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Correspondence, Time, Energy, Uncertainty, Tunnelling, and Collapse of Probability Densities

Gabino Torres–Vega
Physics Department, Cinvestav
Mexico

1. Introduction

Classical and Quantum Mechanics make use of different objects and they also were written on different background spaces. As a consequence, they have followed different paths regarding the mathematical objects that they use. Classical Mechanics makes use of points and functions on a real cotangent space T^*Q of a mechanical system, and makes use of differential geometry as the basic language with which the theory is developed. This is due to the existence of trajectories of single points.

On the other side, Quantum Mechanics makes use of state vectors in a complex Hilbert space, with operators, commutators, and eigenvectors, and makes use of some postulates that look weird from a classical point of view. This is a point of view that was induced by the lack of trajectories and by the use of probabilistic interpretations of state vectors.

There have been efforts to define similar classical and quantum functions that can be compared with each other. Quantum densities were written as functions on phase–space by means of integral transformations. Two of these transformations are the Wigner (Wigner, 1932, Muga & Snider, 1992, Sala, R, Brouard, S, & Muga, JG, 1993, Sala & Muga, 1994, Bracken, 2003) and the Husimi (Husimi, 1940, Torres & Frederick, 1990, 1991) transforms. They provide with a phase–space function that can be used as a classical picture of the quantum probability density. However, these functions are difficult to interpret.

An approach to classical–quantum correspondence uses quantum concepts in Classical Mechanics focusing on the eigenfunctions of the classical Liouville operator. Complex functions are introduced, together with a quantum–like inner product between phase–space functions, into the classical theory, but this leads to some inconveniences like having quantities with no physical interpretation (Koopman, 1931, Jaffé & Brumer, 1984, 1985,, Jaffé, 1988 Woodhouse, 1991). Here, we do not make use of complex quantities at all.

In this chapter, we stay with plain Classical Mechanics and we want to identify some of the classical objects that are the analogue of quantum quantities. These analogues allows us to take a point of view of classical systems similar to the one used in quantum systems. These analogues show that these theories are not that far from each other. In fact, we show that we can handle classical systems in a very similar way as it is done for quantum systems. We define eigenfunctions of classical dynamical variables and use them to define alternative representations of classical quantities and in the calculation of averages and of other quantities.

The way of handling time in classical and quantum physics, as well as in other theories in physics, has been a subject of the interest of many researchers for a long time now (Aharonov et al, 1961, Allcock, 1969, Muga et al, 1998, Muga & Leavens, 2000, Muga, JG, Sala–Mayato, R and Egusquiza, IL (ed), 2002, 2008, Galapon, 2002, Galapon et al, 2004, Isidro, 2005, Torres, 2007, 2009, Delgado et al, 1997, Giannitrapani, 1997, Halliwell, 1999, Hegerfeldt et al, 2004, Kijowski, 1974, Kobe et al, 1993, 1994, Kochański et al, 1999, Leavens, 2002, León, 1997, Rovelli, 1990, 1991). We make use of those developments and further develop and apply those ideas in this chapter.

Looking for classical analogues of quantum objects is of help in clarifying the physical meaning of the latter, and it shows us that we can also make use of the quantum language in the classical realm, taking a path in parallel to the direction that Quantum Mechanics theory has taken.

Earlier treatments of time in Classical Mechanics make use of canonical transformations. However, this type of treatment needs to introduce a "tempus" variable which is not related with physical time and has no physical interpretation. One can see applications of this theory in the treatment of tunnelling through a potential barrier (Razavy, 1967, 1971, Kobe, 2001). Here we do not need to introduce additional variables.

In quantum Mechanics, the description of the evolution of wave packets can be carried out in terms of the eigenfunctions of the Hamiltonian operator. This is the operator that appears in the evolution equation for wave functions. Then, a way of approaching the evolution of probability densities in classical phase–space makes use of the eigenfunctions of the Liouville operator (Jaffé, 1988, Jaffé & Brumer, 1984, 1985). That seems to be a reasonable approach because the classical evolution equation of probability densities is determined by the Liouville operator, precisely. However, some of these eigenfunctions are complex with no physical interpretation. Here, we focus on the eigen surfaces and eigen functions of the dynamical quantities instead.

Conserved quantities have been used to construct directional derivatives (Jaffé, 1988, Jaffé & Brumer, 1985), but nothing has been said about the use of the use of pairs of conjugate variables. Here we propose to also use conjugate dynamical variables as generators of translations in phase–space. With the use of conserved quantities, the motion of phase–space points is kept on the energy shell surface, but with a conjugate function, points can leave that shell.

On the other hand, we are interested on recognising that many of the concepts that are used in the theory of Quantum Mechanics can also be used in the study of classical systems, a point of view which is closer to Quantum Mechanics than other approaches like Geometric Mechanics or Geometric Quantisation. Our approach makes use of eigen objects, operators, and commutators, in a similar way as is done in Quantum Mechanics. This approach will lead to a plausible classical interpretation of the collapse of a quantum wave function; the goal of this paper. A benefit of our approach is that it is of help in the understanding of quantum phenomena.

We will be working with conservative Hamiltonian systems, systems for which Hamilton's equations of motion apply, without an additional "tempus variable" involved.

Throughout the text, we will be considering as a model system the nonlinear oscillator with dimensionless Hamiltonian given by (José and Saletan, 1998)

$$\mathcal{H}(z) = \frac{p^2}{2} + \frac{k}{2}\left(\sqrt{a^2+q^2} - l\right)^2 . \tag{1}$$

This Hamiltonian describes the motion of a bead that slides on a horizontal bar and is acted on by a spring attached to a fixed point a distance a from the bar. The force constant of the spring is k, and l is its natural length, so that $\sqrt{a^2+q^2}-l$ is its stretch. The fixed points for this system are located at

$$q_0 = 0, \pm\sqrt{l^2-a^2}\,, \quad p_0 = 0\,. \tag{2}$$

There is only a minimum of the potential function when $a > l$, and there are two minima and one maximum when $a < l$.
We will also consider an application of our results to the tunnelling through a potential barrier.

