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Analysis of Fuzzy Logic Models

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1. Introduction

One of the most important results of mathematics in the 20th century is the Kolmogorov model of probability and statistics. It gave many impulses for research and develop so in theoretical area as well as in applications in a large scale of subjects.

It is reasonable to ask why the Kolmogorov approach played so important role in the probability theory and in mathematical statistics. In disciplines which have been very successfull for many centuries.

Of course, Kolmogorov stated probability and statistics on a new and very effective foundation - set theory. For the first time in the history basic notions of probability theory have been defined precisely but simply. So a random event has been defined as a subset of a space, a random variable as a measurable function and its mean value as an integral. More precisely, abstract Lebesgue integral. It is hopeful to wait some new stimuls from the fuzzy generalization of the classical set theory. The aim of the chapter is a presentation of some results of the type.

2. Fuzzy systems and their algebraizations

Any subset A of a given space Ω can be identified with its characteristic function

$$\chi_A : \Omega \to \{0,1\}$$

where

$$\chi_A(\omega) = 1,$$

if $\omega \in A$,

$$\chi_A(\omega) = 0,$$

if $\omega \notin A$. From the mathematical point of view a fuzzy set is a natural generalization of χ_A(see [73]). It is a function

$$\varphi_A : \Omega \to [0,1].$$

Evidently any set (i.e. two-valued function on $\Omega, \chi_A \to \{0,1\}$) is a special case of a fuzzy set (multi-valued function), $\varphi_A : \Omega \to [0,1]$.

There are many possibilities for characterizations of operations with sets (union $A \cup B$ and intersection $A \cap B$). We shall use so called Lukasiewicz characterization:

$$\chi_{A \cup B} = (\chi_A + \chi_B) \wedge 1,$$

$$\chi_{A \cap B} = (\chi_A + \chi_B - 1) \vee 0.$$

(Here $(f \vee g)(\omega) = \max(f(\omega), g(\omega)), (f \wedge g)(\omega) = \min(f(\omega), g(\omega))$.) Hence if $\varphi_A, \varphi_B : \Omega \to [0,1]$ are fuzzy sets, then the union (disjunction φ_A or φ_B of corresponding assertions) can be defined by the formula

$$\varphi_A \oplus \varphi_B = (\varphi_A + \varphi_B - 1) \wedge 1,$$

the intersection (conjunction φ_A and φ_B of corresponding assertions) can be defined by the formula

$$\varphi_A \odot \varphi_B = (\varphi_A + \varphi_B - 1) \vee 0.$$

In the chapter we shall work with a natural generalization of the notion of fuzzy set so-called IF-set (see [1], [2]), what is a pair

$$A = (\mu_A, \nu_A) : \Omega \to [0,1] \times [0,1]$$

of fuzzy sets $\mu_A, \nu_A : \Omega \to [0,1]$, where

$$\mu_A + \mu_A \leq 1.$$

Evidently a fuzzy set $\varphi_A : \Omega \to [0,1]$ can be considered as an IF-set, where

$$\mu_A = \varphi_A : \Omega \to [0,1], \nu_A = 1 - \varphi_A : \Omega \to [0,1].$$

Here we have

$$\mu_A + \nu_A = 1,$$

while generally it can be $\mu_A(\omega) + \nu_A(\omega) < 1$ for some $\omega \in \Omega$. Geometrically an IF-set can be regarded as a function $A : \Omega \to \Delta$ to the triangle

$$\Delta = \{(u,v) \in R^2 : 0 \leq u, 0 \leq v, u + v \leq 1\}.$$

Fuzzy set can be considered as a mapping $\varphi_A : \Omega \to D$ to the segment

$$D = \{(u,v) \in R^2; u + v = 1, 0 \leq u \leq 1\}$$

and the classical set as a mapping $\psi : \Omega \to D_0$ from Ω to two-point set

$$D_0 = \{(0,1), (1,0)\}.$$

In the next definition we again use the Lukasiewicz operations.

Definition 1.1. By an IF subset of a set Ω a pair $A = (\mu_A, \nu_A)$ of functions

$$\mu_A : \Omega \to [0,1], \nu_A; \Omega \to [0,1]$$

is considered such that

$$\mu_A + \nu_A \leq 1.$$

We call μ_A the membership function, ν_A the non membership function and

$$A \leq B \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

If $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ are two IF-sets, then we define

$$A \oplus B = ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0),$$

$$A \odot B = ((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1),$$

$$\neg A = (1 - \mu_A, 1 - \nu_A).$$

Denote by $\mathcal{F}$ a family of IF sets such that

$$A, B \in \mathcal{F} \Longrightarrow A \oplus B \in \mathcal{F}, A \odot B \in \mathcal{F}, \neg A \in \mathcal{F}.$$

Example 1.1. Let $\mathcal{F}$ be the set of all fuzzy subsets of a set Ω. If $f : \Omega \to [0,1]$ then we define

$$A = (f, 1 - f),$$

i.e. $\nu_A = 1 - \mu_A$.

Example 1.2. Let $(\Omega, \mathcal{S})$ be a measurable space, $\mathcal{S}$ a σ-algebra, $\mathcal{F}$ the family of all pairs such that $\mu_A : \Omega \to [0,1], \nu_A : \Omega \to [0,1]$ are measurable. Then $\mathcal{F}$ is closed under the operations $\oplus, \odot, \neg$.

Example 1.3. Let $(\Omega, \mathcal{T})$ be a topological space, $\mathcal{F}$ the family of all pairs such that $\mu_A : \Omega \to [0,1], \nu_A : \Omega \to [0,1]$ are continuous. Then $\mathcal{F}$ is closed under the operations $\oplus, \odot, \neg$.

Remark. Of course, in any case $A \oplus B, A \odot B, \neg A$ are IF-sets, if A, B are IF-sets. E.g.

$$A \oplus B = ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0),$$

hence

$$(\mu_A + \mu_B) \wedge 1 + (\nu_A + \nu_B - 1) \vee 0 =$$
$$= ((\mu_A + \mu_B) \wedge 1 + (\nu_A + \nu_B - 1)) \vee ((\mu_A + \mu_B) \wedge 1) =$$
$$= ((\mu_A + \mu_B + \nu_A + \nu_B - 1) \wedge (1 + \nu_A + \nu_B - 1)) \vee ((\mu_A + \mu_B) \wedge 1) \leq$$
$$\leq ((1 + 1 - 1) \wedge (\nu_A + \nu_B)) \vee ((\mu_A + \mu_B) \wedge 1) =$$
$$= (1 \wedge (\nu_A + \nu_B)) \vee ((\mu_A + \mu_B) \wedge 1) \leq$$
$$\leq 1 \vee 1 = 1.$$

Probably the most important algebraic model of multi-valued logic is an MV-algebra ([48],[49]). MV-algebras play in multi-valued logic a role analogous to the role of Boolean algebras in two-valued logic. Therefore we shall present a short information about MV-alegras and after it we shall prove the main result of the section: a possibility to embed the family of IF-sets to a suitable MV-algebra.

Let us start with a simple example.

Example 1.4. Consider the unit interval $[0,1]$ in the set R of all real numbers. It will stay an MV-algebra, if we shall define two binary operations $\oplus, \odot$ on $[0,1]$, one unary operation $\neg$ and the usual ordering $\leq$ by the following way:

$$a \oplus b = \min(a + b, 1),$$

$$a \odot b = \max(a + b - 1, 0),$$

$$\neg a = 1 - a.$$

It is easy to imagine that $a \oplus b$ corresponds to the disjunction of the assertions a, b, $a \odot b$ to the conjunction of a, b and $\neg a$ to the negation of a.

By the Mundici theorem ([48])any MV-algebra can be defined similarly as in Example 1.4, only the group R must be substitute by an arbitrary l-group.

Definition 1.2. By an l-group we consider an algebraic system $(G, +, \leq)$ such that

(i) $(G, +)$ is an Abelian group,

(ii) $(G, \leq)$ is a lattice,

(iii) $a \leq b \Longrightarrow a + c \leq b + c$.

