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Symbolic Determination of Jacobian and Hessian Matrices and Sensitivities of Active Linear Networks by Using Chan-Mai Signal-Flow Graphs

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1. Introduction

Every network synthesis procedure normally includes a first-order or (more rarely) second-order network sensitivity analysis. The main problem here is the evaluation of the corresponding first- or second-order derivatives of network functions with respect to the circuit element values. These derivatives form the network Jacobian (**J**) and Hessian (**H**) matrices, respectively. A variety of methods exist for such an evaluation but most of them are intended for the sensitivity of one network transfer function only. Besides this in many cases it is desirable to find the symbolic expressions of the sensitivities because such a presentation facilitates the element value influence determination. An other useful and important application of the matrices **J** and **H** is in the tasks for optimization of synthesized networks with respect to their sensitivities or other parameters (Korn & Korn, 1968; Wilde,1978).

As it is well known all linear active networks can be modeled by using passive elements and nullator-norator pairs (nullors). The presented paper deals with the application of Chan-Mai signal-flow graphs (CMG) to the determination of the matrices **J** and **H** elements, having in mind the peculiarities of nullors and their influence on the passive element network admittance matrix and on the corresponding CMG. The method developed here is an improved and enlarged version of the approach in (Nenov, 2004). One demonstrates that the method reduces to the obtaining of two (for the elements of **J**) or four (for the elements of **H**) isomorphic Chan-Mai signal-flow graphs.

2. Chan-Mai signal flow graph

It was introduced in graph theory in 1967 (Chan & Mai, 1967). Compared with other kinds of oriented graphs (especially Mason and Coates graphs) the Chan-Mai graph (CMG) holds out a simplest way to the representation the relationships between the dependent and independent quantities in an algebraic equation set. In order to make easier the understanding of the following sections of the paper further we give the procedure for drawing of CMG and the basic formulae related.

Assume the algebraic set

$$\mathbf{A}.\mathbf{X} = \mathbf{Y} \tag{1}$$

is given, where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & . & . \\ a_{n1} & a_{n2} & . & a_{nn} \end{bmatrix} \tag{2}$$

is a square matrix with real or complex entries and

$$\left.\begin{aligned} \mathbf{X} &= \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}_t; \\ \mathbf{Y} &= \begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix}_t \end{aligned}\right\} \tag{3}$$

are the vectors of the dependent and of the independent variables, respectively. The CMG consists of n vertices with sink signals $y_1, y_2, \dots, y_n$, n vertices with source signals $x_1, x_2, \dots, x_n$ and maximum n^2 edges with transmission coefficients a_{ji} directed from the vertex x_i toward the vertex y_j ; $i,j = 1, 2, \dots, n$ – Fig. 1. The calculations on the base of a CMG are connected with the following definitions (Chan & Mai, 1967, Donevsky & Nenov, 1979):

i. By removing all outgoing from the vertex x_i edges and by adding the edges with transmission coefficients y_j from the vertex x_i directed toward the vertices y_j , $j=1, 2, \dots, n$ one obtains the *graph* CMG,*i*;
ii. A *separation (S)* contains all vertices of CMG and a part of edges so that every vertex is incident to only one incoming and one only outgoing edge. The product of the transmission coefficients of all edges in a separations represents the corresponding *separation product (SP)*;
iii. Two edges with transmission coefficients a_{ij} and a_{ji} form a *symmetrical pair*.
iv. An edge which does not belong to a symmetrical pair is an *asymmetrical edge*.

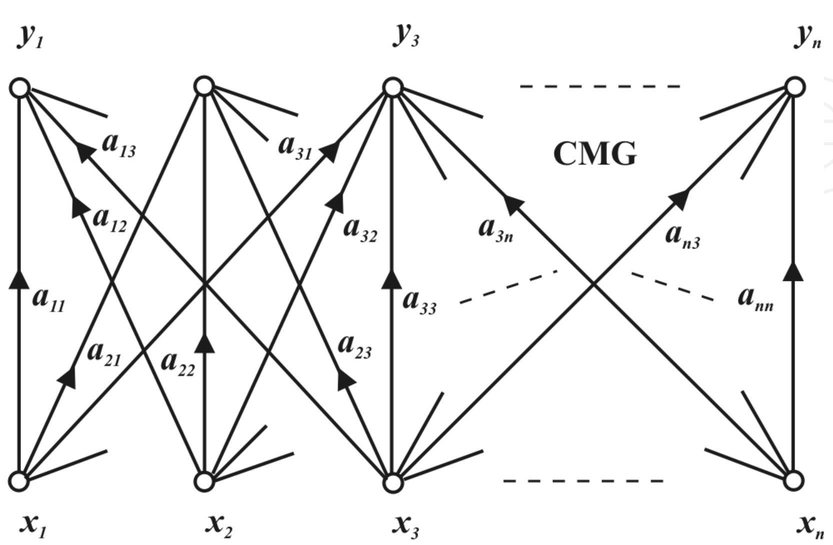

Fig. 1. Chan-Mai Signal-Flow Graph

An arbitrary unknown quantity x_i in $\mathbf{X}$ can be evaluated according to the expression

$$x_i = \frac{\sum_{q=1}^{m} (signSP_q) SP_q(CMG,i)}{\sum_{k=1}^{r} (signSP_k) SP_k(CMG,_k)}, \tag{4}$$

where

$$signSP_l = \begin{cases} (-1)^{N_{s,l}+N_{a,l}-1} & \text{for } N_{a,l} \neq 0 \\ (-1)^{N_{s,l}} & \text{for } N_{a,l} = 0 \end{cases} \tag{5}$$
$$l = q \text{ or } k$$

In (4) and (5) r is the number of the separations in CMG, m is the number of the separations in CMG$_{,i}$, $N_{a,k}$ is the number of all asymmetrical edges in k-th separation of CMG, $N_{s,k}$ is the number of all symmetrical pairs in k-th separation of CMG, N_{aq} is the number of the asymmetrical edges in q-th separation of CMG,i, $N_{s,q}$ is the number of the symmetrical pairs in q-th separation of CMG,i, whereas $SP_q(CMG,i)$ and $SP_k(CMG)$ are the separation products of q-th separation of CMG,i and the separation products of k-th separation of CMG, respectively.

3. Nullor network Chan-Mai signal-flow graph

Suppose that an equivalent nullor network N with $m+1$ nodes, r passive branches and g nullors is given and the nodal equation of its passive part N_p (the part of N which is obtained by removing all nullors) is

$$\mathbf{Y}_p \mathbf{V}_p = \mathbf{I}_p \tag{6}$$

where

$$\mathbf{Y} = \begin{bmatrix} Y_{p,11} & \cdots & Y_{p,1m} \\ . & . & . \\ Y_{p,m1} & \cdots & Y_{p,mm} \end{bmatrix} \tag{7}$$

is the nodal matrix of N_p and

$$\left.\begin{aligned} \mathbf{V}_p &= \begin{bmatrix} V_{p,1} & V_{p,2} & \cdots & V_{p,m} \end{bmatrix}_t; \\ \mathbf{I}_p &= \begin{bmatrix} I_{p,1} & I_{p,2} & \cdots & I_{p,m} \end{bmatrix}_t \end{aligned}\right\} \tag{8}$$

are the nodal voltage and the nodal current vectors of N_p, respectively. Additionally we assume that between the nodes of all node pairs in N only one element or more than one but parallel connected elements exist.

The equation (1) can be represented graphically by using a CMG $\boldsymbol{G}_\mathrm{p}$ (Chan & Mai, 1967). Further, taking into account the peculiarities of the nullators and the norators (Davies, 1966) the graph $\boldsymbol{G}_\mathrm{p}$ can be transformed into the graph $\boldsymbol{G}$ of the actual network N according to the following

Rule 1:

i. When a nullator is connected between the node k in N and the ground node $m+1$ one removes all vertices going out from the vertex V_k of $\boldsymbol{G}_\mathrm{p}$;
ii. When a norator is connected between the node k in N and the ground node $m+1$ one removes all vertices coming into the vertex I_k of $\boldsymbol{G}_\mathrm{p}$;
iii. When a nullator is connected between the nodes k and l in N one unites the vertices V_k and V_l in $\boldsymbol{G}_\mathrm{p}$;
iv. When a norator is connected between the nodes k and l in N one unites the vertices I_k and I_l in $\boldsymbol{G}_\mathrm{p}$.

The so obtained graph CMG $\boldsymbol{G}$ corresponds to the matrix equation

$$\mathbf{YV} = \mathbf{I} \tag{9}$$

where $\mathbf{Y}$ is an $(n \times n)$ nodal admittance matrix of N, $\mathbf{V}$ is the nodal voltage vector of N and $\mathbf{I}$ is the nodal current vector of for $n=m-g$.

4. Jacobian matrix determination

The matrices in (9) have the form:

$$\left.\mathbf{Y} = \begin{bmatrix} Y_{11} & .. & Y_{1i} & .. & Y_{1n} \\ . & . & . & . & . \\ Y_{j1} & .. & Y_{ji} & .. & Y_{jn} \\ . & . & . & . & . \\ Y_{n1} & .. & Y_{nj} & .. & Y_{nn} \end{bmatrix};\ \mathbf{V} = \begin{bmatrix} V_1 & V_2 & \dots & V_n \end{bmatrix}_t;\ \mathbf{I} = \begin{bmatrix} I_1 & I_2 & \dots & I_n \end{bmatrix}_t \right\} \tag{10}$$

In the common case every element Y_{ji} in (7) is an algebraic admittance sum

$$Y_{ji} = \sum_s y_s;\ j,i \in \{1,2,\dots,n\}; s \in \{1,2,\dots,r\}, \tag{11}$$

where y_s is the admittance of s-th branch of the network N_p.

