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Quantum Correlations in Successive Spin Measurements

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1. Introduction

Quantum Mechanics (QM) represents one of the pillars of modern physics: so far a huge amount of theoretical predictions deriving from this theory have been confirmed by very accurate experimental data. No doubts can be raised on the validity of this theory. Nevertheless, even after one century since its birth, many problems related to the interpretation of this theory persist: non-local effects of entangled states, wave function reduction and the concept of measurement in QM, the transition from a microscopic probabilistic world to a macroscopic deterministic word perfectly described by classical mechanics and so on. A possible way out from these problems would be if QM represents a statistical approximation of an unknown deterministic theory, where all observables have well defined values fixed by unknown variables, the so called Hidden Variable Theories (HVT). Therefore, the debate whether QM is a complete theory and probabilities have a non-epistemic character (i.e. nature is intrinsically probabilistic) or whether it is a statistical approximation of a deterministic theory and probabilities are due to our ignorance of some parameters (i.e. they are epistemic) dates to the beginning of the theory itself.

The fundamental paper where this problem clearly emerged appeared in 1935 when Einstein, Podolsky and Rosen asked this question by considering an explicit example (Einstein et al., 1935). For this purpose, they introduced the concept of element of reality according to the following definition: if, without disturbing a system in any way , one can predict with certainty the value of a physical quantity, then there is an element of physical reality corresponding to this quantity. They formulated also the reasonable hypothesis (consistent with special relativity) that every non-local action was forbidden. A theory is complete when it describes every element of reality. They concluded that either one or more of their premises was wrong or Quantum Mechanics was not a complete theory, in the sense that not every element of physical reality had a counterpart in the theory.

This problem led to the search of a "complete theory" by adding hidden variables to the wave function in order to implement realism. For a long time, there was a general belief among quantum physicists that quantum mechanics can not be replaced by some complete theory (HVT) due to Von Neumann's impossibility proof (who imposed an unwarranted constraint on HVT). But in sixties we got two theorems due to J. S. Bell (Bell, 1964) (Bell, 1966) and Kochen and Specker (Kochen & Specker, 1967). These theorems showed that quantum mechanics can

not be replaced by some classes of HVT, namely local and non-contextual HVT. The most celebrated of this kind of HVT was presented by Bohm in 1952 (Bohm, 1952). Bohm, just prior to developing his HV interpretation, introduced a simplified scenario involving two spin-half particles with correlated spins, rather than two particles with correlated positions and momenta as used by EPR. The EPR-Bohm scenario has the advantage of being experimentally accessible.

In 1964 John Bell (Bell, 1964) derived an inequality (which is a statistical result, and is called Bell's inequality BI) using locality and reality assumptions of EPR-Bohm, and showed that the singlet state of two spin-1/2 particles violates this inequality, and hence the contradiction with quantum mechanics.

Contemporary versions of the argument are based on the Clauser, Horne, Shimony and Holt (CHSH) inequality(Clauser et al., 1969), rather than the original inequality used by Bell. There is a very good reason for that. While Bell's argument applied only to the singlet state, the CHSH inequality is violated by all pure entangled states (Gisin & Peres, 1992). Early versions of CHSH inequalities involved only two observers, each one having a choice of two (mutually incompatible) experiments. The various outcomes of each experiment were lumped into two sets, arbitrarily called $+1$ and -1. Possible generalizations involve more than two observers, or more than two alternative experiments for each observer, or more than two distinct outcomes for each experiment. We may consider n-partite systems, each subject to a choice of m v-valued measurements. This gives a total of $(mv)^n$ experimentally accessible probabilities. The set of Bell inequalities is then the set of inequalities that bounds this region of probabilities to those accessible with a local hidden variable model. Thus for each value of n, m and v the set of local realistic theories is a polytopes bounded by a finite set of linear Bell inequalities. The CHSH inequalities apply to a situation $(n, m, v) = (2, 2, 2)$. Gisin *et al* (Gisin & Bechmann-Pasquinucci, 1998) have found a family of Bell inequalities for the case with the number of measurements is arbitrary, i.e. $(n, m, v) = (2, m, 2)$. Collins et al (Collins, Gisin, Linden, Massar & Popescu., 2002) and Kaszlikowski et al (Kaszlikowski et al., n.d.) have produced inequalities for arbitrarily high dimensional systems, i.e. $(n, m, v) = (2, 2, v)$. The most complete study of Bell inequalities is for the case $(n, m, v) = (n, 2, 2)$. n-particle generalizations of the CHSH inequality were first proposed by Mermin (Mermin, 1990), and Belinskii and Klyshko (Belinskii & Klyshko, 1993), and have been extended by Werner and Wolf (Werner & Wolf, 2000), and Zukowski and Brukner (Zuckowski & Brukner, 2002) to give the complete set for two dichotomic observables per site.

On the theoretical side, " violation of Bell's inequalities" had become synonymous with "non-classical correlation", i.e., entanglement. One of the first papers in which finer distinctions were made was the construction of states with the property that they satisfy all the usual assumptions leading to the Bell inequalities, but can still not be generated by a purely classical mechanism (are not "separable" in modern terminology) (Werner, 1989). This example pointed out a gap between the obviously entangled states (violating a Bell inequality) and the obviously non-entangled ones, which are merely classical correlated (separable). In 1995 Popescu (Popescu, 1995) (and later (Bennett et al., 1996)) narrowed this gap considerably by showing that after local operations and classical communication one could "distill" entanglement, leading once again to violations, even from states not violating any Bell inequality initially. To summarize this phase: it became clear that violations of Bell inequalities, while still a good indicator for the presence of non-classical correlations by no means capture all kinds of "entanglement".

Bell inequalities are statistical predictions about measurements made on two particles,

typically photons or particles with spin $\frac{1}{2}$. So some people were trying to show a direct contradiction (which is not a statistical one) of quantum mechanics with local realism. Greenberger, Horne and Zeilinger (GHZ)(Greenberger et al., 1990) found a way to show more immediately, without inequalities, that results of quantum mechanics are inconsistent with the assumptions of EPR. It focuses on just one event, not the statistics of many events. Their proof relies on *eight* dimensional Hilbert space, unlike the case of Bell's theorem, which is valid in *four* dimensions. Heywood and Redhead (Heywood & Redhead, 1983) have provided a direct contradiction (without inequalities) of quantum mechanics with local realism for a particular state of two spin-1 particles. Finally, Hardy (Hardy, 1992) gave a proof of non locality for two particles with spin $\frac{1}{2}$ that only requires a total of *four* dimensions in Hilbert space like Bell's proof but does not require inequalities. This was accomplished by considering a particular experimental setup consisting of two over-lapping Mach-Zehnder interferometers, one for positrons one for electrons, arranged so that if the electron and positron each take a particular path then they will meet and annihilate one another with probability equal to 1. This arrangement is required to produce assymetric entangled state which only exhibits non locality without any use of inequality. The argument has been generalized to two spin s particles by Clifton and Niemann (Clifton & Niemann, 1992) and to N spin half particles by Pagonis and Clinton (Pagonis & Clifton, 1992).

Later, Hardy showed that this kind of non locality argument can be made for almost all entangled states of two spin-$\frac{1}{2}$ particles except for maximally entangled one (Hardy, 1993). This proof was again simplified by Goldstein (Goldstein, 1994) and extended it to the case of bipartite systems whose constituents belong to Hilbert spaces of arbitrary dimensions.

Conceptually, as well as mathematically, space and time are differently described in quantum mechanics. While time enters as an external parameter in the dynamical evolution of a system, spatial coordinates are regarded as quantum mechanical observables. Moreover, spatially separated quantum systems are associated with the tensor product structure of the Hilbert state-space of the composite system. This allows a composite quantum system to be in a state that is not separable regardless of the spatial separation of its components. We speak about *entanglement in space*. On the other hand, time in quantum mechanics is normally regarded as lacking such a structure. Because of different roles time and space play in quantum theory one could be tempted to assume that the notion of *"entanglement in time"* cannot be introduced in quantum physics.

In this chapter we propose and analyze a particular scenario to account for the deviations of QM from 'realism' (defined below), which involves correlations in the outputs of successive measurements of noncommuting operators on a spin-*s* state. The correlations for successive measurements have been used previously by Popescu (Popescu, 1995) in the context of nonlocal quantum correlations, in order to analyze a class of Werner states which are entangled but do not break (bipartite) Bell-type inequality. Although local HVT can simulate the quantum correlations between the outputs of single ideal measurement on each part of the system, it fails to simulate the correlations of the second measurements on each part. Leggett and Garg have used consecutive measurements to challenge the applicability of QM to macroscopic phenomena (Leggett & Garg, 1985). While the temporal Bell inequalities, considered in refs. (Leggett & Garg, 1985) (see also (Paz & Mahler, 1993)), are for histories, we deal here with Bell-type inequalities with predetermined measurement values at different times. The temporal Bell inequalities deal with measurement of the same observable at different times, whereas we deal here with different measurements at different times. Finally there is a large literature on the problem of information of a quantum state that can be

obtained by measuring the same operator successively on a single system. The research in this area is elegantly summarized in (Alter & Yamamoto, n.d.). Bell-type inequality with successive measurements was first considered by (Brukner et al., 2004). They have derived CHSH-type inequality (Clauser et al., 1969) for two successive measurements on an arbitrary state of a single qubit and have shown that every such state would violate that inequality for proper choice of the measurement settings. They have shown that the quantum mechanical correlation for three successive measurements, for any single qubit input state is the product of two consecutive correlations each of which is the correlation of two consecutive measurements – a scenario quite uncommon for spatial correlations. As an application of their approach, they have used the correlations in two successive measurements to overcome the limitations in RAM of a computer to calculate a Boolean function whose input bits are supplied sequentially in time.

We consider and analyze the correlations between the outputs of successive measurements for a general spin S state as against the general qubit state. We show that, for $S > \frac{1}{2}$, the quantum mechanical correlation for three successive measurements is not a product of two successive correlations, that is, the correlations in two successive measurements. We show that for $S = \frac{1}{2}$, the correlation between the outputs of measurements from $n - k$ to n (last k out of n successive measurements) $k = 0, 1, \ldots, n - 1$, depend on the measurement prior to $n - k$, when k is even, while for odd k, these correlations are independent of the outputs of measurements prior to $n - k$. Further, we show that all qubit states break the Bell type inequalities corresponding to n successive measurements, where n is any finite number. Finally, we study Hardy's nonlocality arguments for the correlations between the outputs of n successive measurements for all s-spin measurements. We show that the maximum probability of success of Hardy's argument in the successive measurement is much higher than the spatial ones in a certain sense.

The chapter is organized as follows. In Section 2 we describe the basic scenario in detail. Section 3 formulates the implications of hidden variable theory (HVT) for this scenario in terms of Bell-type inequalities. Section 4 evaluates these inequalities for mixed input states of single spin-s system for two and three successive measurements (considering various values of s). Section 5 deals with n successive measurements on spin-1/2 system. Section 6 explains the logical structure of Hardy's argument on time locality and, we show that no time-local stochastic HVT (SHVT) can simultaneously satisfy Hardy's argument. Finally we conclude with summary and comments in Section 7. Mathematical details are relegated to Appendices A and B.