Definition 1.3. By an MV-algebra we consider an algebraic system $(M, 0, u, \oplus, \odot)$ such that $M = [0, u] \subset G$, where $(G, +, \leq)$ is an l-group, 0 its neutral element, u a positive element, and

$a \oplus b = (a + b) \wedge u$,

$a \odot b = (a + b - u) \vee 0$,

$\neg a = u - a$.

Example 1.5. Let $(\Omega, \mathcal{S})$ be a measurable space, $\mathcal{S}$ a σ-algebra,

$G = \{A = (\mu_A, \nu_A); \mu_A, \nu_A : \Omega \to R\}$,

$A + B = (\mu_A + \mu_B, \nu_A + \nu_B - 1) = (\mu_A + \mu_B, 1 - (1 - \nu_A + 1 - \nu_B))$,

$A \leq B \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B$.

Then $(G, +, \leq)$ is an l-group with the neutral element $\mathbf{0} = (0, 1)$, $A - B = (\mu_A - \mu_B, \nu_A - \nu_B + 1)$, and the lattice operations

$A \vee B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$,

$A \wedge B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

Put $u = (1, 0)$ and define the MV-algebra

$M = \{A \in G; (0, 1) = \mathbf{0} \leq A \leq u = (1, 0)\}$,

$A \oplus B = (A + B) \wedge u =$

$= (\mu_A + \mu_B, \nu_A + \nu_B - 1) \wedge (1, 0) =$

$= ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0,$

$A \odot B = (A + B - u) \vee (0, 1) =$

$$= ((\mu_A + \mu_B, \nu_A + \nu_B - 1) - (1,0)) \vee (0,1) =$$

$$= (\mu_A + \mu_B - 1, \nu_A + \nu_B - 1 - 0 + 1) \vee (0,1) =$$

$$= ((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1),$$

$$\neg A = (1,0) - (\mu_A, \nu_A) =$$

$$= (1 - \mu_A, 0 - \nu_A + 1) =$$

$$= (1 - \mu_A, 1 - \nu_A).$$

Connections with the family of IF-sets (Definition 1.1) is evident. Hence we can formulate the main result of the section.

Theorem 1.1. Let $(\Omega, \mathcal{S})$ be a measurable space, $\mathcal{F}$ the family of all IF-sets $A = (\mu_A, \nu_A)$ be such that μ_A, ν_A are $\mathcal{S}$-measurable. Then there exists an MV-algebra $\mathcal{M}$ such that $\mathcal{F} \subset \mathcal{M}$, the operations $\oplus, \odot$ are extensions of operations on $\mathcal{F}$ and the ordering $\leq$ is an extension of the ordering in $\mathcal{F}$.

Proof. Consider MV-algebra $\mathcal{M}$ constructed in Example 1.5. If $A, B \in \mathcal{F}$, then the operations on $\mathcal{M}$ coincide with the operations on $\mathcal{F}$. The ordering $\leq$ is the same.

Theorem 1.1 enables us in the space of IF-sets to use some results of the well developed probability theory on MV-algebras ([66 - 68]). Of course, some methods of the theory can be generalized in so-called D-posets ([28]). The system $(D \leq, -, 0, 1)$ is called D-poset, if $(D, \leq)$ is partially ordered set with the smallest element 0 and the largest element 1, $-$ is a partially binary operation satisfying the following statements:

1. $b - a$ is defined if and only if $a \leq b$.

2. $a \leq b$ implies $b - a \leq b$ and $b - (b - a) = a$.

3. $a \leq b \leq c$ implies $c - b \leq c - a$ and $(c - a) - (c - b) = b - a$.

3. Probability on IF-events

In IF-events theory an original terminology is used. The main notion is the notion of a state ([21], [22],[57], [58], [61][, [62]). It is an analogue of the notion of probability in the Kolmogorov classical theory. As before $\mathcal{F}$ is the family of all IF-sets $A = (\mu_A, \nu_A)$ such that $\mu_A, \nu_A : (\Omega, \mathcal{S}) \to [0,1]$ are $\mathcal{S}$-measurable.

Definition 2.1. A mapping $m : \mathcal{F} \to [0,1]$ is called a state if the following properties are satisfied:

(i) $m(1_\Omega, 0_\Omega) = 1, m(0_\Omega, 1_\Omega) = 0,$

(ii) $A \odot B = (0_\Omega, 1_\Omega) \Longrightarrow m((A \oplus B)) = m(A) + m(B),$

(iii) $A_n \nearrow A \Longrightarrow m(A_n) \nearrow m(A).$

Of course, also the notion with the name probability has been introduced in IF-events theory.

Definition 2.2. Let $\mathcal{J}$ be the family of all compact intervals in the real line, $\mathcal{J} = \{[a,b]; a, b \in R, a \leq b\}$. Probability is a mapping $P : \mathcal{F} \to \mathcal{J}$ satisfying the following conditions:

(i) $P(1_\Omega, 0_\Omega) = [1, 1], P(0_\Omega, 1_\Omega) = [0, 0]$,

(ii) $A \odot B = (0_\Omega, 1_\Omega) \implies P((A \oplus B)) = P(A) + P(B)$,

(iii) $A_n \nearrow A \implies P(A_n) \nearrow P(A)$.

It is easy to see that the following property holds.

Proposition 2.1. Let $P : \mathcal{F} \to \mathcal{J}, P(A) = [P^\flat(A), P^\sharp(A)]$. Then P is a probability if and only if $P^\flat, P^\sharp : \mathcal{F} \to [0, 1]$ are states.

Hence it is sufficient to characterize only the states ([4], [5], [54]).

Theorem 2.1. For any state $m : \mathcal{F} \to [0, 1]$ there exist probability measures $P, Q : \mathcal{S} \to [0, 1]$ and $\alpha \in [0, 1]$ such that

$$m((\mu_A, \nu_A)) = \int_\Omega \mu_A \, dP + \alpha(1 - \int_\Omega (\mu_A + \nu_A) dQ).$$

Proof. The main instrument in our investigation is the following implication, a corollary of (ii):

$$f, g \in \mathcal{F}, f + g \leq 1 \implies m(f, g) = m(f, 1 - f) + m(0, f + g). \tag{1}$$

We shall define the mapping $P : \mathcal{S} \to [0, 1]$ by the formula $P(A) = m(\chi_A, 1 - \chi_A)$. Let $A, B \in \mathcal{S}, A \cap B = \varnothing$. Then $\chi_A + \chi_B \leq 1$, hence $(\chi_A, 1 - \chi_A) \odot (\chi_B, 1 - \chi_B) = (0, 1)$. Therefore

$$P(A) + P(B) = m(\chi_A, 1 - \chi_A) + m(\chi_B, 1 - \chi_B) =$$

$$= m((\chi_A, 1 - \chi_A) \oplus (\chi_B, 1 - \chi_B)) =$$

$$= m(\chi_A + \chi_B, 1 - \chi_A - \chi_B) = m(\chi_{A \cup B}, 1 - \chi_{A \cup B} = P(A \cup B).$$

Let $A_n \in \mathcal{S}(n = 1, 2, ...), A_n \nearrow A$. Then

$$\chi_{A_n} \nearrow \chi_A, 1 - \chi_{A_n} \searrow 1 - \chi_A,$$

hence by (iii)

$$P(A_n) = m(\chi_{A_n}, 1 - \chi_{A_n}) \nearrow m(\chi_A, 1 - \chi_A) = P(A).$$

Evidently $P(\Omega) = m(\chi_\Omega, 1 - \chi_\Omega) = m((1, 0)) = 1$, hence $P : \mathcal{S} \to [0, 1]$ is a probability measure.

