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Goldberg's Number Influence on the Validity Domain of the Quasi-Linear Approximation of Finite Amplitude Acoustic Waves

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1. Introduction

Nonlinear propagation occurs widely in many acoustic systems, especially in the field of medical ultrasound. Despite the widespread use of ultrasound in diagnosis and therapy, the propagation of ultrasound through biological media was modeled as a linear process for many years. The invalidity of infinitesimal acoustic assumption, at biomedical frequencies and intensities, was demonstrated by Muir and Carstensen (Muir & Carstensen, 1980). It was realized that nonlinear effects are not negligible and must therefore be taken into account in theoretical developments of ultrasound in biomedical research. Indeed, increasing the acoustic frequency or intensity in order to enhance resolution or penetration depth may alter the beam shape in a way not predicted by linear theory.

Nonlinear effects occur more strongly when ultrasound propagates through slightly dissipative liquids such as water or amniotic fluid. As in medical sonography, the full bladder or the pregnant uterus, which may be filled with amniotic fluid, is used as an acoustic window in many types of diagnoses; a special attention is given to slightly dissipative liquids where the possibility of signal distortions has several implications. However, within soft tissues, the tendency for wave distortion to occur is limited by dissipation.

In absorbing medium, nonlinear effects cannot be examined without considering dissipation. The absorption limits the generation of harmonics by decreasing their amplitudes gradually. In addition, as the absorption coefficient increases with frequency, the energy transformation towards frequencies higher than the fundamental frequency (generation of harmonics) can also lead to significant acoustic losses. Nonlinear effects create all higher harmonics from the energy at the insonation frequency, but, due to the absorption of high frequency components, only the lower harmonic orders and the fundamental remain. So, the tendency for wave distortion to occur is limited by dissipation.

Dissipation can have various origins (Sehgal & Greenleaf, 1982): viscosity (resulting from shear motions between fluid particles), thermal conduction (due to the energy loss resulting from thermal conduction between particles) or molecular relaxation (where the molecular equilibrium state is affected by the pressure variations of the acoustic wave propagation).

Nonlinear effects and dissipation are antagonistic phenomena. The nonlinearity mechanism shocks the wave by generating harmonics while dissipation increases with frequency and

attenuates the harmonics resulting from nonlinear effects. The shock length l_s (Enflo & Hedberg, 2002; Naugolnykh & Ostrovosky, 1998) quantifies the influence of the nonlinear phenomena, and it is necessary to define another parameter, denoted Goldberg's number Γ (Goldberg, 1957), when dissipation is added. Γ represents the ratio of the absorption length l_a (the inverse of the absorption coefficient α and corresponds to the beginning of the old age region) to the shock length l_s at which the waveform would shock if absorption phenomena were absent:

$$\Gamma = \frac{l_a}{l_s} = \frac{k\beta M}{\alpha} \tag{1}$$

where k, M, β and α are, respectively, the wave number, the acoustic Mach number, the acoustic nonlinearity parameter and the absorption coefficient.

It should be noted that higher harmonics may turn the wave into shock state. On the other hand, dissipation attenuates higher harmonics much more than lower harmonics, thus making it more difficult for the waves to go into shock.

The dimensionless parameter Γ measures the relative importance of the nonlinear and dissipative phenomena, which are in perpetual competition. Thus, the Goldberg's number is a reliable indicator for any analysis including these two phenomena. An analysis based on the Goldberg's number is important since it is an essential step for solving general problems involving ultrasound waves of finite amplitude.

Nowadays, Tissue Harmonic Imaging (THI) or second harmonic imaging offers several advantages over conventional pulse-echo imaging. Both harmonic contrast and lateral resolution are improved in harmonic mode. Tissue Harmonic Imaging also provides a better signal to noise ratio which leads to better image quality in many applications. The major benefit of Tissue Harmonic Imaging is artifact reduction resulting in less noisy images, making cysts appear clearer and improving visualization of pathologic conditions and normal structures. Indeed, Tissue Harmonic Imaging is widely used for detecting subtle lesions (e.g., thyroid and breast) and visualizing technically- challenging patients with high body mass index.

In order to create images exclusively from the second harmonic, a theoretical review with some mathematical approximations is elaborated, in this chapter, to derive an analytical expression of the second harmonic. The performance of the simplified model of the second harmonic is interesting, as it can provide a simple, useful model for understanding phenomena in diagnostic imaging.

Despite the significant advantages offered by Tissue Harmonic Imaging, theory has been partially explained. A number of works were elaborated over recent decades. Among these, are Trivett and Van Buren (Trivett & Van Buren, 1981) work which have presented an analysis of the generated harmonics based on the generalized Burgers' equation. Significant differences in the calculated harmonic content were found by Trivett and Van Buren when compared with those obtained by Woodsum (Woodsum, 1981). No explanation was given by Trivett and Van Buren to justify their results. In an author's reply, Woodsum seemed to attribute these differences to the high number of terms retained by Trivett and Van Buren in the Fourier series.

Similarly, Haran and Cook (Haran & Cook, 1983) have used the Burgers' equation to elaborate an algorithm for calculating harmonics generation by a finite amplitude plane wave of ultrasound propagating in a lossy and nondispersive medium. Their algorithm accounts for an absorption coefficient of any desired frequency dependence. The variation effect of the absorption coefficient on the second harmonic was demonstrated in a medium similar to carbon tetrachloride. Calculations for several types of tissue and biological fluids were presented. It was shown that for some biological media having a low absorption coefficient, a significant distortion of the plane wave can be observed for large propagation ranges.

