2

Discrete-Time Minimum-Mean-Square-Error Filtering

2.1 Introduction

This chapter reviews the solutions for the discrete-time, linear stationary filtering problems that are attributed to Wiener [1] and Kolmogorov [2]. As in the continuous-time case, a model-based approach is employed. Here, a linear model is specified by the coefficients of the input and output difference equations. It is shown that the same coefficients appear in the system's (frequency domain) transfer function. In other words, frequency domain model representations can be written down without background knowledge of z-transforms.

In the 1960s and 1970s, continuous-time filters were implemented on analogue computers. This practice has been discontinued for two main reasons. First, analogue multipliers and op amp circuits exhibit poor performance whenever (temperature-sensitive) calibrations become out of date. Second, updated software releases are faster to turn around than hardware design iterations. Continuous-time filters are now routinely implemented using digital computers, provided that the signal sampling rates and data processing rates are sufficiently high. Alternatively, continuous-time model parameters may be converted into discrete-time and differential equations can be transformed into difference equations. The ensuing discrete-time filter solutions are then amenable to more economical implementation, namely, employing relatively lower processing rates.

The discrete-time Wiener filtering problem is solved in the frequency domain. Once again, it is shown that the optimum minimum-mean-square-error solution is found by completing the square. The optimum solution is noncausal, which can only be implemented by forward and backward processes. This solution is actually a smoother and the optimum filter is found by taking the causal part.

The developments rely on solving a spectral factorisation problem, which requires pole-zero cancellations. Therefore, some pertinent discrete-time concepts are introduced in Section 2.2 prior to deriving the filtering results. The discussion of the prerequisite concepts is comparatively brief since it mirrors the continuous-time material introduced previously. In Section 2.3 it is shown that the structure of the filter solutions is unchanged – only the spectral factors are calculated differently.

"If we value the pursuit of knowledge, we must be free to follow wherever that search may lead us. The free mind is not a barking dog, to be tethered on a ten foot-chain." *Adlai Ewing Stevenson Jr.*

2.2 Prerequisites

2.2.1 Spaces

Discrete-time real-valued stochastic processes are denoted as $v_k^T = [v_{1,k}^T, v_{2,k}^T, \ldots, v_{n,k}^T]$ and $w_k^T = [w_{1,k}^T, w_{2,k}^T, \ldots, w_{n,k}^T]$, where $v_{i,k}, w_{i,k} \in \mathbb{R}$, $i = 1, \ldots n$ and $k \in (-\infty, \infty)$. The v_k and w_k are said to belong to the space $\mathbb{R}^n$. In this chapter, the vector w denotes the set of w_k over all time k, that is, $w = \{w_k, k \in (-\infty,\infty)\}$. The inner product $\langle v,w \rangle$ of two discrete-time vector processes v and w is defined by

$$\langle v,w \rangle = \sum_{k=-\infty}^{\infty} v_k^T w_k \ . \tag{1}$$

The 2-norm or Euclidean norm of a discrete-time vector process w, $\|w\|_2$, is defined as $\|w\|_2 = \sqrt{\langle w,w \rangle} = \sqrt{\sum_{k=-\infty}^{\infty} w_k^T w_k}$. The square of the 2-norm, that is, $\|w\|_2^2 = \langle w^T w \rangle = \sum_{k=-\infty}^{\infty} w_k^T w_k$ is commonly known as energy of the signal w. The Lebesgue 2-space is denoted by ℓ_2 and is defined as the set of discrete-time processes having a finite 2-norm. Thus, $w \in \ell_2$ means that the energy of w is bounded. See [3] for more detailed discussions of spaces and norms.

2.2.2 Discrete-time Polynomial Fraction Systems

Consider a linear, time-invariant system $\mathcal{G}$ that operates on an input process $w_k \in \mathbb{R}$ and produces an output process $y_k \in \mathbb{R}$, that is, $\mathcal{G} : \mathbb{R} \rightarrow \mathbb{R}$. Suppose that the difference equation for this system is

$$a_n y_{k-n} + a_{n-1} y_{k-n+1} + \ldots + a_1 y_{k-1} + a_0 y_k = b_m w_{k-m} + b_{m-1} w_{k-m+1} + \ldots + b_1 w_{k-1} + b_0 w_k \,, \tag{2}$$

where $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ are real-valued constant coefficients, with $a_n \neq 0$ and zero initial conditions.

Example 1. The difference equation $y_k = 0.1x_k + 0.2\, x_{k-1} + 0.3y_{k-1}$ specifies a system in which the coefficients are $a_0 = 1$, $a_1 = -0.3$, $b_0 = 0.2$ and $b_1 = 0.3$. Note that y_k is known as the current output and y_{k-1} is known as a past output.

2.2.3 The Z-Transform of a Discrete-time Sequence

The two-sided z-transform of a discrete-time process, y_k, is denoted by $Y(z)$ and is defined by

$$Y(z) = \sum_{k=-\infty}^{\infty} y_k z^{-k} \,, \tag{3}$$

where $z = e^{j\omega t}$ and $j = \sqrt{-1}$. Given a process y_k with z-transform $Y(z)$, y_k can be calculated from $Y(z)$ by taking the inverse z-transform of $y(z)$,

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"To live effectively is to live with adequate information." *Norbert Wiener*

$$y_k = \frac{1}{2\pi j}\int_{-e^{jwT}}^{e^{jwT}} Y(z)z^{k-1}dz \ . \tag{4}$$

Theorem 1 Parseval's Theorem:

$$\int_{-\infty}^{-\infty}\left|y_k\right|^2 dk = \frac{1}{2\pi j}\int_{-e^{jwT}}^{e^{jwT}}\left|Y(z)\right|^2 dz \ . \tag{5}$$

That is, the energy in the time domain equals the energy in the frequency domain.

2.2.4 Polynomial Fraction Transfer Functions

In the continuous-time case, a system's differential equations lead to a transfer function in the Laplace transform variable. Here, in discrete-time, a system's difference equations lead to a transfer function in the z-transform variable. Applying the z-transform to both sides of (2) yields the difference equation

$$\begin{aligned}\left(a_n z^{-n} + a_{n-1}z^{-n+1} + \ldots + a_1 z^{-1} + a_0\right)Y(z) \\ = \left(b_m z^{-m} + b_{m-1}z^{-m+1} + \ldots + b_1 z^{-1} + b_0\right)W(z) \ .\end{aligned} \tag{6}$$

Therefore

$$\begin{aligned}Y(z) &= \left[\frac{b_m z^{-m} + b_{m-1}z^{-m+1} + \ldots + b_1 z^{-1} + b_0}{a_n z^{-n} + a_{n-1}z^{-n+1} + \ldots + a_1 z^{-1} + a_0}\right]W(z) \\ &= G(z)W(z) \ ,\end{aligned} \tag{7}$$

where

$$G(z) = \left[\frac{b_m z^{-m} + b_{m-1}z^{-m+1} + \ldots + b_1 z^{-1} + b_0}{a_n z^{-n} + a_{n-1}z^{-n+1} + \ldots + a_1 z^{-1} + a_0}\right] \tag{8}$$

is known as the transfer function of the system. It can be seen that knowledge of the system difference equation (2) is sufficient to identify its transfer function (8).

2.2.5 Poles and Zeros

The numerator and denominator polynomials of (8) can be factored into m and n linear factors, respectively, to give

$$G(z) = \frac{b_m(z-\beta_1)(z-\beta_2)\ldots(z-\beta_m)}{a_n(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_n)} \ . \tag{9}$$

The numerator of $G(z)$ is zero when $z = \beta_i$, $i = 1 \ldots m$. These values of z are called the zeros of $G(z)$. Zeros inside the unit circle are called minimum-phase whereas zeros outside the unit

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"There is no philosophy which is not founded upon knowledge of the phenomena, but to get any profit from this knowledge it is absolutely necessary to be a mathematician." *Daniel Bernoulli*

circle are called non-minimum phase. The denominator of $G(z)$ is zero when $z = \alpha_i$, $i = 1 \ldots n$. These values of z are called the poles of $G(z)$.