The vectors $\mathbf{V}$ and $\mathbf{I}$ correspond to the unknown (dependent) variables and to independent variables of N, respectively and consequently

$$\mathbf{V} = \mathbf{Y}^{-1}\mathbf{I}\,. \tag{12}$$

Let us suppose that the admittance y_s changes its value to

$$y_s^{'} = y_s + dy_s\,. \tag{13}$$

Usually the admittance y_s takes part in several (but no more then four) elements of (7) and then all these elements change their values (Nenov, 2004)

$$\left.\begin{array}{l} Y_{ji}^{'} = Y_{ji} + dY_{ji} = Y_{ji} + \dfrac{\partial Y_{ji}}{\partial y_s} dy_s; \\ j,i \in \{1,2,...,n\}; s \in \{1,2,...,r\} \end{array}\right\} \tag{14}$$

and

$$\mathbf{Y}^{'} = \mathbf{Y} + d\mathbf{Y}. \tag{15}$$

In a common case the admittance y_s influences the admittances Y_{ji} ,Y_{jl} , Y_{ki} and Y_{kl}; $i,j,k,l \in \{1, 2, \ldots, n\}$. Then one obtains

$$\left.\begin{array}{l} d\mathbf{Y} = dy_s \mathbf{K}_s; \\ \mathbf{K}_s = \begin{bmatrix} 0 & . & 0 & . & 0 & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & \dfrac{\partial Y_{ji}}{\partial y_s} & . & \dfrac{\partial Y_{jl}}{\partial y_s} & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & \dfrac{\partial Y_{ki}}{\partial y_s} & . & \dfrac{\partial Y_{kl}}{\partial y_s} & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & 0 & . & 0 & . & 0 \end{bmatrix}; \;\; i,j,k,l \in \{1,2,...,n\} \end{array}\right\}. \tag{16}$$

Note that the values of the derivatives in (16) are 1 or –1 because every admittance y_s takes part in (11) only once. Hence

$$\mathbf{V}^{'} = \mathbf{V} + d\mathbf{V} \tag{17}$$

for

$$d\mathbf{V} = \left(\frac{\partial \mathbf{V}}{\partial Y_{ji}} \frac{\partial Y_{ji}}{\partial y_s} + \frac{\partial \mathbf{V}}{\partial Y_{jl}} \frac{\partial Y_{jl}}{\partial y_s} + \frac{\partial \mathbf{V}}{\partial Y_{ki}} \frac{\partial Y_{ki}}{\partial y_s} + \frac{\partial \mathbf{V}}{\partial Y_{kl}} \frac{\partial Y_{kl}}{\partial y_s} \right) dy_s, \tag{18}$$

or:

$$\left.\begin{array}{l} d\mathbf{V} = dy_s \displaystyle\sum_{pq} \frac{\partial \mathbf{V}}{\partial Y_{pq}} \frac{\partial Y_{pq}}{\partial y_s}; \\ p,q \in \{1,2,...,n\}. \end{array}\right\} \tag{19}$$

By substituting $\mathbf{Y}'$ and $\mathbf{V}'$ in (9) instead $\mathbf{Y}$ and $\mathbf{V}$, respectively, it follows

$$[\mathbf{Y} + d\mathbf{Y}].[\mathbf{V} + d\mathbf{V}] = \mathbf{I}. \tag{20}$$

Having in mind that

$$d\mathbf{Y}d\mathbf{V} \to \mathbf{0} \tag{21}$$

the equation (20) yields

$$\mathbf{Y}d\mathbf{V} = -d\mathbf{Y}\mathbf{V} \tag{22}$$

or

$$d\mathbf{V} = -\mathbf{Y}^{-1}d\mathbf{Y}\mathbf{V}\,. \tag{23}$$

Then we obtain

$$\frac{\partial \mathbf{V}}{\partial y_s}dy_s = -dy_s.\mathbf{Y}^{-1}\mathbf{K}_s\mathbf{V} \tag{24}$$

and the Jacobian matrix (Korn & Korn, 1968). for the change of the admittance y_s is

$$\mathbf{J} = \begin{bmatrix} \frac{\partial V_1}{\partial y_1} & \frac{\partial V_1}{\partial y_2} & .. & \frac{\partial V_1}{\partial y_s} & .. & \frac{\partial V_1}{\partial y_r} \\ \frac{\partial V_2}{\partial y_1} & \frac{\partial V_2}{\partial y_2} & .. & \frac{\partial V_2}{\partial y_s} & .. & \frac{\partial V_2}{\partial y_r} \\ . & . & .. & . & .. & . \\ \frac{\partial V_n}{\partial y_1} & \frac{\partial V_n}{\partial y_2} & .. & \frac{\partial V_n}{\partial y_s} & .. & \frac{\partial V_n}{\partial y_r} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 & .. & \mathbf{J}_s & .. & \mathbf{J}_r \end{bmatrix}, \tag{25}$$

where

$$\mathbf{J}_s = \begin{bmatrix} \frac{\partial V_1}{\partial y_s} & \frac{\partial V_2}{\partial y_s} & .. & \frac{\partial V_n}{\partial y_s} \end{bmatrix}_t . \tag{26}$$

Taking into account (24) and (25) one obtains

$$\left.\begin{aligned} &\mathbf{J}_s = -\mathbf{Y}^{-1}\mathbf{K}_s\mathbf{Y}^{-1}\mathbf{I} = -\mathbf{Y}^{-1}\mathbf{K}_s\mathbf{V} = -\mathbf{Y}^{-1}\mathbf{V}_s; \\ &\mathbf{V}_s = \mathbf{K}_s\mathbf{V}. \end{aligned}\right\} \tag{27}$$

and according to (20) ÷ (22)

$$\mathbf{J} = -\mathbf{Y}^{-1}\begin{bmatrix} \mathbf{K}_1 & .. & \mathbf{K}_s & .. & \mathbf{K}_r \end{bmatrix}\mathbf{V}\,. \tag{28}$$

The expressions (22) show that in order to find the vector $\mathbf{J}_s$ it is necessary to follow the

Rule 2:

i. Find the vector $\mathbf{V}$ by using the CMG $\boldsymbol{G}$;
ii. Evaluate the vector $\mathbf{V}_s$;
iii. Draw a new CMG $\boldsymbol{G}_s$ where the source vertices are the elements of the vector $\mathbf{J}_s$ and the sink vertices are the elements of the vector $\mathbf{V}_s$;
iv. Find the source vertex variables in $\boldsymbol{G}_s$.

Example A

The network N in Fig. 2 is given, where m=6; r=9; g=2. Here obviously V_2=V_3=V_{23}; V_6=0 and we wish to find the vector

$$\mathbf{J}_3 = \left[\frac{\partial V_1}{\partial (sC_3)} \quad \frac{\partial V_{23}}{\partial (sC_3)} \quad \frac{\partial V_4}{\partial (sC_3)} \quad \frac{\partial V_5}{\partial (sC_3)} \right]_t . \tag{29}$$

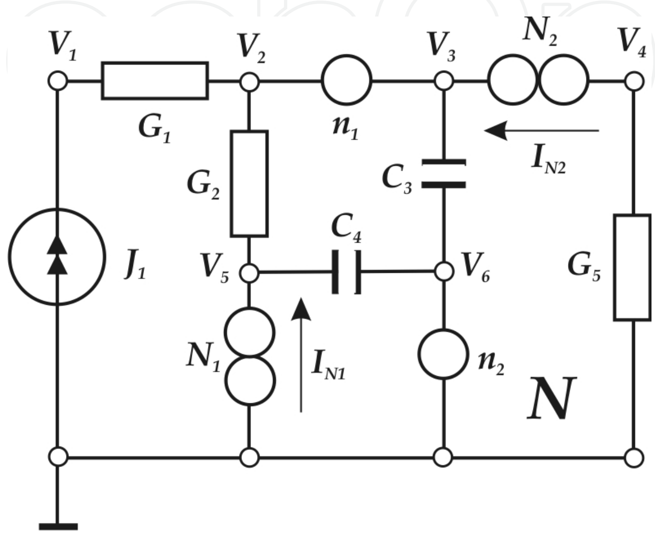

Fig. 2. Nullor Network N

In Fig. 3 the CMG $\boldsymbol{G}_{\mathrm{p}}$ of the passive part of N is drawn (Nenov, 2004). Further following the ***Rule*** 2 we reach to the graph $\boldsymbol{G}$ in Fig. 4 for

$$\left. \begin{array}{l} \mathbf{Y} = \begin{bmatrix} G_1 & -G_1 & 0 & 0 \\ -G_1 & G_1 + G_2 & 0 & -G_2 \\ 0 & sC_3 & G_5 & 0 \\ 0 & -sC_3 & 0 & -sC_4 \end{bmatrix}; \\ \mathbf{V} = \begin{bmatrix} V_1 & V_2 = V_3 = V_{23} & V_4 & V_5 \end{bmatrix}_t ; \mathbf{I} = \begin{bmatrix} J_1 & 0 & 0 & 0 \end{bmatrix}_t . \end{array} \right\} \tag{30}$$