2. Conjugate variables and representations

Let us consider a Poisson manifold $(T^*Q, \{\bullet,\bullet\})$ associated to a classical system, with $\{\bullet,\bullet\}$ the usual Poisson bracket, which for two functions $F(z)$ and $G(z)$ is defined as

$$\{F,G\}(z) = \frac{\partial F(z)}{\partial q^i}\frac{\partial G(z)}{\partial p_i} - \frac{\partial G(z)}{\partial q^i}\frac{\partial F(z)}{\partial p_i}\,, \tag{3}$$

where $z = (q,p)$, $q = (q^1,\ldots,q^n)$, $p = (p_1,\ldots,p_n)$ is a point on T^*Q.
We will be mainly concerned with pairs of conjugate variables and some of the consequences of that relationship between them. Conjugate variables are the variables that are related by a constant Poisson bracket, $\{F,G\}(z) = 1$, in Classical Mechanics and by a constant commutator, $[\hat{F},\hat{G}] = i\hbar$, in Quantum Mechanics, between the corresponding quantum operators $\hat{F}$, $\hat{G}$.
The dynamics of classical systems usually is described in terms of the pairs of conjugate variables q^i and p_i. These pairs of variables are related by a constant Poisson bracket,

$$\{q^i, p_j\} = \delta^i_j\,. \tag{4}$$

The domain in which this relationship is valid is $\mathcal{D} = \mathbb{R}^n \times \mathbb{R}^n$. They are the coordinates for describing the evolution of a classical system, and each variable, usually, take continuous values from $-\infty$ to ∞ or on some subset of it. Time is a parameter in terms of which the motion of point particles can be described.
The quantum position $\hat{Q}$ and momentum $\hat{P}$ operators are related by a constant commutator

$$[\hat{Q}^i, \hat{P}_j] = i\hbar\delta^i_j\,, \tag{5}$$

and their eigenfunctions cannot be normalised in the conjugate space, meaning that they are not part of a Hilbert space. However, these eigenfunctions are used as coordinates (as a representation).
In Quantum Mechanics, the commutator between the time operator $\hat{T}$ and the Hamiltonian operator $\hat{H}$ is assumed to be

$$[\hat{T}, \hat{H}] = i\hbar, \tag{6}$$

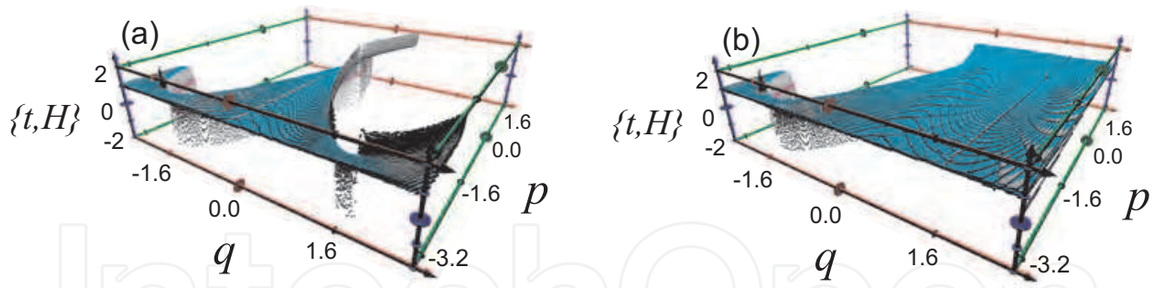


Fig. 1. Values of the Poisson bracket $\{t, H\}(z)$ for the nonlinear oscillator with $a = 1$, $l = 2$, and $k = 9.8$. The turquoise points indicate that the Poisson bracket evaluates to one there, whereas in the red points the value is zero. (a) $X = 0$, and (b) X the location of the right minima of the potential. Dimensionless units.

so that they are a pair of conjugate operators. We find a similar relationship in the classical regime, the Poisson bracket between time and energy is equal to one

$$\{t, H\}(z) = \frac{\partial t}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q^i}\frac{\partial t}{\partial p_i} = \frac{\partial t}{\partial q^i}\frac{\partial q^i}{\partial t} + \frac{\partial p_i}{\partial t}\frac{\partial t}{\partial p_i} = \frac{dt}{dt} = 1\,, \tag{7}$$

where we have made use of Hamilton's equations of motion

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}\,, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}\,, \tag{8}$$

and of the chain rule. We argue that these variables, energy and time, can also be used as an alternative coordinate system. In these coordinates, motion of conservative systems becomes quite simple, one of the variables is kept constant and the other just increases. What we have here is the set of canonical variables of group theory.

An object that is inherent to a Poisson bracket equal to one is the domain in which that equality holds. For the nonlinear oscillator, the values that the Poisson bracket take have been plotted in figure 1, for two choices of reference zero time surface. The way in which those values were obtained is explained below. The domain of the energy, $\mathcal{D}(\mathcal{H})$, is the whole of phase–space, but the domain of time, $\mathcal{D}(t)$, is not. The domain of the Poisson bracket is the intersection of these domains, $\mathcal{D}(\mathcal{H}) \cap \mathcal{D}(t)$, which, in this case, coincides with the domain of time. In Quantum Mechanics, we have to consider the intersection of the domains of $\hat{H}\hat{T}$ and of $\hat{T}\hat{H}$, $\mathcal{D}(\hat{H}\hat{T}) \cap \mathcal{D}(\hat{T}\hat{H})$.

The new coordinates, (t, E), are a bit different from (p, q) because to a value of the energy correspond two values of p_i (usually p_i appears as p_i^2 in the Hamiltonian) so that E is bounded from below. It is then necessary to make the distinction between the cases of positive or negative momentum.