Recently, D'hooge et al. (D'hooge et al., 1999) have analyzed the nonlinear propagation effects of pulses on broadband attenuation measurements and their implications in ultrasonic tissue characterization by using a simple mathematical model based on the numerical solution, in the time domain, of the Burgers' equation. The developed model has been validated by measuring the absorption coefficient of both a tissue-mimicking phantom in vitro and a liver in vivo at several pressure amplitudes using transmission and reflection measurements, respectively.

In the present chapter, the intensity effects on the behavior of the fundamental and the generated second harmonic, by using both the numerical solution of the Burgers' equation and the analytical expressions established with the quasi-linear approximation are examined. An analysis on the validity domain of the fundamental and the second harmonic analytical expressions established with the quasi-linear approximation is elaborated. The deviations resulting from the analytical expressions established with the quasi-linear approximation and the numerical solution of the Burgers' equation are estimated. This investigation is based on Krassilnikov et al. (Krassilnikov et al., 1957) experimental results. These experimental data concern water and glycerol that correspond, respectively, to a weakly dissipative liquid approaching the characteristics of urine or amniotic fluid (Bouakaz et al., 2004) and a strongly dissipative liquid with some similarities to soft tissues.

It should be noted that in this study all derivations are developed entirely in the frequency domain, thus avoiding both the steep waveform problems and the use of FFT, which alternates between time and frequency domains. The utility of the method resides in the ease with which it can be implemented on a digital computer.

2. Theoretical formulation

The description of acoustic waves in a liquid is founded on the theory of motion of a liquid, which is considered to be continuous. In the present investigation, the viscosity and the heat conduction coefficients, although in general are functions of the state variables, are assumed to be constant. The theoretical formulation of the propagation of finite amplitude plane progressive waves in a homogeneous and dissipative liquid is elaborated in section 2.1, and the theoretical model is based on the derivation of a nonlinear partial differential equation in which the longitudinal particle velocity is a function of time and space. In section 2.2, the dimensionless Burgers' equation is presented, which is considered to be among the most exhaustively studied equations in the theory of nonlinear waves.

2.1 Basic equations

The propagation of finite amplitude plane progressive waves in a homogeneous and dissipative liquid is governed by the Burgers' equation. Here, it is assumed that the ultrasonic wave propagates in the positive z direction, and the differential change of the longitudinal particle velocity with respect to z is given by (Enflo & Hedberg, 2002; Naugolnykh & Ostrovosky, 1998):

$$\frac{\partial u(z,\tau)}{\partial z} = \frac{\beta}{c_0^2} u(z,\tau) \frac{\partial u(z,\tau)}{\partial \tau} + \frac{D}{2c_0^3} \frac{\partial^2 u(z,\tau)}{\partial \tau^2} \tag{2}$$

$D = \frac{1}{\rho_0}\left[\left(\frac{4}{3}\mu + \xi\right) + \kappa\left(\frac{1}{c_v} - \frac{1}{c_p}\right)\right]$ is the diffusivity of the sound for a thermoviscous fluid. This parameter is a function of the fluid shear viscosity μ, the fluid bulk viscosity ξ, the thermal conductivity κ, the specific heat at constant volume c_v and the specific heat at constant pressure c_p. The acoustic nonlinearity parameter $\beta = 1 + B/2A$ is function of the nonlinearity parameter of the medium B/A, which represents the ratio of quadratic to linear terms in the isentropic pressure-density relation (Hamilton & Blackstock, 1988; Khelladi et al., 2007, 2009). $\tau = t - z/c_0$ is the retarded time, c_0 is the infinitesimal sound speed and ρ_0 is the undisturbed density of the liquid.

The term on the left hand side of equation (2) is the linear wave propagation. The first term on the right hand side of equation (2) is the nonlinear term that accounts for quadratic nonlinearity producing cumulative effects in progressive plane wave propagation, while the second term represents the loss due to viscosity and heat conduction or any other agencies of dissipation.

Nonlinear propagation in a dissipative liquid is considered using Fourier series expansion. By assuming that the solution of equation (2) is periodic in time with period $2\pi/\omega_0$, the solution can be written as the sum of the fundamental and the generated harmonics. Thus $u(z,\tau)$ can be developed in Fourier series, with amplitudes that are functions of the spatial coordinate z:

$$u(z,\tau) = \sum_{n=1}^{+\infty} [v_n(z)\cos(n\omega_0\tau) + u_n(z)\sin(n\omega_0\tau)] \tag{3}$$

ω_0 is the characteristic angular frequency and v_n, u_n are the Fourier coefficients of the nth harmonic.

When complex notation is used, equation (3) changes to (Haran & Cook, 1983; Ngoc & Mayer, 1987):

$$u(z,\ \tau) = \sum_{n=-\infty}^{+\infty} W_n(z)\ e^{in\omega_0\tau} \tag{4}$$

The complex amplitude can be expressed as $W_n = w_n e^{in\varphi_n}$ where w_n, φ_n correspond respectively to the amplitude and the phase of the nth harmonic, and $i^2 = -1$. Note that $W_{-n}^* = W_n$, * symbolizes the complex conjugate.