Example 2. Consider a system described by the difference equation $y_k + 0.3y_{k-1} + 0.04y_{k-2} = w_k + 0.5w_{k-1}$. It follows from (2) and (8) that the corresponding transfer function is given by

$$G(z) = \frac{1+0.5z^{-1}}{1+0.3z^{-1}+0.04z^{-2}} = \frac{z^2+0.5z}{z^2+0.3z+0.04} = \frac{z(z+0.5)}{(z-0.1)(z+0.4)}$$

which possesses poles at $z = 0.1, -0.4$ and zeros at $z = 0, -0.5$.

2.2.6 Polynomial Fraction Transfer Function Matrix

In the single-input-single-output case, it is assumed that $w(z)$, $G(z)$ and $y(z) \in \mathbb{R}$. In the multiple-input-multiple-output case, $G(z)$ is a transfer function matrix. For example, suppose that $w(z) \in \mathbb{R}^m$, $y(z) \in \mathbb{R}^p$, then $G(z) \in \mathbb{R}^{p\times m}$, namely

$$G(z) = \begin{bmatrix} G_{11}(z) & G_{12}(s) & .. & G_{1m}(z) \\ G_{21}(z) & G_{22}(s) & & \\ \vdots & & \ddots & \vdots \\ G_{p1}(z) & & .. & G_{pm}(z) \end{bmatrix}, \tag{10}$$

where the components $G_{ij}(z)$ have the polynomial transfer function form within (8) or (9).

2.2.7 State-Space Transfer Function Matrix

The polynomial fraction transfer function matrix (10) can be written in the state-space representation

$$G(z) = C(zI - A)^{-1}B + D, \tag{11}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times p}$.

Example 3. For a state-space model with $A = -0.5$, $B = C = 1$ and $D = 0$, the transfer function is $G(z) = (z^{-1} - 0.5)^{-1}$.

Example 4. For state-space parameters $A = \begin{bmatrix} -0.33 & -0.04 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 0.2 & -0.04 \end{bmatrix}$ and $D = 1$, the use of Cramer's rule, that is, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, yields the transfer function $G(z) = \frac{z(z+0.5)}{(z-0.1)(z+0.4)}$.

"A mathematician is a blind man in a dark room looking for a black cat which isn't there." *Charles Robert Darwin*

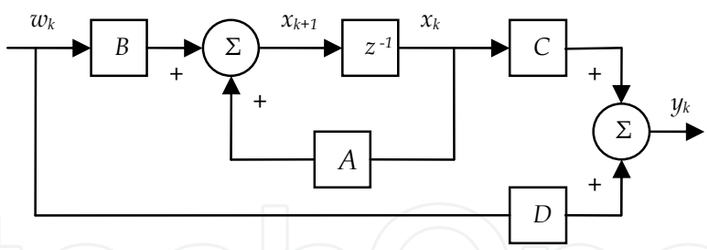

Figure 1. Discrete-time state-space system.

2.2.8 State-Space Realisation

The state-space transfer function matrix (11) can be realised as a discrete-time system $\mathcal{G}: \mathbb{R}^m \rightarrow \mathbb{R}^p$

$$x_{k+1} = Ax_k + Bw_k, \tag{12}$$

$$y_k = Cx_k + Dw_k, \tag{13}$$

where $w_k \in \mathbb{R}^m$ is an input sequence, $x_k \in \mathbb{R}^n$ is a state vector and $y_k \in \mathbb{R}^p$ is an output. This system is depicted in Fig. 1. It is assumed that w_k is a zero-mean, stationary process with $E\{w_j w_k^T\} = Q\delta_{jk}$, where $\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$ is the Kronecker delta function. In most applications, discrete-time implementations are desired, however, the polynomial fraction transfer function or state-space transfer function parameters may be known in continuous-time. Therefore, two methods for transforming continuous-time parameters to discrete-time are set out below.

2.2.9 The Bilinear Approximation

Transfer functions in the z-plane can be mapped exactly to the s-plane by substituting $z = e^{sT_s}$, where s = jw and T_S is the sampling period. Conversely, the substitution

$$\begin{aligned} s &= \frac{1}{T_S}\log(z) \\ &= \frac{2}{T_s}\left[\frac{z-1}{z+1} + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \frac{1}{5}\left(\frac{z-1}{z+1}\right)^5 + \frac{1}{7}\left(\frac{z-1}{z+1}\right)^7 + \ldots\right] \end{aligned} \tag{14}$$

can be used to map s-plane transfer functions into the z-plane. The bilinear transform is a first order approximation to (14), namely,

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"I do not like it, and I am sorry I ever had anything to do with it." *Erwin Rudolf Josef Alexander Schrödinger*

$$s \approx \frac{2}{T_S}\left[\frac{z-1}{z+1}\right]. \tag{15}$$

Example 5. Consider the continuous-time transfer function $H(s) = (s + 2)^{-1}$ with $T_S = 2$. Substituting (15) yields the discrete-time transfer function $H(z) = (3z + 1)^{-1}$. The higher order terms within the series of (14) can be included to improve the accuracy of converting a continuous-time model to discrete time.

2.2.10 Discretisation of Continuous-time Systems

The discrete-time state-space parameters, denoted here by $\{A_D, B_D, C_D, D_D, Q_D, R_D\}$, can be obtained by discretising the continuous-time system

$$\dot{x}(t) = A_C(t) + B_C w(t)\,, \tag{16}$$

$$y(t) = C_C x(t) + D_C w(t)\,, \tag{17}$$

$$z(t) = y(t) + v(t)\,, \tag{18}$$

where $E\{w(t)w^T(\tau)\} = Q_C\delta(t-\tau)$ and $E\{v(t)v^T(\tau)\} = R_C\delta(t-\tau)$. Premultiplying (16) by $e^{-A_C t}$ and recognising that $\frac{d}{dt}(e^{-A_C t}x(t)) = e^{-A_C t}\dot{x}(t) - e^{-A_C t}A_C x(t)$ yields

$$\frac{d}{dt}(e^{-A_C t}x(t)) = e^{-A_C t}B_C w(t)\,. \tag{19}$$

Integrating (19) results in

$$e^{-A_C t}x(t) - e^{-A_C t_0}x(t_0) = \int_{t_0}^{t} e^{-F\tau}B_C w(\tau)d\tau \tag{20}$$

and hence

$$\begin{aligned} x(t) &= e^{A_C(t-t_0)}x(t_0) + e^{A_C t}\int_{t_0}^{t} e^{-F\tau}B_C w(\tau)d\tau \\ &= e^{A_C(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{F(t-\tau)}B_C w(\tau)d\tau \end{aligned} \tag{21}$$

is a solution to the differential equation (16). Suppose that $x(t)$ is available at integer k multiples of T_s. Assuming that $w(t)$ is constant during the sampling interval and substituting $t_0 = kT_s$, $t = (k+1)T_s$ into (21) yields

$$x((k+1)T_s) = e^{A_C T_s}x(kT_s) + \int_{kT_s}^{(k+1)T_s} e^{A_C((k+1)T_s-\tau)}B_C d\tau w(kT_s)\,. \tag{22}$$

With the identifications $x_k = x(kT_s)$ and $w_k = w(kT_s)$ in (22), it can be seen that