In general, the explicit expression in terms of z of one of the variables related by a constant Poisson bracket is known but not the other, as is the case of energy (known) and time (not known). Below, we will show how to generate the unknown one (time) using the equations of motion. We can generate the unknown variable because they are related by the Poisson bracket precisely.

In Quantum Mechanics we have a similar situation. Usually, quantum dynamics is analysed in coordinate or momentum representations, but we can also change to energy or time

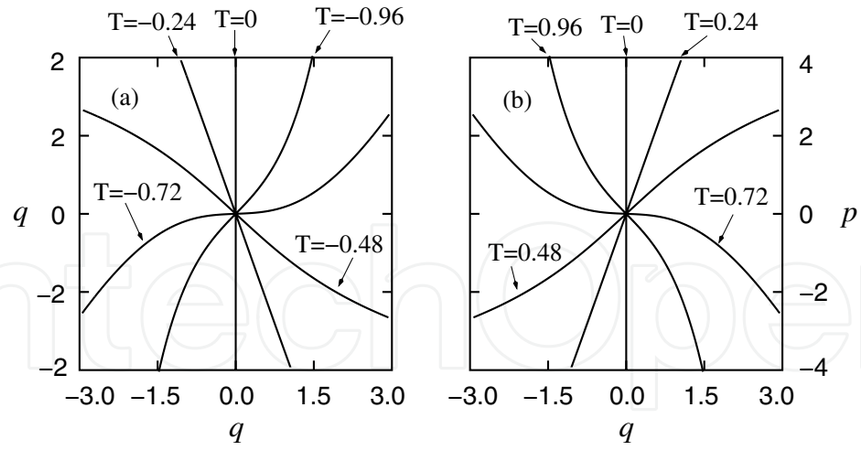


Fig. 2. Time surfaces for the nonlinear oscillator with $a = 1$, $l = 2$, and $k = 9.8$. (a) Negative times and (b) positive times. The initial curve is $q = 0$. Dimensionless units.

representations. The energy representation is well known (the energy eigenstates) but the time representation is not. However, because time is conjugate to energy, we can define time eigenstates and then use them as an alternative representation of quantum states, in the same way as it is done in classical systems (see below).

3. Generating the time coordinate system

Let us consider a cotangent space T^*Q with coordinates $z = (q, p)$. Given the Hamiltonian $H(z)$ of a classical system, Hamilton's equations of motion

$$\frac{dz}{dt} = X_H \ , \quad X_H = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q} \right) \ , \tag{9}$$

use the conjugate variable to the Hamiltonian, time, as a parameter for describing the motion of a particle on the energy shell. The origin of time on the integral lines of equation (9) is chosen arbitrarily and, usually, it is different for each integral line.

We can generate a time coordinate system in T^*Q so that we can have the time variable in terms of (p, q). A point in cotangent space T^*Q can be propagated according to the dynamical system defined by equations (9). These points will move along the surfaces of constant H so that the value of H does not changes but the value of t does. In order to get a coordinate system for time in T^*Q, we define constant t surfaces in T^*Q. A hypersurface $\Sigma_0(z)$ that crosses the constant H surfaces is chosen as the reference, the origin of t, and by propagating it we will obtain surfaces $\Sigma_t(z)$ corresponding to other values of time, so obtaining the desired coordinate system for t in T^*Q.

We will use the surface $q^1 = X$ as the initial time surface, and we will make a distinction on the sign of the momentum of these points giving rise to two time eigensurfaces:

$$\Sigma^{\pm}_{t=0}(z) = \{ z | q^1 = X, \ \pm p_1 > 0 \} \ . \tag{10}$$

In figures 2 and 3 there are examples of the surfaces that comprise the time coordinate system in the case of the nonlinear oscillator. The initial surfaces are $q^1 = X$, with $X = 0, 1.5$. Under this convention, the time values for each point in phase–space are shown in Fig. 4. They

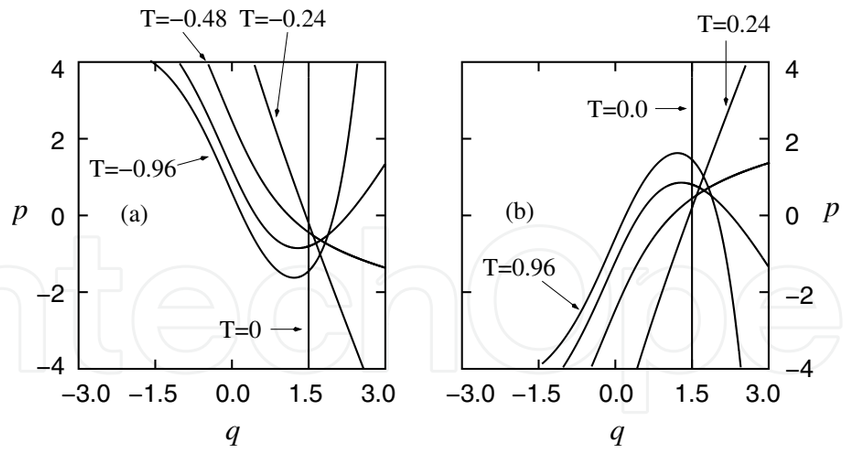


Fig. 3. Time surfaces for the nonlinear oscillator with $a = 1$, $l = 2$, and $k = 9.8$. (a) Negative times and (b) positive times. The initial curve is $q = 1.5$. Dimensionless units.

are the shortest times that it would take to a particle with z as final position to arrive at or depart from $q^1 = X$. These values are the classical analogue of the quantum time operator $\hat{T}$. Note that not all points in T^*Q will arrive or depart from $q^1 = X$. For this reason, some regions of T^*Q are not part of the domain of the Poisson bracket equal to one (see Fig. 1). Points on the separatrix move quite slow as seen on the plot. Time reversal symmetry, i.e. the transformation $(t, p) \to (-t, -p)$ is evident in these figures.