For the easiest derivation, equation (4) is substituted into equation (2) (Haran & Cook, 1983; Ngoc & Mayer, 1987):

$$\frac{\partial W_n}{\partial z} = i\frac{\beta\omega_0}{c_0^2}\sum_{m=-\infty}^{+\infty}(n-m)W_m W_{n-m} - \alpha n^2 W_n \tag{5}$$

where $\alpha = \frac{D\omega_0^2}{2c_0^3}$

Equation (5) describes the amplitude variation of the nth harmonic in the propagation direction z. The summation over m expresses nonlinear interactions among various spectral components caused by the energy transfer process, while the other term accounts for loss due to dissipation relative to the nth harmonic.

Equation (5) is rewritten in another form (Haran & Cook, 1983; Ngoc & Mayer, 1987):

$$\frac{\partial W_n}{\partial z} = i\frac{\beta\omega_0}{c_0^2}\left[\sum_{m=1}^{n-1} m W_m W_{n-m} + \sum_{m=n}^{+\infty} n W_m W_{m-n}^*\right] - \alpha n^2 W_n \tag{6}$$

By using the real notation, knowing that $W_n = \frac{v_n - iu_n}{2}$ and $W_{-n} = \frac{v_n + iu_n}{2}$, equation (6) yields two coupled partial differential equations governing the behavior of the components v_n and u_n as a function of the spatial coordinate z (Aanonsen et al., 1984; Hamilton et al., 1985):

$$\frac{\partial v_n}{\partial z} = \frac{\beta\omega_0}{2c_0^2}\left[\sum_{m=1}^{n-1} m(u_m v_{n-m} + v_m u_{n-m}) - \sum_{m=n}^{+\infty} n(v_m u_{m-n} - u_m v_{m-n})\right] - \alpha n^2 v_n \tag{7}$$

$$\frac{\partial u_n}{\partial z} = \frac{\beta\omega_0}{2c_0^2}\left[\sum_{m=1}^{n-1} m(u_m u_{n-m} - v_m v_{n-m}) - \sum_{m=n}^{+\infty} n(u_m u_{m-n} + v_m v_{m-n})\right] - \alpha n^2 u_n \tag{8}$$

For a sinusoidal source condition, $u(0,\tau) = u_0 \sin(\omega_0\tau)$ (Aanonsen et al., 1984; Hamilton et al., 1985; Hedberg, 1999; Menounou & Blackstock, 2004), equation (3) becomes:

$$u(z,\tau) = \sum_{n=1}^{+\infty} u_n(z)\sin(n\omega_0\tau) \tag{9}$$

Equation (8) is then written more simply as:

$$\frac{\partial u_n}{\partial z} = \frac{\beta\omega_0}{2c_0^2}\left[\sum_{m=1}^{n-1} m u_m u_{n-m} - \sum_{m=n}^{+\infty} n u_m u_{m-n}\right] - \alpha n^2 u_n \tag{10}$$

The incremental change of the particle velocity can be approximated by the first order truncated power series (Haran & Cook, 1983; Ngoc et al., 1987):

$$u(z+\Delta z,t)=u(z,t)+\frac{\partial u(z,t)}{\partial z}\Delta z \tag{11}$$

By combining equations (10) and (11), an iterative description of finite amplitude plane wave propagation in a homogeneous and dissipative liquid, is obtained:

$$u_n(z+\Delta z)=u_n(z)+\frac{\beta\omega_0}{2c_0^2}\left[\sum_{m=1}^{n-1}mu_m(z)u_{n-m}(z)-\sum_{m=n}^{+\infty}nu_m(z)u_{m-n}(z)\right]\Delta z-\alpha n^2u_n(z)\Delta z \tag{12}$$

The first summation term on the right hand side of equation (12) represents the contribution of lower order harmonics to the nth harmonic, while the second one is associated with the contribution of higher order harmonics. According to the sign of each contribution the nth harmonic energy can be enhanced or decreased. The last term in this equation represents losses undergone by the nth harmonic.

Generally, the absorption coefficient α depends on the propagation medium characteristics and the insonation frequency. For the considered viscous fluids, this frequency dependence is quadratic with frequency and can be represented by (Smith & Beyer, 1948; Willard, 1941):

$$\alpha=\alpha_0f^2 \tag{13}$$

where α_0 depends upon the nature of the liquid, and $f=\omega_0/2\pi$ is the insonation frequency.

Therefore the Goldberg's number Γ, increases with the amplitude of excitation and decreases with frequency.

Equation (12) becomes:

$$u_n(z+\Delta z)=u_n(z)+\frac{\beta\omega_0}{2c_0^2}\left[\sum_{m=1}^{n-1}mu_m(z)u_{n-m}(z)-\sum_{m=n}^{+\infty}nu_m(z)u_{m-n}(z)\right]\Delta z-\alpha_nu_n(z)\Delta z \tag{14}$$

where $\alpha_n=\alpha_0n^2f^2$

Equation (14) allows the determination of the nth harmonic amplitude at the location $z+\Delta z$ in terms of all harmonics at the preceding spatial coordinate z. This derivation requires an appropriate truncation of the finite series on the right hand side of equation (14) to ensure a negligibly small error in the highest harmonic of interest and to maintain some acceptable accuracy.