"In the fields of observation, chance favours only the mind that is prepared." *Louis Pasteur*

$$A_D = e^{A_C T_s} \ , \tag{23}$$

$$B_D = \int_{kT_s}^{(k+1)T_s} e^{A_C((k+1)T_s - \tau)} B_C d\tau \ . \tag{24}$$

The τ within the definite integral (24) varies from kT_s to $(k+1)Ts$. For a change of variable $\lambda = (k+1)T_s - \tau$, the limits of integration become $\lambda = T_s$ and $\lambda = 0$, which results in the simplification

$$\begin{aligned} B_D &= -\int_{T_s}^{0} e^{A_C \lambda} B_C d\lambda \\ &= \int_{0}^{T_s} e^{A_C \lambda} B_C d\lambda \ . \end{aligned} \tag{25}$$

Denoting $E\{w_j w_k^T\} = Q_D \delta_{jk}$ and using (25) it can be shown that [4]

$$Q_D = \int_{0}^{T_s} e^{A_C \lambda} B_C Q_C B_C^T e^{A_C \lambda} d\lambda \ . \tag{26}$$

The exponential matrix is defined as

$$e^{A_C t} = I + A_C t + \frac{A_C^2 t^2}{2} + \cdots + \frac{A_C^N t^N}{N!} \ , \tag{27}$$

which leads to

$$A_D = I + A_C T_s + \frac{(A_C T_s)^2}{2!} + \frac{(A_C T_s)^3}{3!} + \frac{(A_C T_s)^4}{4!} + \cdots \ , \tag{28}$$

$$B_D = T_s + \frac{A_C T_s^2}{2!} + \frac{A_C^2 T_s^3}{3!} + \frac{A_C^3 T_s^4}{4!} + \cdots \ , \tag{29}$$

$$Q_D = B_C Q_C B_C^T T_s + \frac{(A_C B_C Q_C B_C^T + B_C Q_C B_C^T A_C^T) T_s^2}{2!} + \cdots \ . \tag{30}$$

It is common practice ([4] – [6]) to truncate the above series after terms linear in T_s. Some higher order terms can be retained in applications where parameter accuracy is critical. Since the limit as $N \to \infty$ of $T_s^N / N!$ is 0, the above series are valid for any value of T_s. However, the sample period needs to be sufficiently small, otherwise the above discretisations will be erroneous. According to the Nyquist-Shannon sampling theorem, the sampling rate is required to be at least twice the highest frequency component of the continuous-time signal. In respect of (17), the output map may be written as

$$y(kT_s) = C_C x(kT_s) + D_C w(kT_s) \tag{31}$$

"We are more easily persuaded, in general, by the reasons we ourselves discover than by those which are given to us by others." *Blaise Pascal*

and thus

$$C_D = C_C\,, \tag{32}$$

$$D_D = D_C\,. \tag{33}$$

Following the approach of [7], it is assumed that the continuous-time signals are integrated between samples, for example, the discretised measurement noise is $v(kT_s) = \frac{1}{T_s}\int_{kT_s}^{(k+1)T_s} v(\tau)d\tau$. Then the corresponding measurement noise covariance is

$$R_D = \frac{1}{T_s^2}\int_{kT_s}^{(k+1)T_s} R_C d\tau = \frac{1}{T_s} R_C\,. \tag{34}$$

In some applications, such as inertial and satellite navigation [8], the underlying dynamic equations are in continuous-time, whereas the filters are implemented in discrete-time. In this case, any underlying continuous-time equations together with (28) – (30) can be calculated within a high rate foreground task, so that the discretised state-space parameters will be sufficiently accurate. The discrete-time filter recursions can then be executed within a lower rate background task.

2.2.11 Asymptotic Stability

Consider a discrete-time, linear, time-invariant system $\mathcal{G}$ that operates on an input process w and produces an output process y. The system $\mathcal{G}$ is said to be asymptotically stable if the output remains bounded, that is, $y \in \ell_2$, for any input $w \in \ell_2$. Two equivalent conditions for $\mathcal{G}$ to be asymptotically stable are as follows.

(i) The i eigenvalues of the system's state matrix are inside the unit circle, that is, for A_i of (11), $\left|\lambda_i(A)\right| < 1$.

(ii) The i poles of the system's transfer function are inside the unit circle, that is, for α_i of (9), $\left|\alpha_i\right| < 1$.

Example 6. A state-space system having $A = -0.5$, $B = C = 1$ and $D = 0$ is stable, since $\lambda(A) = 0.5$ is in the unit circle. Equivalently, the corresponding transfer function $G(z) = (z + 0.5)^{-1}$ has a pole at $z = -0.5$ which is inside the unit circle and so the system is stable.

2.2.12 Adjoint Systems

Let $\mathcal{G} : \mathbb{R}^p \rightarrow \mathbb{R}^q$ be a linear system operating on the interval $[0, T]$. Then $\mathcal{G}^H : \mathbb{R}^q \rightarrow \mathbb{R}^p$, the adjoint of $\mathcal{G}$, is the unique linear system such that, for all $\alpha \in \mathbb{R}^q$ and $w \in \mathbb{R}^p$, $\langle \alpha, \mathcal{G} w \rangle = \langle \mathcal{G}^H \alpha, w \rangle$. The following derivation is a simplification of the time-varying version that appears in [9].

"Eighty percent of success is showing up." *(Woody) Allen Stewart Konigsberg*

Lemma 1 (State-space representation of an adjoint system): *Suppose that a discrete-time linear time-invariant system* $\mathcal{G}$ *is described by*

$$x_{k+1} = Ax_k + Bw_k, \tag{35}$$

$$y_k = Cx_k + Dw_k, \tag{36}$$

with $x_0 = 0$. *The adjoint* $\mathcal{G}^H$ *is the linear system having the realisation*

$$\zeta_{k-1} = A^T\zeta_k - C^T\alpha_k, \tag{37}$$

$$\beta_k = -B^T\zeta_k + D^T\alpha_k, \tag{38}$$

with $\zeta_T = 0$.

Proof: *The system (35) – (36) can be written equivalently*

$$\begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix}\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} 0(t) \\ y(t) \end{bmatrix} \tag{39}$$

with $x_0 = 0$. *Thus*

$$\langle y, \mathcal{G}w\rangle = \left\langle \begin{bmatrix} \zeta \\ \alpha \end{bmatrix}, \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix} \right\rangle \tag{40}$$

$$= \sum_{k=1}^{N}\zeta_k^T x_{k+1} - \sum_{k=1}^{N}\zeta_k^T(Ax_k + Bw_k) + \sum_{k=1}^{N}\alpha_k^T(Cx_k + Dw_k)$$

$$= \left\langle \begin{bmatrix} z^{-1}I - A^T & C^T \\ -B^T & D^T \end{bmatrix}\begin{bmatrix} \zeta \\ \alpha \end{bmatrix}, \begin{bmatrix} x \\ w \end{bmatrix} \right\rangle$$

$$= \langle \mathcal{G}^H\boldsymbol{\alpha}, \boldsymbol{w}\rangle \tag{41}$$

where $\mathcal{G}^H$ *is given by (37) – (38).* □

Thus, the adjoint of a discrete-time system having the parameters $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a system with parameters $\begin{bmatrix} A^T & -C^T \\ -B^T & D^T \end{bmatrix}$. Adjoint systems have the property $(\mathcal{G}^H)^H = \mathcal{G}$. The adjoint of

"There is something fascinating about science. One gets such wholesale returns of conjecture out of such a trifling investment of fact." *Samuel Langhorne Clemens aka. Mark Twain*

the transfer function matrix $G(z)$ is denoted as $G^H(z)$ and is defined by the transfer function matrix

$$G^H(z) = G^T(z^{-1}) . \tag{42}$$

Example 7. Suppose that a system $\mathcal{G}$ has the state-space parameters A = - 0.5 and $B = C = D = 1$. From Lemma 1, an adjoint system has the state-space parameters $A = -0.5$, $B = C = -1$, $D = 1$ and the corresponding transfer function is $G^H(z) = 1 + (z^{-1} + 0.5)^{-1} = (3z + 2)(z + 2)^{-1}$, which is unstable and non-minimum-phase. Alternatively, the adjoint of $G(z) = 1 + (z + 0.5)^{-1} = (z + 1.5)(z + 0.5)^{-1}$ can be obtained using (42), namely, $G^H(z) = G^T(z\text{-}1) = (3z + 2)(z + 2)^{-1}$.