2. Basic scenario

Consider the following sequence of measurements. A quantum particle with spin s, prepared in the initial state ρ_0, is sent through a cascade of Stern-Gerlach (SG) measurements for the spin components along the directions given by the unit vectors $\hat{a}_1, \hat{a}_2, \hat{a}_3, \ldots, \hat{a}_n$ (i.e., measurement of observables of the form $\vec{S}.\hat{a}$, where $\vec{S} = (S_x, S_y, S_z)$ is the vector of spin angular momentum operators S_x, S_y, S_z and $\hat{a}$ is a unit vector from $\mathbb{R}^3$). Each measurement has $2s + 1$ possible outcomes. For the i-th measurement, we denote these outcomes (eigenvalues) by $\alpha_i \in \{s, s - 1, \ldots, -s\}$. We denote by $\langle \alpha_i \rangle$ the quantum mechanical (ensemble) average $\langle \vec{S} \cdot \hat{a}_i \rangle$, by $\langle \alpha_i \alpha_j \rangle$ the average $\langle (\vec{S} \cdot \hat{a}_i)(\vec{S} \cdot \hat{a}_j) \rangle$ etc.

Each of the $(2s + 1)^n$ possible outcomes, which one gets after performing n consecutive measurements, corresponds to a particular combination of the results of the measurements at previous $n - 1$ steps and the result of the measurement at the n-th step. The probability

of each of these $(2s + 1)^n$ outcomes is the joint probability for such combinations. Note that even though the spin observables $\vec{S} \cdot \hat{a}_1$, $\vec{S} \cdot \hat{a}_2$, ..., $\vec{S} \cdot \hat{a}_n$, whose measurements are being performed at times t_1, t_2, ..., t_n respectively (with $t_1 < t_2 < \ldots < t_n$) do not commute, above-mentioned joint probabilities for the outcomes are well defined because each of these spin observables act on different states (Fine, 1982) (Anderson et al., 2005) (Ballentine, 1990). We emphasize that this is the joint probability for the results of n actual measurements and not a joint probability distribution for hypothetical simultaneous values of n noncommuting observables. Moreover, various sub-beams (*i.e.*, wave functions) emerging from every Stern-Gerlach apparatus (corresponding to $(2s + 1)$ outcomes) in every stage of measurement are separated without any overlap or recombination between them. In other words, the eigen wave packet $\psi_{s-j,t_i,\hat{a}_i}(x)$, corresponding to the eigen value $s - j$ of the observable $\vec{S} \cdot \hat{a}_i$, measured at time t_i, will not have any part in the regions where the SG setups, for measurement of the observables $\vec{S} \cdot \hat{a}_{i+1,s}$, $\vec{S} \cdot \hat{a}_{i+1,s-1}$, ..., $\vec{S} \cdot \hat{a}_{i+1,s-j+1}$, $\vec{S} \cdot \hat{a}_{i+1,s-j-1}$, ..., $\vec{S} \cdot \hat{a}_{i+1,-s}$, are situated. We further assume that, between two successive measurements, the spin state does not change with time i.e. $\vec{S}$ commutes with the interaction Hamiltonian, if any. Also, throughout the string of measurements, no component (*i.e.*, sub-beam) is blocked. It is to be mentioned here that the time of each of the measurements are measured by a common clock.

3. Implications of HVT

HVT assumes that in every possible state of the system, all observables have well defined (sharp) values (Redhead, 1987). On the measurement of an observable in a given state, the value possessed by the observable in that state (and no other value) results. To gain compatibility with QM and the experiments, a set of 'hidden' variables is introduced which is denoted collectively by λ. For given λ, the values of all observables are specified as the values of appropriate real valued functions defined over the domain Λ of possible values of hidden variables. For the spin observable $\vec{S} \cdot \hat{a}$, we denote the value of $\vec{S} \cdot \hat{a}$ in the QM (spin) state $|\psi\rangle$ by α. Considered as a function, $\alpha : \Lambda \to R$, we represent the value of $\vec{S} \cdot \hat{a}$ when the hidden variables have the value λ by $\alpha(\lambda)$. More generally, we may require that a value of λ gives the probability density $p(\alpha|\lambda)$ over the values of α rather than specifying the value of α (stochastic HVT). We denote the probability density function for the hidden variables in the state $|\psi\rangle$ by ρ_ψ ($\rho_\psi(\lambda)d\lambda$ measures the probability that the collective hidden variable lies in the range λ to $\lambda + d\lambda$). Then the average value of $\vec{S} \cdot \hat{a}$ in the state $|\psi\rangle$ is

$$\langle \alpha \rangle = \int_\Lambda \alpha(\lambda)\rho_\psi(\lambda)d\lambda, \tag{1}$$

where the integration is over Λ defined above. In the general case (SHVT)

$$\langle \alpha \rangle = \int_\Lambda \alpha p(\alpha|\lambda)\rho_\psi(\lambda)d\lambda. \tag{2}$$

We now analyze the consequences of SHVT for our scenario. In general, the outputs of kth and lth experiments may be correlated so that,

$$p(\alpha_i, \hat{a}_k \& \alpha_j, \hat{a}_\ell) \neq p(\alpha_i; \hat{a}_k)p(\alpha_j; \hat{a}_l). \tag{3}$$

However, in SHVT we suppose that these correlations have a common cause represented by a stochastic hidden variable λ so that

$$p(\alpha_i, \hat{a}_k \& \alpha_j, \hat{a}_\ell | \lambda) = p(\alpha_i, \hat{a}_k | \lambda) p(\alpha_j, \hat{a}_l | \lambda). \tag{4}$$

As a consequence of equation (4), the probability $p(\alpha_i, \hat{a}_k | \lambda)$ obtained in a measurement ($\vec{S} \cdot \hat{a}_k$ say) performed at time t_k is independent of any other measurement ($\vec{S} \cdot \hat{a}_l$ say) made at some earlier or later time t_l. This is called locality in time (Leggett & Garg, 1985) (Brukner et al., 2004).

One should note that for a two dimensional quantum mechanical system, one can always assign values (deterministically or probabilistically) to the observables with the help of a HVT. Once the measurement is done, the system will be prepared in an output state (namely, an eigenstate of the observable), and the earlier HVT may or may not work to reproduce the values of the observables to be measured on that output state (prepared after the first measurement). We have considered possibility of existence of a HVT for every input qubit state which can reproduce the measurement outcomes of n successive measurements.

Equation (4) is the crucial equation expressing the fundamental implication of SHVT to the successive measurement scenario. We now obtain the Bell type inequalities from equation (4) which can be compared with QM. Here we assume that in HVT all probabilities corresponding to outputs of measurements account for the possible changes in the values of the observable being measured, (due to the interaction of the measuring device and the system), occurring in the previous measurements.

Now $\langle \alpha_i \alpha_j \rangle$ is the expectation value of obtaining the outcome α_i in the measurement of the observable $\vec{S} \cdot \hat{a}_i$ at time t_i as well as the outcome α_j in the measurement of the observable $\vec{S} \cdot \hat{a}_j$ at later time t_j. Due to the HVT, we must have (dropping $\hat{a}_k, \hat{a}_\ell$)

$$\langle \alpha_i \alpha_j \rangle = \int \rho(\lambda) E(\alpha_i, \alpha_j, \lambda) d\lambda, \tag{5}$$

where

$$E(\alpha_i, \alpha_j, \lambda) = \sum_{\alpha_i, \alpha_j} \alpha_i \alpha_j p(\alpha_i, \alpha_j | \lambda) = \sum_{\alpha_i} \alpha_i p(\alpha_i | \lambda) \sum_{\alpha_j} \alpha_j p(\alpha_j | \lambda)$$

$$= E(\alpha_i, \lambda) E(\alpha_j, \lambda) \tag{6}$$

by equation (4). Now let us consider the case of two successive measurements, with options $\hat{a}_1, \hat{a}_1'$ and $\hat{a}_2, \hat{a}_2'$ respectively for measuring spin components. In each run of the experiment, a random choice between $\{\hat{a}_1, \hat{a}_1'\}$ and $\{\hat{a}_2, \hat{a}_2'\}$ is made. Define θ_i ($i = 1, 1'$) to be the angle between $\hat{a}_i$ and the positive z-axis, θ_{ij} ($i = 1, 1'$ and $j = 2, 2'$) is the angle between $\hat{a}_j$ and $\hat{a}_i$. Using condition (6) and the result (Shimony, n.d.) (Jarrett, 1984)

$$-2s^2 \leq xy + xy' + x'y - x'y' \leq 2s^2, \quad x, y, x', y' \in \{-s, -s+1, \ldots, s-1, s\},$$

we obtain

$$-2s^2 \leq E(\alpha_1, \alpha_2, \lambda) + E(\alpha_1, \alpha_2', \lambda) + E(\alpha_1', \alpha_2, \lambda) - E(\alpha_1', \alpha_2', \lambda) \leq 2s^2. \tag{7}$$

Multiplying by $\rho(\lambda) d\lambda$ and integrating over Λ, we get the CHSH-type inequality (Clauser et al., 1969) (involving the hidden variable λ) corresponding to performing two successive measurements of spin-s observables on a spin-s initial state:

$$|\langle BI \rangle| = \frac{1}{2} |\langle \alpha_1 \alpha_2 \rangle + \langle \alpha_1 \alpha_2' \rangle + \langle \alpha_1' \alpha_2 \rangle - \langle \alpha_1' \alpha_2' \rangle| \leq s^2. \tag{8}$$

Similarly, using the algebraic fact

$$-2s^3 \leq xyz' + xy'z + x'yz - x'y'z' \leq 2s^3,$$

where

$$x, y, z, x', y', z' \in \{-s, -s+1, \ldots, s-1, s\}$$

and[1]

$$E(\alpha_i, \alpha_j, \alpha_k, \lambda) = E(\alpha_i, \lambda)E(\alpha_j, \lambda)E(\alpha_k, \lambda),$$

we can prove Mermin-Klyshko Inequality (MKI) (Mermin, 1990), (Belinskii & Klyshko, 1993) for three successive measurements,

$$|\langle MKI \rangle| = \frac{1}{2}|\langle \alpha_1 \alpha_2 \alpha_3' \rangle + \langle \alpha_1 \alpha_2' \alpha_3 \rangle + \langle \alpha_1' \alpha_2 \alpha_3 \rangle - \langle \alpha_1' \alpha_2' \alpha_3' \rangle| \leq s^3. \tag{9}$$

Let $|\langle MKI' \rangle| \leq s^3$, where $|\langle MKI' \rangle|$ is obtained from equation (9) by interchanging primes with non-primes in MKI. It is easily shown that

$$|\langle SI \rangle| = |\langle MKI \rangle + \langle MKI' \rangle| \leq |\langle MKI \rangle| + |\langle MKI' \rangle| \leq 2s^3. \tag{10}$$

This is the Svetlichny inequality (SI) (Svetlichny, 1987),(Seevinck & Svetlichny, 2002),(Collins, Gisin, Popescu, Roberts & Scarani, 2002).