Now we prove two identities. First the implication:

$$A_1, ..., A_n \in \mathcal{S}, \alpha_1, ..., \alpha_n \in [0, 1], A_i \cap A_j = \varnothing (i \neq j) \implies$$

$$m(\sum_{i=1}^n \alpha_i \chi_{A_i}, 1 - \sum_{i=1}^n \alpha_i \chi_{A_i}) = \sum_{i=1}^n m(\alpha_i \chi_{A_i}, 1 - \alpha_i \chi_{A_i}). \tag{2}$$

It can be proved by induction. The second identity is the following

$$0 \leq \alpha, \beta \leq 1 \implies m(\alpha \beta \chi_A, 1 - \alpha \beta \chi_A) = \alpha m(\beta \chi_A, 1 - \beta \chi_A). \tag{3}$$

First it can be proved by induction the equality

$$qm(\frac{1}{q}\beta\chi_A, 1 - \frac{1}{q}\beta\chi_A) = m(\beta\chi_A, 1 - \beta\chi_A)$$

holding for every $q \in N$. Therefore

$$m(\frac{1}{q}\beta\chi_A, 1 - \frac{1}{q}\beta\chi_A) = \frac{1}{q}m(\beta\chi_A, 1 - \beta\chi_A)$$

$$m(\frac{p}{q}\beta\chi_A, 1 - \frac{p}{q}\beta\chi_A) = \frac{p}{q}m(\beta\chi_A, 1 - \beta\chi_A),$$

hence (3) holds for every rational α. Let $\alpha \in R, \alpha \in [0,1]$. Take $\alpha_n \in Q, \alpha_n \nearrow \alpha$. Then

$$\alpha_n\chi_A \nearrow \alpha\chi_A, 1 - \alpha_n\chi_A \searrow 1 - \alpha\chi_A.$$

Therefore

$$m(\alpha\beta\chi_A, 1 - \alpha\beta\chi_A) = \lim_{n\to\infty} m(\alpha_n\beta\chi_A, 1 - \alpha_n\beta\chi_A) =$$

$$= \lim_{n\to\infty} \alpha_n m(\beta\chi_A, 1 - \beta\chi_A) = \alpha m(\beta\chi_A, 1 - \beta\chi_A),$$

hence, (3) is proved, too. Particularly, if we give $\beta = 1$, then

$$m(\alpha\chi_A, 1 - \alpha\chi_A) = \alpha m(\chi_A, 1 - \chi_A).$$

Let $f : \Omega \to [0,1]$ be simple, $\mathcal{S}$-measurable, i.e.

$$f = \sum_{i=1}^{n} \alpha_i\chi_{A_i}, A_i \in \mathcal{S}(i = 1, ., ,, n), A_i \cap A_j = \emptyset(i \neq j).$$

Combining (2), (3), and the definition of P we obtain

$$m(f, 1 - f) = \sum_{i=1}^{n} m(\alpha_i\chi_{A_i}, 1 - \alpha_i\chi_{A_i}) =$$

$$= \sum_{i=1}^{n} \alpha_i m(\chi_{A_i}, 1 - \chi_{A_i}) =$$

$$= \sum_{i=1}^{n} \alpha_i P(A_i) = \int_{\Omega} f dP,$$

hence

$$m(f, 1 - f) = \int_{\Omega} f dP,$$

for any $f : \Omega \to [0,1]$simple. If $f : \Omega \to [0,1]$ is an arbitrary $\mathcal{S}$-measurable function, then there exists a sequence (f_n) of simple measurable functions such that $f_n \nearrow f$. Evidently, $1 - f_n \searrow 1 - f$. Therefore

$$m(f, 1 - f) = \lim_{n\to\infty} m(f_n, 1 - f_n) = \lim_{n\to\infty} \int_{\Omega} f_n dP = \int_{\Omega} f dP,$$

hence

$$m(f, 1 - f) = \int_\Omega dP, \tag{4}$$

for any measurable $f : \Omega \to [0, 1]$.

Now take our attention to the second term $m(0, f + g)$ in the right side of the equality (1). First define $M : \mathcal{S} \to [0, 1]$ by the formula

$$M(A) = m(0, 1 - \chi_A).$$

As before it is possible to prove that M is a measure. Of course,

$$M(\Omega) = m(0, 0) = \alpha \in [0, 1].$$

Define $Q : \mathcal{S} \to [0, 1]$ by the formula

$$m(0, 1 - \chi_A) = \alpha Q(A).$$

As before, it is possible to prove

$$m(0, 1 - f) = \alpha \int_\Omega f dQ,$$

for any $f : \Omega \to [0, 1]$ measurable, or

$$m(0, h) = \alpha \int_\Omega (1 - h) dQ, \tag{5}$$

for any $h : \Omega \to [0, 1], \mathcal{S}$-measurable. Combining (1), (4), and (5) we obtain

$$m(A) = m((\mu_A, \nu_A) = m((\mu_A, 1 - \mu_A)) + m((0, \mu_A + \nu_A))$$

$$= \int_\Omega \mu_A dP + \alpha (1 - \int_\Omega (\mu_A + \nu_A) dQ).$$

A simple consequence of the representation theorem is the following property of the mapping $P - \alpha Q : \mathcal{S} \to R$.

Proposition 2.2. Let $P, Q : \mathcal{S} \to [0, 1]$ be the probabilities mentioned in Theorem 2.1, α is the corresponding constant. Then

$$P(A) - \alpha Q(A) \geq 0$$

for any $A \in \mathcal{S}$.

Proof. Put $B = (0, 0), C = (\chi_A, 0)$. Then $B \leq C$, hence $m(0, 0) \leq m(\chi_A, 0)$. Therefore

$$\alpha = m(0, 0) \leq m(\chi_A, 0) = P(A) + \alpha(1 - Q(A)).$$

Theorem 1.1 is an embedding theorem stating that every IF-events algebra $\mathcal{F}$ can be embedded to and MV-algebra $\mathcal{M}$. Now we shall prove that any state $m : \mathcal{F} \to [0, 1]$ can be extended to a state $\overline{m} : \mathcal{M} \to [0, 1]$ ([63]).

Theorem 2.2. Let $\mathcal{M} \supset \mathcal{F}$ be the MV-algebra constructed in Theorem 1.1. Then every state $m : \mathcal{F} \to [0, 1]$ can be extended to a state $\overline{m} : \mathcal{M} \to [0, 1]$.

Proof. It is easy to see that any element $(\mu_A, \nu_A) \in \mathcal{M}$ can be presented in the form

$$(\mu_A, \nu_A) \odot (0, 1 - \nu_A) = (0, 1),$$

$$(\mu_A, 0) = (\mu_A, \nu_A) \oplus (0, 1 - \nu_A).$$

If $(\mu_A, \nu_A) \in \mathcal{F}$, then

$$m((\mu_A, 0)) = m((\mu_A \nu_A)) + m((0, 1 - \nu_A)).$$

Generally, we can define $\overline{m} : \mathcal{M} \to [0, 1]$ by the formula

$$\overline{m}((\mu_A, \nu_A)) = m((\mu_A, 0)) - m((0, 1 - \nu_A)),$$

so that $\overline{m}$ is an extension of m. Of course, we must prove that $\overline{m}$ is a state. First we prove that $\overline{m}$ is additive.