2.2.13 Causal Systems

A causal system is a system whose output depends exclusively on past and current inputs and outputs.

Example 8. Consider $x_{k+1} = 0.3x_k + 0.4x_{k-1} + w_k$. Since the output x_{k+1} depends only on past states x_k, x_{k-1}, and past inputs w_k, this system is causal.

Example 9. Consider $x_k = 0.3x_{k+1} + 0.4x_k + w_{k+1}$. Since the output x_k depends on future outputs x_{k+1} and future w_{k+1} inputs, this system is non-causal.

2.2.14 Realising Unstable System Components

Unstable system components are termed unrealisable because their outputs are not in ℓ_2, that is, they are unbounded. In other words, unstable systems cannot produce a useful output. However, an unstable causal component can be realised as a stable non-causal or backwards component. Consider the system $\mathcal{G}$ (35) – (36) in which the eigenvalues of A all lie outside the unit circle. In this case, a stable adjoint system $\beta = \mathcal{G}^H \alpha$ can be realised by the following three-step procedure.

(i) Time-reverse the input signal α_k, that is, construct α_τ, where $\tau = N - k$ is a time-to-go variable.

(ii) Realise the stable system $\mathcal{G}^T$

$$\zeta_{\tau+1} = A^T\zeta_\tau + C^T\alpha_\tau , \tag{43}$$

$$\beta_\tau = B^T\zeta_\tau + D^T\alpha_\tau , \tag{44}$$

with $\zeta_T = 0$.

(iii) Time-reverse the output signal β_τ , that is, construct β_k .

Thus if a system consists of a cascade of stable and unstable components, it can be realised by a combination of causal and non-causal components. This approach will be exploited in the realisation of smoothers subsequently.

"I've lost my faith in science." *Ruth Elizabeth (Bette) Davis*

Example 10. Suppose that it is desired to realise the system $G(z) = G_2^H(z)G_1(z)$, in which $G_1(z)$ = $(z + 0.6)^{-1}$ $G_2^H(z) = z(0.9z+1)^{-1}$, that is, $G_2(z) = (z+0.9)^{-1}$. This system can be realised using the processes shown in Fig. 2.

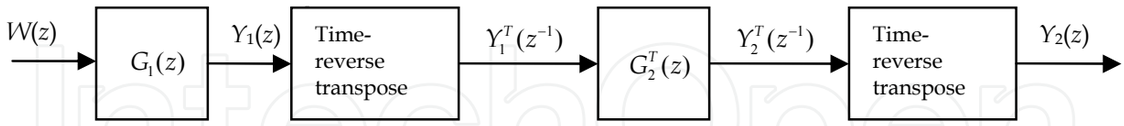

Figure 2. Realising an unstable $G(z) = G_2^H(z)G_1(z)$.

2.2.15 Power Spectral Density

Consider again a linear, time-invariant system $y = \mathcal{G}w$ and its corresponding transfer function matrix $G(z)$. Then $\Phi_{yy}(z)$, the power spectral density of y, is given by

$$\Phi_{yy}(z) = GQG^H(z)\,, \tag{45}$$

which has the property $\Phi_{yy}(z) = \Phi_{yy}(z^{-1})$. From Parseval's Theorem (5), the average total energy of $y(t)$ is given by

$$\int_{-e^{jwT}}^{e^{jwT}} \Phi_{yy}(z)dz = \int_{-\infty}^{\infty} \left|y_k\right|^2 dk = \left\|y(t)\right\|_2^2 = E\{y^T(t)y(t)\}\,, \tag{46}$$

which equals the area under the power spectral density curve.

2.2.16 Spectral Factorisation

To avoid confusion with the z-transform variable, denote the noisy measurements of $y(z) = G(z)w(z)$ by

$$u(z) = y(z) + v(z)\,, \tag{47}$$

where $v(z) \in \mathbb{R}^p$ is the z-transform of an independent, zero-mean, stationary, white measurement noise process with $E\{v_j v_k^T\} = R\delta_{jk}$. Let

$$\Phi_{uu}(z) = GQG^H(z) + R \tag{48}$$

denote the spectral density matrix of the measurements $u(t)$. A discrete-time transfer function is said to be minimum phase if its zeros lies inside the unit circle. Conversely, transfer functions having outside-unit-circle-zeros are known as non-minimum phase.

Suppose that $\Phi_{uu}(z)$ is a spectral density matrix of transfer functions possessing equal order numerator and denominator polynomials that do not have roots on the unit circle. Then the spectral factor matrix $\Delta(z)$ satisfies the following.

"Knowledge advances by steps, and not by leaps." *Baron Thomas Macaulay*

(i) $\Delta(z)\,\Delta^H(z) = \Phi_{uu}(z)$.
(ii) $\Delta(z)$ is causal, that is, the poles of $\Delta(z)$ are inside the unit circle.
(iii) $\Delta^{-1}(z)$ is causal, that is, the zeros of $\Delta(z)$ which are the poles of $\Delta^{-1}(z)$ are inside the unit circle.

The problem of spectral factorisation within discrete-time Wiener filtering problems is studied in [10]. The roots of the transfer function polynomials need to be sorted into those inside the unit circle and those outside the unit circle. Spectral factors can be found using Levinson-Durbin and Schur algorithms, Cholesky decomposition, Riccati equation solution [11] and Newton-Raphson iteration [12].

Example 11. Applying the Bilinear Transform (15) to the continuous-time low-pass plant $G(s) = (s + 1)^{-1}$ for a sample frequency of 2 Hz yields $G(z) = 0.2(z+1)(z-0.6)^{-1}$. With $Q = R = 1$, the measurement spectral density (48) is $\Phi_{uu}(z) = \dfrac{(1.08z - 0.517)}{(z - 0.6)} \times \dfrac{(-0.517z + 1.08)}{(-0.6z - 1.0)}$. By inspection, $\Delta(z) = (1.08z - 0.517)(z - 0.6)^{-1}$ has inside-unit-circle-poles and zeros that satisfy $\Delta(z)\Delta^H(z) = \Phi_{uu}(z)$.

Example 12. Consider the high-pass plant $G(z) = 4.98(z - 0.6)(z + 0.99)^{-1}$ and $Q = R = 1$. The spectral density is $\Phi_{uu}(z) = \dfrac{(5.39z - 2.58)}{(z + 0.99)} \times \dfrac{(-2.58z + 5.39)}{(0.99z - 1.0)}$. Thus the stable, minimum phase spectral factor is $\Delta(z) = (5.39z - 2.58)(z + 0.99)^{-1}$, since it has inside-unit-circle-poles and zeros.