In the hypothesis of the quasi-linear approximation, all the harmonics of higher order than two can be neglected in the numerical solution of the Burgers' equation, so equation (14) changes to:

$$\begin{cases} \dfrac{\partial u_1(z)}{\partial z} = -\dfrac{\beta\omega_0}{2c_0^2} u_1(z)u_2(z) - \alpha_1 u_1(z) \\ \dfrac{\partial u_2(z)}{\partial z} = \dfrac{\beta\omega_0}{2c_0^2} u_1^2(z) - \alpha_2 u_2(z) \end{cases} \tag{15}$$

where $\alpha_1 = \alpha_0 f^2$ and $\alpha_2 = 4\alpha_0 f^2 = 4\alpha_1$ denote the absorption coefficients of the fundamental and the second harmonic, respectively.

In many situations, the experimental studies are based on pressure measurements. Knowing that the ratio of the nth harmonic pressure to the associated particle velocity is given by $p_n(z,t) = \rho_0 c_0 u_n(z,t)$ (Germain et al., 1989); equation (15) is rewritten as:

$$\begin{cases} \dfrac{\partial p_1(z)}{\partial z} = -\dfrac{\beta\omega_0}{2\rho_0 c_0^3} p_1(z)p_2(z) - \alpha_1 p_1(z) \\ \dfrac{\partial p_2(z)}{\partial z} = \dfrac{\beta\omega_0}{2\rho_0 c_0^3} p_1^2(z) - \alpha_2 p_2(z) \end{cases} \tag{16}$$

If $p_2(z) << \dfrac{2P_0}{\Gamma}$, then $\dfrac{\beta\omega_0}{2\rho_0 c_0^3} p_1(z)p_2(z)$ can be neglected comparatively to $\alpha_1 p_1(z)$. The acoustic pressure of the fundamental can be written as (Gong et al., 1989; Thuras et al., 1935):

$$p_1(z) = P_0 e^{-\alpha_1 z} \tag{17}$$

where P_0 is the characteristic pressure amplitude (the value of the fundamental pressure at $z = 0$).

Equation (16) becomes:

$$\frac{\partial p_2(z)}{\partial z} = hP_0^2 e^{-2\alpha_1 z} - \alpha_2 p_2(z) \tag{18}$$

with $h = \dfrac{\beta\omega_0}{2\rho_0 c_0^3}$

The solution of equation (18) is easily obtained. Knowing that for $z = 0$ $p_2(0) = 0$, the acoustic pressure of the second harmonic component can be expressed as (Cobb, 1983; Thuras et al., 1935):

$$p_2(z) = hP_0^2 \left(\frac{e^{-\alpha_2 z} - e^{-2\alpha_1 z}}{2\alpha_1 - \alpha_2} \right) \tag{19}$$

Moreover, if the term $(\alpha_2 - 2\alpha_1)z << 1$, an approximation of equation (19) can be made (Bjørnø, 2002; Cobb, 1983; Zhang et al.,1991):

$$p_2(z) = hP_0^2 z e^{-(\alpha_1 + \alpha_2/2)z} \tag{20}$$

2.2 Dimensionless equations

For theoretical analysis as well as for numerical implementation, it is more convenient to define dimensionless variables, by using the characteristic particle velocity U_0, the characteristic time $1/\omega_0$ and the lossless plane wave shock formation length l_s:

$$U = \frac{u}{U_0},\ \theta = \omega_0 \tau \ \text{ and } \ \sigma = \frac{z}{l_s} \tag{21}$$

where U, θ and σ are, respectively, the dimensionless longitudinal particle velocity, the dimensionless time and the dimensionless propagation path.

Insertion of equation (21) into the Burgers' equation (equation (2)), gives the dimensionless equation (Bjørnø, 2002; Fenlon, 1971; Hedberg, 1994):

$$\frac{\partial U(\sigma,\theta)}{\partial \sigma} = U(\sigma,\theta)\frac{\partial U(\sigma,\theta)}{\partial \theta} + \Gamma^{-1}\frac{\partial^2 U(\sigma,\theta)}{\partial \theta^2} \tag{22}$$

The dimensionless amplitude of the nth harmonic at the dimensionless location $\sigma + \Delta\sigma$ in terms of all harmonics at the preceding dimensionless location σ can be written as:

$$U_n(\sigma+\Delta\sigma) = U_n(\sigma) + \frac{1}{2}\left[\sum_{m=1}^{n-1} m U_m(\sigma) U_{n-m}(\sigma) - \sum_{m=n}^{+\infty} n U_m(\sigma) U_{m-n}(\sigma)\right]\Delta\sigma - n^2\Gamma^{-1}U_n(\sigma)\Delta\sigma \tag{23}$$

With this dimensionless notation, the acoustic pressure of the fundamental and the second harmonic can be expressed as:

$$p_1(\sigma) = P_0 e^{-\alpha_1 l_s \sigma} \tag{24}$$

$$p_2(\sigma) = \frac{1}{2}P_0\left(\frac{e^{-\alpha_2 l_s\sigma} - e^{-2\alpha_1 l_s\sigma}}{(2\alpha_1 - \alpha_2)\, l_s}\right) \tag{25}$$

In the case of $(\alpha_2 - 2\alpha_1) l_s \sigma << 1$, equation (25) becomes:

$$p_2(\sigma) = \frac{1}{2}P_0\sigma e^{-(\alpha_1+\alpha_2/2) l_s\sigma} \tag{26}$$

3. Numerical experiments and discussions

Krassilnikov et al. (Krassilnikov et al., 1957) experimental data for water and for glycerol are used in order to simulate the amplitude of the first two harmonics, by using both the numerical solution of the Burgers' equation (equation (23)) and the analytical expressions established with the quasi-linear approximation (equations (24), (25) and (26)). Table 1 lists material properties.