For n successive measurements on spin s system, we define the MK polynomials recursively as follows:

$$M_1 = \alpha_1, \ M_1' = \alpha_1', \tag{11}$$

$$M_n = \frac{1}{2}M_{n-1}(\alpha_n + \alpha_n') + \frac{1}{2}M_{n-1}'(\alpha_n - \alpha_n'), \tag{12}$$

where M_n' are obtained from M_n by interchanging all primed and non-primed α's. The recursive relation (12) gives, for all $1 \leq k \leq n-1$ (Collins, Gisin, Popescu, Roberts & Scarani, 2002) ,(Cabello, 2002a):

$$M_n = \frac{1}{2}M_{n-k}(M_k + M_k') + \frac{1}{2}M_{n-k}'(M_k - M_k'). \tag{13}$$

In particular, we have

$$M_2 = BI = \frac{1}{2}(\alpha_1 \alpha_2 + \alpha_1' \alpha_2 + \alpha_1 \alpha_2' - \alpha_1' \alpha_2'), \tag{14}$$

$$M_3 = MKI = \frac{1}{2}(\alpha_1 \alpha_2 \alpha_3' + \alpha_1 \alpha_2' \alpha_3 + \alpha_1' \alpha_2 \alpha_3 - \alpha_1' \alpha_2' \alpha_3'). \tag{15}$$

We now show that in HVT,

$$|\langle M_n \rangle| \leq s^n. \tag{16}$$

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[1] This is obtained by using equation (4) and the similar argument as has been used in deriving equation (6).

First note that (16) is true for $n = 2, 3$ (equations (8), (9)). Suppose it is true for $n = k$ i.e. $Max|\langle M_k \rangle| = s^k$. Now

$$|\langle M_{k+1} \rangle| = \frac{1}{2}|\langle M_k \alpha_{k+1} \rangle + \langle M_k \alpha'_{k+1} \rangle + \langle M'_k \alpha_{k+1} \rangle - \langle M'_k \alpha'_{k+1} \rangle|.$$

Since HVT applies here we can use (4) to get

$$|\langle M_{k+1} \rangle| = \frac{1}{2}|\langle M_k \rangle (\langle \alpha_{k+1} \rangle + \langle \alpha'_{k+1} \rangle) + \langle M'_k \rangle (\langle \alpha_{k+1} \rangle - \langle \alpha'_{k+1} \rangle)|.$$

This implies, by induction hypothesis (and using the fact that $\max|\langle M_2 \rangle| = s^2$), that

$$\max |\langle M_{k+1} \rangle| = s \max |\langle M_k \rangle| = s^{k+1}.$$

This result is derived for n spin-s particles by Cabello (Cabello, 2002a).
We now define a quantity, denoted by η_n, which will be required later on. η_n, is the ratio between maximum $|\langle M_n \rangle|$ given by quantum correlation between n successive measurement's outputs and the maximal classical one,

$$\eta_n = \frac{max|\langle M_n \rangle_{QM}|}{s^n}. \tag{17}$$

4. Mixed input state for arbitrary spin

4.1 Two successive measurements (BI)
We first deal with the case when input state is a mixed state whose eigenstates coincide with those of $\vec{S} \cdot \hat{a}_0$ for some $\hat{a}_0$ whose eigenvalues we denote by $\alpha_0 \in \{-s, \cdots s\}$. For spin $1/2$ this is the most general mixed state because given any density operator ρ_0 for spin $1/2$ (corresponding to some point within the Bloch sphere), we can find an $\hat{a}_0$ such that the eigenstates of $\vec{S} \cdot, \hat{a}_0$ and ρ_0 coincide. However, for $s > 1/2$, our choice forms a restricted class of mixed states. We note that these are the only states accessible via SG experiments. Thus we have

$$\rho_0 = \sum_{\alpha_0} p_{\alpha_0} |\vec{S} \cdot \hat{a}_0, \alpha_0 \rangle \langle \vec{S} \cdot \hat{a}_0, \alpha_0|; \quad \left(\sum_{\alpha_0} p_{\alpha_0} = 1 \right) \tag{18}$$

After the first measurement along $\hat{a}_1$, the resulting state of the system is

$$\rho_1 = \sum_{\alpha_1} M^\dagger_{\alpha_1} \rho_0 M_{\alpha_1}, \tag{19}$$

where

$$M^\dagger_{\alpha_1} = M_{\alpha_1} = |\vec{S} \cdot \hat{a}_1, \alpha_1 \rangle \langle \vec{S} \cdot \hat{a}_1, \alpha_1|.$$

Now $\langle \alpha_1 \alpha_2 \rangle_{QM}$ is the expectation value (according to QM) that given the initial state ρ_0 (given in equation (18)), the 1st measurement along $\hat{a}_1$ will give rise to any value $\alpha_1 \in \{-s, -s + 1, \ldots, s-1, s\}$, and then, on the after-measurement state ρ_1 (given in equation (19)), if one

performs measurement along $\hat{a}_2$, one of the values $\alpha_2 \in \{-s, -s+1, \ldots, s-1, s\}$ will arise. So

$$\langle \alpha_1 \alpha_2 \rangle_{QM} = Tr(\rho_1 \vec{S} \cdot \hat{a}_1 \vec{S} \cdot \hat{a}_2) =$$
$$\sum_{\alpha_0 \alpha_1 \alpha_2} p_{\alpha_0} \alpha_1 \alpha_2 |\langle \vec{S} \cdot \hat{a}_0, \alpha_0 | \vec{S} \cdot \hat{a}_1, \alpha_1 \rangle|^2 |\langle \vec{S} \cdot \hat{a}_1, \alpha_1 | \vec{S} \cdot \hat{a}_2, \alpha_2 \rangle|^2. \tag{20}$$

Note that, since $\vec{S} \cdot \hat{a}_i$ are complete observables, all of whose eigenvalues are non degenerate, the probabilities factorize like those of a Markov chain (Beck & Graudenz, 1992). Every factor in (20) corresponds to the transition amplitude between two successive measurements. By equation (A.12), we get

$$\langle \alpha_1 \alpha_2 \rangle = \frac{1}{2} \cos \theta_{12} [A \cos^2 \theta_1 + B], \tag{21}$$

where

$$A = 3\chi - s(s+1), \quad B = s(s+1) - \chi, \quad \chi = \sum_{\alpha_0 = -s}^{+s} \alpha_0^2 p_{\alpha_0}.$$

It is to be noted that although A can have positive and negative values, B will always be positive. Moreover, for all $\theta \in [0, 2\pi]$, if $A \geq 0$, $A\cos^2\theta_1 + B$ is always positive and if $A < 0$, $A\cos^2\theta_1 + B \geq B + A = 2\chi \geq 0$. We now have the following expression for the quantity BI, appeared in equation (8):

$$BI = \frac{1}{4}\{(A\cos^2\theta_1 + B)(\cos\theta_{12} + \cos\theta'_{12}) + (A\cos^2\theta'_1 + B)(\cos\theta''_{12} - \cos\theta'''_{12})\}$$
$$= \frac{3\chi - s(s+1)}{4}\left\{\cos^2\theta_1(\cos\theta_{12} + \cos\theta'_{12}) + \cos^2\theta'_1(\cos\theta''_{12} - \cos\theta'''_{12})\right\}$$
$$+ \frac{s(s+1) - \chi}{4}\left\{(\cos\theta_{12} + \cos\theta'_{12}) + (\cos\theta''_{12} - \cos\theta'''_{12})\right\}, \tag{22}$$

where (according to Appendix A) θ_1 is the angle between $\hat{a}_0$ and $\hat{a}_1$, θ'_1 is the angle between $\hat{a}_0$ and $\hat{a}'_1$, θ_{12} is the angle between $\hat{a}_1$ and $\hat{a}_2$, θ'_{12} is the angle between $\hat{a}_1$ and $\hat{a}'_2$, θ''_{12} is the angle between $\hat{a}'_1$ and $\hat{a}_2$, θ'''_{12} is the angle between $\hat{a}'_1$ and $\hat{a}''_2$. We have used, in Eq (22), the expressions for A and B in terms of χ and s. Note that the second term in equation (22) (*i.e.*, the term with the factor $\frac{s(s+1)-\chi}{4}$) is similar to the expression for $\langle \alpha_1 \alpha_2 \rangle + \langle \alpha_1 \alpha'_2 \rangle + \langle \alpha'_1 \alpha_2 \rangle - \langle \alpha'_1 \alpha'_2 \rangle$ corresponding to the CHSH inequality (Clauser et al., 1969). And hence, its maximum value will occur when we choose all the four vectors $\hat{a}_1$, $\hat{a}'_1$, $\hat{a}_2$, $\hat{a}'_2$ on the same plane. But we also have to take care about maximization of the first term in equation (22) and that might require these four vectors to be on different planes. In order to resolve this issue, we now consider the spherical-polar co-ordinates (θ_1, ϕ_1), (θ'_1, ϕ'_1), (θ_2, ϕ_2), (θ'_2, ϕ'_2) of the vectors $\hat{a}_1$, $\hat{a}'_1$, $\hat{a}_2$, $\hat{a}'_2$ respectively, where $\theta_1, \theta'_1, \theta_2, \theta'_2 \in [0, \pi]$ and $\phi_1, \phi'_1, \phi_2, \phi'_2 \in [0, 2\pi]$. Then BI has the form

$$BI = \frac{1}{4}(A\cos^2\theta_1 + B)[\cos\theta_1(\cos\theta_2 + \cos\theta'_2) + \sin\theta_1\sin\theta_2\cos(\phi_1 - \phi_2)$$
$$+ \sin\theta_1\sin\theta'_2\cos(\phi_1 - \phi'_2)] + \frac{1}{4}(A\cos^2\theta'_1 + B)[\cos\theta'_1(\cos\theta_2 - \cos\theta'_2)$$
$$+ \sin\theta'_1\sin\theta_2\cos(\phi'_1 - \phi_2) - \sin\theta'_1\sin\theta'_2\cos(\phi'_1 - \phi'_2)]. \tag{23}$$

Here also the maximum value of $|BI|$ will occur when all the vectors $\hat{a}_1$, $\hat{a}_1'$, $\hat{a}_2$, $\hat{a}_2'$ lie on the same plane. This is obtained by:

$$\frac{\partial BI}{\partial \phi_1} = \frac{\partial BI}{\partial \phi_2} = \frac{\partial BI}{\partial \phi_1'} = \frac{\partial BI}{\partial \phi_2'} = 0 \Rightarrow \phi_1 = \phi_1' = \phi_2 = \phi_2'.$$

In that case, the maximum value of $|BI|$ will occur when $\theta_1' = \pi - \theta_1$, $\theta_2 = \pi/2$, $\theta_2' = 0$ and (correspondingly) the quantity

$$\eta_2 = \frac{|BI|}{s^2} = \left(\frac{1}{2s^2}\right)|(\sin\theta_1 + \cos\theta_1)|(A\cos^2\theta_1 + B) \tag{24}$$

is maximized over all possible values of θ_1 ($A\cos^2\theta_1 + B \geq 0$). If $\eta_2 > 1$, the correlations for two successive measurements violate the CHSH-type inequality (8), and hence a contradiction with the above-mentioned HVT. In fact $\frac{\partial \eta_2}{\partial \theta_1} = 0$ implies that

$$B\tan^3\theta_1 + (2A - B)\tan^2\theta_1 + (3A + B)\tan\theta_1 - (A + B) = 0. \tag{25}$$

Real roots (for $\tan\theta_1$) of this equation give values of θ_1 for which η_2 is maximum. The maximum value of η_2 is evaluated at these θ_1's.