Because Y_{32}=sC_3; Y_{42}= -sC_3 and

$$\frac{\partial Y_{32}}{\partial (sC_3)} = 1; \ \frac{\partial Y_{32}}{\partial (sC_3)} = -1. \tag{31}$$

from (16) and (31) we have

$$\left. \begin{array}{l} \mathbf{K}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}; \\ \mathbf{V}_3 = \mathbf{K}_3 \mathbf{V} = \begin{bmatrix} 0 & 0 & V_{23} & -V_{23} \end{bmatrix}_t \end{array} \right\} \tag{32}$$

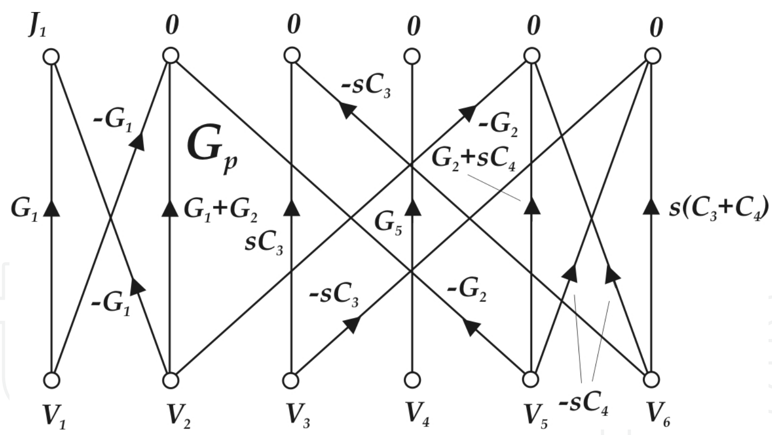

Fig. 3. CM Signal-Flow Graph $\boldsymbol{G_p}$

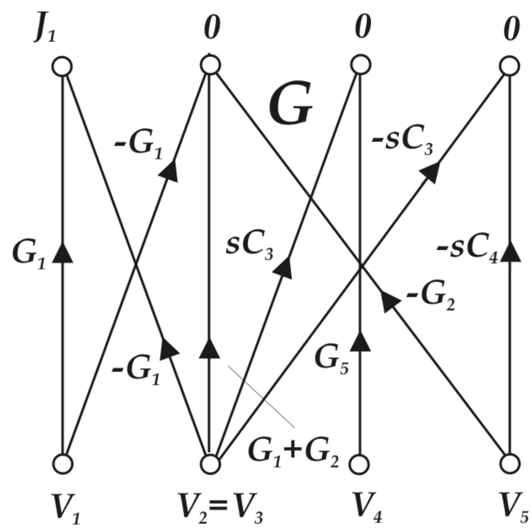

Fig. 4. CM Signal-Flow Graph $\boldsymbol{G}$

Obviously, in the case we have to find the voltage V_{23} only. For this purpose a CM graph $\boldsymbol{G}_{23}$ is drawn (Fig. 5).

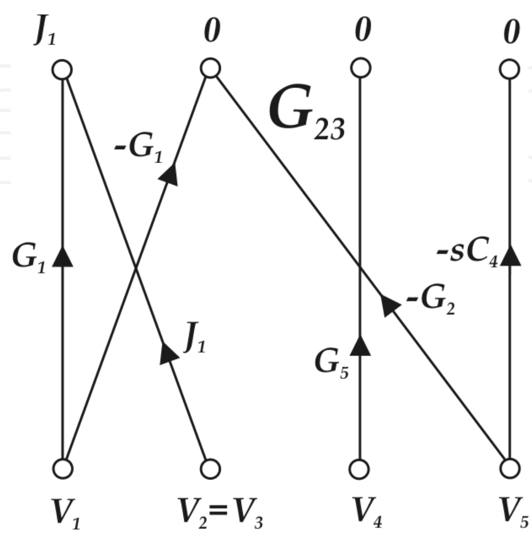

Fig. 5. CM Signal-Flow Graph $\boldsymbol{G_{23}}$

According to (4) and (5) for the separations of the graph $\boldsymbol{G}$ (Fig. 6) we obtain

$$\left.\begin{aligned} &SP_1 = -G_1 sC_4 G_5 (G_1 + G_2); N_{a,1} = 4; N_{s,1} = 0; \\ &SP_2 = -G_1^2 sC_4 G_5; N_{a,2} = 2; N_{s,2} = 1; \\ &SP_3 = G_2 sC_3 G_5 : N_{a,3} = 2; N_{s,3} = 1 \end{aligned}\right\} \tag{33}$$

and for the unique separation of the graph $\boldsymbol{G}_{23}$ (Fig. 7):

$$SP_{23,1} = J_1 G_1 sC_4 G_5; N_{a,23,1} = 2; N_{s,23,1} \tag{34}$$

Then the formulae (4) and (5) yield

$$V_{23} = \frac{J_1 C_4}{G_2 (C_3 + C_4)}\,. \tag{35}$$

Having in mind (26) ÷ (29) we have

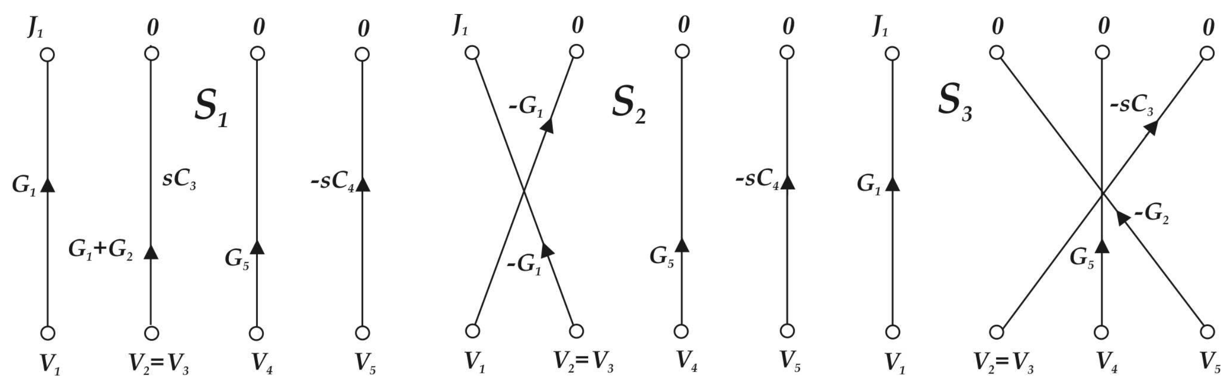

Fig. 6. Separations S_1, S_2 and S_3 of CM Graph $\boldsymbol{G}$

$$\mathbf{J}_3 = -\mathbf{Y}^{-1}\mathbf{V}_3 = \mathbf{Y}^{-1}(-\mathbf{V}_3) \tag{36}$$

and following ***Rule 2*** one draws the CM graph $\boldsymbol{G}_{J3}$ (Fig. 8). Obviously, the graphs $\boldsymbol{G}$ and $\boldsymbol{G}_{J3}$ have one and the same structure and consequently the expressions (33) hold for the source vertex quantities in (29) also. But for the nominator polynomials in (4) we have to draw according the ***Rule 2*** four new CM graphs – $\boldsymbol{G}_{J3,1}$, $\boldsymbol{G}_{J3,23}$, $\boldsymbol{G}_{J3,4}$ and $\boldsymbol{G}_{J3,5}$ – Fig. 9.