The quantum procedure that can be used to generate time eigenstates is very similar to the classical one. The quantum initial state, in momentum space (in one dimension, for simplicity),

$$\langle p|t = 0\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-ipX} , \tag{11}$$

is the equivalent to the line $q = X$ in phase–space. The squared modulus of this state in momentum space is constant for all values of p, but it is a delta function centred at $q = X$ in coordinate space. This is the reference state for time. The propagation of it, i.e. the state

$$\langle p|t\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-it\hat{H}/\hbar} e^{-ipX} , \tag{12}$$

is the time coordinate system that can be used for a time representation. These states cannot be normalised.

Quantum time eigenstates have been in use for a long time now without realizing it. Let us rewrite the expression for a wave packet in the coordinate representation in terms of the momentum wave packet as (unless otherwise stated the integrals are taken from $-\infty$ to ∞)

$$\begin{aligned} \psi(x;t) &= \frac{1}{\sqrt{2\pi\hbar}} \int dp\; e^{ipx} \psi(p;t) = \frac{1}{\sqrt{2\pi\hbar}} \int dp\; e^{ipx} e^{-it\hat{H}/\hbar} \psi(p) \\ &= \frac{1}{\sqrt{2\pi\hbar}} \int dp\; \psi(p) e^{it\hat{H}/\hbar} e^{ipx} = \frac{1}{\sqrt{2\pi\hbar}} \int dp\; \psi(p) (e^{-it\hat{H}/\hbar} e^{-ipx})^* \\ &= \langle t|\psi\rangle , \end{aligned} \tag{13}$$

which is the inner product between the time eigenstate $|t\rangle$ and a ket $|\psi\rangle$.

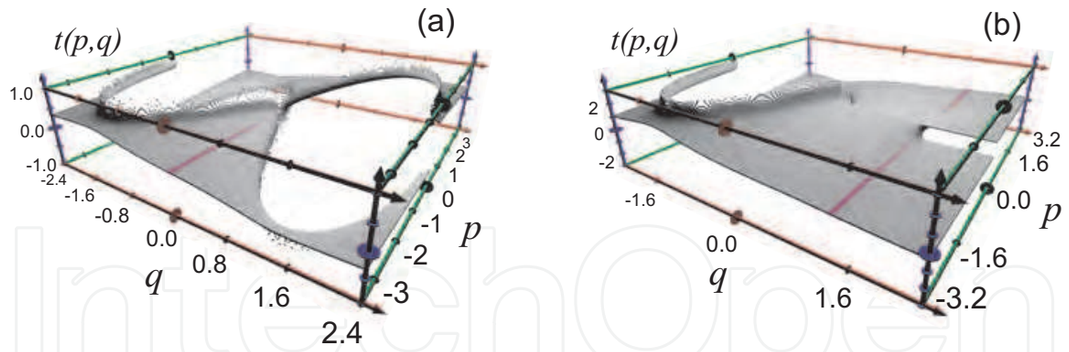


Fig. 4. Time values $t(z)$ for the nonlinear oscillator with $a = 1$, $l = 2$, and $k = 9.8$. (a) $X = 0$ and (b) $X = 1.5$. The points coloured in cyan indicate the zero time points. Dimensionless units.

4. Time and energy eigenfunctions

Quantum energy eigenstates $|\epsilon\rangle$ are the states characterised by just being multiplied by the corresponding eigenvalue ϵ after the application of the Hamiltonian operator, i.e. $\hat{H}|\epsilon\rangle = \epsilon|\epsilon\rangle$. We can take as the classical analogue of the quantum energy eigenstate to a unit density with the constant energy shells as support,

$$\nu_\epsilon^\pm(z) = \delta(z - \Sigma_\epsilon^\pm(z)) \tag{14}$$

where

$$\Sigma_\epsilon^\pm(z) = \{z | H(z) = \epsilon,\ \pm p_1 > 0\}\ . \tag{15}$$

If we evaluate the Hamiltonian function on the support $\Sigma_\epsilon^\pm(z)$ of this function, we will obtain the value ϵ. The density $\nu_\epsilon^\pm(z)$ is the classical analogue of the quantum density $|\langle q|\epsilon^\pm\rangle|^2$. Now, a unit density with the time eigen surfaces $\Sigma_t^\pm(z)$ as support,

$$\nu_t^\pm(z) = \delta(z - \Sigma_t^\pm(z))\ ,$$

is the analogue of the squared magnitude of the quantum time eigenfunction $|\langle q|t\rangle|^2$. In the realm of functions on T^*Q, the classical time eigen density is generated by starting with $\nu_{t=0}^\pm(z)$ and propagating it with the classical propagator as

$$\nu_t^\pm(z) = e^{-t\mathcal{L}(z)}\nu_{t=0}^\pm(z)\ , \quad \mathcal{L}(z) = X_H \cdot \nabla\ , \tag{16}$$

where the vector field is $X_H = (\partial H/\partial p, -\partial H/\partial q)$. If we evaluate time on the support $\Sigma_t^\pm(z)$ of the eigen density $\nu_t^\pm(z)$ we will get the value t.

The unboundedness of the eigensurfaces of a dynamical variable usually implies a problem with the normalisation of functions with them as a support so that they cannot become probability densities. However, we need to include that type of variables to have a representation of quantities in T^*Q. These eigen densities, $\nu_\epsilon(z)$ and $\nu_t(z)$ are the classical analogues of the quantum representation vectors $\langle\epsilon|$, and $\langle t|$, respectively. But recall that some of these quantum vectors are not part of the Hilbert space.