Let $A = (\mu_A, \nu_A) \in \mathcal{M}, B = (\mu_B, \nu_B) \in \mathcal{M}, A \odot B = (0, 1)$, hence

$$((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1) = (0, 1),$$

$$\mu_A + \mu_B \leq 1, 1 - \nu_A + 1 - \nu_B \leq 1.$$

Therefore

$$\overline{m}(A) + \overline{m}(B) = \overline{m}(\mu_A, \nu_A) + \overline{m}(\mu_B, \nu_B)$$

$$= m(\mu_A, 0) - m(0, 1 - \nu_A) + m(\mu_B, 0) - m(0, 1 - \nu_B) =$$

$$= m(\mu_A + \mu_B, 0) - m(0, 1 - \nu_A - \nu_B) =$$

$$= m(\mu_A + \mu_B, \nu_A + \nu_B) = \overline{m}(A \oplus B).$$

Before the continuity of $\overline{m}$ we shall prove its monotonicity. Let $A \leq B$, i.e. $\mu_A \leq \mu_A, \nu_A \geq \nu_B$. Then by Theorem 2.1

$$\overline{m}(A) = m(\mu_A, 0) - m(0, 1 - \nu_A) =$$

$$= \int_\Omega \mu_A dP + \alpha(1 - \int_\Omega (\mu_A + 0) dQ - \int_\Omega 0 dP - \alpha(1 - \int_\Omega (0 + 1 - \nu_A) dQ) =$$

$$= \int_\Omega \mu_A dP + \alpha(1 - \int_\Omega (\mu_A + \nu_A) dQ).$$

Therefore

$$\overline{m}(B) - \overline{m}(A) = \int_\Omega \mu_B dP + \alpha - \alpha \int_\Omega \mu_B dQ - \alpha \int_\Omega \nu_B dQ -$$

$$- (\int_\Omega \mu_A dP + \alpha - \alpha \int_\Omega \mu_A dQ - \alpha \int_\Omega \nu_A dQ) =$$

$$= \int_\Omega (\mu_B - \mu_A) dP - \alpha \int_\Omega (\mu_B - \mu_A) dQ + \alpha \int_\Omega (\nu_A - \nu_B) dQ.$$

Of course, as an easy consequence o Proposition 2.1 we obtain the inequality

$$\int_\Omega f dP - \alpha \int_\Omega f dQ \geq 0$$

for any non-negative measurable $f : \Omega \to R$. Therefore

$$\overline{m}(B) - \overline{m}(A) = \int_\Omega f dP - \alpha \int_\Omega f dQ + \alpha \int_\Omega (\nu_A - \mu_B) dQ \geq 0.$$

Finally let $A_n = (\mu_{A_n}, \nu_{A_n}) \in \mathcal{M}$, $A = (\mu_A, \nu_A) \in \mathcal{M}$, $A_n \nearrow A$, i.e. $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$. We have

$$\overline{m}(A_n) = \int_\Omega \mu_{A_n} dP - \alpha \int_\Omega \mu_{A_n} dQ + \alpha - \alpha \int_\Omega \nu_{A_n} dQ \nearrow$$

$$\nearrow \int_\Omega \mu_A dP - \alpha \int_\Omega \mu_A dQ + \alpha - \alpha \int_\Omega \nu_A dQ = \overline{m}(A).$$

4. Observables

In the classical probability there are three main notions:

probability = measure

random variable = measurable function

mean value = integral.

The first notion has been studied in the previous section. Now we shall define the second two notions.

Classically a random variable is such function $\xi : (\Omega, \mathcal{S}) \to R$ that $\xi^{-1}(A) \in \mathcal{S}$ for any Borel set $A \in \mathcal{B}(R)$ (here $\mathcal{B}(R) = \sigma(\mathcal{J})$ is the σ-algebra generated by the family $\mathcal{J}$ of all intervals). Now instead of a σ-algebra $\mathcal{S}$ we have the family $\mathcal{F}$ of all IF-events, hence we must give to any Borel set A an element of $\mathcal{F}$. Of course, instead of random variable we shall use the term observable ([15], [16], [18], [32], [35]).

Definition 3.1. An observable is a mapping

$$x : \sigma(\mathcal{J}) \to \mathcal{F}$$

satisfying the following conditions:

(i)
$$x(R) = (1, 0), x(\varnothing) = (0, 1),$$

(ii)
$$A \cap B = \varnothing \implies x(A) \odot x(B) = (0, 1), x(A \cup B) = x(A) \oplus x(B),$$

(iii)
$$A_n \nearrow A \implies x(A_n) \nearrow x(A).$$

Proposition 3.1. If $x : \sigma(\mathcal{J}) \to \mathcal{F}$ is an observable, and $m : \mathcal{F} \to [0, 1]$ is a state, then

$$m_x = m \circ x : \sigma(\mathcal{J}) \to [0, 1]$$

defined by

$$m_x(A) = m(x(A))$$

is a probability measure.

Proof. First
$$m_x(R) = m(x(R)) = m((1,0)) = 1.$$
If $A \cap B = \varnothing$, then $x(A) \odot x(B) = (0,1)$, hence
$$m_x(A \cup B) = m(x(A \cup B)) = m((x(A) \oplus x(B)) =$$
$$= m(x(A)) + m(x(B)) = m_x(A) + m_x(B).$$
Finally, $A_n \nearrow A$ implies $x(A_n) \nearrow x(A)$, hence
$$m_x(A_n) = m(x(A_n)) \nearrow m(x(A)) = m_x(A).$$

Proposition 3.2. Let $x : \sigma(\mathcal{J}) \to \mathcal{F}$ be an observable, $m : \mathcal{F} \to [0,1]$ be a state. Define $F : R \to [0,1]$ by the formula
$$F(u) = m(x((-\infty, u))).$$
Then F is non-decreasing, left continuous in any point $u \in R$,
$$\lim_{u \to \infty} F(u) = 1, \ \lim_{u \to -\infty} F(u) = 0.$$

Proof. If $u < v$, then
$$x((-\infty, v)) = x((-\infty, u)) \oplus x((u,v)) \geq x((-\infty, u)),$$
hence
$$F(v) = m((-\infty, v)) \geq m(x((-\infty, u))) = F(u),$$
F is non decreasing. If $u_n \nearrow u$, then
$$x((-\infty, u_n)) \nearrow x((-\infty, u)),$$
hence
$$F(u_n) = m(x((-\infty, u_n))) \nearrow m(x((-\infty, u))) = F(u),$$
F is left continuous in any $u \in R$. Similarly $u_n \nearrow \infty$ implies
$$x((-\infty, u_n)) \nearrow x((-\infty, \infty)) = (1, 0).$$
Therefore
$$F(u_n) = m(x((-\infty, u_n))) \nearrow m((1,0))) = 1$$
for every $u_n \nearrow \infty$, hence $\lim_{u \to \infty} F(u) = 1$. Similarly we obtain
$$u_n \searrow -\infty \implies -u_n \nearrow \infty,$$
hence
$$m(x((u_n, -u_n))) \nearrow m(x(R)) = 1.$$
Now
$$1 = \lim_{n \to \infty} F(-u_n) = \lim_{n \to \infty} m(x((u_n, -u_n))) + \lim_{n \to \infty} F(u_n) =$$
$$= 1 + \lim_{n \to \infty} F(u_n),$$

hence $\lim_{n \to \infty} F(u_n) = 0$ for any $u_n \searrow -\infty$.

Of course, we must describe also the random vector $T = (\xi, \eta) : \Omega \to R^2$. We have

$$T^{-1}(C \times D) = \xi^{-1}(C) \cap \eta^{-1}(D).$$

In the IF case we shall use product of functions instead of intersection of sets ([47], [56], [68]).

Definition 3.2. The product $A.B$ of two IF-events $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ is the IF set

$$A.B = (\mu_A.\mu_B, 1 - (1 - \nu_A).(1 - \nu_B)) = (\mu_A.\mu_B, \nu_A + \nu_B - \nu_A.\nu_B).$$

Definition 3.3. Let $x_1, ..., x_n : \sigma(\mathcal{J}) \to \mathcal{F}$ be observables. By the joint observable of $x_1, ..., x_n$ we consider a mapping $h : \sigma(\mathcal{J}^n) \to \mathcal{F}(\mathcal{J}^n$ being the set of all intervals of R^n) satisfying the following conditions:

(i) $h(R^n) = (1, 0)$

(ii) $A \cap B = \varnothing \implies h(A) \odot h(B) = (0, 1)$, and $h(A \cup B) = h(A) \oplus h(B)$,

(iii) $A_n \nearrow A \implies h(A_n \nearrow h(A)$,

(iv) $h(C_1 \times C_2 \times ... \times C_n) = x_1(C_1).x_2(C_2).....x_n(C_n)$, for any $C_1, C_2, ..., C_n \in \mathcal{J}$.