According to Krassilnikov et al. (Krassilnikov et al., 1957) experimental work, the absorption coefficient is a quadratic function of frequency. The absorption coefficient is that obtained from an infinitesimal acoustic excitation, even though the acoustic intensity increases. In the

case of water $\alpha_0 = 0.23\ 10^{-13}\ Np.m^{-1}.Hz^{-2}$ and for glycerol $\alpha_0 = 26\ 10^{-13}\ Np.m^{-1}.Hz^{-2}$ (Krassilnikov et al., 1957).

Nonlinear effects occur more strongly when ultrasound propagates through slightly dissipative liquids, so a special attention is given to a propagation medium characterized by a Goldberg number greater than unity. In this case, when the waveform approaches the shock length, nonlinear effects dominate dissipation phenomena. The amplitude of the generated harmonics increases at the expense of the fundamental component. After the shock length, absorption limits the generation of harmonics by decreasing theirs amplitudes gradually with the propagation path. For this reason, all the simulations of the first two harmonics are plotted as a function of the dimensionless location σ up to unity and for several values of the acoustic intensity. Moreover, all the shock lengths for several intensities are greater than $19.8\,\text{cm}$ (Table 2). As in biomedical diagnosis the region of interest (ROI) is about $20\,\text{cm}$, it is absolutely useless to explore beyond $\sigma = 1$ and the selected range $0 \le \sigma \le 1$ is amply appropriate for this kind of investigation.

It should be pointed out that the shock length l_s depends on the medium characteristics ρ_0, c_0, β and on the external parameters such as the insonation frequency and the amplitude of excitation. In this study, the insonation frequency is fixed at $2\,\text{MHz}$, thus the shock length for a given medium will depend only upon the amplitude of excitation.

Among all the configurations presented in this study, including various acoustic intensities and two analyzed mediums, only one case is sensitive in biomedical diagnostic and must be analyzed with extreme caution. Indeed, a more favorable situation where nonlinear effects have sufficient time to be entirely established corresponds to the case of water, for which the acoustic intensity is equal to $4.7\,\text{W/cm}^2$ and as a consequence a shock length equal to $19.8\,\text{cm}$. As the generation of harmonics occurs while moving away from the source and approaching the shock length, the greatest signal distortion may occur in the range of interest. Moreover, the irradiation of living tissue with shock waves in diagnostic processes appears risky since the damage and exposure criteria for these radiations have not been delineated.

It should be noted that all the simulations are made with intensities of $0.2 - 4.7\,\text{W/cm}^2$ (Table 2), which correspond to breast lesion diagnosis (Nightingale et al., 1999).

It will be stated by the derivation of the Goldberg number that water surpasses any tissue in its ability to produce extremely distorted waveforms even at relatively low intensity. So, a special attention is given to this liquid where the possibility of distortion occurring has several implications. Indeed, water can generate extreme waveform distortion compared to glycerol, as indicated by the Goldberg's number for water, which is 200 times larger than that of glycerol for an acoustic intensity of about $0.2\,\text{W/cm}^2$ (Table 2).

Parameters	Temperature (°C)	Density $\rho_0(kg/m^3)$	Sound velocity $C_0(m/s)$	Acoustic nonlinearity parameter β
Water	20	998	1481	3.48
Glycerol	20	1260	1980	5.4

Table 1. Material properties.

Water			
Temperature (°C)	Intensity (W/cm²)	Shock length (m)	Goldberg Number
20	0.34	0.739	14.7
20	2	0.304	35.7
20	4.7	0.198	54.9
Glycerol			
Temperature (°C)	Intensity (W/cm²)	Shock length (m)	Goldberg Number
22	0.2	1.443	0.07
21	2.5	0.408	0.24
19	4.5	0.304	0.32

Table 2. Goldberg's number for water and glycerol with intensities of $0.2 - 4.7\ \mathrm{W/cm^2}$ and an insonation frequency of $2\ \mathrm{MHz}$.

Initially, the ultrasonic wave is taken to be purely sinusoidal with a frequency of $2\ \mathrm{MHz}$ in the two considered media. Only the fundamental wave exists at the starting location $\sigma = 0$, and the other harmonic modes are generated as the wave propagates from the source. Through an iterative method, the value of the Goldberg number is inserted into the Burgers' equation in order to determine its numerical solution (Table 2). 40 harmonics are retained to simulate the numerical solution of the Burgers' equation which is considered, in the deviation calculus, as an exact solution.

For a better readability and interpretation of the obtained numerical data, a symbol with a defined shape and type is inserted on the graphic layout of the analyzed functions. All the following simulations exploit equations (23), (24), (25) and (26) corresponding respectively to the numerical solution of the Burgers' equation, the quasi-linear approximation of the acoustic pressure of the fundamental, the quasi-linear approximation of the second harmonic and the quasi-linear approximation of the approximated second harmonic.

The simulations relating to water and glycerol are represented in all figures (a) and (b), respectively.