We find that for $s = \frac{1}{2}$, $\chi = 1/4$ for all ρ_0, and so $A = 0$, $B = 1/4$. So, from equation (24), we have $\eta_2 = \sin\theta_1 + \cos\theta_1$. Therefore equation (25) becomes $\tan^3\theta_1 - \tan^2\theta_1 + \tan\theta_1 - 1 = 0$, whose only one real solution is $\tan\theta_1 = 1$. So $\theta_1 = \pi/4$ or $5\pi/4$. $\theta_1 = \pi/4$ gives the maximum possible value $\eta_2 = \sqrt{2} > 1$. Thus all possible spin-1/2 states break BI for (proper choices of) two successive measurements. This can be compared with the measurement correlations corresponding to measurement of spin observables on space-like separated two particles scenario where only the entangled pure states break BI while not all entangled mixed states break it (Werner, 1989).

From now on, we will use the range of values of the quantity $\xi \equiv \chi/s^2$ to identify the parametric region of the initial density matrix ρ_0 where the inequality (8) will be violated. Thus we see that for all spin-1/2 input states ρ_0, $\xi = 1$.

For a spin-1 system, we first consider all input states ρ_0 none of which have a contribution of $S_z = 0$ eigenstate. In this case $\chi = 1$, $A = B = 1$. So $\eta_2 = (1/2)(\sin\theta_1 + \cos\theta_1)(\cos^2\theta_1 + 1)$ and equation (25) takes the form $\tan^3\theta_1 + \tan^2\theta_1 + 4\tan^3\theta_1 - 2 = 0$. The only real root of this equation is $\tan\theta_1 \approx 0.433$. Thus the maximum possible value of η_2 is (using equation (24)) 1.2112 (approximately). Thus we see that all input spin-1 states ρ_0, none of which has a component along $|S_z = 0\rangle$, break BI (equation (8)) for proper choice of the observables.

Next, for $s = 1$, we consider the state ρ_0, for which $p_{\alpha_0=0} = 1$, i.e., $\rho_0 = |\vec{S} \cdot \hat{a}_0, 0\rangle\langle\vec{S} \cdot \hat{a}_0, 0|$. In this case, $\chi = 0$, and so $A = -2$, $B = 2$, $\xi = 0$. Then equation (25) takes the form $2\tan^3\theta_1 - 6\tan^2\theta_1 - 4\tan\theta_1 = 0$. It has three real solutions, which corresponds to $\theta_1 = 0$ (or π), 74.3165^o (approx.), 150.6836^o (approx.). The maximum possible value of η_2 occurs at $\theta_1 = 74.3165^o$, and the corresponding value is given by $\eta_2 \approx 1.1428$. Thus the state $\rho_0 = |\vec{S} \cdot \hat{a}_0, 0\rangle\langle\vec{S} \cdot \hat{a}_0, 0|$ breaks the BI (equation (8)).

For $s = 1$, when $0 < p_{\alpha_0=0} \equiv p_0$ (say) < 1, we have $\chi = 1 - p_0 = \xi$, $A = 1 - 3p_0 = 3\xi - 2$ and $B = 1 + p_0 = 2 - \xi$. We then have $\eta_2 = (1/2)|\sin\theta_1 + \cos\theta_1|\{(3\xi - 2)\cos^2\theta_1 + (2 - \xi)\}$ (by equation (24)), and equation (25) becomes $(2 - \xi)\tan^3\theta_1 + (7\xi - 6)\tan^2\theta_1 + 4(2\xi - 1)\tan\theta_1 - 2\xi = 0$. In this case, one can show numerically that the BI will break (i.e., $\eta_2 > 1$) if and only if either $0 < \xi < 0.33$ or $0.77 < \xi < 1$ (equivalently, either $0.67 < p_0 < 1$ or $0 < p_0 < 0.23$).

Thus we see that, when $s = 1$, only those input states ρ_0 (given in equation (18)) will break BI for each of which $p_{\alpha_0=0} \in [0, 0.23) \cup (0.67, 1]$. Next we consider the situations where $s > 1$. Note that, by definition (true for all s),

$$\zeta = (p_s + p_{-s}) + (p_{s-1} + p_{-s+1})\left(1 - \frac{1}{s}\right)^2 + (p_{s-2} + p_{-s+2})\left(1 - \frac{2}{s}\right)^2 + \ldots,$$

where $0 \le p_s, p_{-s}, p_{s-1}, p_{-s+1}, p_{s-2}, p_{-s+2}, \ldots \le 1$ and $\sum_{\alpha_0=-s}^{s} p_{\alpha_0} = 1$. Therefore, we must have $0 \le \zeta \le 1$. In this case, equations (24) and (25) respectively take the forms

$$\eta_2 = \frac{1}{2s^2}\,|\sin\theta_1 + \cos\theta_1|\left[\{3\zeta s^2 - s(s+1)\}\cos^2\theta_1 + \{s(s+1) - \zeta s^2\}\right], \tag{26}$$

$$\{s(s+1) - \zeta s^2\}\tan^3\theta_1 + \{7\zeta s^2 - 3s(s+1)\}\tan^2\theta_1 + \\ \{8\zeta s^2 - 2s(s+1)\}\tan\theta_1 - 2\zeta s^2 = 0. \tag{27}$$

Let us first consider the input states of the form

$$\rho_0 = p_s|\vec{S}\cdot\hat{a}_0, s\rangle\langle\vec{S}\cdot\hat{a}_0, s| + p_{-s}|\vec{S}\cdot\hat{a}_0, -s\rangle\langle\vec{S}\cdot\hat{a}_0, -s|, \tag{28}$$

with $p_s + p_{-s} = 1$ and $p_{s-1}, p_{-s+1}, p_{s-2}, p_{-s+2}, \ldots = 0$. Thus we see here that $\zeta = 1$. Also equations (26) and (27) have respectively been turned into the forms

$$\eta_2 = (1/2s)|\sin\theta_1 + \cos\theta_1|\{(2s-1)\cos^2\theta_1 + 1\}, \tag{29}$$

$$\tan^3\theta_1 + (4s-3)\tan^2\theta_1 + (6s-2)\tan\theta_1 - 2s = 0. \tag{30}$$

As here $s > 1$, therefore the last equation will have only one positive root and the other two roots will be complex. The positive root will correspond to an angle $\theta_1^{max}(s) \in (0, \pi/4)$ for which it can be shown that $\eta_2^{max}(s) \equiv \eta_2(\theta_1^{max}(s)) > 1$ for all $s > 1$. Hence, in this case, BI is violated. If ρ_0 has contribution from neither of the states corresponding to $\alpha_0 = \pm s$ (*i.e.*, $p_s = p_{-s} = 0$), we have

$$\zeta = (p_{s-1} + p_{-s+1})\left(1 - \frac{1}{s}\right)^2 + (p_{s-2} + p_{-s+2})\left(1 - \frac{2}{s}\right)^2 + \ldots \le \left(1 - \frac{1}{s}\right)^2.$$

From equation (24) it follows that

$$\eta_2 = \frac{1}{\sqrt{2}s}\,|\sin(\theta_1 + \pi/4)|\left[(s + 1 - s\zeta) + (3s\zeta - s - 1)\cos^2\theta_1\right]$$

$$< \frac{1}{\sqrt{2}s} \times 1 \times [(s + 1 - s\zeta) + (3s\zeta - s - 1) \times 1] = \sqrt{2}\zeta \le \sqrt{2}\left(1 - \frac{1}{s}\right)^2.$$

But the quantity $\sqrt{2}(1 - 1/s)^2$ is less than 1 for all $s = 1/2, 1, 3/2, \ldots, 6$. Therefore, for $s > 1$, if the initial state ρ_0 has contribution from neither of the states corresponding to $\alpha_0 = \pm s$, BI will be satisfied for all $s \le 6$. Thus we see that whenever $s \in \{3/2, 2, 5/2, \ldots, 6\}$, in order that ρ_0 violates BI, the associated quantity ζ must have values near 1. In table 1, we have given the ranges of values of ζ (obtained numerically) for which BI is violated, starting from $s = 1/2$. The case when $s \to \infty$ has also been considered in table 1.

s	ξ	s	ξ	s	ξ
$\frac{1}{2}$	$\xi = 1$	$\frac{5}{2}$	$0.847 \le \xi \le 1$	$\frac{9}{2}$	$0.858 \le \xi \le 1$
1	$0 \le \xi \le 0.33$ and $0.77 \le \xi \le 1$	3	$0.851 \le \xi \le 1$	5	$0.859 \le \xi \le 1$
$\frac{3}{2}$	$0.824 \le \xi \le 1$	$\frac{7}{2}$	$0.854 \le \xi \le 1$	$\frac{11}{2}$	$0.860 \le \xi \le 1$
2	$0.84 \le \xi \le 1$	4	$0.856 \le \xi \le 1$	6	$0.862 \le \xi \le 1$
				∞	$0.87 \le \xi \le 1$

Table 1. The ranges of ξ, for which BI is violated.

s	η_2	s	η_2	s	η_2
$\frac{1}{2}$	$\sqrt{2}$	$\frac{5}{2}$	1.1638	$\frac{9}{2}$	1.1538
1	1.2112	3	1.1599	5	1.1526
$\frac{3}{2}$	1.1817	$\frac{7}{2}$	1.1572	$\frac{11}{2}$	1.1517
2	1.17	4	1.1553	6	1.1509
				∞	1.143

Table 2. The maximum violation of BI for different spin values for two successive measurements.