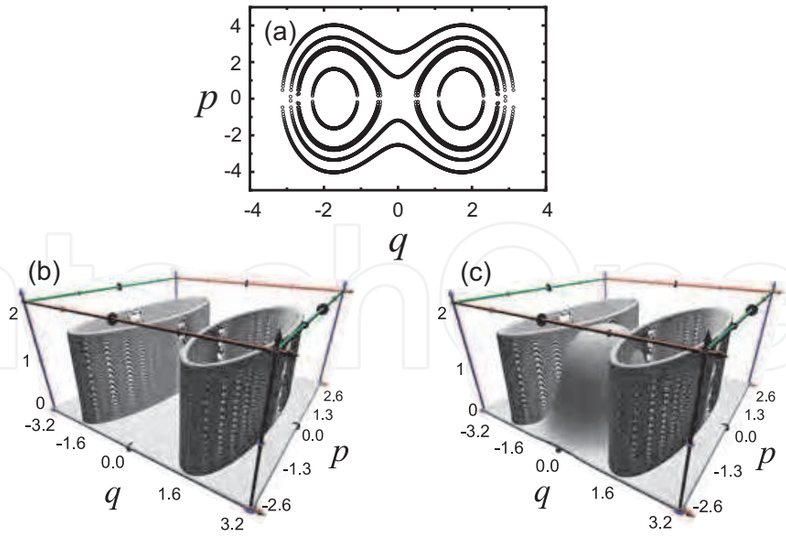


Fig. 5. Classical energy densities corresponding to the quantum eigenvalues for the nonlinear oscillator. (a) Energy curves in phase–space, (b) classical eigendensity for one energy value, and (c) the overlap between the energy eigendensity and a Gaussian density. Dimensionless units.

5. Uncertainty of conjugate eigenfunctions

In Quantum Mechanics, the average of an operator $\hat{F}$ when the system is in one of its eigenfunctions $|f\rangle$ is the corresponding eigenvalue f, i.e. $\langle f|\hat{F}|f\rangle = f$, and the width vanishes, i.e. $\Delta\hat{F} = \sqrt{\langle f|\hat{F}^2|f\rangle - \langle f|\hat{F}|f\rangle^2} = 0$. However, for the conjugate operator $\hat{G}$, the average $\langle f|\hat{G}|f\rangle$ and width $\Delta\hat{G} = \sqrt{\langle f|\hat{G}^2|f\rangle - \langle f|\hat{G}|f\rangle^2}$, in the states $|f\rangle$, are not defined. A property in agreement with Heisenberg's uncertainty principle, $(\Delta\hat{F})^2(\Delta\hat{G})^2 \geq \hbar^2\langle[\hat{F},\hat{G}]/i\hbar\rangle^2/4$, because when one of the observables, in this case $\hat{F}$. is well defined, the conjugate observable $\hat{G}$ becomes undefined.

Note that we also observe this characteristic on the eigensurfaces of dynamical variables of classical systems. In the classical case, with the nonlinear oscillator as an example (see figure 5), on the energy eigen surfaces $\Sigma_\epsilon^\pm(z)$, the energy is well defined with no dispersion, i.e. there is a zero width in energy. However, they span values of time ranging from minus infinity to infinity. And vice-versa, the time eigen density has zero width in time but they include an unbounded set of energy values (see figures 2 and 3). Then, we can say that the classical eigendensities $\nu_\epsilon^\pm(z)$ and $\nu_t^\pm(z)$ correspond to the squared magnitude of the quantum energy, $|\epsilon^\pm\rangle$, and time, $|t^\pm\rangle$, eigenstates, respectively,

$$\nu_\epsilon^\pm(z) \leftrightarrow |\epsilon^\pm\rangle\langle\epsilon^\pm| \ , \quad \nu_t^\pm(z) \leftrightarrow |t^\pm\rangle\langle t^\pm| \ . \tag{17}$$

In Fig. 5, we show few classical energy curves, for the nonlinear oscillator, with energy values equal to the quantum eigenvalues. Note that the separatrix is excluded from these values. There is also, a schematic representation of a classical unit density with one of the energy curves as support.

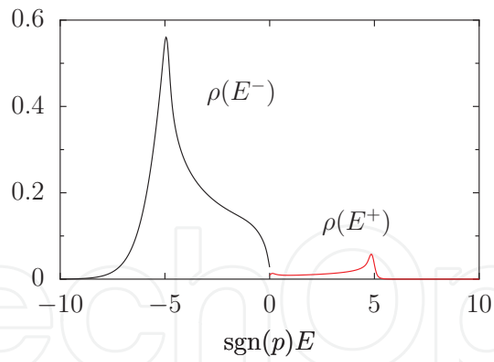


Fig. 6. Classical energy representation $\rho(\epsilon^{\pm})$ of a Gaussian probability density $e^{-(q-q_0)^2/2\sigma^2-2\sigma^2(p-p_0)^2}/\sqrt{2\pi}\sigma$ centred at $(q,p)=(0,-1)$ for the nonlinear oscillator with $a=1$, $l=2$, $k=9.8$, and $\sigma=.8$. The peaks are located at the energy of the separatrix. This is the classical analogue of the squared magnitude of the energy representation of a quantum state $|\langle\epsilon^{\pm}|\psi\rangle|^2$. Dimensionless units.

6. Energy and time representations

With the help of the classical energy eigendensities (14), the energy representation of a probability density $\rho(z)$ is defined as

$$\rho(\epsilon^{\pm}) = \int dz\ \nu_{\epsilon}^{\pm}(z)\rho(z)\ . \tag{18}$$

Note that we have reduced the representation from one with $2n$ variables, namely p_i and q^i, to one with only one variable, $\epsilon^{\pm}$. Then, it is not possible to recover the original density $\rho(z)$ from the reduced one $\rho(\epsilon^{\pm})$. An example of these reduced densities is found in Fig. 6 for the nonlinear oscillator.