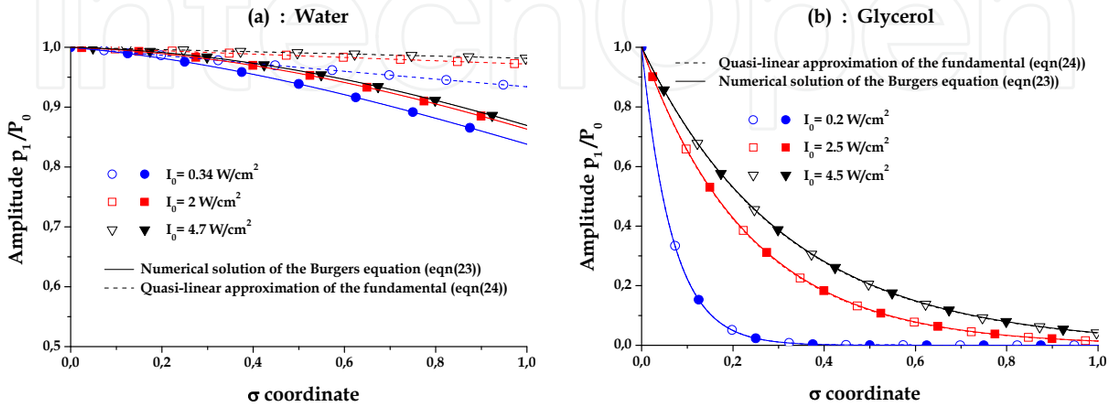

Fig. 1. Pressure amplitude P_1/P_0 versus the σ coordinate.

The amplitude p_1/P_0 (figure 1a, figure 1b) and the amplitude p_2/P_0 (figure 2a, figure 2b) increase with Γ (Table 2). So, the effect of the increased acoustic intensity is to enhance the amplitude of the fundamental and also that of the second harmonic.

In the hypothesis of linear acoustics, increasing the absorption coefficient leads systematically to a decrease of the wave amplitude. The finite amplitude waves do not obey to the same principle because nonlinear effects and dissipation are two phenomena in perpetual contest. The interplay between these two phenomena developed along the propagation path is not simply an additive effect as normally assumed in linear acoustics. Therefore, a measure of whether nonlinear effects or absorption will prevail is the Goldberg's number Γ. The larger Γ is, the more nonlinear effects dominate. Whereas for values of $\Gamma < 1$, absorption is so strong that no significant nonlinear effects occur. Thus the calculation of the Goldberg's number is required to quantify the amplitude of the generated harmonics.

By taking water as an example, the most significant amplitude of the generated harmonic, for various values of intensity, corresponds to the highest Goldberg's number (figure 2a). This is in perfect agreement with physical phenomena that take place in the analyzed medium. Indeed, a high Goldberg number corresponds to a predominance of the nonlinearity phenomenon as compared to dissipation, which represents the main factor of amplitude decrease. This situation is also apparent for glycerol (figure 2b).

For a slightly dissipative liquid, it can be seen that the second harmonic component grows cumulatively with increasing the normalized length σ at the expense of the fundamental (figure 1a, figure 2a). Its growth begins to taper off at the location of the initial shock formation, beyond this location the curves decay as expected. So, the nonlinearity mechanism is a bridge that facilitates the energy exchange among different harmonic modes. An increase of the Goldberg's number enhances the transfer of energy from the fundamental to higher harmonics and between harmonics themselves. Thus, the generated harmonics can only follow the evolution of the fundamental which gives them birth.

However, for a strongly dissipative medium, the absorption is so strong that significant nonlinear effects do not occur. Indeed, the old age region begins at a range smaller than the shock length and once nonlinear effects take place, absorption dominates the behavior of the fundamental and the generated harmonic (figure 1b, figure 2b). In absorbing media, the exchange of energy is more complicated, because absorption diminishes amplitude with increasing the propagation path and acts as a low pass filter that reduces the energy of higher harmonics (figure 2b).

The evaluation of the relative deviation, for each analytical expression in relation to the numerical solution of the Burgers' equation, is carried out in the following way:

$$Deviation(\%) = \frac{\left| \text{analytical } e\text{xpression- numerical solution(Burgers)} \right|}{\text{numerical solution(Burgers)}} \times 100 \qquad (27)$$

The relative deviation, on the selected range, of the analytical expression of the fundamental component (equation (24)) in relation to the numerical solution of the Burgers' equation is less than 4% for glycerol (figure 3b).

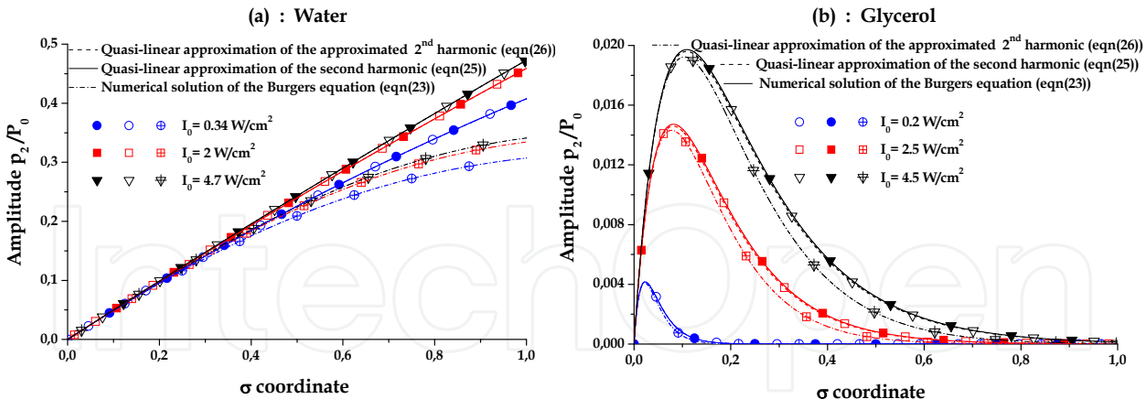

Fig. 2. Pressure amplitude p_2/P_0 versus the σ coordinate.