The maximum violation of Bell inequality, characterized by η_2, decreases monotonically with s. Table 2 summarizes (obtained numerically) the maximum allowed value of η_2 for each s. We see from this table that for all spin values s, BI is broken. Note that there is a sharp decrease in η_2 from $s = \frac{1}{2}$ to $s = 1$, while η_2 decreases slowly as s increases from 1. A possible reason is that, for $s = 1/2$, all states break BI while for $s \ge 1$, only a fraction of spin states break it. We now consider a case where the initial state ρ_0^{max} (given in equation (28)) is contaminated by the maximally noisy state, resulting in the state

$$\rho(f) = (1 - f)\rho_0^{\max} + \frac{f}{2s + 1}I, \tag{31}$$

where the positive parameter f (≤ 1) is the probability of the noise contamination of the state $\rho_0^{\max}$. Proceeding as before (see equation (21)), we get

$$\langle \alpha_1 \alpha_2 \rangle = \frac{1}{2}\cos\theta_{12}[A'\cos^2\theta_1 + B'] \tag{32}$$

where

$$A' = (1 - f)(2s - 1)s; \quad B' = (1 - f)s + \frac{2}{3}f(s + 1)s,$$

which leads to

$$\eta_{noise} = \left(\frac{1}{2s^2}\right)(\sin\theta_1 + \cos\theta_1)(A'\cos^2\theta_1 + B'). \tag{33}$$

Using the maximization procedure (i.e., taking $\frac{\partial \eta_{noise}}{\partial \theta_1} = 0$), $\tan\theta_1$ for maximum η_{noise} is given by a real root of

$$B'\tan^3\theta_1 + (2A' - B')\tan^2\theta_1 + (3A' + B')\tan\theta_1 - (A' + B') = 0. \tag{34}$$

The range of f for which $\eta_{noise} > 1$ is tabulated in table 3. Note that for $s = \frac{1}{2}$ the state corresponding to $f = 1$ (the random mixture) also breaks BI! Of course we have already

s	f	s	f	s	f
$\frac{1}{2}$	$0 \leq f \leq 1$	$\frac{5}{2}$	$f < 0.287$	$\frac{9}{2}$	$f < 0.239$
1	$f < 0.696$	3	$f < 0.267$	5	$f < 0.234$
$\frac{3}{2}$	$f < 0.395$	$\frac{7}{2}$	$f < 0.254$	$\frac{11}{2}$	$f < 0.230$
2	$f < 0.321$	4	$f < 0.245$	6	$f < 0.227$
				∞	$f < 0.195$

Table 3. The range of the noise f over which BI is violated.

shown that for $s = \frac{1}{2}$, BI is broken for all states. This indicates that the notion of "classicality", compatible with the usual local HVT, is different in nature from the notion of classicality that would arise from the non-violation of BI here.

Table 3 answers the question, "what is the maximum fraction of noise that can be added to $\rho_0^{\max}$, which maximally breaks BI, so that the state has stronger than "classical correlations[2]?" We see that the corresponding fraction of noise (*i.e.*, for which BI is violated) decreases monotonically with s, or with the dimension of the Hilbert space. This may be compared with the results of Collins and Popescu (Collins & Popescu, 2001) who found that the nonlocal character of the correlations between the outcomes of measurements performed on entangled systems separated in space is robust in the presence of noise. They showed that, for any fraction of noise, by taking the Hilbert space of large enough dimension, one can find bipartite entangled states giving nonlocal correlations. These results have been obtained by considering two successive measurements on each part of the system. On the other hand, in the present case of successive measurements on the single spin state, we see that the fraction of noise that can be added so that the quantum correlations continue to break Bell inequality, falls off monotonically with s, or the dimension of the Hilbert space. For $s = \frac{1}{2}$ all fractions $f \leq 1$ are allowed, while for large s, $f < 0.195$.

4.2 Three successive measurements (MKI)

We again assume the input state to be given by equation (18). Using equation (A.19) :

$$\langle \alpha_1 \alpha_2 \alpha_3 \rangle = \frac{1}{16} \cos \theta_{23} \{ \cos \theta_1 [M \cos^2 \theta_{12} + N] + R[3 \cos^2 \theta_{12} - 1] \} \tag{35}$$

where

$$M = \sum_{\alpha_0 = -s}^{+s} p_{\alpha_0} \alpha_0 [9\alpha_0^2 + s(s+1) - 3],$$

$$N = \sum_{\alpha_0 = -s}^{+s} p_{\alpha_0} \alpha_0 [5s(s+1) - 3\alpha_0^2 + 1],$$

$$R = \sum_{\alpha_0 = -s}^{+s} p_{\alpha_0} \alpha_0 [5\alpha_0^2 - 3s(s+1) + 1],$$

θ_1 is the angle between $\hat{a}_0$ and $\hat{a}_1$ (measured with respect to the right-handed system $(\hat{a}_0, \hat{a}_1, (\hat{a}_0 \times \hat{a}_1)/|\hat{a}_0 \times \hat{a}_1|)$), θ_{12} is the angle between $\hat{a}_1$, $\hat{a}_2$ (measured with respect to the right-handed system $(\hat{a}_1, \hat{a}_2, (\hat{a}_1 \times \hat{a}_2)/|\hat{a}_1 \times \hat{a}_2|)$), etc.

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[2] *i.e.*, correlations obeying "realism" and "locality in time", as described in section 3.

s	η_3	s	η_3	s	η_3
$\frac{1}{2}$	$\sqrt{2}$	$\frac{5}{2}$	1.1736	$\frac{9}{2}$	1.1634
1	1.2178	3	1.1698	5	1.1621
$\frac{3}{2}$	1.1907	$\frac{7}{2}$	1.1670	$\frac{11}{2}$	1.1610
2	1.1793	4	1.1650	6	1.1601
				∞	1.1527

Table 4. The maximum violation of MKI for different spin values for three successive measurements.

We now consider the pure state $|\vec{S} \cdot \hat{a}_0, s\rangle\langle\vec{S} \cdot \hat{a}_0, s|$ instead of considering the most general state ρ_0, given in equation (18). So here $M = s(2s-1)(5s+3)$, $N = s(2s^2 + 5s + 1)$, and $R = s(2s-1)(s-1)$. Substituting the correlations like that in equation (35) in the MKI (given in equation (9)), using the above-mentioned values of M, N, R, and then finding out the conditions (numerically) for which $\eta_3 \equiv |MKI|/s^3$ is maximized, we get the maximum possible η_3-values for different spins as summarized in table 4.

We see that $\eta_3 > 1$ for all spins and $\eta_3 > \eta_2$ except $s = \frac{1}{2}$, while $\eta_3 = \eta_2 = \sqrt{2}$ for $s = 1/2$. Also η_3, like η_2, decreases monotonically with s. It is interesting, in the case of two and three successive measurements of spin s prepared in a pure state, the maximum violation of BI and MKI tends to a constant for arbitrary large s.

$$\eta_3(s \to \infty) = 1.153 \text{ (approx.)},$$

$$\eta_2(s \to \infty) = 1.143 \text{ (approx.)}.$$

It is thus seen that large quantum numbers do not guarantee "classical" (as defined in this chapter) behavior.

It is straightforward to check that, three successive measurements satisfy Svetlichny Inequality (SI) (equation (10)). The reason is that, for all s, the settings of the measurement directions which maximize MKI' are obtained from those which maximize MKI by interchanging primes on the corresponding unit vectors. Thus these two settings are incompatible so that we cannot get a single set of measurement directions, which maximize both MKI and MKI'. In fact, for all s, the measurement directions which maximize MKI (MKI') correspond to $MKI' = 0$ ($MKI = 0$).

We now consider the situation of three consecutive observations but two-fold correlations for two measurements $\vec{S}.\hat{a}_1$ and $\vec{S}.\hat{a}_3$ performed, say, at time t_1 and t_3, but where an additional measurement ($\vec{S}.\hat{a}_2$) is performed at time t_2 lying between t_1 and t_3 ($t_1 < t_2 < t_3$).
By substituting Eq (A.20) in Bell type inequality Eq(8) and simplifying, we obtain:

$$|BI| = \frac{1}{2}|[\cos\theta_{32} + \cos\theta_{3'2}]\langle\alpha_1\alpha_2\rangle + [\cos\theta_{32} - \cos\theta_{3'2}]\langle\alpha_1'\alpha_2\rangle|$$

$$\leq \frac{1}{2}|[\cos\theta_{32} + \cos\theta_{3'2}]||\langle\alpha_1\alpha_2\rangle| + |[\cos\theta_{32} - \cos\theta_{3'2}]||\langle\alpha_1'\alpha_2\rangle|$$

$$\leq \cos\theta_{32}s^2 \leq s^2. \tag{36}$$

We have used $max|\langle\alpha_1\alpha_2\rangle| = max|\langle\alpha_1'\alpha_2\rangle| = s^2$.
So, the correlation function (36) for a given measurement performed at t_2 cannot violate the Bell type inequality for measurements at t_1 and t_3. Therefore, any measurement performed at time t_2 "disentangles" events at time t_1 and t_3 if $t_1 < t_2 < t_3$ (Brukner et al., 2004).

5. n successive measurements for Spin-$\frac{1}{2}$

5.1 Violation Mermin-Klyshko Inequality (MKI)

We consider now n successive measurements in direction $\vec{S} \cdot \hat{a}_i$, $(i = 1, 2, 3, \ldots, n)$ on a spin $s = \frac{1}{2}$ particle in a mixed state. For simplicity we take the eigenvalues to be $\alpha_k = \pm 1$, *i.e.*, the eigenvalues of σ_z are taken here as ± 1 instead of $\pm (1/2)$. We also write $|\alpha_k\rangle$ for $|\vec{S} \cdot \hat{a}_k, \alpha_k\rangle$. The initial state is taken as

$$\rho_0 = p_{+1}|\alpha_0 = +1\rangle\langle\alpha_0 = +1| + p_{-1}|\alpha_0 = -1\rangle\langle\alpha_0 = -1|. \tag{37}$$

For a spin-$\frac{1}{2}$ system, we have

$$|\langle\alpha_{k-1}|\alpha_k\rangle|^2 = \frac{1}{2}(1 + \alpha_{k-1}\alpha_k \cos\theta_{k-1,k}) \tag{38}$$

where $\cos\theta_{k-1,k} = \hat{a}_{k-1} \cdot \hat{a}_k$ for $k = 1, 2, \ldots, n$. So, given the input state $|\alpha_0\rangle$, the (joint) probability that the measurement outcomes will be $\alpha_1 \in \{+1, -1\}$ in the first measurement, $\alpha_2 \in \{+1, -1\}$ in the second measurement, ..., $\alpha_n \in \{+1, -1\}$ in the n-th measurement, will be given by

$$p(\alpha_1, \alpha_2, \cdots, \alpha_n) = \frac{1}{2^n}\prod_{i=1}^{n}(1 + \alpha_{i-1}\alpha_i \cos\theta_{i-1,i}). \tag{39}$$

Thus we see that given the input state $\rho_0 = \sum_{\alpha_0 = \pm 1} p_{\alpha_0}|\alpha_0\rangle\langle\alpha_0|$, the average output state after n successive measurements will be given by

$$\rho_n = \sum_{\alpha_0, \alpha_1, \ldots, \alpha_n = \pm 1} p_{\alpha_0} p(\alpha_1, \alpha_2, \ldots, \alpha_n)|\alpha_n\rangle\langle\alpha_n|.$$