Recall that the quantum energy representation is obtained as the inner product between the energy eigenstate, $|\epsilon\rangle$, and the wave function, $|\psi\rangle$, separated into negative and positive momentum parts, as

$$\psi(\epsilon^{\pm}) = \int_0^{\infty} dp\langle\epsilon|\pm p\rangle\langle\pm p|\psi\rangle\ . \tag{19}$$

Thus, the squared magnitude of this quantity is the analogue to the classical energy representation of a probability density of equation (18). Since the quantum spectrum is discrete for the nonlinear oscillator, only few points will be found in the energy representation, as can be seen in Fig. 7.

For a classical time representation of a probability density, we calculate the overlap between the time eigendensity Eq. (16) with a probability density $\rho(z)$

$$\rho^{\pm}(t) = \int dz\ \nu_{t}^{\pm}(z)\rho(z)\ . \tag{20}$$

This is also a reduced representation of the density $\rho(z)$ because it only depends on one variable. For the nonlinear oscillator, an example of this representation is shown in Fig. 8. The time representation depends on the zero–time reference surface $\Sigma_{t=0}(z)$. Different regions

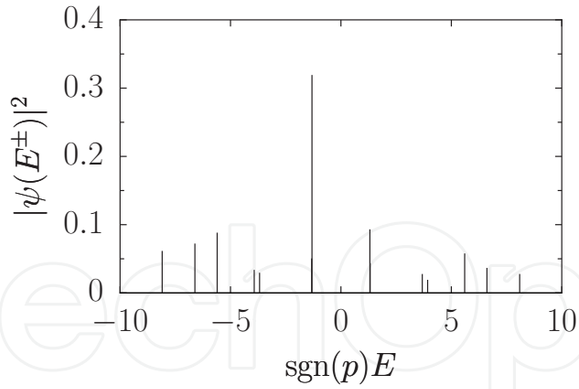


Fig. 7. Quantum energy representation of the Gaussian state $e^{-(q-q_0)^2/4\sigma^2 - ip_0(q-q_0/2)}/\sqrt{\sigma\sqrt{2\pi}}$ centred at $(q,p) = (0,-1)$, with $\sigma = 0.2$, and for the nonlinear oscillator. Dimensionless units.

of phase–space are available depending upon the choice of X, the location of the zero–time reference surface.
A quantum time representation is obtained in a similar way: the wave packet $|\psi\rangle$ is projected onto the time eigenstates $|t^{\pm}\rangle$ of equation (12),

$$\psi(t^{\pm}) = \int_0^{\infty} \langle t| \pm p\rangle\langle \pm p|\psi\rangle \ . \tag{21}$$

The squared magnitude of this projection is the desired quantum time probability density. For the nonlinear oscillator, the quantum time probability density looks like the ones shown in Fig. 9.
The time width of a given probability density that can be calculated with our procedure is a static property. It is a consequence of a probability density having a non zero width on phase–space in a particular set of coordinates. Other possibility is the time dependence and time width due to the actual motion of the system.
The following is not possible for quantum systems, but a joint representation is obtained with the joint eigen surfaces and densities of energy and time,

$$\Sigma^{\pm}_{\epsilon\tau}(z) = \{z|H(z) = \epsilon,\ t(z) = \tau,\ \pm p_1 > 0 \text{ when } \tau = 0\} \ , \tag{22}$$

$$\nu^{\pm}_{\epsilon\tau}(z) = \delta(z - \Sigma^{\pm}_{\epsilon\tau}(z)) \ . \tag{23}$$

The energy-time representation of a classical probability density is then given by

$$\rho^{\pm}(\epsilon,\tau) = \int dz\ \nu^{\pm}_{\epsilon\tau}(z)\rho(z) \ . \tag{24}$$

This representation has no quantum counterpart because energy and time cannot be determined simultaneously in quantum systems. An example of this representation is found in Fig. 10 for the nonlinear oscillator.
We can say that functions like the classical energy or time eigenstates have been in use for a long time now in an unnoticed way. We can rewrite a time dependent probability density in

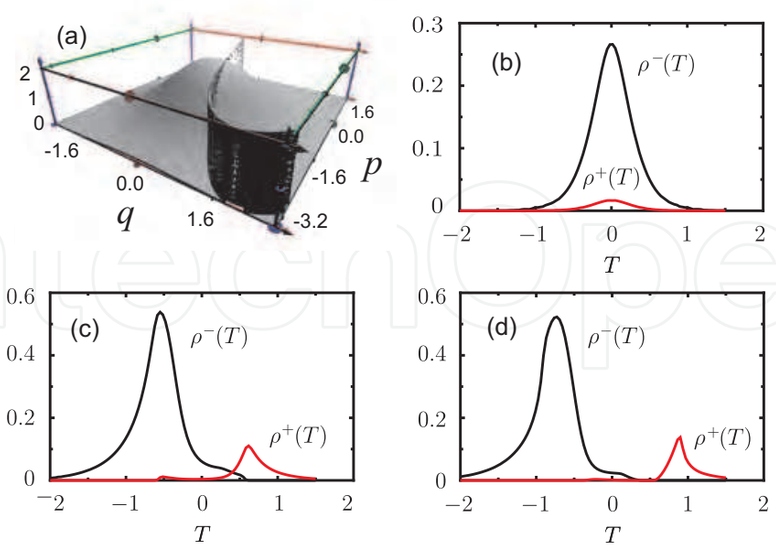


Fig. 8. Time eigendensity and representation for the nonlinear oscillator with $a = 1$, $l = 2$, and $k = 9.8$. (a) Schematic representation of the overlap between a time eigenfunction and a Gaussian probability density centred at $(q, p) = (0, -1)$, and the time representation $\rho^{\pm}(T)$ for (b) $X = 0$, (c) X the positive fixed point, and (d) for $X = 2.2$. Dimensionless units.