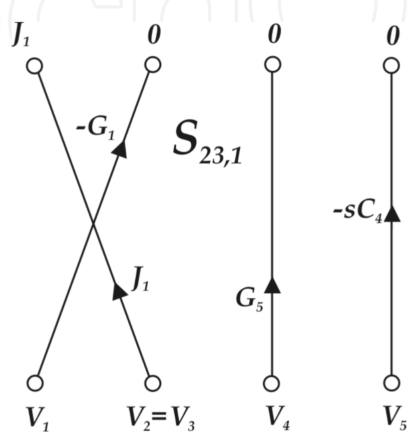

Fig. 7. Separation $S_{23,1}$ of CM Graph $\boldsymbol{G}_{23}$

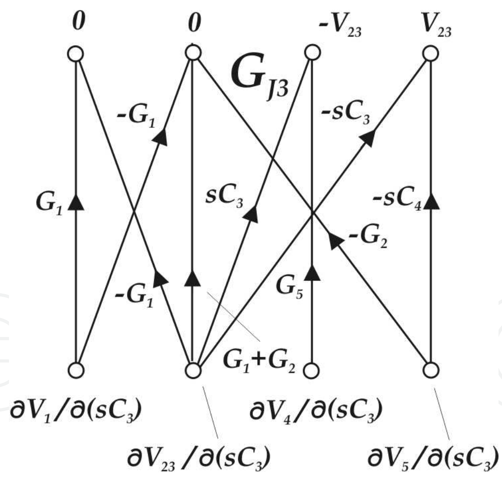

Fig. 8. CM Graph G_{J3}

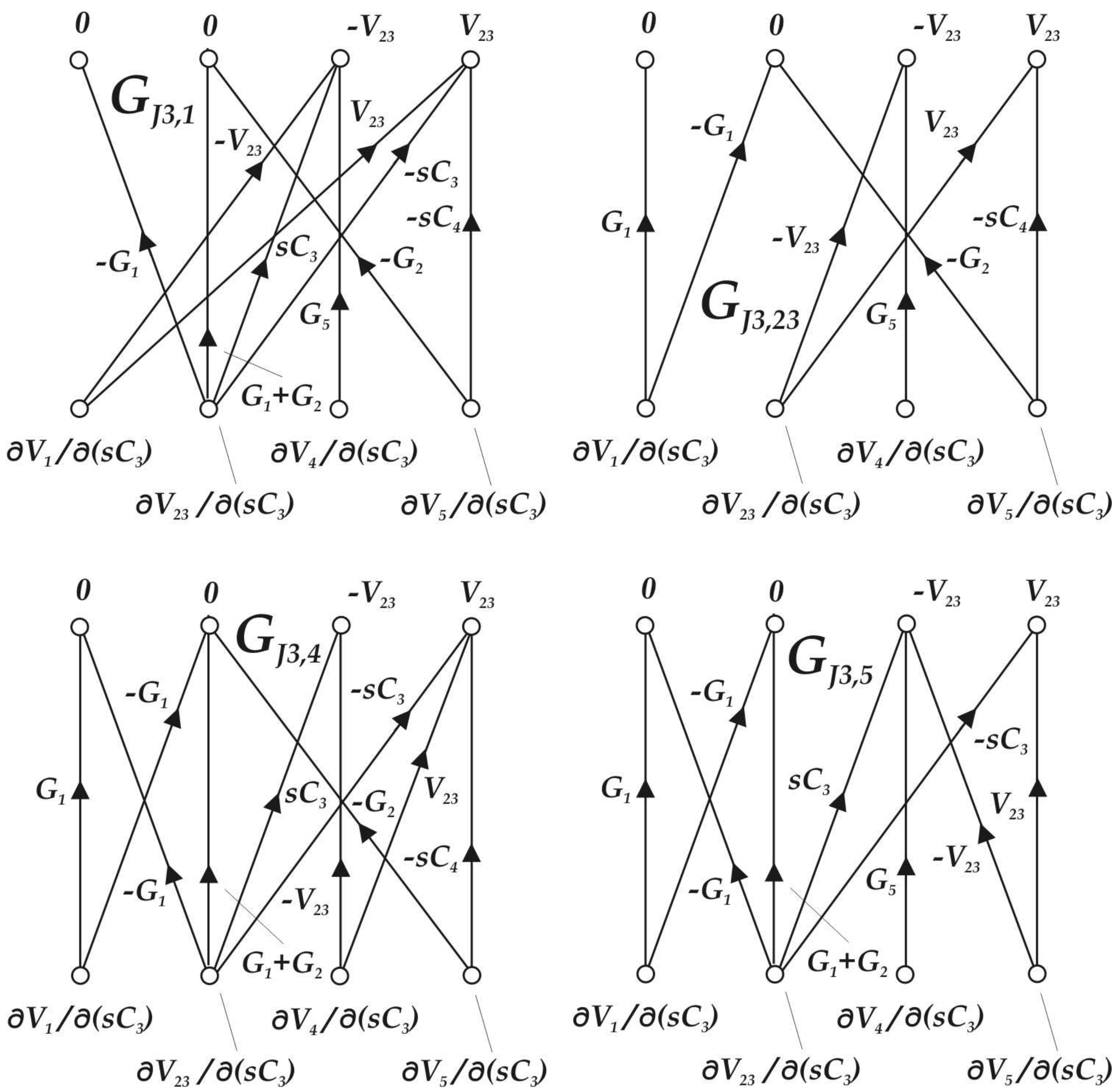

Fig. 9. CM Graphs $G_{J3,1}$,$G_{J3,23}$, $G_{J3,4}$ and $G_{J3,5}$

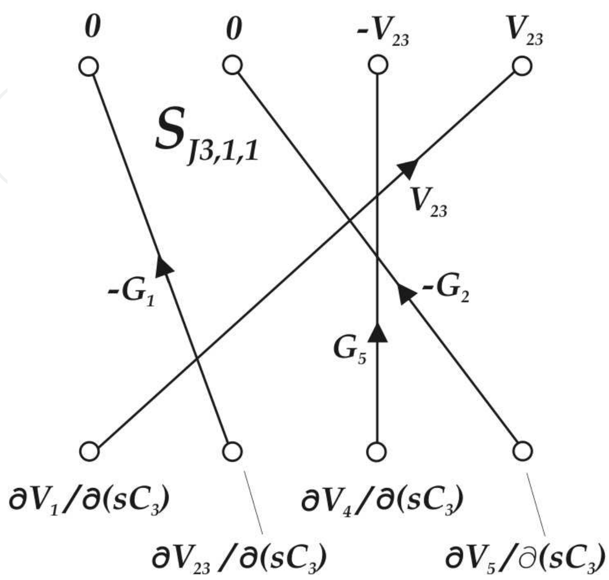

Fig. 10. Separation $S_{J3,1,1}$ of CM Graph $G_{J3,1}$

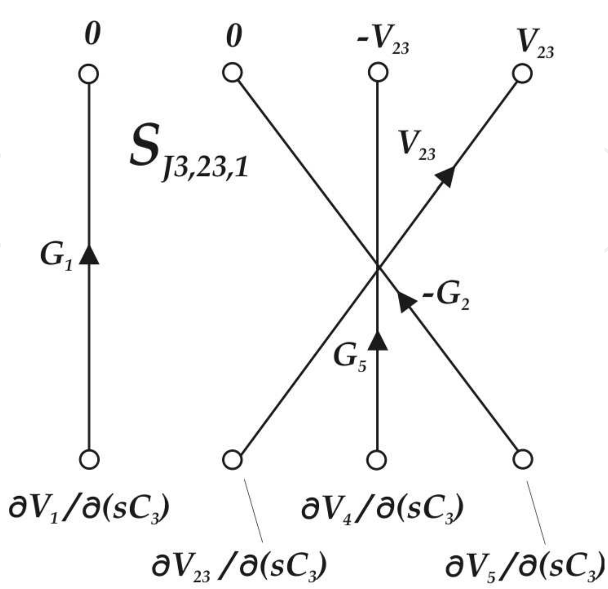

Fig. 11. Separation $S_{J3,23,1}$ of CM Graph $G_{J3,23}$

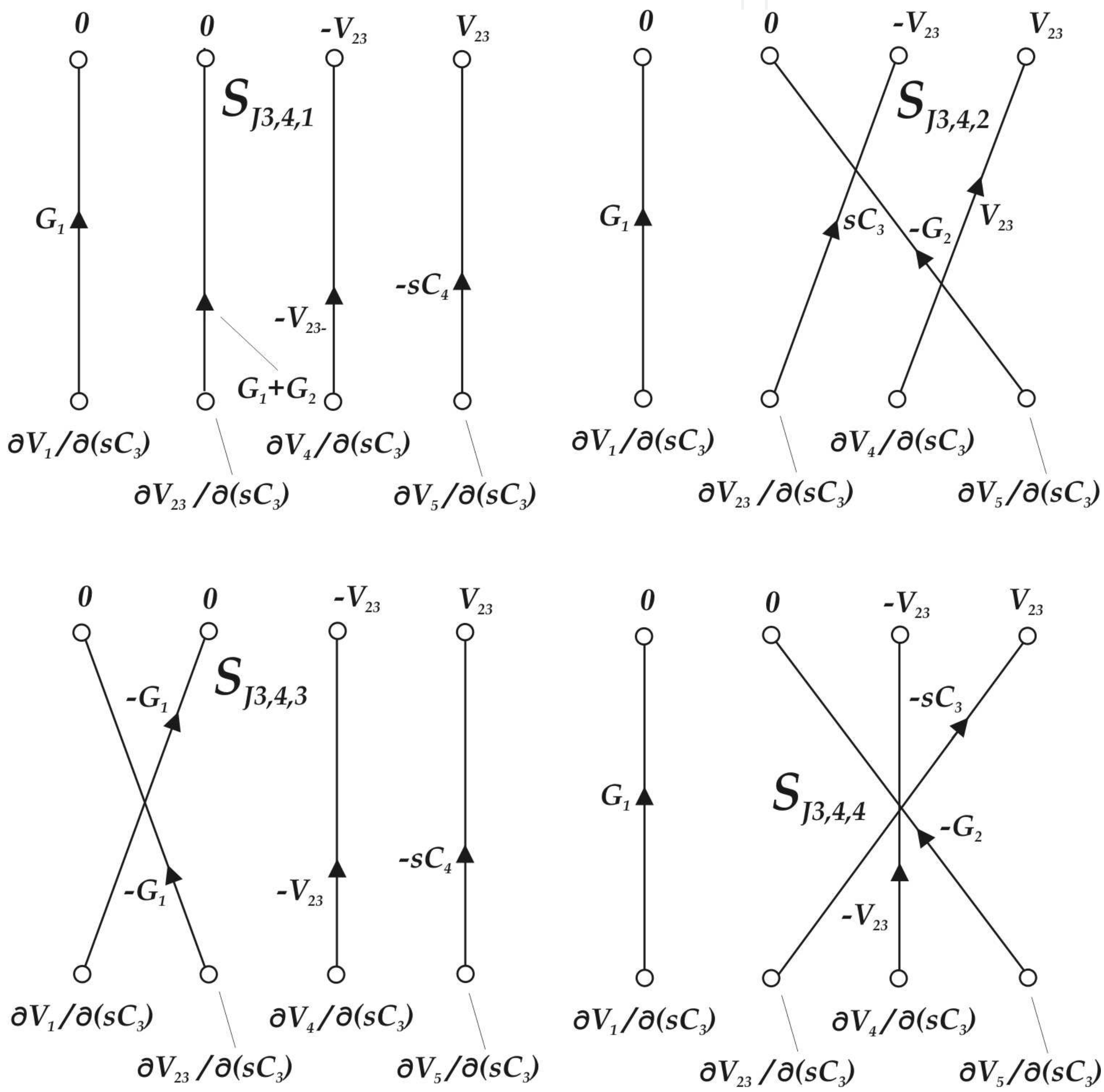

Fig. 12. Separations $\boldsymbol{S}_{J3,4,1}$, $\boldsymbol{S}_{J3,4,2}$, $\boldsymbol{S}_{J3,4,3}$ and $\boldsymbol{S}_{J3,4,4}$ of CM Graph $\boldsymbol{G}_{J3,4}$

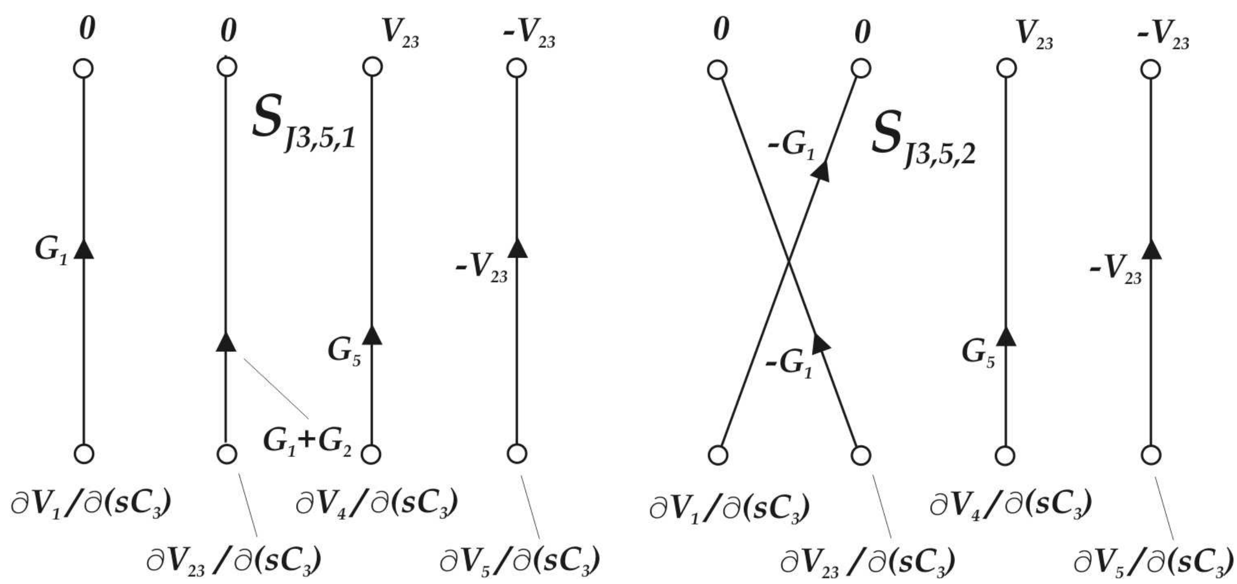

Fig. 13. Separations $S_{J3,5,1}$ *and* $S_{J3,5,2}$ of CM Graph $G_{J3,5}$

From Fig. 9 ÷ Fig. 13 it follows

$$\left.\begin{array}{l}
SP_{J3,1,1} = G_1G_2G_5V_{23};\ N_{a,J3,1,1} = 4;\ Ns,_{J3,1,1} = 0; \\
SP_{J3,23,1} = -G_1G_2G_5V_{23};\ N_{a,J3,23,1} = 4;\ Ns,_{J3,23,1} = 1; \\
SP_{J3,4,1} = G_1(G_1+G_2)sC_4V_{23};\ N_{a,J3,4,1} = 4;\ Ns,_{J3,4,1} = 0; \\
SP_{J3,4,2} = -G_1G_2sC_3V_{23};\ N_{a,J3,4,2} = 4;\ N_{s,J3,4,2} = 0; \\
SP_{J3,4,3} = G_1^2sC_4V_{23};\ N_{a,J3,4,3} = 2;\ N_{s,J3,4,3} = 1; \\
SP_{J3,4,4} = -G_1G_2sC_3V_{23};\ N_{a,J3,4,4} = 2;\ N_{s,J3,4,4} = 1; \\
SP_{J3,5,1} = G_1(G_1+G_2)C_5V_{23};\ N_{a,J3,5,1} = 4;\ N_{s,J3,5,1} = 0; \\
SP_{J3,5,2} = G_1^2G_5V_{23};\ N_{a,J3,5,2} = 2;\ N_{s,J3,5,2} = 1;
\end{array}\right\} \quad (37)$$

Than by substituting (35) in (36) and by taking into consideration (33) from (4) and (5) we obtain the vector $\mathbf{J}_3$:

$$\mathrm{J}_3 = \left[\frac{J_1C_4}{G_2s(C_3+C_4)^2} \quad \frac{J_1C_4}{G_2s(C_3+C_4)^2} \quad \frac{J_1C_4^2}{G_2G_5(C_3+C_4)^2} \quad \frac{J_1C_4}{G_2s(C_3+C_4)^2} \right]_t \quad (38)$$

5. Hessian matrix determination

In many practical cases it is necessary and useful to find not only the first-order derivatives of a network function or variable (for example voltage V_w) among n variables with respect to some parameter (for example y_s) but their second-order derivatives with respect to the same or to an other parameter (for example y_t), too.

The matrix formed from all possible second-order derivatives of V_w with respect to the simultaneous changes of two parameters