Thus, for a strongly dissipative liquid, equation (24) can be considered as a good approximation of equation (23). In fact, in this case the Goldberg's number is lower than unity (Table 2); then dissipation becomes important and dominates nonlinear effects.

As for water, the relative deviation of the analytical expression of the fundamental component (equation (24)) in relation to the numerical solution of the Burgers' equation is about 12% at $\sigma = 1$ (figure 3a). It should be noted that for water, the deviations increase with Γ (figure 3a). Indeed, in this case nonlinear effects become important ($p_2(z)$ much greater than $2P_0/\Gamma$) and the analytical expression of the fundamental established with the quasi-linear approximation is not valid.

For glycerol, the relative deviation of the analytical expression of the second harmonic (equation (25)) in relation to the numerical solution of the Burgers' equation is much weaker than that resulting from equation (26) (figure 4b). As an example, for $\sigma \approx 0.1$ the deviation obtained from equation (25) is lower than 1%, and that produced by equation (26) can reach 40%.

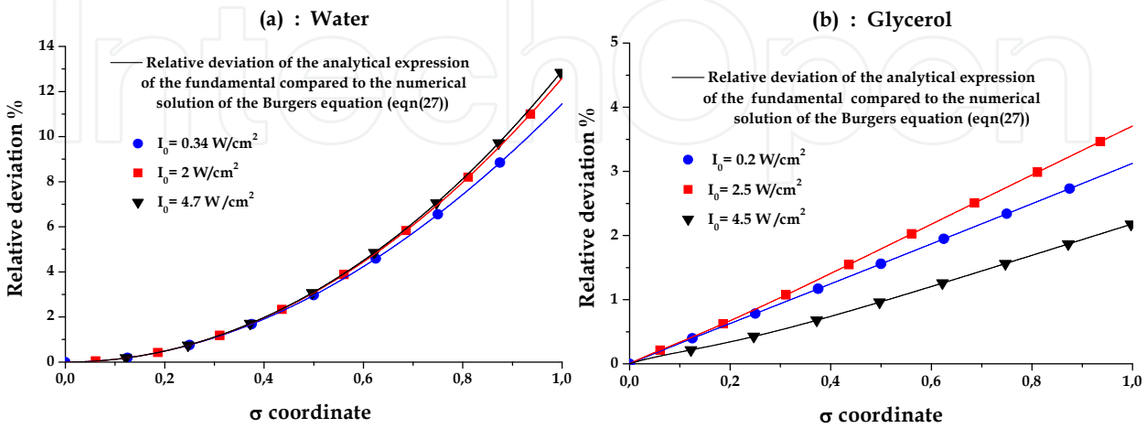

Fig. 3. Relative deviation of the analytical expression of the fundamental compared to the numerical solution of the Burgers' equation versus the σ coordinate.

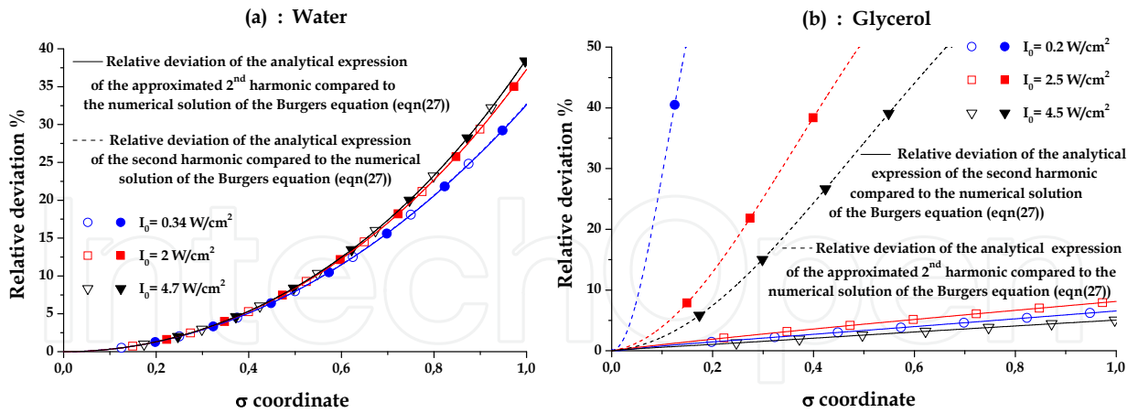

Fig. 4. Relative deviation of the respectively analytical expression of the second harmonic and the approximated second harmonic compared to the numerical solution of the Burgers' equation versus the σ coordinate.

So, for a strongly dissipative liquid, equation (25) is a good approximation of the numerical solution of the Burgers' equation (figure 4b). But, the equivalence of equations (25) and (26) is not checked (figure 4b). Indeed, equation (26) is a good approximation of equation (25) only if $(\alpha_2 - 2\alpha_1) l_s \sigma$ is weak comparatively to unity.