Then, for n successive measurements on spin-1/2 system,

$$\begin{aligned}
\langle\alpha_{n-1}\alpha_n\rangle_{QM} &= \sum_{\alpha_{n-1}, \alpha_n = \pm 1} \alpha_{n-1}\{\text{coeff. of } |\alpha_{n-1}\rangle\langle\alpha_{n-1}| \text{ in } \rho_{n-1}\}\alpha_n |\langle\alpha_{n-1}|\alpha_n\rangle|^2 \\
&= \sum_{\alpha_0 = \pm 1} p_{\alpha_0} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_n = \pm 1} \alpha_{n-1}\alpha_n p(\alpha_1, \alpha_2, \cdots, \alpha_n) \\
&= \sum_{\alpha_0 = \pm 1} p_{\alpha_0} 2^{-n} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_n = \pm 1} \prod_{i=1}^{n} \alpha_{n-1}\alpha_n(1 + \alpha_{i-1}\alpha_i \cos\theta_{i-1,i}) \\
&= \cos\theta_{n-1,n} \tag{40}
\end{aligned}$$

by equation (39). Further

$$\begin{aligned}
\langle\alpha_n\rangle_{QM} &= \sum_{\alpha_n = \pm 1} \alpha_n \{\text{coeff. of } |\alpha_n\rangle\langle\alpha_n| \text{ in } \rho_n\} \\
&= \sum_{\alpha_0 = \pm 1} p_{\alpha_0} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_n = \pm 1} \alpha_n p(\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&= \sum_{\alpha_0 = \pm 1} p_{\alpha_0} 2^{-n} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_n = \pm 1} \prod_{i=1}^{n} \alpha_n(1 + \alpha_{i-1}\alpha_i \cos\theta_{i-1,i}) \\
&= (p_{+1} - p_{-1})\cos\theta_1 \cos\theta_{12} \cdots \cos\theta_{n-1,n} \tag{41}
\end{aligned}$$

where $\theta_1 \equiv \theta_{0,1}$, $\theta_{12} \equiv \theta_{1,2}$, etc. Now equations (40) and (41) give,

$$\langle \alpha_n \rangle_{QM} = \langle \alpha_1 \rangle_{QM} \langle \alpha_2 \alpha_3 \rangle_{QM} \cdots \langle \alpha_{n-1} \alpha_n \rangle_{QM}. \tag{42}$$

Further,

$$\langle \alpha_{n-k} \cdots \alpha_n \rangle_{QM} =$$
$$\sum_{\alpha_0} p_{\alpha_0} 2^{-n} \sum_{\alpha_1, \alpha_2, \dots, \alpha_n = \pm 1} \prod_{i=1}^{n} (\alpha_{n-k} \cdots \alpha_n)(1 + \alpha_{i-1} \alpha_i \cos \theta_{i-1,i}) =$$
$$\begin{cases} \langle \alpha_1 \rangle_{QM} \langle \alpha_2 \alpha_3 \rangle_{QM} \cdots \langle \alpha_{n-1} \alpha_n \rangle_{QM} & k \text{ even} \\[2mm] \langle \alpha_{n-k} \alpha_{n-k+1} \rangle_{QM} \langle \alpha_{n-k+2} \alpha_{n-k+3} \rangle_{QM} \cdots \langle \alpha_{n-1} \alpha_n \rangle_{QM} & k \text{ odd} \end{cases}$$
$$\tag{43}$$

All of the above results are inherently quantum and are not compatible with HVT (see the discussion in the next paragraph). The first two results ((41) and (42)) are the special cases of the last result (43) for $k = 1$ and $k = 0$ (with $\alpha_0 = 1$). If the number of variables (which are averaged) is odd (i.e. k is even) the average depends on the measurements prior to $(n - k)$, while in the other case the average does not depend on the measurements prior to $(n - k)$. For example, for two successive measurements (taking $n = 2$ and $k = 1$), gives $\langle \alpha_1 \alpha_2 \rangle = \cos \theta_{12}$, which is independent of the initial state. On the other hand, for three successive measurements (taking $n = 3$ and $k = 2$), we have $\langle \alpha_1 \alpha_2 \alpha_3 \rangle = \langle \alpha_1 \rangle \langle \alpha_2 \alpha_3 \rangle$ – showing its dependence on the initial state (as $\langle \alpha_1 \rangle = (p_{+1} - p_{-1}) \cos \theta_1$ depends upon the initial state $\rho_0 = \sum_{\alpha_0 = \pm 1} p_{\alpha_0} |\alpha_0\rangle \langle \alpha_0|$). Moreover, the correlation $\langle \alpha_1 \alpha_2 \alpha_3 \alpha_4 \rangle_{QM}$ for four successive measurements (for example) turns out to be dependent only on the two 'disjoint' correlations $\langle \alpha_1 \alpha_2 \rangle_{QM}$ and $\langle \alpha_3 \alpha_4 \rangle_{QM}$. In general, we have:

$$\langle \alpha_1 \alpha_2, \dots, \alpha_{2p} \rangle = \langle \alpha_1 \alpha_2 \rangle \langle \alpha_3 \alpha_4 \rangle \dots \langle \alpha_{2p-1} \alpha_{2p} \rangle \tag{44}$$

and

$$\langle \alpha_1 \alpha_2, \dots, \alpha_{2p+1} \rangle = \langle \alpha_1 \rangle \langle \alpha_2 \alpha_3 \rangle \dots \langle \alpha_{2p} \alpha_{2p+1} \rangle. \tag{45}$$

Interestingly if $\hat{a}_0 \perp \hat{a}_1$ so that $\cos \theta_1 = 0$ (and so, $\langle \alpha_1 \rangle_{QM} = 0$) or, if the initial state is the random mixture $(1/2) \sum_{\alpha_0 = \pm 1} |\alpha_0\rangle \langle \alpha_0|$ (and so $\langle \alpha_1 \rangle = 0$), then for all even k,

$$\langle \alpha_{n-k} \cdots \alpha_n \rangle_{QM} = 0$$

and so

$$\langle \alpha_1 \alpha_2 \cdots \alpha_{n=2p+1} \rangle_{QM} = 0.$$

We shall now show that for n successive experiments (with $n > 1$), QM violates the inequality $|\langle MKI \rangle| \leq s^3$ (see equation (9)) up to $\sqrt{2}$ for $s = 1/2$ systems. We take the eigenvalues to be $\alpha_k = \pm 1$ so $|\langle M_k \rangle|_{HVT} \leq 1$). We have already shown that for $n = 2$ and $n = 3$, the corresponding MKI's are violated (section 4). Now, we know that temporal two-fold correlations $\langle \alpha_{k-1} \alpha_k' \rangle$, $\langle \alpha_{k-1}' \alpha_k \rangle$, $\langle \alpha_{k-1} \alpha_k \rangle$ and $\langle \alpha_{k-1}' \alpha_k' \rangle$ are independent on the previous measurements. So by using equations (43) and (13) we find that

$$|\langle M_k \rangle| = \frac{1}{2} |\langle M_{k-2} \rangle [\langle \alpha_{k-1} \alpha_k' \rangle + \langle \alpha_{k-1}' \alpha_k \rangle] + \langle M_{k-2}' \rangle [\langle \alpha_{k-1} \alpha_k \rangle - \langle \alpha_{k-1}' \alpha_k' \rangle]|.$$
$$\tag{46}$$

We now consider the spherical-polar co-ordinates $(\theta_{k-1}, \phi_{k-1})$, $(\theta'_{k-1}, \phi'_{k-1})$, (θ_k, ϕ_k), (θ'_k, ϕ'_k) of the vectors $\hat{a}_{k-1}$, $\hat{a}'_{k-1}$, $\hat{a}_k$, $\hat{a}'_k$ respectively, where all $\theta \in [0, \pi]$ and all $\phi \in [0, 2\pi]$. Then $|\langle M_k \rangle|$ has the form

$$
\begin{aligned}
|\langle M_k \rangle| = \frac{1}{2}|\langle M_{k-2} \rangle [&\cos\theta_{k-1}\cos\theta'_k + \sin\theta_{k-1}\sin\theta'_k \cos(\phi_{k-1} - \phi'_k) \\
&+ \cos\theta'_{k-1}\cos\theta_k + \sin\theta'_{k-1}\sin\theta_k \cos(\phi'_{k-1} - \phi_k)] \\
&+ \langle M'_{k-2} \rangle [\cos\theta_{k-1}\cos\theta_k + \sin\theta_{k-1}\sin\theta_k \cos(\phi_{k-1} - \phi_k) \\
&- \cos\theta'_{k-1}\cos\theta'_k - \sin\theta'_{k-1}\sin\theta'_k \cos(\phi'_{k-1} - \phi'_k)]|.
\end{aligned}
\tag{47}
$$

We know from

$$
\frac{\partial |\langle M_k \rangle|}{\partial \phi_{k-1}} = \frac{\partial |\langle M_k \rangle|}{\partial \phi_k} = \frac{\partial |\langle M_k \rangle|}{\partial \phi'_{k-1}} = \frac{\partial |\langle M_k \rangle|}{\partial \phi'_k} = 0 \Rightarrow \phi_{k-1} = \phi'_{k-1} = \phi_k = \phi'_k.
$$

The maximum value of $|\langle M_k \rangle|$ will occur when all the vectors $\hat{a}_{k-1}$, $\hat{a}'_{k-1}$, $\hat{a}_k$, $\hat{a}'_k$ lie on the same plane. We obtain:

$$
\begin{aligned}
|\langle M_k \rangle| &\leq \frac{1}{2}|\langle M_{k-2} \rangle [\cos(\theta_{k-1} - \theta'_k) + \cos(\theta'_{k-1} - \theta_k)] \\
&\quad + \langle M'_{k-2} \rangle [\cos(\theta_{k-1} - \theta_k) - (\cos\theta'_{k-1} - \theta'_k)]| \\
&\leq \frac{1}{2}|\langle M_{k-2} \rangle||[\cos(\theta_{k-1} - \theta'_k) + \cos(\theta'_{k-1} - \theta_k)] \\
&\quad + \frac{1}{2}|\langle M'_{k-2} \rangle||[\cos(\theta_{k-1} - \theta_k) - (\cos\theta'_{k-1} - \theta'_k)]|.
\end{aligned}
\tag{48}
$$

By substituting $x = \theta_{k-1} - \theta'_k$, $y = \theta'_{k-1} - \theta_k$, $z = \theta_{k-1} - \theta_k$ and $\theta'_{k-1} - \theta'_k = x + y - z$ in above-equation and by using

$$
\frac{\partial |\langle M_k \rangle|}{\partial x} = \frac{\partial |\langle M_k \rangle|}{\partial y} = \frac{\partial |\langle M_k \rangle|}{\partial z} = 0,
$$

we get $x = y = -z = \pi/4$. Finally, by using the fact $|\langle M_k \rangle| + |\langle M'_k \rangle| \leq 2$, we obtain:

$$
|\langle M_k \rangle| \leq \frac{\sqrt{2}}{2}\{|\langle M_{k-2} \rangle + \langle M'_{k-2} \rangle|\} \leq \sqrt{2}.
\tag{49}
$$

One can get this result by induction hypothesis. Thus, we conclude that QM violates the MKI ($|\langle M_n \rangle| \leq 1$) for n successive measurements upto $\sqrt{2}$, *i.e.*,

$$
\eta_n = \sqrt{2}.
\tag{50}
$$

5.2 Violation Scarani-Gisin inequality (SCI)

Although in contrast to correlations in space there are no genuine multi-mode correlations in time, we will see that temporal correlations can be stronger than spatial ones in a certain sense. We denote by $\max[B_{QM}^{space}(i,j)]$ the maximal value of the Bell expression for qubits i and j (Bell inequality is obtained by Eq(8)). Scarani and Gisin (Scarani & Gisin, 2001) found an interesting bound that holds for arbitrary state of three qubits:

$$
\max[B_{QM}^{space}(1,2)] + \max[B_{QM}^{space}(2,3)] \leq 2.
\tag{51}
$$

Physically, this means that no two pairs of qubits of a three-qubit system can violate the CHSH inequalities simultaneously. This is because if two systems are highly entangled, they can not be entangled highly to another systems. Let us denote by $\max[B_{QM}^{time}(i,j)]$ the maximal value of the Bell expression for two consecutive observations of a single qubit at times i and j. Since quantum correlations between two successive measurements do not depend on the initial state (see Eq (40)), one can obtain:

$$\left[B_{QM}^{time}(k-1,k)\right] + [B_{QM}^{time}(k,k+1)] =$$

$$\frac{1}{2}\left[\cos\theta_{k-1,k} + \cos\theta_{k-1,k'} + \cos\theta_{k-1',k} - \cos\theta_{k-1',k'}\right] +$$

$$\frac{1}{2}\left[\cos\theta_{k,k+1} + \cos\theta_{k,k+1'} + \cos\theta_{k',k+1} - \cos\theta_{k',k+1'}\right]. \tag{52}$$