$$\mathbf{H}_w = \begin{bmatrix} \frac{\partial^2 V_w}{\partial y_1^2} & \frac{\partial^2 V_w}{\partial y_1 \partial y_2} & \cdots & \frac{\partial^2 V_w}{\partial y_1 \partial y_n} \\ \frac{\partial^2 V_w}{\partial y_2 \partial y_1} & \frac{\partial^2 V_w}{\partial y_2^2} & \cdots & \frac{\partial^2 V_w}{\partial y_2 \partial y_n} \\ . & . & . & . \\ \frac{\partial^2 V_w}{\partial y_n \partial y_1} & \frac{\partial^2 V_w}{\partial y_n \partial y_2} & \cdots & \frac{\partial^2 V_w}{\partial {y_n}^2} \end{bmatrix} \tag{39}$$

is the Hessian matrix or briefly Hessian (Korn & Korn, 1968, Wilde, 1978). Obviously for a network one exists a variety of Hessian matrices – every one matrix corresponds to a definite network function or variable.

The results obtained in section 3. can be applied to the derivation of a Hessian matrix as it will be explained below. By differentiating the vector $\mathbf{J}_s$ in (27) with respect to the admittance y_t one obtains

$$\left.\begin{aligned} &\frac{\partial \mathbf{J}_s}{\partial y_t} = \frac{\partial^2 \mathbf{V}}{\partial y_s \partial y_t} = -\left[\frac{\partial \mathbf{Y}^{-1}}{\partial y_t}\mathbf{K}_s\mathbf{Y}^{-1} + \mathbf{Y}^{-1}\frac{\partial}{\partial y_t}\left(\mathbf{K}_s\mathbf{Y}^{-1}\right)\right]\mathbf{I}; \\ &s,t \in \{1,2,\ldots,n\}. \end{aligned}\right\} \tag{40}$$

Because the elements in **Y** depend linearly on the network element admittances and their derivatives with respect to the parameter y_s equal 1, -1 or 0 it holds

$$\frac{\partial \mathbf{K}_s}{\partial y_t} = 0; \forall s; \frac{\partial \mathbf{Y}}{\partial y_t} = \mathbf{K}_t \tag{41}$$

and from (40) it follows

$$\left.\begin{aligned} &\frac{\partial^2 \mathbf{V}}{\partial y_s \partial y_t} = \mathbf{Y}^{-1}\left(\mathbf{K}_t\mathbf{Y}^{-1}\mathbf{K}_s + \mathbf{K}_s\mathbf{Y}^{-1}\mathbf{K}_t\right)\mathbf{Y}^{-1}\mathbf{I} = \\ &= \mathbf{Y}^{-1}\mathbf{K}_{st}\mathbf{V} = \mathbf{Y}^{-1}\mathbf{V}_{st}; \\ &\mathbf{K}_{st} = \mathbf{K}_t\mathbf{Y}^{-1}\mathbf{K}_s + \mathbf{K}_s\mathbf{Y}^{-1}\mathbf{K}_t; \\ &\mathbf{V}_{st} = \mathbf{K}_{st}\mathbf{V}. \end{aligned}\right\} \tag{42}$$

The last result compared with the formulae (27) and (28) shows that we can find the vector $\partial^2\mathbf{V}/\partial y_s \partial y_t$ in principle by using the same approach as for $\partial\mathbf{V}/\partial y_s$ in section 3.

