u)du. \tag{37}$$

The determining equation (see (58)) for the Noether point symmetries corresponding to L in (37) is

$$X^{[1]}(L) + LD(\tau) = D(B), \tag{38}$$

where X is given by

$$X = \tau(t,u,v)\frac{\partial}{\partial t} + \xi(t,u,v)\frac{\partial}{\partial u} + \eta(t,u,v)\frac{\partial}{\partial v}, \tag{39}$$

with first extension (59)

$$X^{[1]} = X + (\dot{\xi} - \dot{u}\dot{\tau})\frac{\partial}{\partial \dot{u}} + (\dot{\eta} - \dot{v}\dot{\tau})\frac{\partial}{\partial \dot{v}}, \tag{40}$$

where $\dot{\tau}, \dot{\xi}$ and $\dot{\eta}$ denote total time derivatives of τ, ξ and η respectively. Proceeding as in Section 3, (see details of computations in (58)) we obtain the following seven cases:

Case 1. $n \neq 0$, $f(u)$ and $g(v)$ arbitrary but not of the form contained in cases 3, 4, 5 and 6.

We find that $\tau = 0$, $\xi = 0$, $\eta = 0$, $B =$ constant and we conclude that there is no Noether point symmetry.

Noether point symmetries exist in the following cases.

Case 2. $n = 0$, $f(u)$ and $g(v)$ arbitrary.

We obtain $\tau = 1$, $\xi = 0$, $\eta = 0$ and $B =$ constant. Therefore we have a single Noether symmetry generator

$$X_1 = \frac{\partial}{\partial t} \tag{41}$$

with the Noetherian integral given by

$$I = \dot{u}\dot{v} + \int f(u)du + \int g(v)dv.$$

Case 3. $f(v)$ and $g(u)$ constants. We have eight Noether point symmetries associated with the standard Lagrangian for the corresponding system (35)-(36) and this case is well-known.

Case 4. $f = \alpha v + \beta,\ \ g = \gamma u + \lambda,$ where α, β, γ and λ are constants, with $\alpha \neq 0$ and $\gamma \neq 0$.

There are three subcases, namely

4.1. For all values of $n \neq 0, 2$, we obtain $\tau = 0$, $\xi = a(t)$, $\eta = l(t)$ and $B = t^n \dot{l} u + t^n \dot{a} v - \lambda \int t^n a\, dt - \beta \int t^n l dt + C_1$, C_1 a constant. Therefore we obtain Noether point symmetry

$$X_1 = a(t)\frac{\partial}{\partial u} + l(t)\frac{\partial}{\partial v}, \tag{42}$$

where $a(t)$ and $l(t)$ satisfy the second-order coupled Lane-Emden system

$$\ddot{l} + \frac{n}{t}\dot{l} + \gamma a = 0,\ \ddot{a} + \frac{n}{t}\dot{a} + \alpha l = 0. \tag{43}$$

The first integral in this case is given by

$$I_1 = t^n \dot{l} u + t^n \dot{a} v - \lambda \int t^n a\, dt - \beta \int t^n l dt - a t^n \dot{v} - l t^n \dot{u}.$$

4.2. $n = 2$. In this subcase the Noether symmetries are X_1 given by the operator (42) and

$$X_2 = \frac{\partial}{\partial t} - u t^{-1}\frac{\partial}{\partial u} - v t^{-1}\frac{\partial}{\partial v}. \tag{44}$$

The value of B for the operator X_2 is given by $B = uv$.

The associated first integral for X_2 is given by

$$I_2 = uv + \frac{\alpha}{2}t^2 v^2 + \frac{\gamma}{2}t^2 u^2 + ut\dot{v} + vt\dot{u} + t^2\dot{u}\dot{v}.$$

In this subcase, we note that the first integral corresponding to X_1 is subsumed in Case 4.1 above with β, $\lambda = 0$.

4.3. $n = 0$. Here the Noether operators are X_1 given by the operator (42) and

$$X_2 = \frac{\partial}{\partial t},\ \text{with } B = C_2, C_2 \text{ a constant.} \tag{45}$$

This reduces to Case 2.

We note also that the first integral associated with X_1 is contained in Case 4.1 above where $a(t)$ and $l(t)$ satisfy the coupled system

$$\ddot{l} + \gamma a = 0,\ \ddot{a} + \alpha l = 0. \tag{46}$$

Case 5. $f = \alpha v^r$, $g = \beta u^m$, $m \neq -1$ and $r \neq -1$ where α, β are constants, with $\alpha \neq 0$ and $\beta \neq 0$.

There are three subcases,viz.,

5.1. If $n = \dfrac{2m+2r+mr+3}{rm-1}$, $rm \neq 1, m \neq -1, m \neq 1$ and $r \neq -1$, we obtain $\tau = t$, $\xi = -\dfrac{(1+n)}{m+1}u$, $\eta = -\dfrac{(1+n)}{r+1}v$ and $B =$ constant.