By selecting,

$$\theta_{k-1,k} = \theta_{k-1,k'} = \theta_{k-1',k} = \theta_{k,k+1} = \theta_{k,k+1'} = \theta_{k',k+1} = \frac{\pi}{4}$$

and

$$\theta_{k-1',k'} = \theta_{k',k+1'} = \frac{3\pi}{4},$$

we obtain:

$$\max\left[B_{QM}^{time}(k-1,k)\right] + \max[B_{QM}^{time}(k,k+1)] = \sqrt{2} + \sqrt{2} = 2\sqrt{2} > 2. \tag{53}$$

Thus, although there are no genuine three-fold temporal correlations, a specific combination of two-fold correlations can have values that are not achievable with correlations in space for any three-qubit system. In fact, one would need two pairs of maximally entangled two-qubit states to achieve the bound in (53). Also note that the local realistic bound is 2, which is equal to the bound in (51). Similar conclusion can be obtained for the sum of n successive measurements.

$$\max\left[B_{QM}^{time}(1,2)\right] + \max\left[B_{QM}^{time}(2,3)\right] + \ldots + \max\left[B_{QM}^{time}(n-1,n)\right] = n\sqrt{2} > n. \tag{54}$$

5.3 Violation chained Bell inequalities (CHI)

Generalized CHSH inequalities may be obtained by providing more than two alternative experiments to each process. We consider two successive measurements on a spin-$\frac{1}{2}$ particle in a mixed state, such that the first experiment can measure spin component along one of the directions $\hat{a}_1, \hat{a}_3, \ldots, \hat{a}_{2n-1}$ and the second experiment along one of the directions $\hat{b}_2, \hat{b}_4, \ldots, \hat{b}_{2n}$. The results of these measurements are called α_r ($r = 1, 3, \ldots, 2n-1$) and β_s ($s = 2, 4, \ldots, 2n$), respectively, and their values are ± 1 (in unit if $\hbar/2$). We have a generalized CHSH inequality (Braunstein & Caves, 1990),(Peres, 1993):

$$CBI = \frac{1}{2}\left|\langle\alpha_1\beta_2\rangle + \langle\beta_2\alpha_3\rangle + \langle\alpha_3\beta_4\rangle + \ldots + \langle\alpha_{2n-1}\beta_{2n}\rangle - \langle\beta_{2n}\alpha_1\rangle\right| \leq n-1$$

$$\tag{55}$$

This upper bound is violated by quantum correlations in two successive measurements, increasingly with larger n. In order to obtain the maximum value above-inequality, we consider the spherical-polar co-ordinates (θ_k, ϕ_k),($k = 1, 3, \ldots, 2n-1$) of the vectors

$\hat{a}_1, \hat{a}_3, \ldots, \hat{a}_{2n-1}$ and ($k = 2, 4, \ldots, 2n$) for vectors $\hat{b}_2, \hat{b}_4, \ldots, \hat{b}_{2n}$. The maximum value $|CBI|$ will occur when all the vectors lie on the same plane. this is because of:

$$\frac{\partial(CBI)}{\partial \phi_k} = 0 \Rightarrow \phi_1 = \phi_2 = \ldots = \phi_n.$$

After partial differential over all θ_k, we get:

$$\frac{\partial(CBI)}{\partial \theta_k} = 0 \Rightarrow \theta_{12} = \theta_{23} = \ldots = \theta_{2n-1,2n} = \theta.$$

Therefore, we obtain:

$$CBI = (2n - 1)\cos\theta - \cos(2n - 1)\theta. \tag{56}$$

So,

$$\frac{\partial(CBI)}{\partial \theta} = 0 \Rightarrow \theta = \pi/2n. \tag{57}$$

By substituting , we obtain:

$$CBI = (2n - 1)\cos\frac{\text{ß}}{2n} - \cos\frac{(2n - 1)\text{ß}}{2n} = 2n\cos\frac{\text{ß}}{2n} \tag{58}$$

We know $\cos(\frac{\pi}{2n})$ tends to $(1 - \frac{\pi^2}{8n^2})$ for $n \longrightarrow \infty$. Therefore the maximum CBI can be made arbitrarily close to $2n$.

5.4 Violation Bell inequalities involving Tri and Bi-measurements correlations

It would be interesting to consider Bell inequalities involving both two and three successive measurement correlations. The simplest way of obtaining such an inequality would be by adding genuinely bipartite correlations to the tripartite correlations considered in Mermin's inequality. For instance, a straightforward calculation would allow us to prove that any local realistic theory must satisfy the following inequality (Cabello, 2002b):

$$-5 \leq \langle \alpha_1 \alpha_2 \alpha_3' \rangle - \langle \alpha_1 \alpha_2' \alpha_3' \rangle - \langle \alpha_1' \alpha_2 \alpha_3' \rangle - \langle \alpha_1' \alpha_2' \alpha_3 \rangle -$$
$$\langle \alpha_1 \alpha_2' \rangle - \langle \alpha_1 \alpha_3' \rangle - \langle \alpha_2 \alpha_3 \rangle \leq 3 \tag{59}$$

A numerical calculation shows that both the GHZ and W states give a same maximal violation of the inequality (59). However, if we assign a higher weight to the bipartite correlations appearing in the inequality, then we can reach a Bell inequality such as

$$-8 \leq \langle \alpha_1 \alpha_2 \alpha_3' \rangle - \langle \alpha_1 \alpha_2' \alpha_3' \rangle - \langle \alpha_1' \alpha_2 \alpha_3' \rangle - \langle \alpha_1' \alpha_2' \alpha_3 \rangle -$$
$$2\langle \alpha_1 \alpha_2' \rangle - 2\langle \alpha_1 \alpha_3' \rangle - 2\langle \alpha_2 \alpha_3 \rangle \leq 4, \tag{60}$$

which is violated by the W state but not by GHZ state (Cabello, 2002b). It is not difficult to show that three successive measurements correlations for spin 1/2 break the hybrid Bell inequalities.

$$-5.34 \leq \langle \alpha_1 \alpha_2 \alpha_3' \rangle - \langle \alpha_1 \alpha_2' \alpha_3' \rangle - \langle \alpha_1' \alpha_2 \alpha_3' \rangle - \langle \alpha_1' \alpha_2' \alpha_3 \rangle -$$
$$\langle \alpha_1 \alpha_2' \rangle - \langle \alpha_1 \alpha_3' \rangle - \langle \alpha_2 \alpha_3 \rangle \leq 3.8 \tag{61}$$

and

$$-8.2 \leq \langle \alpha_1 \alpha_2 \alpha_3' \rangle - \langle \alpha_1 \alpha_2' \alpha_3' \rangle - \langle \alpha_1' \alpha_2 \alpha_3' \rangle - \langle \alpha_1' \alpha_2' \alpha_3 \rangle -$$
$$2\langle \alpha_1 \alpha_2' \rangle - 2\langle \alpha_1 \alpha_3' \rangle - 2\langle \alpha_2 \alpha_3 \rangle \leq 4.8. \tag{62}$$

So two successive measurements correlations are relevant to those of three successive measurements. This behavior is analogous to three particle W state (Cabello, 2002b).

6. Hardy's argument for n successive measurements for all spin-s measurements

Hardy's nonlocality argument is considered weaker than Bell inequalities in the bipartite case, as every maximally entangled state of two spin-$\frac{1}{2}$ particles violates Bell's inequality maximally but none of them satisfies Hardy-type nonlocality conditions. The scenario in successive spin measurements is quite different, however. We showed in previous sections that all n successive spin-s measurements break Bell-type inequalities, in contrast to the bipartite case, where only the entangled states break it. In this section, we prove that all n successive spin-s measurements satisfy Hardy-type argument conditions. Consider four yes/no-type events A, A', B and B', where A and A' may happen at time t_1, and B and B' may happen at another time, t_2 ($t_2 > t_1$). The joint probability that, at the first time (t_1), A and, at the second time (t_2), B are " yes" is 0. The joint probability that, at the first time (t_1), A is "no" and, at the second time (t_2), B' is "yes" is 0. The joint probability that, at the first time (t_1), A' is "yes" and, at the second time (t_2), B is "no", is 0. The joint probability that both A' and B' are "yes" is nonzero. We can write this as follows:

$$p(A = +1, B = +1) = 0,$$
$$p(A = -1, B' = +1) = 0,$$
$$p(A' = +1, B = -1) = 0,$$
$$p(A' = +1, \acute{B} = +1) = p \neq 0. \tag{63}$$

We show that these four statements are not compatible with time-local realism. The nonzero probability appearing in the argument is the measure of violation of time-local realism. It is interesting that two successive s-spin measurements violate time-local realism. We deal with the case where the input state is a pure state whose eigenstates coincide with those of $s.\hat{a}_0$ for some $\hat{a}_0$ whose eigenvalues we denote $\alpha_0 = j$. Hardy's argument for a system of n successive spin-s measurements, in it's minimal form (Parasuram & Ghosh, n.d.), is given by following conditions:

$$p(s.\hat{a}_1 = j, s.\hat{a}_2 = j, \ldots, s.\hat{a}_n = j) = 0, \tag{64}$$
$$p(s.\hat{a}_1 = j - 1, s.\hat{a}'_2 = j, \ldots, s.\hat{a}'_n = j) = 0,$$
$$p(s.\hat{a}_1 = j - 2, s.\hat{a}'_2 = j, \ldots, s.\hat{a}'_n = j) = 0,$$
$$\cdot$$
$$\cdot$$
$$p(s.\hat{a}_1 = -j, s.\hat{a}'_2 = j, \ldots, s.\hat{a}'_n = j) = 0,$$
$$\cdot$$
$$\cdot$$
$$\cdot \tag{65}$$
$$p(s.\hat{a}'_1 = j, \ldots s.\hat{a}_l = j - 1, \ldots, s.\hat{a}'_n = j) = 0,$$
$$p(s.\hat{a}'_1 = j, \ldots, s.\hat{a}_l = j - 2, \ldots, s.\hat{a}'_n = j) = 0,$$

$$\tag{66}$$

$$.$$
$$.$$
$$p\ (s.\hat{a}'_1 = j, \ldots s.\hat{a}'_l = -j, \ldots, s.\hat{a}'_n = j) = 0, \tag{67}$$
$$.$$
$$.$$
$$.$$
$$.$$
$$p\ (s.\hat{a}'_1 = j, s.\hat{a}'_2 = j, \ldots, s.\hat{a}_n = j - 1) = 0,$$
$$p\ (s.\hat{a}'_1 = j, s.\hat{a}'_2 = j, \ldots, s.\hat{a}_n = j - 2) = 0,$$
$$.$$
$$.$$
$$p\ (s.\hat{a}'_1 = j, s.\hat{a}'_2 = j, \ldots, s.\hat{a}_n = -j) = 0,$$
$$p\ (s.\hat{a}'_1 = j, s.\hat{a}'_2 = j, \ldots, s.\hat{a}'_n = j) = p. \tag{68}$$