However here we must pay attention to the obtaining of the matrix $\mathbf{K}_{st}$: In the common case the matrices $\mathbf{K}_s$ and $\mathbf{K}_t$ contain more than one nonzero element (1 or –1). Hence we can expressed each of them as a sum of no more then four addends

$$\mathbf{K}_s = \sum_a \mathbf{K}_{s,a};\ \mathbf{K}_t = \sum_b \mathbf{K}_{t,b};\ a,b \le 4\ , \tag{43}$$

where each of the matrices $\mathbf{K}_{s,a}$ and $\mathbf{K}_{t,b}$ has only one nonzero element. Then as a result the expression of $\mathbf{K}_{st}$ in (42) is a sum of products of the kind

$$\mathbf{K}_{s,a}\mathbf{Y}^{-1}\mathbf{K}_{t,b} \text{ and } \mathbf{K}_{t,b}\mathbf{Y}^{-1}\mathbf{K}_{s,a};\ \forall a,b.. \tag{44}$$

The products in (44) are square matrices with only one nonzero element which is a definite element of $\mathbf{Y}^{-1}$. Let, for example, the nonzero element for the left-side matrix in (44) is on i-th row and on j-th column and the similar element for the right-side matrix is on k-th row and on l-th column. Then it is easy to see that the corresponding product in (44) contains the element $\pm z_{u,v}$; $u, v \in \{1, 2, \ldots, n\}$ on i-th row and on l-th column, where $z_{u,v}$ is an element of $\mathbf{Y}^{-1}$. The upper (lower) sign of this element holds for equal (non equal) signs of nonzero elements of $\mathbf{K}_{s,a}$ and $\mathbf{K}_{t,b}$ in (44), respectively.

The matrix $\mathbf{Y}^{-1}$ can be evaluated by using an auxiliary CM graph $\boldsymbol{G_0}$ too. For this purpose let us consider the equation

$$\mathbf{X} = \mathbf{Y}^{-1}\mathbf{E}, \tag{45}$$

where

$$\left.\mathbf{Y}^{-1} = \begin{bmatrix} z_{11} & z_{12} & \cdot & z_{1n} \\ z_{21} & z_{22} & \cdot & z_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ z_{n1} & z_{n2} & \cdot & z_{nn} \end{bmatrix};\ \mathbf{X} = \begin{bmatrix} x_1 & x_2 & . & x_n \end{bmatrix}_t;\ \mathbf{E} = \begin{bmatrix} e_1 & e_2 & . & e_n \end{bmatrix}_t.\right\}. \tag{46}$$

After multiplying in (45) for $\mathbf{X}$ one follows

$$\mathbf{X} = \begin{bmatrix} z_{11}e_1 + z_{12}e_2 + \ldots + z_{1n}e_n \\ z_{21}e_1 + z_{22}e_2 + \ldots + z_{2n}e_n \\ . \\ z_{n1}e_1 + z_{n2}e_2 + \ldots + z_{nn}e_n \end{bmatrix}. \tag{47}$$

This means that if the CM graph $\boldsymbol{G_0}$ corresponds to (45) the multipliers of $e_1, e_2, \ldots, e_n$ for every element of $\mathbf{X}$ are elements of $\mathbf{Y}^{-1}$. Note that in real cases a limited number of the elements of $\mathbf{Y}^{-1}$ are necessary only. Hence for determination of an element of the Hessian matrix $\mathbf{H}_{st}$ we can form the following:

Rule 3:

i. Draw the CM graph $\boldsymbol{G_p}$ of the nullor network under consideration;
ii. Transform the graph $\boldsymbol{G_p}$ into the graph $\boldsymbol{G}$, according to the *Rule 1* in section 2. and compose the vectors $\mathbf{V}$ and $\mathbf{I}$;
iii. Determine the vector $\mathbf{V}$ from $\boldsymbol{G}$;
iv. Write the matrices $\mathbf{K}_s$ and $\mathbf{K}_t$;
v. Determine the matrix $\mathbf{Y}^{-1}$ by using the auxiliary CM graph $\boldsymbol{G_0}$;
vi. Determine the matrix $\mathbf{K}_{st}$;
vii. Determine the matrix $\mathbf{V}_{st}$;
viii. Draw a CM graph $\boldsymbol{G}_{st}$ in accordance with $\mathbf{V}_{st}$;
ix. Determine the elements of the vector $\partial^2\mathbf{V}/\partial y_s\partial y_t$ from $\boldsymbol{G}_{st}$.

Note that by following the above sequence we obtain *2n* elements of *n* Hessian matrices simultaneously, *because* $\partial^2\mathbf{V}/\partial y_s\partial y_t = \partial^2\mathbf{V}/\partial y_t\partial y_s$ – Fig. 14.

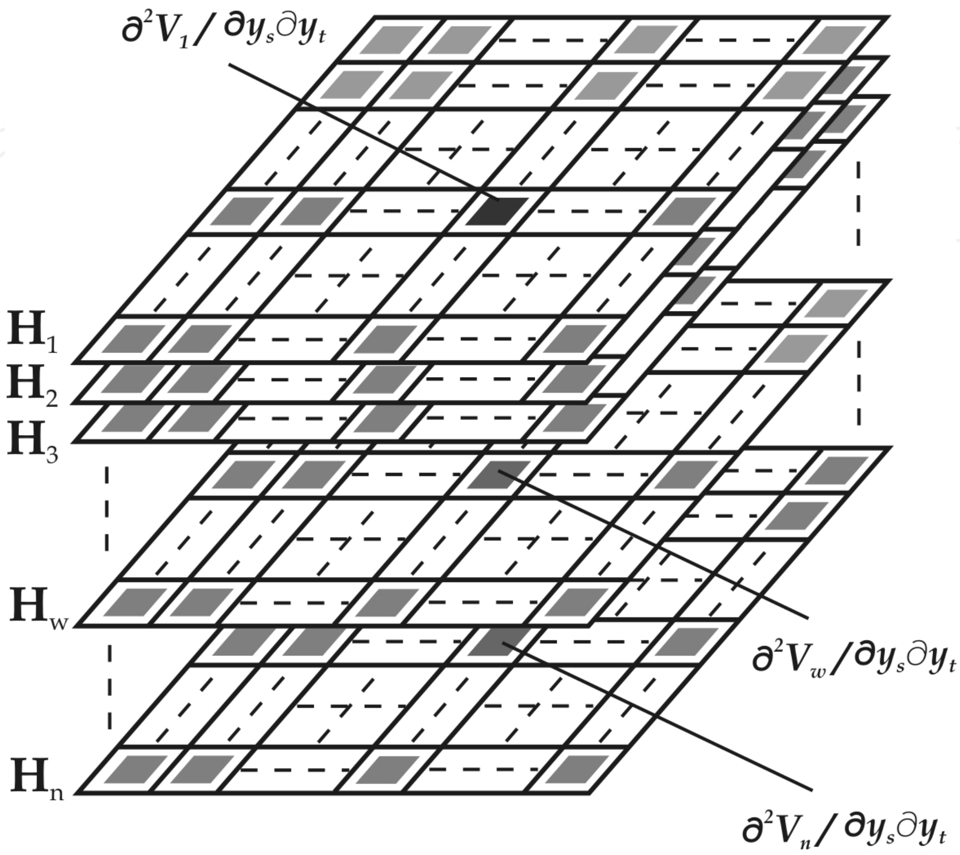

Fig. 14. A Set of *n* Hessian Matrices

Example B

Suppose that we want to determine the vector $\partial^2\mathbf{V}/\partial(sC_3)\partial(sC_4)$ for the network *N* in Fig.2. Because the items **i**, **ii** and **iii** of the ***Rule 3*** were fulfilled in the **Example A** we have to continue further: Here the matrices $\mathbf{K}_3$ and $\mathbf{K}_4$ are

$$\left.\begin{array}{l}
\mathbf{K}_{31} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};\ \mathbf{K}_{32} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}; \\
\mathbf{K}_3 = \mathbf{K}_{31} + \mathbf{K}_{32};\ \ \mathbf{K}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}
\end{array}\right\} \tag{48}$$

and from (42) ÷ (46) it follows

$$\mathbf{K}_{34} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z_{24} \\ 0 & -z_{43}+z_{44} & 0 & z_{24} \end{bmatrix}. \tag{49}$$

We can find the nonzero elements of $\mathbf{K}_{34}$ by using the auxiliary CM graph G_0 drawn in Fig. 15. By comparing (47) with (49) one settles we need only these addends of elements x_2 and x_4 in (47) that content the quantities e_4 and e_3, e_4, respectively. According to the Chan-Mai procedure we draw the graphs $G_{0,2}$ and $G_{0,4}$ - Fig. 16 and Fig. 17.