Thus we obtain a single Noether point symmetry

$$X = t\frac{\partial}{\partial t} - \frac{(1+n)}{m+1}u\frac{\partial}{\partial u} - \frac{(1+n)}{r+1}v\frac{\partial}{\partial v} \tag{47}$$

with the associated first integral

$$I = \beta t^{n+1}\frac{u^{m+1}}{m+1} + \alpha t^{n+1}\frac{v^{r+1}}{r+1} + \frac{(n+1)}{m+1}t^n u\dot{v} + \frac{(n+1)}{r+1}t^n v\dot{u} + t^{n+1}\dot{u}\dot{v}.$$

We now consider the case when $m = -1$ and $r = -1$, in Case 5. Here we have two subcases

Case 5.2. $n = 0$, $(m = -1, r = -1)$.

This case provides us with two Noether symmetries namely,

$$X_1 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} \text{ and } X_2 = \frac{\partial}{\partial t} \text{ with } B = 0 \text{ for both cases.} \tag{48}$$

We obtain the Noetherian first integrals corresponding to X_1 and X_2 as

$$I_1 = \dot{u}v - u\dot{v},\ I_2 = \dot{u}\dot{v} + \ln u + \ln v,$$

respectively.

Case 5.3. $n = -1$ $(m = -1, r = -1)$.

Here we obtain two Noether symmetry operators, viz.,

$$X_1 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} \text{ with } B = 0 \text{ and } X_2 = t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u} \\ \text{with } B = -2\ln t \tag{49}$$

and first integrals associated with X_1 and X_2 are given by

$$I_1 = \dot{u}vt^{-1} - u\dot{v}t^{-1}, I_2 = -2\ln t + \ln u + \ln v - 2u\dot{v}t^{-1} + \dot{u}\dot{v},$$

respectively.

Case 6. $f = \alpha\exp(\beta v) + \lambda$, $g = \delta\exp(\gamma u) + \sigma$, $\alpha, \beta, \lambda, \gamma, \delta$, and σ are constants, with $\alpha \neq 0, \beta \neq 0, \delta \neq 0, \gamma \neq 0$.

There are two subcases. They are

6.1. If $n = 1$, $\lambda = 0$ and $\sigma = 0$, we obtain $\tau = t$, $\xi = -\dfrac{2}{\gamma}$, $\eta = -\dfrac{2}{\beta}$ and $B = C_3$, C_3 a constant. Therefore we have a single Noether point symmetry

$$X_1 = t\frac{\partial}{\partial t} - \frac{2}{\gamma}\frac{\partial}{\partial u} - \frac{2}{\beta}\frac{\partial}{\partial v} \tag{50}$$

and this results in the first integral

$$I = t^2\dot{u}\dot{v} + \frac{\alpha t^2}{\beta}\exp(\beta v) + \frac{\delta t^2}{\gamma}\exp(\gamma u) + \frac{2}{\gamma}t\dot{v} + \frac{2}{\beta}t\dot{u}.$$

6.2. If $n = 0$, $\lambda = 0$ and $\sigma = 0$, we deduce that $\tau = 1$, $\xi = 0$, $\eta = 0$ and $B = C_4$, C_4 a constant. The Noether operator is given by

$$X_1 = \frac{\partial}{\partial t}. \tag{51}$$

This reduces to Case 2.

Case 7. $f = \alpha \ln v + \beta$, $g = \gamma \ln u + \lambda$, where α, β, γ and λ are constants with $\alpha \neq 0$, $\gamma \neq 0$. If $n = 0$, we obtain $\tau = 1$, $\xi = 0$, $\eta = 0$ and $B = C_5$, C_5 a constant. This reduces to Case 2.

5. Concluding remarks

In this Chapter we gave a brief history of the Lane-Emden-Fowler equation and its applications in various fields. Several methods have been employed by scientists to solve the Lane-Emden-Fowler equation. Various generalizations of the Lane-Emden-Fowler equations were given which can be found in the literature. Also we gave the extension of the Lane-Emden equation to the System of Lane-Emden equations. We presented the complete Lie symmetry group classification of a generalized Lane-Emden-Fowler equation and performed the Lie and Noether symmetry analysis of this problem. It should be noted that Lie symmetry method is the most powerful tool to solve nonlinear differential equations. Finally, we classified a generalized coupled Lane-Emden system with respect to the standard first-order Lagrangian according to its Noether point symmetries and obtained first integrals for the corresponding Noether operators.

6. Acknowledgement

The author thanks Dr Ben Muatjetjeja for fruitful discussions.

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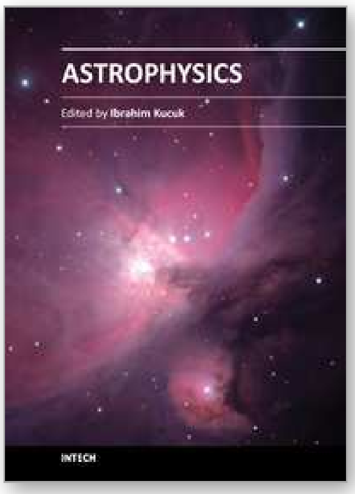

Astrophysics
Edited by Prof. Ibrahim Kucuk

ISBN 978-953-51-0473-5
Hard cover, 398 pages
Publisher InTech
Published online 30, March, 2012
Published in print edition March, 2012

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