Theorem 3.1. ([63]) For any observables $x_1, ..., x_n : \sigma(\mathcal{J}) \to \mathcal{F}$ there exists their joint observable $h : \sigma(\mathcal{J}^n) \to \mathcal{F}$.

Proof. We shall prove it for $n = 2$. Consider two observables $x, y : \sigma(\mathcal{J}) \to \mathcal{F}$. Since $x(A) \in \mathcal{F}$, we shall write

$$x(A) = (x^\flat(A), 1 - x^\sharp(A))$$

and similarly

$$y(B) = (y^\flat(B), 1 - y^\sharp(B)).$$

By the definition of product we obtain

$$x(C).y(D) = (x^\flat(C).y^\flat(D), 1 - x^\sharp(C).y^\sharp(D)).$$

Therefore, we shall construct similarly

$$h(K) = (h^\flat(K), 1 - h^\sharp(K))$$

Fix $\omega \in \Omega$ and define $\mu, \nu : \sigma(\mathcal{J}) \to [0, 1]$ by

$$\mu(A) = x^\flat(A)(\omega), \nu(B) = y^\flat(B)(\omega).$$

Let $\mu \times \nu$ be the product of the probability measures μ, ν. Put

$$h^\flat(K)(\omega) = \mu \times \nu(K).$$

Then

$$h^\flat(C \times D)(\omega) = \mu(C).\nu(D) = x^\flat(C).y^\flat(D)(\omega)$$

hence

$$h^\flat(C \times D) = x^\flat(C).y^\flat(D).$$

Analogously

$$h^\sharp(C \times D) = x^\sharp(C).y^\sharp(D).$$

If we define

$$h(A) = (h^\flat(A), 1 - h^\sharp(A)), A \in \sigma(\mathcal{J}^2),$$

then

$$h(C \times D) = (x^\flat(C).y^\flat(D), 1 - x^\sharp(C).y^\sharp(D)) = x(C).y(D).$$

Now we shall present two applications of the notion of the joint observable. The first is the definition of function of a finite sequence of observables, e.g. their sum. In the classical case

$$\xi + \eta = g \circ T : \Omega \to R$$

where $g(u,v) = u + v, T(\omega) = (\xi(\omega), \eta(\omega))$. Hence $\xi + \eta$ can be defined by the help of pre-images:

$$(\xi + \eta)^{-1} = T^{-1} \circ g^{-1} : \mathcal{B}(R) \to \mathcal{S}.$$

Definition 3.4. Let $x_1, ..., x_n : \mathcal{B}(R) \to \mathcal{F}$ be observables, $g : R_n \to R$ be a measurable function. Then we define

$$g(x_1, ..., x_n) : \mathcal{B}(R) \to \mathcal{F}$$

by the formula

$$g(x_1, ..., x_n)(C) = h(g^{-1}(C)), C \in \mathcal{B}(R),$$

where $h : \mathcal{B}(R^n) \to \mathcal{F}$ is the joint observable of the observables $x_1, ..., x_n$.

Example 3.1. $x_1 + ... + x_n : \mathcal{B}(R) \to \mathcal{F}$ is the observable defined by the formula $(x_1 + ... + x_n)(C) = h(g^{-1}(C))$, where $h : \mathcal{B}(R^n) \to \mathcal{F}$ is the joint observable of $x_1, ..., x_n$, and $g : R^n \to R$ is defined by the equality $g(u_1, ..., u_n) = u_1 + ... + u_n$.

The second application of the joint observable is in the formulation of the independency.

Definition 3.5. Let $m : \mathcal{F} \to [0,1]$ be a state, $(x_n)_{n=1}^\infty$ be a sequence of observables, $h_n : \sigma(\mathcal{J}^n) \to \mathcal{F}$ be the joint observable of $x_1, ..., x_n (n = 1, 2, ...)$. Then $(x_n)_{n=1}^\infty$ is called independent, if

$$m(h_n(C_1 \times C_2 \times ... \times C_n)) = m(x_1(C_1)).m(x_2(C_2)).....m(x_n(C_n))$$

for any $n \in N$ and any $C_1, ..., C_n \in \sigma(\mathcal{J})$.

Now let us return to the notion of mean value of an observable. In the classical case

$$E(g \circ \xi) = \int_\Omega g \circ \xi dP = \int_R g dF$$

where F is the distribution function of ξ.

Definition 3.6. Let $x : \mathcal{B}(R) \to \mathcal{F}$ be an observable, $m : \mathcal{F} \to [0,1]$ be a state, $g : R \to R$ be a measurable function, F be the distribution function of x $(F(t) = m(x((-\infty, t))))$. Then we

define the mean value $E(g \circ x)$ by the formula

$$E(g \circ x) = \int_R g dF$$

if the integral exists.

Example 3.2. Let x be discrete, i.e. there exist $x_i \in R, p_i \in (0,1], i = 1, ..., k$ such that

$$F(t) = \sum_{x_i < t} p_i.$$

Then

$$E(x) = \int_R t dF(t) = \sum_{i=1}^{k} x_i p_i.$$

The second classical case is the continuous distribution, where

$$F(t) = \int_{-\infty}^{t} \varphi(u) du.$$

Then

$$E(x) = \int_R t dF(t) = \int_{-\infty}^{\infty} t \varphi(t) dt.$$

Example 3.3. Let us compute the dispersion

$$\sigma^2(x) = E(g \circ x),$$

where

$$g(u) = (u - a)^2, a = E(x).$$

Here we have two possibilities. The first

$$\sigma^2 = \int_R (t - a)^2 dF(t)$$

i.e.

$$\sigma^2(x) = \sum_{i=1}^{k} (x_i - a)^2 p_i$$

in the discrete case, and

$$\sigma^2(x) = \int_{-\infty}^{\infty} (t - a)^2 \varphi(t) dt$$

in the continuous case. The second possibility is the equality

$$\sigma^2(x) = E((x - a)^2) = E(x^2) - 2aE(x) + E(a^2) =$$

$$= E(x^2) - a^2, a = E(x).$$

Since $a = E(x)$ is known, it is sufficient to compute $E(x^2)$. In the case we have $g(t) = t^2$, hence

$$E(x^2) = \int_R g(t) dF(t) = \int_R t^2 dF(t).$$

In the discrete case we have

$$E(x^2) = \sum_{1=i}^{k} x_i^2 p_i,$$

in the continuous case we obtain

$$E(x^2) = \int_{-\infty}^{\infty} t^2 \varphi(t) dt.$$

5. Sequences

In the section we want to present a method for studying of limit properties of some sequences $(x_n)_n, x_n : \mathcal{B}(R) \to \mathcal{F}$ of observables ([7], [25], [31], [32], [49]). The main idea is a representation of the given sequence by a sequence of random variables $(\xi_n)_n, \xi_n :$ $(\Omega, \mathcal{S}, P) \to R$. Of course, the space $(\Omega, \mathcal{S})$ depends on a concrete sequence $(x_n)_n$, for different sequences various spaces $(\Omega, \mathcal{S}, P)$ can be obtained.

The main instrument is the Kolmogorov consistency theorem ([67]). It starts with a sequence of probability measures $(\mu_n)_n, \mu_n : \sigma(\mathcal{J}_n) \to [0, 1]$ such that

$$\mu_{n+1} | \sigma(\mathcal{J}_n) \times R = \mu_n$$

i. e. $\mu_{n+1}(A \times R) = \mu_n(A)$ for any $A \in \sigma(\mathcal{J}_n)$ (consistency condition). Let $\mathcal{C}$ be the family of all cylinders in the space R^N, i. e. such sets $A \subset R^N$ that

$$A = \{(t_n)_n; (t_1, ..., t_k) \in B\},$$

where $k \in N, B \in \mathcal{B}(R^k) = \sigma(\mathcal{J}^k)$. Then by the Kolmogorov consistency theorem there exists exactly one probability measure

$$P : \sigma(\mathcal{C}) \to [0, 1]$$

such that

$$P(A) = \mu_k(B). \tag{6}$$

If we denote by π_n the projection $\pi_n : R^N \to R^n$,

$$\pi_n((t_i)_{i=1}^{\infty}) = (t_1, t_2, ..., t_n),$$

then we can formulate the assertion (6) by the equality

$$P(\pi_n^{-1}(B)) = \mu_n(B), \tag{7}$$

for any $B \in \mathcal{C}$.