$$z_{24} = z_{44} = -\frac{1}{s(C_3 + C_4)}; z_{43} = 0\,. \tag{50}$$

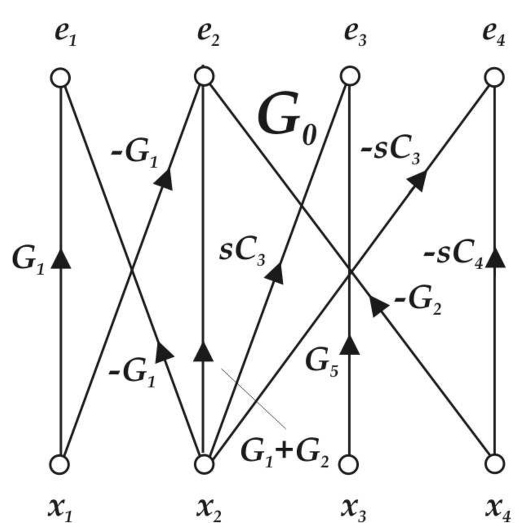

Fig. 15. The auxiliary CM graph G_0

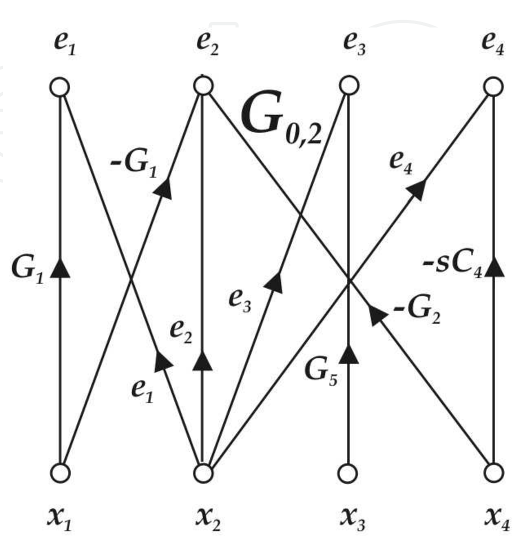

Fig. 16. The CM graph $G_{0,2}$

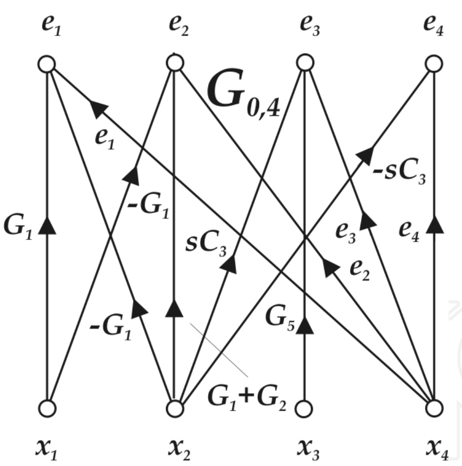

Fig. 17. The CM graph $G_{0,4}$

From Fig. 18 ÷ Fig. 19 we obtain the products

$$\left.\begin{array}{l}
SP_{0,2,1} = -G_1 sC_4 G_5 e_2;\ N_{a,0,2,1} = 4;\ N_{s,0,2,1} = 0;\\
SP_{0,2,2} = G_1 sC_4 G_5 e_1;\ N_{a,0,2,2} = 2;\ N_{s,0,2,2} = 1;\\
SP_{0,2,3} = -G_1 G_2 G_5 e_4;\ N_{a,0,2,3} = 2;\ N_{s,0,2,3} = 1;\\
SP_{0,4,1} = G_1 (G_1 + G_2) G_5 e_4;\ N_{a,0,4,1} = 4;\ N_{s,0,4,1} = 0;\\
SP_{0,4,2} = G_1^2 G_5 e_4;\ N_{a,0,4,2} = 2;\ N_{s,0,4,2} = 1;\\
SP_{0,4,3} = -G_1 sC_3 G_5 e_2;\ N_{a,0,4,3} = 2;\ N_{s,0,4,3} = 1;
\end{array}\right\}. \tag{51}$$

Note that with the exception of the sink and source quantities the graph G_0 is isomorphic to the graph G in Fig. 4. That is why the expressions (33) remain valid for the denominator in (4) also. Than for the elements of the vector (47) from (51) and (33) it follows:

$$\left.\begin{array}{l}
x_2 = \dfrac{G_1 sC_4 G_5 e_1 + G_1 sC_4 G_5 e_2 - G_1 G_2 G_5 e_4}{G_1 G_2 G_5 s(C_3 + C_4)};\\
x_4 = \dfrac{-G_1 sC_3 G_5 e_2 - G_1 G_2 G_5 e_4}{G_1 G_2 G_5 s(C_3 + C_4)}
\end{array}\right\} \tag{52}$$

or taking into consideration (46) and (47)

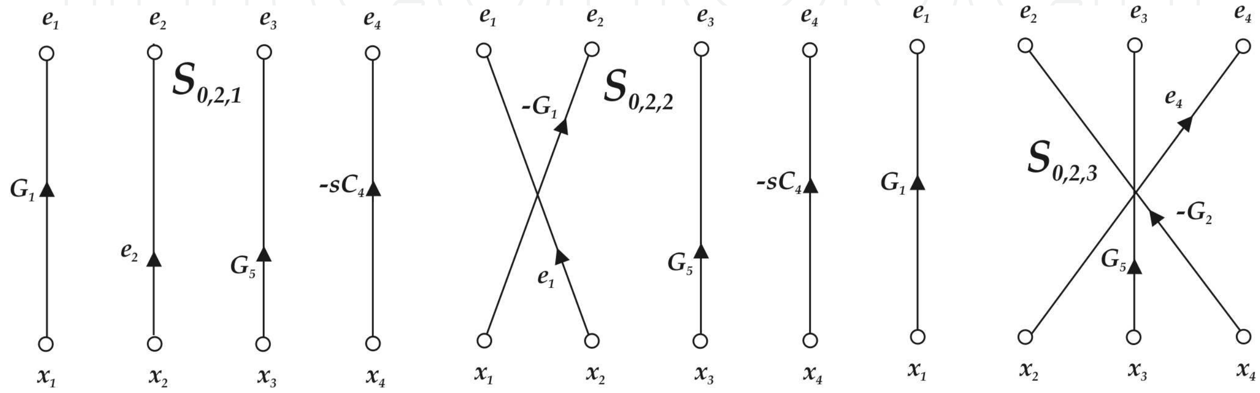

Fig. 18. Separations $S_{0,2,1}$, $S_{0,2,2}$ and $S_{0,2,3}$ of CM graph $G_{0,2}$

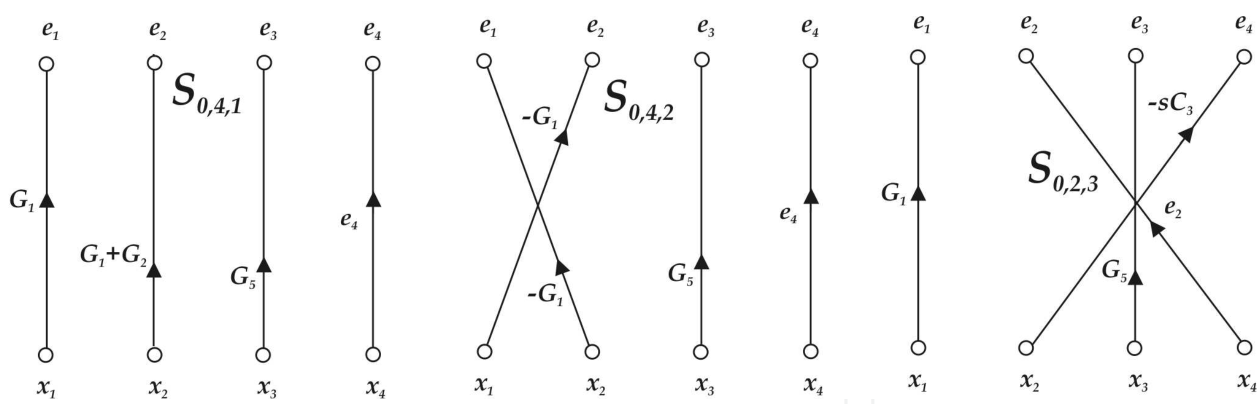

Fig. 19. Separations $\boldsymbol{S}_{0,4,1}$, $\boldsymbol{S}_{0,4,2}$ and $\boldsymbol{S}_{0,4,3}$ of CM graph $\boldsymbol{G}_{0,4}$

$$\left.z_{24} = z_{44} = -\frac{1}{s(C_3 + C_4)};\ z_{43} = 0\right\}. \tag{53}$$