In the case of water, the relative deviation of the analytical expression of the second harmonic (equation (25)) in relation to the numerical solution of the Burgers' equation is about 40% at $\sigma = 1$ (figure 4a). In fact, the determination of the analytical expression of the second harmonic is based on the analytical expression of the fundamental. As in the case of a slightly dissipative medium a noticeable deviation between $p_1(\sigma)$ and the numerical solution of the Burgers' equation is observed, the deviation of the analytical expression of the second harmonic in relation to the numerical solution of the Burgers' equation becomes more significant. These deviations increase with Γ (figure 4a). Moreover in this case, $(\alpha_2 - 2\alpha_1) l_s \sigma$ is weak comparatively to unity and equations (25) and (26) are equivalent. Consequently, the preceding comments are also applicable for the analytical expression of the approximated second harmonic (equation (26)) (figure 4a).

According to this study, all these obtained solutions are valid, since the measurement is made near the source; otherwise some assumptions must be taken into account in the analysis of the propagation of finite amplitude acoustic waves in liquids. In addition, the analytical expressions precision depends essentially on the Goldberg's number value.

Moreover, for a strongly dissipative medium, the analytical expressions of the fundamental and second harmonic (equations (24) and (25)) can constitute a good approximation of the numerical solution of the Burgers' equation.

For a slightly dissipative medium, the analytical expressions established show discrepancies when compared to the numerical solution of the Burgers' equation. Indeed, equation (24) assumes that the differential variation of the fundamental component with respect to the spatial coordinate is only proportional to the product of the absorption coefficient and the

acoustic pressure of the fundamental ($\frac{\partial p_1(z)}{\partial z} = -\alpha_1 p_1(z)$). This hypothesis is not always checked (equation (16)).

As mentioned at the beginning of this chapter, the performance of the simplified model (equation (26)) is interesting, as it can provide a simple, useful model for understanding phenomena in diagnostic imaging. In fact, tissue harmonic imaging offers several unique advantages over conventional imaging. The greater clarity, contrast and details of the harmonic images are evident and have been quantitatively verified, like the ability to identify suspected cysts... Despite the significant advantages offered by harmonic imaging, theory has been only partially explained. According to the theoretical development established in this chapter, equation (26) is valid only if $p_2(\sigma) << 2P_0/\Gamma$ and $(\alpha_2 - 2\alpha_1) l_s \sigma << 1$. Not taking into account these assumptions can generate erroneous numerical results.

On the other hand, as the finite amplitude method is based on pressure measurements of the finite amplitude wave distortion during its propagation, the analytical expressions of the fundamental (equation (24)), the second harmonic (equation (25)) and the approximated second harmonic (equation (26)) lead also to the measurement of the acoustic nonlinearity parameter β. However, this method necessitates an accurate model taking into account diffraction effects (Labat et al., 2000; Gong et al., 1989; Zhang et al., 1991). The omission of this phenomenon can explain the discrepancies observed of the nonlinearity parameter values measured by the finite amplitude method compared to those achieved by the thermodynamic method (Law et al. 1983; Plantier et al., 2002; Sehgal et al., 1984; Zhang & Dunn, 1991). The latter is potentially very accurate. The major advantage of the thermodynamic method is that it does not depend on the characteristics of the acoustic field (Khelladi et al., 2007, 2009).

4. Conclusion

The validity domain of the fundamental and the second harmonic analytical expressions established with the quasi-linear approximation can be preset only on the derivation of the Goldberg's number, which can be considered as a reliable indicator for any analysis incorporating nonlinear effects and dissipation.

The obtained numerical results illustrate that the analytical expressions of the fundamental and the second harmonic established with the quasi-linear approximation provide a good approximation of the numerical solution of the Burgers' equation for a propagation medium characterized by a Goldberg number that is small compared to unity.

In the other hand, for a propagation medium characterized by a Goldberg number greater than unity, the analytical expressions of the fundamental and the second harmonic already established with the quasi-linear approximation are not checked and must be redefined.

For that purpose, future studies will concentrate on a new mathematical formulation of the fundamental and second harmonic for a propagation medium characterized by a Goldberg number that is large compared to unity.

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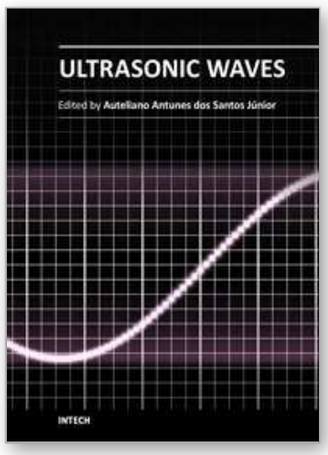

Ultrasonic Waves
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Ultrasonic waves are well-known for their broad range of applications. They can be employed in various fields of knowledge such as medicine, engineering, physics, biology, materials etc. A characteristic presented in all applications is the simplicity of the instrumentation involved, even knowing that the methods are mostly very complex, sometimes requiring analytical and numerical developments. This book presents a number of state-of-the-art applications of ultrasonic waves, developed by the main researchers in their scientific fields from all around the world. Phased array modelling, ultrasonic thrusters, positioning systems, tomography, projection, gas hydrate bearing sediments and Doppler Velocimetry are some of the topics discussed, which, together with materials characterization, mining, corrosion, and gas removal by ultrasonic techniques, form an exciting set of updated knowledge. Theoretical advances on ultrasonic waves analysis are presented in every chapter, especially in those about modelling the generation and propagation of waves, and the influence of Goldberg's number on approximation for finite amplitude acoustic waves. Readers will find this book ta valuable source of information where authors describe their works in a clear way, basing them on relevant bibliographic references and actual challenges of their field of study.

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