Theorem 4.1. Let m be a state on a space $\mathcal{F}$ of all IF-events. Let $(x_n)_n$ be a sequence of observables, $x_n : \mathcal{B}(R) \to \mathcal{F}$, and let $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ be the joint observable of $x_1, ..., x_n, n = 1, 2, ...$ If we define $\mu_n : \mathcal{B}(R^n) \to [0, 1]$ by the equality

$$\mu_n = m \circ h_n,$$

then $(\mu_n)_n$ satisfies the consistency condition

$$\mu_{n+1} | (\sigma(\mathcal{J}_n) \times R) = \mu_n.$$

Proof. Let $C_1, C_2, ..., C_n \in \mathcal{B}(R)$. Then by Definition 3.3. and Definition 3.1

$$\mu_{n+1}(C_1 \times C_2 \times ... \times C_n \times R) = m(x_1(C_1)).x_2(C_2).....x_n(C_n).x_{n+1}(R)) =$$

$$= m(x_1(C_1)).x_2(C_2).....x_n(C_n).(1,0)) =$$

$$= m(x_1(C_1)).x_2(C_2).....x_n(C_n)) =$$

$$= \mu_n(C_1 \times C_2 \times ... \times C_n),$$

hence $\mu_{n+1}|(\mathcal{J}_n \times R) = \mu_n|\mathcal{J}_n$. Of course, if two measures coincide on $\mathcal{J}_n$ then they coincide on $\sigma(\mathcal{J}_n)$, too.

Now we shall formulate a translation formula between sequences of observables in $(\mathcal{F}, m)$ and corresponding random variables in $(R^N, \sigma(\mathcal{C}), P)$ ([67]).

Theorem 4.2. Let the assumptions of Theorem 4.1 be satisfied. Let $g_n : R^n \to R$ be Borel measurable functions $n = 1, 2, ...$ Let $\mathcal{C}$ be the family of all cylinders in $R^N, \xi_n : R^N \to R$ be defined by the formula $\xi_n((t_i)_i) = t_n$,

$$\eta_n : R^N \to R, \eta_n = g_n(\xi_1., ..., \xi_n),$$

$$y_n : \mathcal{B}(R^n) \to \mathcal{F}, y_n = h_n \circ g_n^{-1}.$$

Then

$$P(\eta_n^{-1}(B)) = m(y_n(B))$$

for any $B \in \mathcal{B}(R)$.

Proof. Put $A = g_n^{-1}(B)$. By Theorem 4.1.

$$m(y_n(B)) = m(h_n(g_n^{-1}(B))) = P(\pi_n^{-1}(g_n^{-1}(B))) =$$

$$= P((g_n \circ \pi_n)^{-1}(B)) = P(\eta_n^{-1}(B)).$$

As an easy corollary of Theorem 4.2 we obtain a variant of central limit theorem. In the classical case

$$\lim_{n \to \infty} P(\{\omega; \frac{\frac{1}{n}\sum_{i=1}^n \xi_i(\omega) - a}{\frac{\sigma}{\sqrt{n}}} < t\}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-\frac{u^2}{2}} du$$

Of course, we must define for observables the element

$$(\frac{\sqrt{n}}{\sigma} \sum_{i=1}^n x_i - a)(-\infty, t)$$

It is sufficient to put

$$g_n(u_1, ..., u_n) = \frac{\sqrt{n}}{\sigma} \sum_{i=1}^n u_i - a$$

Theorem 4.3. Let $(x_n)_n$ be a sequence of square integrable, equally distributed, independent observables, $E(x_n) = a, \sigma^2(x_n) = \sigma^2 (n = 1, 2, ...)$. Then

$$\lim_{n \to \infty} m(\frac{\frac{1}{n}\sum_{i=1}^n x_i - a}{\frac{\sigma}{\sqrt{n}}}(-\infty, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-\frac{u^2}{2}} du$$

Proof. We shall use the notation from the last two theorems. Then for $C \in \sigma(\mathcal{J})$

$$m(x_n(C)) = m(h_n(R \times ... \times R \times C) = P(\pi_n^{-1}(R \times ... \times R \times C)) = P(\xi_n^{-1}(C)),$$

hence

$$E(\xi_n) = \int_{-\infty}^{\infty} t dP_{\xi_n}(t) = \int_{-\infty}^{\infty} t dm_{x_n}(t) = E(x_n) = a,$$

and

$$\sigma^2(\xi_n) = \sigma^2(x_n) = \sigma^2.$$

Moreover,

$$P(\xi_1^{-1}(C_1) \cap ... \cap \xi_n^{-1}(C_n)) = P(\pi_n^{-1}(C_1 \times ... \times C_n)) =$$

$$= m(h_n(C_1 \times ... \times C_n) = m(x_1(C_1)).....m(x_n(C_n)) = P(\xi_1^{-1}(C_1)).....P(\xi_n^{-1}(C_n)),$$

hence $\xi_1, ..., \xi_n$ are independent for every n. Put $g_n(u_1, ..., u_n) = \frac{\sqrt{n}}{\sigma} \sum_{i=1}^{n} u_i - a$. By Theorem 4.2. we have

$$m(\frac{\sqrt{n}}{\sigma}(\sum_{i=1}^{n} x_i - a)((-\infty, t)) = m(h_n(g_n^{-1}(-\infty, t))) = m(y_n((-\infty, t)) =$$

$$= P(\eta_n^{-1}((-\infty, t))) = P(\{(\omega); \frac{\sqrt{n}}{\sigma} \sum_{i=1}^{n} \xi_i(\omega) - a < t\}).$$

Therefore by the classical central limit theorem

$$\lim_{n \to \infty} m(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i - a}{\frac{\sigma}{\sqrt{n}}}(-\infty, t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{u^2}{2}} du$$

Let us have a look to the previous theorem from another point of view, say, categorial. We had

$$\lim_{n \to \infty} P(\eta_n^{-1}((-\infty, t)) = \phi(t)$$

We can say that $(\eta_n)_n$ converges to ϕ in distribution. Of course, there are important possibilities of convergencies, at least in measure and almost everywhere.

A sequence $(\eta_n)_n$ of random variables (= measurable functions) converges to 0 in measure $\mu : \mathcal{S} \to [0, 1]$, if

$$\lim_{n \to \infty} \mu(\eta^{-1}(-\varepsilon, \varepsilon)) = 0$$

for every $\varepsilon > 0$. And the sequence converges to 0 almost everywhere, if

$$\lim_{n \to \infty} P(\cap_{p=1}^{\infty} \cup_{k=1}^{\infty} \cap_{n=k}^{\infty} \eta_n^{-1}((-\frac{1}{p}, \frac{1}{p})) = 1$$

Certainly, if $\eta_n(\omega) \to 0$, then

$$\forall \varepsilon > 0 \exists k \forall n > k : -\varepsilon < \eta(\omega) < \varepsilon$$

If we instead of ε use $\frac{1}{p}$, $p \in N$, then $\eta_n(\omega) \to 0$ if and only if

$$\forall p \exists k \forall n > k : \omega \in \eta_n^{-1}((-\frac{1}{p}, \frac{1}{p})).$$

And $\eta_n \to 0$ almost everywhere, if the set $\{\omega; \eta(\omega) \to 0\}$ has measure 1.