2.2.17 Calculating Causal Parts

Suppose that a discrete-time transfer function has the form

$$\begin{aligned} G(z) &= c_0 + \sum_{i=1,|a_i|<1}^{n} \frac{d_i}{z - a_i} + \sum_{j=1,|b_j|>1}^{m} \frac{e_j}{z - b_j} \\ &= c_0 + G_{iucp}(z) + G_{oucp}(z), \end{aligned} \tag{49}$$

where $c_0, d_i, e_j \in \mathbb{R}$, $G_{iucp}(z) = \sum_{i=1,|a_i|<1}^{n} \frac{d_i}{z - a_i}$ is the sum of partial fractions having inside-unit-circle-poles and $G_{oucp}(z) = \sum_{j=1,|b_j|>1}^{m} \frac{e_j}{z - b_j}$ is the sum of partial fractions having outside-unit-circle-poles. Assume that the roots of $G(z)$ are distinct and do not lie on the unit circle. In this case the partial fraction coefficients d_i and e_i within (49) can be calculated from the numerator and denominator polynomials of $G(z)$ via $d_i = (z - a_i)G(z)\big|_{z=a_i}$ and $e_j = (z - b_j)G(z)\big|_{z=b_j}$. Previously, in continuous-time, the convention was to define constants to be causal. This is consistent with ensuring that the non-causal part of the discrete-time

"Knowledge rests not upon truth alone, but on error also." *Carl Gustav Jung*

transfer function is zero at $z = 0$. Thus, the non-causal part of $G(z)$, denoted by $\{G(z)\}_-$, is obtained as

$$\{G(z)\}_- = G_{oucp}(z) - G_{oucp}(0) \tag{50}$$

and the causal part of $G(z)$, denoted by $\{G(z)\}_+$,is whatever remains, that is,

$$\begin{aligned} \{G(z)\}_+ &= G(z) - \{G(z)\}_- \\ &= c_0 + G_{iucp}(z) + G_{oucp}(0). \end{aligned} \tag{51}$$

Hence, the causal part of transfer function can be found by carrying out the following three steps.

(i) If the transfer function is not strictly proper, that is, if the order of the numerator not less than the degree of the denominator, perform synthetic division to extract the constant term.

(ii) Expand out the (strictly proper) transfer function into the sum of partial fractions (49).

(iii) Obtain the causal part from (51), namely, take the sum of the constant term, the partial fractions with inside-unit-circle-poles and the partial fractions with outside-unit-circle-poles evaluated at $z = 0$.

Example 13. Consider the strictly proper transfer function $G(z) = \dfrac{3z+3.2}{z^2+2.6z+1.2} = \dfrac{3z+3.2}{(z+0.6)(z+2)} = \dfrac{1}{z+0.6} + \dfrac{2}{z+2}$. It follows from (50) and (51) that $\{G(z)\}_- = \dfrac{2}{z+2} - 1 = \dfrac{-z}{z+2}$ and $\{G(z)\}_+ = \dfrac{1}{z+0.6} + 1 = \dfrac{z+1.6}{z+0.6}$, respectively. It is easily verified that $G(z) = \{G(z)\}_+ + \{G(z)\}_-$.

Example 14. Consider the proper transfer function $G(z) = \dfrac{2z^2+8.2z+5.6}{z^2+2.6z+1.2}$. Carrying out synthetic division results in $G(z) = 2 + \dfrac{1}{z+0.6} + \dfrac{2}{z+2}$. It follows from (50) and (51) that $\{G(z)\}_- = \dfrac{2}{z+2} - 1 = \dfrac{-z}{z+2}$ and $\{G(z)\}_+ = \dfrac{1}{z+0.6} + 1 + 2 = \dfrac{3z+2.8}{z+0.6}$, respectively.

"The beginning of knowledge is the discovery of something we do not understand." *Frank Patrick Herber*

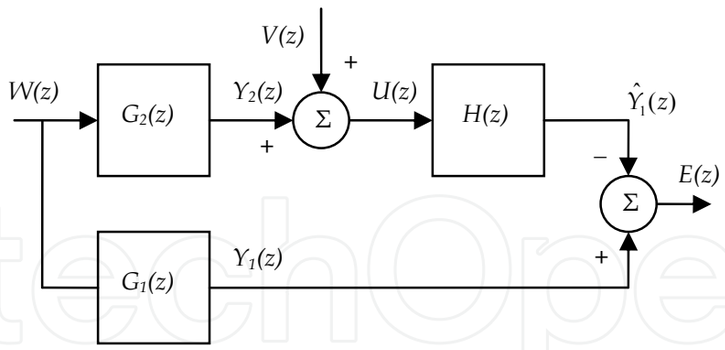

Figure 3. The general z-domain filtering problem.

2.3 Minimum-Mean-Square-Error Filtering

2.3.1 Filter Derivation

This section derives the optimal non-causal minimum-mean-square-error solution for the problem configuration of Fig. 3. The derivation is identical to the continuous-time case which is presented in Chapter 1. It is assumed that the parameters of the stable transfer function $G_2(z) = C_2(zI - A)^{-1}B + D_2$ are known. Let $Y_2(z)$, $W(z)$, $V(z)$ and $U(z)$ denote the z-transform of a system's output, process noise, measurement noise and observations, respectively. Then it follows from (47) that the z-transform of the measurements is

$$U(z) = Y_2(z) + V(z)\,. \tag{52}$$

Consider a fictitious reference system $G_1(z) = C_1(zI - A)^{-1}B + D_1$ as shown in Fig. 3. The problem is to design a filter transfer function $H(z)$ to calculate estimates $\hat{Y}_1(z) = H(z)U(z)$ of $Y_1(z)$ so that the energy $\int_{-j\infty}^{j\infty} E(z)E^H(z)dz$ of the estimation error

$$E(z) = \hat{Y}_1(z) - Y_1(z) \tag{53}$$

is minimised. It can be seen from Fig. 3 that the estimation error is generated by the system

$$E(z) = -\begin{bmatrix} H(z) & H_2G(z) - G_1(z) \end{bmatrix} \begin{bmatrix} V(z) \\ W(z) \end{bmatrix}. \tag{54}$$

The error power spectrum density matrix is given by the covariance of $E(z)$, that is,

$$\Phi_{ee}(z) = E(z)E^H(z) \tag{55}$$

"I shall try to correct errors when shown to be errors; and I shall adopt new views so fast as they shall appear to be true views." *Abraham Lincoln*

$$= \begin{bmatrix} H(z) & HG_2(z) - G_1(z) \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} H^H(z) \\ G_2^H H^H(z) - G_1^H(z) \end{bmatrix}$$

$$= G_1 Q G_1^H(z) - G_1 Q G_2^H H^H(z) - H G_2 Q G_1^H(z) + H \Delta\Delta^H H^H(z),$$

where

$$\Delta\Delta^H(z) = G_2 Q G_2^H(z) + R \tag{56}$$

is the spectral density matrix of the measurements. Completing the square within (55) yields

$$\begin{aligned} \Phi_{ee}(z) = {} & G_1 Q G_1^H(z) - G_1 Q G_2^H (\Delta\Delta^H)^{-1} G_2 Q G_1^H(z) \\ & + (H\Delta(z) - G_1 Q G_2^H \Delta^{-H}(z))(H\Delta(z) - G_1 Q G_2^H \Delta^{-H}(z))^H, \end{aligned} \tag{57}$$

in which $\Delta^{-H}(z) = (\Delta^H)^{-1}(z)$. It follows that the total energy of the error signal can be expressed as

$$\begin{aligned} \int_{-e^{jwT}}^{e^{jwT}} \Phi_{ee}(z)dz = {} & \int_{-e^{jwT}}^{e^{jwT}} G_1 Q G_1^H(z) - G_1 Q G_2^H (\Delta\Delta^H)^{-1} G_2 Q G_1^H(z) dz \\ & + \int_{-e^{jwT}}^{e^{jwT}} (H\Delta(z) - G_1 Q G_2^H \Delta^{-H}(z))(H\Delta(z) - G_1 Q G_2^H \Delta^{-H}(z))^H dz. \end{aligned} \tag{58}$$

The first term on the right-hand-side of (58) is independent of $H(z)$ and represents a lower bound of $\int_{-e^{jwT}}^{e^{jwT}} \Phi_{ee}(z)dz$. The second term on the right-hand-side of (58) may be minimised by a judicious choice for $H(z)$.