phase–space in terms of the phase–space and time eigenfunctions. The following shows this

$$\begin{aligned}\rho(z;t) &= \int dz' \delta(z'-z)\rho(z';t) = \int dz' \delta(z'-z) e^{t\mathcal{L}(z')}\rho(z') \\ &= \int dz' \rho(z') e^{-t\mathcal{L}(z')}\delta(z'-z) + \text{b.t.} \\ &= \int dz' \rho(z')\nu_{zt}(z') + \text{b.t.}\,, \end{aligned} \tag{25}$$

where

$$\nu_{zt}(z') = e^{-t\mathcal{L}(z')}\delta(z'-z)\,, \tag{26}$$

and b.t. stands for the boundary terms arising from the integration by parts (throughout the text partial integration is used with the assumption that contributions from the boundaries always vanish). These terms usually evaluate to zero because probability densities vanish at the boundaries. What we have here is the motion of single points of phase–space weighted by the density at the initial place of the points. This is a way of moving functions on phase–space according to the motion of phase–space eigendensities, an integral operator form of the classical propagator $e^{t\mathcal{L}(z)}$.

The classical time representation $\rho(\tau)$ can be written in terms of the initial time eigendensities and the probability density at time τ, or in terms of the eigendensities at time τ and the initial

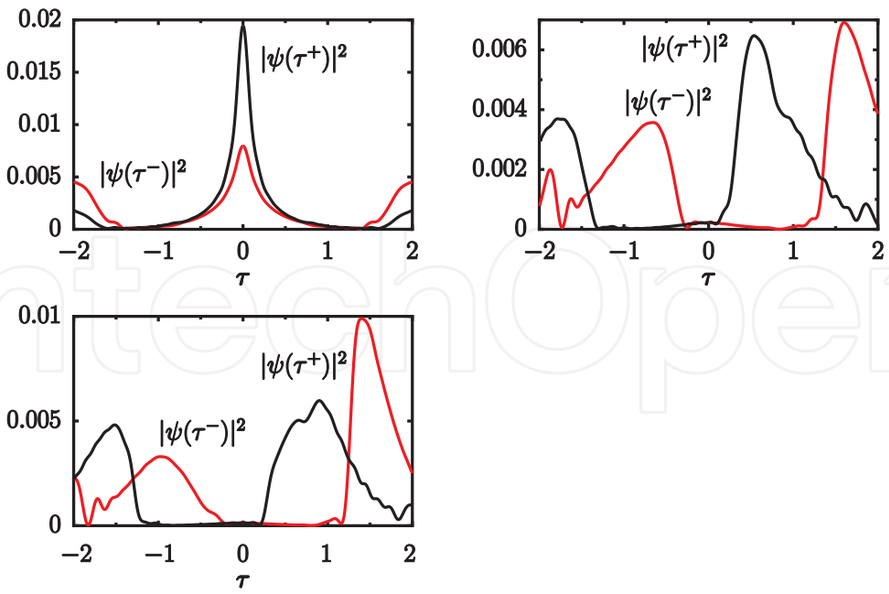


Fig. 9. Quantum time representation of the Gaussian state $e^{-(q-q_0)^2/4\sigma^2 - ip_0(q-q_0/2)}/\sqrt{\sigma\sqrt{2\pi}}$ centred at $(q,p) = (0,-1)$, with $\sigma = 0.2$, and for the nonlinear oscillator. The zero–time reference state is located at (a) $X = 0$, (b) X the location of the right fixed point, and (b) $X = 2$. Dimensionless units.

probability density,

$$\begin{aligned}\rho(\tau) &= \int dz\ \nu_\tau(z)\rho(z) = \int dz\ \rho(z)e^{-\tau\mathcal{L}_H}\delta(z-\Sigma_{\tau=0}(z)) \\ &= \int dz\ \delta(z-\Sigma_{\tau=0}(z))e^{\tau\mathcal{L}_H}\rho(z) + \text{b.t.} = \int dz\ \delta(z-\Sigma_{\tau=0}(z))\rho(z;t) + \text{b.t.} \\ &= \int dz\ \nu_{\tau=0}(z)\rho(z;t) + \text{b.t.}\ . \end{aligned} \tag{27}$$

Then, we do not need to propagate the time eigendensity and the probability density, only the evolution of one of them is enough.

7. Collapse of probability densities

In the process known as the quantum collapse of wave functions, a quantum system represented by the wave function $|\psi\rangle$ ends up in the eigenstate $|g\rangle$ of the operator $\hat{G}$, with probability $|\langle g|\psi\rangle|^2$, after a measurement of the quantity represented with the operator $\hat{G}$. This is a postulate of Quantum Mechanics introduced to ensure a continuity of measurements when the same property of a quantum system is measured several times. This postulate seems a bit awkward at first sight, but it can be understood in classical terms, as we will see in this section.

A classical image of the quantum collapse process is one in which an apparatus selects from the particles of an ensemble $\rho(z)$ only the ones with a z that gives the observed value g of $G(z)$, reducing the domain of $\rho(z)$ from T^*Q to $\Sigma_g(z)$, i.e. $\rho(z) \to \rho(\Sigma_g)$. The probability of

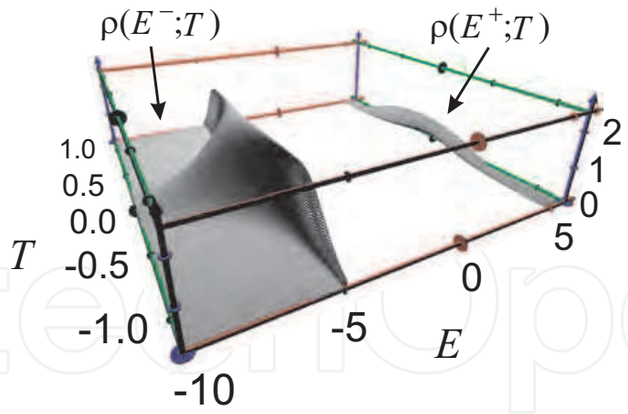


Fig. 10. Classical energy–time representation of a Gaussian density, centred at $(q, p) = (0, -1)$, for the nonlinear oscillator with $a = 1$, $l = 2$, $k = 9.8$, $X = 0$, and $\sigma = .8$. Dimensionless units.

measuring the value g is $\rho(g) = \int dz \nu_g(z)\rho(z)$. Since the support of ρ is reduced to Σ_g, an afterwards measurement of $G(z)$ will return the expected value of g.
An example of this is the following. Let us assume that a beam of particles is travelling to the right and that when this beam crosses the origin of coordinates, a disc with a hole in it is rotating letting the crossing of only a part of the beam. The selected particles can be labelled as having $t = 0$. We can assert that the selected particles will have a time t afterwards, and a wide range of values of energy.