Definition 4.1. A sequence $(y_n)_n$ of observables

(i) converges in distribution to a function $F : R \to R$, if

$$\lim_{n \to \infty} m(y_n((-\infty, t))) = F(t)$$

for every $t \in R$;

(ii) it converges to 0 in state $m : \mathcal{F} \to [0, 1]$, if

$$\lim_{n \to \infty} m(y_n((-\varepsilon, \varepsilon))) = 0$$

for every $\varepsilon > 0$;

(iii) it converges to 0 m-almost everywhere, if

$$\lim_{p \to \infty} \lim_{k \to \infty} \lim_{i \to \infty} m(\wedge_{n=k}^{k+i} y_n(-\frac{1}{p}, \frac{1}{p})) = 0.$$

Theorem 4.4. Let $(y_n)_n$ be a sequence of observables, $(\eta_n)_n$ be the sequence of corresponding random variables. Then

(i) $(y_n)_n$ converges to $F : R \to R$ in distribution if and only if $(\eta_n)_n$ converges to F;

(ii) $y_n)_n$ converges to 0 in state $m : \mathcal{F} \to [0, 1]$ if and only if $(\eta_n)_n$ converges to 0 in measure $P : \mathcal{S} \to [0, 1]$

(iii) if $(\eta_n)_n$ converges P-almost everywhere to 0, then $(y_n)_n$ m-almost everywhere converges to 0.

The details can be found in [66]. Many applications of the method has been described in [25], [31], [35], [37], [39] , [52].

6. Conditional probability

Conditional entropy (of A with respect to B) is the real number $P(A|B)$ such that

$$P(A \cap B) = P(B)P(A|B).$$

When A, B are independent, then $P(A|B) = P(A)$, the event A does not depend on the ocuring of event B. Another point of view:

$$P(A \cap B) = \int_B P(A|B)dP.$$

The number $P(A|B)$ can be regarded as a constant function, Constant functions are measurable with respect to the σ-algebra $\mathcal{S}_0 = \{\emptyset, \Omega\}$.

Generally $P(A|\mathcal{S}_0)$ can be defined for any σ-algebra $\mathcal{S}_0 \subset \mathcal{S}$, as an $\mathcal{S}_0$-measurable function such that

$$P(A \cap C) = \int_C P(A|\mathcal{S}_0) dP, C \in \mathcal{S}_o.$$

If $\mathcal{S}_0 = \mathcal{S}$, then we can put $P(A|\mathcal{S}_0) = \chi_A$, since χ_A is $\mathcal{S}_0$-measurable, and

$$\int_C \chi_A dP = P(A \cap C).$$

An important example of $\mathcal{S}_0$ is the family of all pre-images of a random variable $\xi : \Omega \to R$

$$\mathcal{S}_0 = \{\xi^{-1}(B); B \in \sigma(\mathcal{J})\}.$$

In this case we shall write $P(A|\mathcal{S}_0) = P(A|\xi)$, hence

$$\int_C (P(A|\xi) dP = P(A \cap C), C = \xi^{-1}(B), B \in \sigma(\mathcal{J}).$$

By the transformation formula

$$P(A \cap \xi^{-1}(B)) = \int_{\xi^{-1}(B)} g \circ \xi dP = \int_B g dP_\xi, B \in \sigma(\mathcal{J})$$

And exactly this formulation will be used in our IF-case,

$$m(A.x(B)) = \int_B p(A|x) dm_x = \int_B p(A|x) dF.$$

Of course, we must first prove the existence of such a mapping $p(A|x) : R \to R$ ([34], [70], [72]). Recall that the product of IF-events is defined by the formula

$$K.L = (\mu_K.\mu_L, \nu_K + \nu_L - \nu_K.\nu_L).$$

Theorem 5.1. Let $x : \sigma(\mathcal{J}) \to \mathcal{F}$ be an observable, $m : \mathcal{F} \to [0,1]$ be a state, and let $A \in \mathcal{F}$. Define $\nu : \sigma(\mathcal{J}) \to [0,1]$ by the equality

$$\nu(B) = m(A.x(B)).$$

Then ν is a measure.

Proof. Let $B \cap C = \emptyset, B, C \in \mathcal{B}(R) = \sigma(\mathcal{J})$. Then $x(B).x(C) = (0,1)$, hence

$$A.(x(B) \oplus x(C)) = (A.x(B)) \oplus (A.x(C)),$$

and therefore

$$\nu(B \cup C) = m(A.x(B \cup C)) = m(A.(x(B) \oplus x(C)) = m((A.x(B)) \oplus (A.x(C))) =$$

$$= m(A.x(B)) + m(A.x(C)) = \nu(B) + \nu(C).$$

Let $B_n \nearrow B$. Then $x(B_n) \nearrow x(B)$, hence $A.x(B_n) \nearrow A.x(B)$. Therefore

$$\nu(B_n) = m(A.x(B_n)) \nearrow m(A.x(B)) = \nu(B).$$

Theorem 5.2. Let $x : \sigma(\mathcal{J}) \to \mathcal{F}$ be an observable, $m : \mathcal{F} \to [0, .1]$ be a state, and let $A \in \mathcal{F}$. Then there exists a Borel measurable function $f : R \to R$ (i. e. $B \in \sigma(\mathcal{J}) \Longrightarrow f^{-1}(B) \in \sigma(\mathcal{J})$) such that

$$m(A.x(B)) = \int_B f dm_x$$

for any $B \in \sigma(\mathcal{J})$. If g is another such a function, then

$$m_x(\{u \in R; f(x) \neq g(x)\}) = 0.$$

Proof. Define $\mu, \nu : \sigma(\mathcal{J}) \to [0, 1]$ by the formulas

$$\mu(B) = m_x(B) = m(x(B)), \nu(B) = m(A.x(B)).$$

Then $\mu, \nu : \sigma(\mathcal{J}) \to [0, 1]$ are measures, and $\nu \leq \mu$.

By the Radon - Nikodym theorem there exists exactly one function $f : R \to R$ (with respect to the equality μ- almost everywhere) such that

$$m(A.x(B)) = \nu(B) = \int_B f d\mu = \int_B f dm_x, B \in \sigma(\mathcal{J}).$$

Definition 5.1. Let $x : \sigma(\mathcal{J}) \to \mathcal{F}$ be an observable $A \in \mathcal{F}$. Then the conditional probability $p(A|x) = f$ is a Borel measurable function (i. e. $B \in \mathcal{J} \Longrightarrow f^{-1}(B) \in \sigma(\mathcal{J})$) such that

$$\int_B p(A|x) dm_x = m(A.x(B))$$

for any $B \in \sigma(\mathcal{J})$.

7. Algebraic world

At the end of our communication we shall present two ideas. The first one is in some algebraizations of the product

$$A.B = (\mu_A.\mu_B, \nu_A + \nu_B - \nu_A.\nu_B).$$

The second idea is a presentation of a dual notion to the notion of IF-event.

In MV-algebras the product was introduced independently in [56] and [47]. Let us return to Definition 1.3 and Example 1.5.