Now we return to (42) and (49) and obtain

$$\left.\mathbf{K}_{34} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{s(C_3 + C_4)} \\ 0 & -\dfrac{1}{s(C_3 + C_4)} & 0 & -\dfrac{1}{s(C_3 + C_4)} \end{bmatrix};\ \mathbf{V}_{34} = \mathbf{K}_{34}\mathbf{V} = \mathbf{K}_{34}\begin{bmatrix} V_1 \\ V_{23} \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{V_5}{s(C_3 + C_4)} \\ -\dfrac{V_{23} + V_5}{s(C_3 + C_4)} \end{bmatrix}\right\}. \tag{54}$$

$$\left.\begin{aligned}
&SP_{H,34,1,1} = -G_1 G_2 G_5 \frac{V_{23} + V_5}{s(C_3 + C_4)};\ N_{a,J3,1,1} = 4;\ Ns_{,J3,1,1} = 0;\\
&SP_{H,34,23,1} = G_1 G_2 G_5 \frac{V_{23} + V_5}{s(C_3 + C_4)};\ N_{a,J3,23,1} = 4;\ Ns_{,J3,23,1} = 1;\\
&SP_{H,34,4,1} = -G_1(G_1 + G_2)sC_4 \frac{V_5}{s(C_3 + C_4)};\ N_{a,J3,4,1} = 4;\ Ns_{,J3,4,1} = 0;\\
&SP_{H,34,4,2} = G_1 G_2 sC_3 \frac{V_{23} + V_5}{s(C_3 + C_4)};\ N_{a,J3,4,2} = 4;\ N_{s,J3,4,2} = 0;\\
&SP_{H,34,4,3} = -G_1^2 sC_4 \frac{V_5}{s(C_3 + C_4)};\ N_{a,J3,4,3} = 2;\ N_{s,J3,4,3} = 1;\\
&SP_{H,34,4,4} = G_1 G_2 sC_3 \frac{V_5}{s(C_3 + C_4)};\ N_{a,J3,4,4} = 2;\ N_{s,J3,4,4} = 1;\\
&SP_{H,34,5,1} = -G_1(G_1 + G_2)C_5 \frac{V_{23} + V_5}{s(C_3 + C_4)};\ N_{a,J3,5,1} = 4;\ N_{s,J3,5,1} = 0;\\
&SP_{H,34,5,2} = -G_1^2 G_5 \frac{V_{23} + V_5}{s(C_3 + C_4)};\ N_{a,J3,5,2} = 2;\ N_{s,J3,5,2} = 1;
\end{aligned}\right\}. \tag{55}$$

In order to determine the second derivatives of the vector $\partial^2\mathbf{V}/\partial(sC_3)\partial(sC_4)$ and having in mind (42) one draws the CM graph $\boldsymbol{G}_{H,34}$ shown in Fig. 20. In the case we have a simplification of the analysis on the base of the graph $\boldsymbol{G}_{H,34}$ because the substantial difference between $\boldsymbol{G}_{H,34}$ and $\boldsymbol{G}_{J3}$ consists in the sink and source vertex signal expressions – instead of $-V_{23}$ and V_{23} in $\boldsymbol{G}_{J3}$ the corresponding signals in $\boldsymbol{G}_{H,34}$ are $V_5/s(C_3+C_4)$ and $-(V_{23}+V_5)/s(C_3+C_4)$. Owing to this peculiarity further we use directly (37) after substituting sink vertex signals, namely:

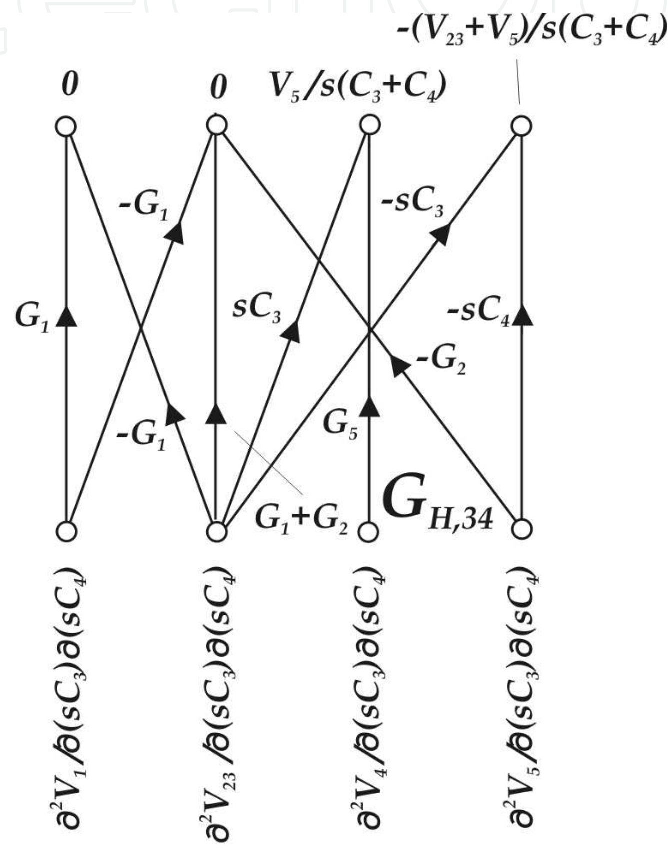

Fig. 20. CM Graph $\boldsymbol{G}_{H,34}$

The voltage V_5 can be find similarly to V_{23} from CM graph $\boldsymbol{G}$ and it is:

$$V_5 = -\frac{C_3 J_1}{s(C_3+C_4)} \,. \tag{56}$$

Then by using (33), (35), (55) and (56) one obtains the vector

$$\frac{\partial^2\mathbf{V}}{\partial(sC_3)\partial(sC_4)} = \left[\frac{J_1(C_4-C_3)}{G_2 s^2 (C_3+C_4)^3} \quad \frac{J_1(C_4-C_3)}{G_2 s^2 (C_3+C_4)^3} \quad -2\frac{J_1 C_3 C_4}{sG_2 G_5 (C_3+C_4)^3} \quad \frac{J_1(C_4-C_3)}{G_2 s^2 (C_3+C_4)^3}\right]_t, \tag{57}$$

Its elements are a part of elements in the Hessian matrices $\mathbf{H}_1$, $\mathbf{H}_{23}$, $\mathbf{H}_4$ and $\mathbf{H}_5$ with respect to the admittances sC_3 and sC_4.

6. First and second-order quadratic sensitivity sums

The sensitivity is an important parameter for the evaluation of practical suitability of electrical networks. For this purpose usually one uses the first-order sensitivity and the second-order sensitivity, defined by the well known formulae (Cederbaum, 1984; Chua & Lin, 1975)

$$S_x^F = \frac{\partial F}{\partial x} . \frac{x}{F} ; \tag{58}$$

and

$$S_{x,y}^F = \frac{\partial^2 F}{\partial x \partial y} . \frac{xy}{F} , \tag{59}$$

respectively and where F is a network function or variable and x, y are changeable network element parameters.

Obviously, the derivatives in these expressions can be determined according to the above described method based on Chan-Mai signal-flow graphs. Besides very often we are interested in a global index as a quadratic sum of sensitivities (first- or second-order):

$$\sum_i (S_x^{F_i})^2 = \sum_i \left(\frac{\partial F_i}{\partial x} . \frac{x}{F_i} \right)^2 \tag{60}$$

and

$$\sum_i (S_{x,y}^{F_i})^2 = \sum_i \left(\frac{\partial^2 F_i}{\partial x \partial y} . \frac{xy}{F_i} \right)^2 , \tag{61}$$

where i ϵ {1, 2, …, n} .

Without loss of generality further we assume that the functions F_i are the elements of the voltage vector **V.** Then the sum (60) can be derived with the help of the expressions of the corresponding Jacobian matrix subvectors $\mathbf{J}_i$ and of the voltage vector **V**:

$$\left. \begin{aligned} &\sum_i (S_x^{V_i})^2 = x^2 \mathbf{J}_{i,t} (\mathbf{M}^{-1})^2 \mathbf{J}_i ; \\ &\mathbf{M} = diag\{V_1,\ V_2,\ \ldots,\ V_i,\ \ldots,\ V_n\} \end{aligned} \right\} . \tag{62}$$

If from the elements of the Hessian matrices $\mathbf{H}_i$ one forms the vector

$$\mathbf{h}_{xy} = \begin{bmatrix} h_{1,xy} & h_{2,xy} & \ldots & h_{i,xy} & \ldots & h_{n,xy} \end{bmatrix}_t ; \ h_{i,xy} = \frac{\partial^2 V_i}{\partial x \partial y} \tag{63}$$

the sum (61) can be rewritten as

$$\sum_i (S_{xy}^{V_i})^2 = x^2 y^2 \mathbf{h}_{i,t} (\mathbf{M}^{-1})^2 \mathbf{h}_i \ . \tag{64}$$

7. Conclusions

A topological method for obtaining the Jacobian and Hessian matrices and their use for quadratic first- or second-order sensitivity sums calculation of active networks is presented. It is based on the replacement of the investigated network N by using a nullor equivalent circuit and on the representation of the circuit passive part N_p by a Chan-Mai signal-flow graph $\boldsymbol{G}_p$. The Jacobian and the Hessian matrix elements of the nullor network can be obtained by means of the some dependent variables of some Chan-Mai graphs derived from $\boldsymbol{G}$. The substantial advantage of the method consists in the use mainly of isomorphic graphs. Two examples illustrate the proposed method.

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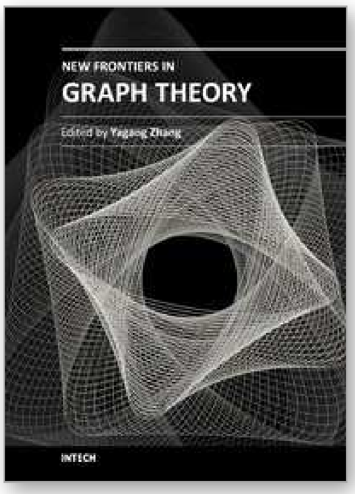

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