Theorem 1: *The optimal solution for the above linear time-invariant estimation problem with measurements (52) and error (53) is*

$$H(z) = G_1 Q G_2^H \Delta^{-H} \Delta^{-1}(z), \tag{59}$$

which minimises $\int_{-e^{jwT}}^{e^{jwT}} \Phi_{ee}(z)dz$.

Proof*: The result follows by setting* $H\Delta(z) - G_1 Q G_2^H \Delta^{-H}(z)$ *equal to the zero matrix within (58).* □

By Parseval's theorem, the minimum mean-square-error solution (59) also minimises $\|e(z)\|_2^2$. The solution (59) is non-causal because the factor $G_2^H(\Delta^H)^{-1}(z)$ possesses outside-unit-circle poles. This optimal non-causal solution is actually a smoother, which can be realised by a combination of forward and backward processes.

"I think anybody who doesn't think I'm smart enough to handle the job is underestimating." *George Walker Bush*

The transfer function matrix of the optimal causal solution or filter is obtained by setting the setting the causal part of $H\Delta(z) - G_1QG_2^H\Delta^{-H}(z)$ equal to the zero matrix, resulting in $\{H(z)\Delta^{-1}(z)\}_+ = \{G_1QG_2^H(\Delta^H)^{-1}\}_+$, that is $H(z)\Delta^{-1}(z) = \{G_1QG_2^H(\Delta^H)^{-1}\}_+$, which implies

$$H(z) = \left\{G_1QG_2^H(\Delta^H)^{-1}\right\}_+ \Delta^{-1}(z) . \tag{60}$$

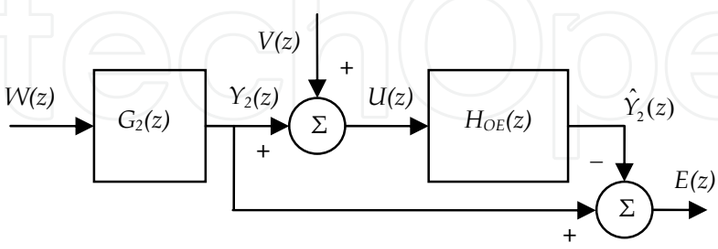

Figure 4. The z-domain output estimation problem.

2.3.2 Output Estimation

In output estimation, it is desired to estimate the output $Y_2(z)$ from the measurements $U(z)$, in which case the reference system is the same as the generating system, as shown in Fig. 4. The optimal non-causal solution (59) with $G_1(z) = G_2(z)$ becomes

$$H_{OE}(z) = G_2QG_2^H\Delta^{-H}\Delta^{-1}(z) . \tag{61}$$

Substituting $G_2QG_2^H(z) = \Delta\Delta^H(z) - R$ into (61) leads to the alternative form

$$\begin{aligned} H_{OE}(z) &= (\Delta\Delta^H - R)(\Delta\Delta^H)^{-1}(z) \\ &= I - R\Delta^{-H}\Delta^{-1}(z) . \end{aligned} \tag{62}$$

The solutions (61) and (62) are non-causal since $G^H(z)$ and $\Delta^{-H}(z)$ are non-causal. The optimal smoother or non-causal filter for output estimation is obtained by substituting $G_1(z) = G_2(z)$ into (60), namely,

$$H_{OE}(z) = \left\{G_2QG_2^H\Delta^{-H}\right\}_+ \Delta^{-1}(z) . \tag{63}$$

An alternative form arises by substituting $GQG^H(z) = \Delta\Delta^H(z) - R$ into (63), which results in

$$\begin{aligned} H_{OE}(z) &= \{\Delta(z) - R\Delta^{-H}\}_+\Delta^{-1}(z) \\ &= I - R\{\Delta^{-H}\}_+\Delta^{-1}(z) . \end{aligned} \tag{64}$$

In [10], it is recognised that $\{\Delta^H(z)\}_+ = \lim_{z\to\infty}\Delta(z)$, which is equivalent to $\{\Delta^H(z)\}_+ = \Delta^H(0)$. It follows that

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"There is much pleasure to be gained from useless knowledge." *Bertrand Arthur William Russell*

$$H_{OE}(z) = I - R\Delta^{-H}(0)\Delta^{-1}(z)\,, \tag{65}$$

which eliminates the need for calculating causal parts.

Example 15. Consider $G_2(z) = (z + 0.2)(z + 0.5)^{-1}$ together with $R = Q = 1$. The spectral factor is $\Delta(z) = (1.43z + 0.489)(z + 0.5)^{-1}$, which leads to $G_2QG_2^H\Delta^{-H}(z) = (0.2z^2 + 1.04z + 0.2)(0.489z^2 + 1.67z + 0.716)^{-1}$ and $\{G_2QG_2^H\Delta^{-H}(z)\}_+ = (0.734z + 0.14)(z + 0.5)^{-1}$. Hence, from (63), $H_{OE}(z) = (0.513z + 0.098)(z + 0.341)^{-1}$. The same solution can be calculated using $\Delta^{-H}(0) = 0.698$ within (65).

When the measurement noise becomes negligibly small, the output estimator approaches a short circuit, that is,

$$\lim_{R\to 0, e^{j\omega T}\to 0} \left|H_{OE}(z)\right| = I\,, \tag{66}$$

The above observation can be verified by substituting $R = 0$ into (65). This asymptote is consistent with intuition, that is, when the measurements are perfect, output estimation will be superfluous.

Example 16. Substituting R = 0.001 within Example 15 yields the filter H(z) = (0.999z + 0.2)(z + 0.2)-1, which illustrates the low measurement noise asymptote (66).

2.3.3 Input Estimation

In input estimation or equalisation problems, $G_2(z)$ is known as the channel model and it is desired to estimate the input process $w(t)$, as depicted in Fig. 5. The simplification of the optimum non-causal solution (59) for the case of $G_1(z) = I$ is

$$H_{IE}(z) = QG_2^H\Delta^{-H}\Delta^{-1}(z)\,, \tag{67}$$

Assume that: the channel model $G_2(z)$ is proper, that is, the order of the numerator is the same as the order of the denominator; and that the channel model $G_2(z)$ is stable and minimum phase, that is, its poles and zeros are inside the unit circle. The causal equaliser for proper, stable, minimum-phase channels is obtained by substituting $G_1(z) = I$ into (60)

$$\begin{aligned} H_{IE}(z) &= \{QG_2^H\Delta^{-H}\}_+\Delta^{-1}(z) \\ &= QG_2^H(0)\Delta^{-H}(0)\Delta^{-1}(z)\,. \end{aligned} \tag{68}$$

Under the above assumptions, the causal equaliser may be written equivalently as

$$H_{IE}(z) = \{G_2^{-1}G_2QG_2^H\Delta^{-H}\}_+\Delta^{-1}(z) \tag{69}$$

$$= \{G_2^{-1}(\Delta\Delta^H - R)\Delta^{-H}\}_+\Delta^{-1}(z)$$

$$= G_2^{-1}(I - R\{\Delta^{-H}\}_+\Delta^{-1}(z)) \tag{70}$$

"He who knows nothing is closer to the truth than he whose mind is filled with falsehoods and errors." *Thomas Jefferson.*

$$= G_2^{-1}(I - R\Delta^{-H}(0)\Delta^{-1}(z))$$

Thus, the equaliser is equivalent to a product of the channel inverse and the output estimator. It follows that when the measurement noise becomes negligibly small, the equaliser estimates the inverse of the system model, that is,

$$\lim_{R\to 0} H_{IE}(z) = G_2^{-1}(z)\ , \tag{71}$$

The above observation follows by substituting $R = 0$ into (69). In other words, if the channel model is invertible and signal to noise ratio is sufficiently high, the equaliser will estimate $w(t)$. When measurement noise is present then the solution trades off channel inversion and filtering. In the high measurement noise case, the equaliser approaches an open circuit, that is,

$$\lim_{Q\to 0, e^{j\omega T}\to 0} \left|H_{IE}(z)\right| = 0\ . \tag{72}$$

The above observation can be verified by substituting $\Delta\Delta^H = R$ into (70). Thus, when the equalisation problem is dominated by measurement noise, the estimation error is minimised by ignoring the data.