Definition 6.1. An MV-algebra with product is a pair $(M, .)$, where M is an MV-algebra, and . is a commutative and associative binary operation on M satisfying the following conditions:

(i) $1.a = a$

(ii) $a.(b \odot \neg c) = (a.b) \odot \neg(a.c).$

Example 6.1. Let $\mathcal{M} \supset \mathcal{F}$ be the MV-algebra defined in Theorem 1.1 (Example1.5). Then $\mathcal{M}$ with the product $A.B = (\mu_A \mu_B, \nu_A + \nu_B - \nu_A \nu_B)$ is an MV-algebra with product. Indeed,

$$(1,0).(\mu_A, \nu_A) = (1.\mu_A, 0 + \nu_A - 0.\nu_A) = (\mu_A, \nu_A).$$

Moreover

$$(\mu_A, \nu_A).((\mu_B, \nu_B) \odot (1 - \mu_C, 1 - \nu_C)) =$$
$$= (\mu_A((\mu_B - \mu_C) \vee 0), \nu_A + (\nu_B - \nu_C + 1) \wedge 1 - \nu_A((\nu_B + 1 - \nu_C) \wedge 1)).$$

On the other hand

$$((\mu_A, \nu_A).(\mu_B, \nu_B)) \odot (\neg(\mu_A, \nu_A).(\mu_C, \nu_C)) =$$
$$((\mu_A(\mu_B - \mu_C)) \vee 0, (\nu_A + (\nu_B - \nu_C + 1) - \nu_A(\nu_B + 1 - \nu_C)) \wedge 1).$$

Denote

$$\nu_B - \nu_C + 1 = k.$$

If $1 \leq k$, then

$$\nu_A + k \wedge 1 - \nu_A(k \wedge 1) = \nu_A + 1 - \nu_A = 1,$$
$$(\nu_A + k - \nu_A k) \wedge 1 = (\nu_A + k(1 - \nu_A)) \wedge 1 = 1.$$

If $k < 1$, then

$$\nu_A + k \wedge 1 - \nu_A k \wedge 1 = \nu_A + k - \nu_A k,$$
$$(\nu_A + k - \nu_A k) \wedge 1 = \nu_A + k - \nu_A k,$$

hence actually

$$A.(B \odot \neg C) = (A.B) \odot (\neg(A.C)).$$

Similarly as in Section 1 we can define a product in D-posets, we shall name such D-posets Kôpka D-posets.

Definition 6.2. A Kôpka D-poset is a pair $(D, *)$, where D is a D-poset, and $*$ is a commutative and associative operation on D satisfying the following conditions:

1. $\forall a \in D : a * 1 = a$;

2. $\forall a, b \in D, a \leq b, \forall c \in D : a * c \leq b * c$;

3. $\forall a, b \in D : a - (a * b) \leq 1 - b$;

4. $\forall (a_n)_n \subset D, a_n \nearrow a, \forall b \in D : a_n * b \nearrow a * b$.

Evidently every IF-family $\mathcal{F}$ can be embedded to an MV-algebra with product and it is a special case of a Kôpka D-poset, hence any result from the Kôpka D-poset theory can be applied to our IF-events theory ([26], [64]).

Now let us consider a theory dual to the IF-events theory, theory of IV-events. A prerequisity of IV-theory is in the fact that it considers natural ordering and operations of vectors. On the other hand the IV-theory is isomorphic to the IF-theory ([65],[43]).

Definition 6.3. Let $(\Omega, \mathcal{S})$ be a measurable space, $\mathcal{S}$ be a σ-algebra. By an IV-event a pair $\overline{A} = (\overline{\mu}_A, \overline{\nu}_A) : \Omega \to [0,1]^2$ is considered such that

$$\overline{A} \leq \overline{B} \iff \overline{\mu}_A \leq \overline{\mu}_B, \overline{\nu}_A \leq \overline{\nu}_B;$$

$$\overline{A} \boxplus \overline{B} = ((\overline{\mu}_A + \overline{\mu}_B) \wedge 1, (\overline{\nu}_A + \overline{\nu}_B) \wedge 1);$$
$$\overline{A} \boxdot \overline{B} = ((\overline{\mu}_A + \overline{\mu}_B - 1) \vee 0, (\overline{\nu}_A + \overline{\nu}_B - 1) \vee 0).$$

Denote by $\mathcal{V}$ the family of all IV-events. By an IV-state a map $\overline{m} : \mathcal{V} \to [0,1]$ is considered such that the following properties are satisfied:

(i) $\overline{m}((0,0)) = 0, \overline{m}((1,1)) = 1;$

(ii) $\overline{A} \boxdot \overline{B} = (0,0) \implies \overline{m}(\overline{A} \boxplus \overline{B}) = \overline{m}(\overline{A}) + \overline{m}(\overline{B});$

(iii) $\overline{A}_n \nearrow \overline{A} \implies \overline{m}(\overline{A}_n) \nearrow \overline{m}(\overline{A}).$

Theorem 6.1. Let $\mathcal{V}$ be the family of all IV-events (with respect to $\Omega, \mathcal{S}$)), $\overline{m} : \to [0,1]$ be an IV-state. Define

$$\mathcal{F} = \{(\overline{\mu}_A, 1 - \overline{\nu}_A); (\overline{\mu}_A, \overline{\nu}_A) \in \mathcal{V}\},$$
$$m : \mathcal{F} \to [0,1], m((\mu_A, \nu_A)) = 1 - \overline{m}(\mu_A, 1 - \nu_A)),$$
$$\varphi : \mathcal{V} \to \mathcal{F}, \varphi((\overline{\mu}_A, \overline{\nu}_A)) = (\overline{\mu}_A, 1 - \overline{\nu}_A).$$

Then $\mathcal{F}$ is the family of all IF-events (with respect to $(\Omega, \mathcal{S})$, m is an IF-state and φ is an isomorphism such that

$\varphi((0,0)) = (0,1), \varphi((1,0)) = (1,1),$

$\varphi(\overline{A} \boxdot \overline{B}) = \varphi(\overline{A}) \odot \varphi(\overline{B}),$

$\varphi(\overline{A} \boxplus \overline{B}) = \varphi(\overline{A}) \oplus \varphi(\overline{B}),$

$\varphi(\neg \overline{A}) = \neg \varphi(\overline{A}),$

$\overline{m}(\overline{A})) = m(\varphi(\overline{A})), \overline{A} \in \mathcal{V}.$

Proof. It is almost straightforward. Of course, the using of the family $\mathcal{V}$ is more natural and the results can be applied immediately to probability theory on $\mathcal{F}$.

8. Conclusion

The structures studied in this chapter have two aspects: the first one is practical, the second theoretical one. Fuzzy sets and their generalization - Atanassov intuitionistic fuzzy sets - in both directions new possibilities give.

From the practical point of view we can recommend e. g. [1], [9], [69]. Of course, the whole IF - theory can be motivated by practical problems and applications (see[10],[44 - 46], [53]).

The main contribution of the presented theory is a new point of view on human thinking and creation. We consider algebraic models for multi valued logic: IF-events, and more generally MV-algebras, D-posets, and effect algebras. They are important for many valued logic as Boolean algebras for two valued logic. Of course, we presented also some results about entropy ([11], [12], [40 - 42], [59]), or inclusion - exclusion principle ([6], [26], [30])for an illustration. But the more important idea is in building the probability theory on IF-events.

The theoretical description of uncertainty has two parts in the present time : objective - probability and statistics, and subjective - fuzzy sets. We show that both parts can be considered together.

9. Acknowledgement

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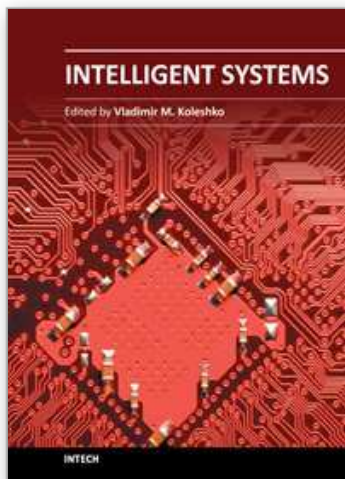
Intelligent Systems

Edited by Prof. Vladimir M. Koleshko

This book is dedicated to intelligent systems of broad-spectrum application, such as personal and social biosafety or use of intelligent sensory micro-nanosystems such as "e-nose", "e-tongue" and "e-eye". In addition to that, effective acquiring information, knowledge management and improved knowledge transfer in any media, as well as modeling its information content using meta-and hyper heuristics and semantic reasoning all benefit from the systems covered in this book. Intelligent systems can also be applied in education and generating the intelligent distributed eLearning architecture, as well as in a large number of technical fields, such as industrial design, manufacturing and utilization, e.g., in precision agriculture, cartography, electric power distribution systems, intelligent building management systems, drilling operations etc. Furthermore, decision making using fuzzy logic models, computational recognition of comprehension uncertainty and the joint synthesis of goals and means of intelligent behavior biosystems, as well as diagnostic and human support in the healthcare environment have also been made easier.

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