8. An application to the tunnelling through a potential barrier

A straightforward application of the results found in previous sections is the determination of tunnelling times through a potential barrier. There are several ways of calculating tunnelling transmission coefficients (Muga, 1991, Wigner, 1972, del Barco, 2007, Kobe, 2001) and tunnelling times. We consider the calculation of tunnelling times that makes use of time averages (Wigner, 1972, del Barco, 2007). Other approaches introduce a "tempus" variable, a variable which is not related to physical time (Kobe, 2001).
Time eigenfunctions provide a sound basis for the use of a constant coordinate in the calculations of time averages that Wigner and other authors use (Wigner, 1972, del Barco, 2007). Our results also show that it is not necessary to consider "initial conditions giving the state of the system for all times but only for a single value of one of the spatial coordinates" as Wigner required, because it is the time eigenstate the quantity that has that property, and it can be determined.
In Quantum Mechanics, it is common the use of the average time, when the system is in the state $|\psi\rangle$, in the determination of tunnelling times. According to equation (13), the average time, at fixed position $q^1 = X$, can be written in terms of the time eigenstates as follows

$$\langle t(X)\rangle \equiv \frac{\int dt\ |\psi(X;t)|^2 t}{\int dt\ |\psi(X;t)|^2} = \frac{\int dt\ |\langle t|\psi\rangle|^2 t}{\int dt\ |\langle t|\psi\rangle|^2} = \frac{\int dt\ \langle\psi|t\rangle t\langle t|\psi\rangle}{\int dt\ \langle\psi|t\rangle\langle t|\psi\rangle},$$

with X fixed. We note that the time eigenkets can be used to form projection $\hat{P}_X$ and time $\hat{T}_X$ operators with them. These operators project onto the subspace that is available to wave

functions that can depart or arrive at $q^1 = X$. They are defined as

$$\hat{P}_X \equiv \int dt\ |t\rangle\langle t| \ , \quad \hat{T}_X \equiv \int dt\ |t\rangle t\langle t| \ . \tag{28}$$

With these operators, the time average, at fixed position, is written as the ratio

$$\langle t(X)\rangle = \frac{\langle \hat{T}_X\rangle}{\langle \hat{P}_X\rangle} \ . \tag{29}$$

For the classical case, and according to equation (20), the average of time, for fixed $q^1 = X$, is defined as

$$\bar{t}(X) = \frac{\int dt\ t\rho(t)}{\int dt\ \rho(t)} = \frac{\int dt\ t \int dz\ \nu_t(z)\rho(z)}{\int dt \int dz\ \nu_t(z)\rho(z)} \ . \tag{30}$$

We note that we need of two functions. One that collects the points that can arrive or depart from $q^1 = X$,

$$P_X(z) \equiv \int dt\ \nu_t(z) \ , \tag{31}$$

and another,

$$t_X(z) \equiv \int dt\ t\ \nu_t(z) \ , \tag{32}$$

which can be considered as a classical time probe function for the subspace determined by the initial time surface $\Sigma_{\tau=0}(z)$. With these definitions, the time average can be written as follows

$$\bar{t}(X) = \frac{\langle t_X\rangle}{\langle P_X\rangle} \ , \tag{33}$$

a result which is similar to the quantum average, equation(29).
The interpretation of the time distributions is that the time eigenstates are used as probe functions that identify the amount of probability that has a particular value of time and sums those contributions.
Thus, Wigner's annotation concerns the time eigendensities. But this is no problem at all because we know them for all time and single value of one of the spatial coordinates, initially.

9. Remarks

We can say that conjugate variables can be used to generate pairs of coordinate systems in the phase–space of classical systems. The eigensurfaces of these variables might cover an unbounded region of T^*Q and then can be used to write other dynamical quantities in terms of them. A unit density with these eigensurfaces as support cannot be normalised and then cannot be used as probability densities in T^*Q.
A similar thing occurs in quantum systems. The eigenstates of pairs of conjugate operators can be used as vectors with which quantum states and operators can be represented. In general the coordinate, momentum, or time eigenstates cannot be normalised. They are not part of

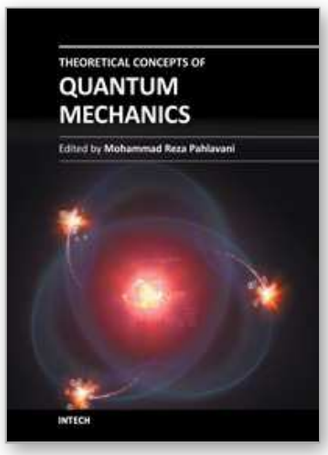
the Hilbert space, so that they cannot be a wave function. However, they are needed so that we can have a representation of other dynamical quantities.
There are many other aspects of Quantum Mechanics that can be analysed in the classical realm. The classical analysis in the terms done in this chapter is useful because it shows that many of the objects found in Quantum theory are also present in the Classical theory, and that they are of help in the understanding of quantum phenomena.
In future work we will study other properties of quantum systems, like the meaning of the Pauli theorem (Pauli, 1926), a theorem that prevents the existence of an hermitian time operator.

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