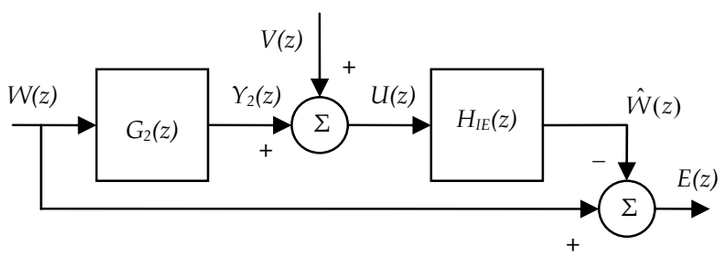

Figure 5. The z-domain input estimation problem.

Example 17. Consider the high-pass plant $G_2(s) = 100(s + 0.1)(s + 10)^{-1}$. Application of the bilinear transform for a sample frequency of 2 Hz yields $G_2(z) = (29.2857z - 27.8571)(z + 0.4286)^{-1}$. With $Q = 1$ and R = 0.001, the spectral factor is $\Delta(z) = (29.2861z + - 27.8568)(z + 0.4286)^{-1}$. From (67), $H_{IE}(z) = (z + 0.4286)(29.2861z - 27.8568)^{-1}$, which is high-pass and illustrates (71).

Example 18. Applying the bilinear transform for a sample frequency of 2 Hz to the low-pass plant $G_2(z) = (s + 10)(s + 0.1)^{-1}$ results in $G_2(z) = (3.4146z - 1.4634)(z - 0.9512)^{-1}$. With $Q = 1$ and R = 0.001, the spectral factor is $\Delta(z) = (3.4151z + 1.4629)(z - 0.9512)^{-1}$. From (67), $H_{IE}(z) = (z - 0.9512)(3.4156z + 1.4631)^{-1}$, which is low pass and is consistent with (71).

"They say that time changes things, but you actually have to change them yourself." *Andy Warhol*

	ASSUMPTIONS	MAIN RESULTS
Signals and systems	$E\{w_k\} = E\{W(z)\} = E\{v_k\} = E\{V(z)\} = 0$. $E\{w_k w_k^T\} = E\{W(z)W^T(z)\} = Q > 0$ and $E\{v_k v_k^T\} = E\{V(z)V^T(z)\} = R > 0$ are known. A, B, C_1, C_2, D_1 and D_2 are known. $G_1(z)$ and $G_2(z)$ are stable, *i.e.*, $\lvert\lambda_i(A)\rvert < 1$.	$G_1(z) = C_1(zI - A)^{-1}B + D_1$ $G_2(z) = C_2(zI - A)^{-1}B + D_2$
Spectral factorisation	$\Delta(z)$ and $\Delta^{-1}(z)$ are stable, *i.e.*, the poles and zeros of $\Delta(z)$ are inside the unit circle.	$\Delta\Delta^H(z) = G_2 Q G_2^H(z) + R$
Non-causal solution		$H(z) = G_1 Q G_2^H (\Delta^H)^{-1}\Delta^{-1}(z)$
Causal solution		$H(z) = \left\{G_1 Q G_2^H (\Delta^H)^{-1}\right\}_+ \Delta^{-1}(z)$

Table 1. Main results for the discrete-time general filtering problem.

2.4 Conclusion

Systems are written in the time-domain as difference equations

$$a_n y_{k-n} + a_{n-1} y_{k-n+1} + \ldots + a_1 y_{k-1} + a_0 y_k = b_m w_{k-m} + b_{m-1} w_{k-m+1} + \ldots + b_1 w_{k-1} + b_0 w_k ,$$

which can expressed as polynomial transfer functions in the z-transform variable

$$Y(z) = \left[\frac{b_m z^{-m} + b_{m-1} z^{-m+1} + \ldots + b_1 z^{-1} + b_0}{a_n z^{-n} + a_{n-1} z^{-n+1} + \ldots + a_1 z^{-1} + a_0}\right] W(z) = G(z)W(z) .$$

It can be seen that knowledge of a system's differential equation is sufficient to identify its transfer function. The optimal Wiener solution minimises the energy of the error and the mean-square-error and the main results are summarised in Table 1. The noncausal (or smoother) solution has unstable factors and can only be realised by a combination of forward and backward processes.

It is noted that $\{\Delta^H(z)\}_+ = \lim_{z\to\infty}\Delta(z) = \Delta^H(0)$, which can simplify calculating causal parts. For example, in output estimation problems where $G_1(z) = G_2(z)$, the minimum-mean-square-

"Time is a great teacher, but unfortunately it kills all its pupils." *Louis Hector Berlioz*

error solution is $H_{OE}(z) = I - R\Delta^{-H}(0)\Delta^{-1}(z)$. In the single-input-single-output case, when the measurement noise becomes negligible, the output estimator approaches a short circuit. Conversely, when the single-input-single-output problem is dominated by measurement noise, the output estimator approaches an open circuit.

In input estimation problems, $G_1(z) = I$. If the channel model is invertible, the optimal causal equaliser is given by $H_{IE}(z) \quad H_{IE}(z) = QG_2^H(0)\Delta^{-H}(0)\Delta^{-1}(z)$. When the measurement noise becomes negligible, that is, $\Delta^{-1}(z) \approx G_2^H(z)$, the optimal equaliser approaches the channel inverse. Conversely, when the problem is dominated by measurement noise, the equaliser approaches an open circuit.

2.5 Problems

Problem 1. Consider the error spectral density matrix

$$\Phi_{ee}(z) = [H\Delta - G_1QG_2^H(\Delta^H)^{-1}][H\Delta - G_1QG_2^H(\Delta^H)^{-1}]^H(z) + [G_1QG_1^H - G_1QG_2^H(\Delta\Delta^H)^{-1}G_2QG_1^H](z).$$

(a) Derive the optimal non-causal solution.
(b) Derive the optimal causal filter from (a).
(c) Derive the optimal non-causal output estimator.
(d) Derive the optimal causal filter from (c).
(e) Derive the optimal non-causal input estimator.
(f) Derive the optimal causal equaliser assuming that the channel inverse exists.

Problem 2. Derive the asymptotes for the following single-input-single-output estimation problems.

(a) Non-causal output estimation at $R = 0$.
(b) Non-causal output estimation at $Q = 0$.
(c) Causal output estimation at $R = 0$.
(d) Causal output estimation at $Q = 0$.
(e) Non-causal input estimation at $R = 0$.
(f) Non-causal input estimation at $Q = 0$.
(g) Causal input estimation at $R = 0$.
(h) Causal input estimation at $Q = 0$.

Problem 3. In respect of the output estimation problem with $G(z) = (z - \beta)(z - \alpha)^{-1}$, $\alpha = -0.3$, $\beta = -0.5$ and Q=1, verify the following.

(a) $R = 10$ yields $H(z) = (0.0948z + 0.0272)(z + 0.4798)^{-1}$.
(b) $R = 1$ yields $H(z) = (0.5059z + 0.1482)(z + 0.3953)^{-1}$.
(c) $R = 0.1$ yields $H(z) = (0.90941z + 0.2717)(z + 0.3170)^{-1}$.
(d) $R = 0.01$ yields $H(z) = (0.9901z + 0.2969)(z + 0.3018)^{-1}$.
(e) $R = 0.001$ yields $H(z) = (0.9990z + 0.2997)(z + 0.3002)^{-1}$.

"If I have ever made any valuable discoveries, it has been owing more to patient attention, that to any other talent." *Isaac Newton*

Problem 4. In respect of the input estimation problem with $G(z) = (z - \beta)(z - \alpha)^{-1}$, $\alpha = -0.9$, $\beta = -0.1$ and Q=1, verify the following.

(a) $R = 10$ yields $H(z) = (z + 0.1)(11.5988z + 1.9000)^{-1}$.
(b) $R = 1$ yields $H(z) = (z + 0.1)(2.4040z + 1.0000)^{-1}$.
(b) $R = 0.1$ yields $H(z) = (z + 0.1)(1.2468z + 0.9100)^{-1}$.
(d) $R = 0.01$ yields $H(z) = (z + 0.1)(1.0381z + 0.9010)^{-1}$.
(e) $R = 0.001$ yields $H(z) = (z + 0.1)(1.043z + 0.9001)^{-1}$.

2.6 Glossary

The following terms have been introduced within this section.

k	The integer-valued time variable. For example, $k \in (-\infty, \infty)$ and $k \in (0, \infty)$ denote $-\infty < k < \infty$ and $0 \le k < \infty$, respectively.
$w_k \in \mathbb{R}^n$	A discrete-time, real-valued, n-element stochastic input signal.
w	The set of w_k over a prescribed interval.
$\mathcal{G} : \mathbb{R}^p \to \mathbb{R}^q$	A linear system that operates on a p-element input signal and produces a q-element output signal.
$y = \mathcal{G}w$	The output of a linear system $\mathcal{G}$ that operates on an input signal w.
A, B, C, D	Time-invariant state space matrices of appropriate dimension. The system $\mathcal{G}$ is assumed to have the realisation $x_{k+1} = Ax_k + Bw_k$, $y_k = Cx_k + Dw_k$ in which w_k is known as the process noise or input signal.
v_k	A stationary stochastic measurement noise signal.
δ_{jk}	The Kronecker delta function.
Q and R	Time-invariant covariance matrices of stochastic signals w_k and v_k, respectively.
$Y(z)$	The z-transform of a continuous-time signal y_k.
$G(z)$	The transfer function matrix of the system $\mathcal{G}$. For example, the transfer function matrix of the system $x_{k+1} = Ax_k + Bw_k$, $y_k = Cx_k + Dw_k$ is given by $G(z) = C(zI - A)^{-1}B + D$.
$\langle v, w \rangle$	The inner product of two discrete-time signals v and w which is defined by $\langle v, w \rangle = \sum_{k=-\infty}^{\infty} v_k^T w_k$.
$\|w\|_2$	The 2-norm of the discrete-time signal w which is defined by $\|w\|_2 = \sqrt{\langle w, w \rangle} = \sqrt{\sum_{k=-\infty}^{\infty} w_k^T w_k}$.
ℓ_2	The set of continuous-time signals having finite 2-norm, which is known as the Lebesgue 2-space (see [3]).

"If your result needs a statistician then you should design a better experiment." *Baron Ernest Rutherford*

Asymptotic stability	A linear discrete-time system $\mathcal{G}$ is said to be asymptotically stable if its output $y \in \ell_2$ for any $w \in \ell_2$. If the real parts of the state matrix eigenvalues are inside the unit circle or equivalently if the real part of transfer function's poles are inside the unit circle then the system is stable.
T_s	Sample period.
$\mathcal{G}^H$	The adjoint of $\mathcal{G}$. The adjoint of a system having the state-space parameters $\{A, B, C, D\}$ is a system parameterised by $\{ A^T, -C^T, -B^T, D^T\}$.
$G^H(z)$	The adjoint (or Hermitian transpose) of the transfer function matrix $G(z)$.
$\Phi_{ee}(z)$	The spectral density matrix of the measurements e.
$\Delta(z)$	The spectral factor of $\Phi_{uu}(z)$ which satisfies $\Delta\Delta^H(z) = GQG^H(z) + R$. For brevity denote $\Delta^{-H}(z) = (\Delta^H)^{-1}(z)$.
$G^{-1}(z)$	The inverse of the transfer function matrix $G(z)$.
$G^{-H}(z)$	The inverse of the adjoint transfer function matrix $G^H(z)$.
$\{G(z)\}_+$	The causal part of the transfer function matrix $G(z)$.
$H(z)$	Transfer function matrix of the minimum mean-square-error solution.
$H_{OE}(z)$	Transfer function matrix of the minimum mean-square-error solution specialised for output estimation.
$H_{IE}(z)$	Transfer function matrix of the minimum mean-square-error solution specialised for input estimation.

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"There's two possible outcomes: if the result confirms the hypothesis, then you've made a discovery. If the result is contrary to the hypothesis, then you've made a discovery." *Enrico Fermi*

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"One of the greatest discoveries a man makes, one of his greatest surprises, is to find he can do what he was afraid he couldn't." *Henry Ford*

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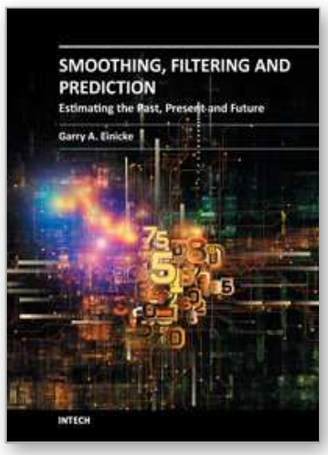

Smoothing, Filtering and Prediction - Estimating The Past, Present and Future

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This book describes the classical smoothing, filtering and prediction techniques together with some more recently developed embellishments for improving performance within applications. It aims to present the subject in an accessible way, so that it can serve as a practical guide for undergraduates and newcomers to the field. The material is organised as a ten-lecture course. The foundations are laid in Chapters 1 and 2, which explain minimum-mean-square-error solution construction and asymptotic behaviour. Chapters 3 and 4 introduce continuous-time and discrete-time minimum-variance filtering. Generalisations for missing data, deterministic inputs, correlated noises, direct feedthrough terms, output estimation and equalisation are described. Chapter 5 simplifies the minimum-variance filtering results for steady-state problems. Observability, Riccati equation solution convergence, asymptotic stability and Wiener filter equivalence are discussed. Chapters 6 and 7 cover the subject of continuous-time and discrete-time smoothing. The main fixed-lag, fixed-point and fixed-interval smoother results are derived. It is shown that the minimum-variance fixed-interval smoother attains the best performance. Chapter 8 attends to parameter estimation. As the above-mentioned approaches all rely on knowledge of the underlying model parameters, maximum-likelihood techniques within expectation-maximisation algorithms for joint state and parameter estimation are described. Chapter 9 is concerned with robust techniques that accommodate uncertainties within problem specifications. An extra term within Riccati equations enables designers to trade-off average error and peak error performance. Chapter 10 rounds off the course by applying the afore-mentioned linear techniques to nonlinear estimation problems. It is demonstrated that step-wise linearisations can be used within predictors, filters and smoothers, albeit by forsaking optimal performance guarantees.

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