First, we prove here that all time-local SHVTs predict $p = 0$. Suppose that a time-local SHVT reproducing, in accordance with Eq.(4), the quantum predictions exist. Accordingly, if we consider, for example, Eq.(64), we must have

$$p(s.\hat{a}'_1 = j, \ldots, s.\hat{a}_l = j - 1, \ldots, s.\hat{a}'_n = j)$$
$$= \int_\Lambda d\lambda \rho(\lambda) p_\lambda (s.\hat{a}'_1 = j, \ldots s.\hat{a}_l = j - 1, \ldots, s.\hat{a}'_n = j)$$
$$= \int_\Lambda d\lambda \rho(\lambda) p_\lambda (s.\hat{a}'_1 = j) \ldots p_\lambda (s.\hat{a}_l = j - 1) \ldots p_\lambda (s.\hat{a}'_n = j)$$
$$= 0, \tag{69}$$

where the second equality is implied by the time-locality condition of Eq.(4). The last equality in Eq.(69) can be fulfilled if and only if the product $p_\lambda (s.\hat{a}'_1 = j) \ldots p_\lambda (s.\hat{a}'_n = j)$ vavishes every time within Λ. An equivalent result holds for Eqs.(64-69), leading to:

$$p_\lambda\ (s.\hat{a}_1 = j) p_\lambda (s.\hat{a}_2 = j) \ldots p_\lambda (s.\hat{a}_n = j) = 0, \tag{70}$$
$$p_\lambda\ (s.\hat{a}_1 = j - 1) p_\lambda (s.\hat{a}'_2 = j) \ldots p_\lambda (s.\hat{a}'_n = j) = 0,$$
$$p_\lambda\ (s.\hat{a}_1 = j - 2) p_\lambda (s.\hat{a}'_2 = j) \ldots p_\lambda (s.\hat{a}'_n = j) = 0,$$
$$.$$
$$.$$
$$.$$
$$p_\lambda\ (s.\hat{a}_1 = -j) p_\lambda (s.\hat{a}'_2 = j) \ldots p_\lambda (s.\hat{a}'_n = j) = 0, \tag{71}$$
$$.$$
$$.$$
$$.$$
$$p_\lambda\ (s.\hat{a}'_1 = j) \ldots p_\lambda (s.\hat{a}_l = j - 1) \ldots p_\lambda (s.\hat{a}'_n = j) = 0,$$
$$p_\lambda\ (s.\hat{a}'_1 = j) \ldots p_\lambda (s.\hat{a}_l = j - 2) \ldots p_\lambda (s.\hat{a}'_n = j) = 0,$$
$$.$$
$$.$$
$$p_\lambda\ (s.\hat{a}'_1 = j) \ldots p_\lambda (s.\hat{a}_l = -j) \ldots p_\lambda (s.\hat{a}'_n = j) = 0, \tag{72}$$

.
.
.

$$p_\lambda \ (s.\hat{a}_1' = j)p_\lambda(s.\hat{a}_2' = j) \ldots p_\lambda(s.\hat{a}_n = j - 1) = 0,$$
$$p_\lambda \ (s.\hat{a}_1' = j)p_\lambda(s.\hat{a}_2' = j) \ldots p_\lambda(s.\hat{a}_n = j - 2) = 0,$$

.
.
.

$$p_\lambda \ (s.\hat{a}_1' = j)p_\lambda(s.\hat{a}_2' = j) \ldots p_\lambda(s.\hat{a}_n = -j) = 0, \tag{73}$$
$$p_\lambda \ (s.\hat{a}_1' = j)p_\lambda(s.\hat{a}_2' = j) \ldots p_\lambda(s.\hat{a}_n' = j) = p \neq 0, \tag{74}$$

where the first $2jn + 1$ equations are supposed to hold almost every time within Λ, while the last equation has to be satisfied in a subset of Λ whose measure according to the distribution $\rho(\lambda)$ is nonzero. To prove the more general result that no conceivable time-local SHVT can simultaneously satisfy Eqs.(70)-(74), a manipulation of those equations is required. To this end, let us sum all equations in each set. We obtain

$$(1 - p_\lambda(s.\hat{a}_1 = j)) \left[p_\lambda(s.\hat{a'}_2 = j) \ldots p_\lambda(s.\hat{a}_n' = j) \right] = 0,$$

.
.
.

$$(1 - p_\lambda(s.\hat{a}_l = j)) \left[p_\lambda(s.\hat{a}_1' = j) \ldots p_\lambda(s.\hat{a}_n' = j) \right] = 0,$$

.
.
.

$$(1 - p_\lambda(s.\hat{a}_n = j)) \left[p_\lambda(s.\hat{a}_1' = j) \ldots p_\lambda(s.\hat{a}_{n-1}' = j) \right] = 0. \tag{75}$$

Now let us partition the set of hidden variables Λ and define the following subsets $A_1, A_2, \ldots A_n$, and B as:

$$A_1 = \{\lambda \in \Lambda | p_\lambda(s.\hat{a}_1 = j) = 0\},$$

.
.
.

$$A_l = \{\lambda \in \Lambda | p_\lambda(s.\hat{a}_l = j) = 0\}, \tag{76}$$

.
.
.

$$A_n = \{\lambda \in \Lambda | p_\lambda(s.\hat{a}_n = j) = 0\},$$
$$B = \Lambda - \{A_1 \cup A_2 \cup \ldots \cup A_n\}. \tag{77}$$

We have that, for all λ belonging to B, $p_\lambda(s.\hat{a}_1 = j)p_\lambda(s.\hat{a}_2 = j) \ldots p_\lambda(s.\hat{a}_n = j) \neq 0$. If set B had a nonzero measure according to the distribution ρ, that is, if $\int_B d\lambda\rho(\lambda) \neq 0$, there would be violation of Eq.(70) and, consequently, of Eq.(64). Therefore, to fulfill Eq.(70), the set $A_1 \cup A_2 \cup \ldots \cup A_n$ must coincide with Λ apart from a set of zero measure, and we are left only with hidden variables belonging to either A_1 or A_2 or $\ldots$ or A_n. If λ

belongs to A_l, then, by definition, $p_\lambda(s.\hat{a}_l = j) = 0$, so that Eq.(75) can be satisfied only if $p_\lambda(s.\hat{a}'_1 = j) \ldots p_\lambda(s.\hat{a}'_n = j) = 0$. Hence, for any $\lambda \in \{A_1 \cup A_2 \cup \ldots \cup A_n\}$, we obtain a result leading to a contradiction of Eq.(74), which requires that there is a set of nonzero ρ measure within Λ where both probabilities do not vanish. To summarize, we have shown that it is not possible to exhibit any time-local hidden-variable model, satisfying Hardy's logic for n successive measurements.

Now, we show that in quantum theory for the n successive spin measurement, sometimes $p > 0$. So, we consider n successive measurements in directions $s.\hat{a}_i$ $(i = 1, 2, \ldots, n)$ on spin-s particles. For a spin-s system, we have (see Appendix-B):

$$|\langle \alpha_{k-1}|\alpha_k \rangle| = |\langle s.\hat{a}_{k-1}|s.\hat{a}_k \rangle| = d^{(s)}_{\alpha_{k-1},\alpha_k}(\beta_k - \beta_{k-1}), \tag{78}$$

where β_k is the angle between the $\hat{a}_k$ and the $+z$ axes. So, given the input state $|\alpha_0\rangle$, the (joint) probability that the measurement outcomes will be $|\alpha_1\rangle \in \{+j, \ldots, -j\}$ in the first measurement, $|\alpha_2\rangle \in \{+j, \ldots, -j\}$ in the second measurement,..., $|\alpha_n\rangle \in \{+j, \ldots, -j\}$ in the n-th measurement, is given by

$$\begin{aligned} p(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \Pi^n_{k=1}|\langle \alpha_{k-1}|\alpha_k \rangle| \\ &= \Pi^n_{k=1}d^2_{\alpha_{k-1},\alpha_k}(\beta_k - \beta_{k-1}). \end{aligned} \tag{79}$$

We deal with the case where the input state is a pure state whose eigenstates coincide with those of $\vec{S}.\hat{a}_0$ for some $\hat{a}_0$ whose eigenvalues we denote $\alpha_0 = j$. Now, by substituting Eq.(79) in the minimal form of Hardy's argument [Eqs.(64)-(68)], we have

$$d^2_{jj}(\beta_1)d^2_{jj}(\beta_2 - \beta_1)\ldots d^2_{jj}(\beta_n - \beta_{n-1}) = 0, \tag{80}$$

$$d^2_{j,j-1}(\beta_1)d^2_{j,j-1}(\beta'_2 - \beta_1)\ldots d^2_{jj}(\beta'_n - \beta'_{n-1}) = 0,$$

$$d^2_{j,j-2}(\beta_1)d^2_{j,j-2}(\beta'_2 - \beta_1)\ldots d^2_{jj}(\beta'_n - \beta'_{n-1}) = 0,$$

.

.

$$d^2_{j,-j}(\beta_1)d^2_{j,-j}(\beta'_2 - \beta_1)\ldots d^2_{jj}(\beta'_n - \beta'_{n-1}) = 0,$$

.

.

$$d^2_{jj}(\beta'_1)\ldots d^2_{j,j-1}(\beta_l - \beta'_{l-1})d^2_{j,j-1}(\beta'_{l+1} - \beta_l)\ldots$$
$$d^2_{jj}(\beta'_n - \beta'_{n-1}) = 0,$$

$$d^2_{jj}(\beta'_1)\ldots d^2_{j,j-2}(\beta_l - \beta'_{l-1})d^2_{j,j-2}(\beta'_{l+1} - \beta_l)\ldots$$
$$d^2_{jj}(\beta'_n - \beta'_{n-1}) = 0,$$

.

.

$$d^2_{jj}(\beta'_1)\ldots d^2_{j,-j}(\beta_l - \beta'_{l-1})d^2_{j,-j}(\beta'_{l+1} - \beta_l)\ldots$$
$$d^2_{jj}(\beta'_n - \beta'_{n-1}) = 0, \tag{81}$$

.
.
.

$$d^{\,2}_{\,jj}(\beta'_1)d^2_{jj}(\beta'_2 - \beta'_1)\dots d^2_{j,j-1}(\beta_n - \beta'_{n-1}) = 0,$$

$$d^{\,2}_{\,jj}(\beta'_1)d^2_{jj}(\beta'_2 - \beta'_1)\dots d^2_{j,j-2}(\beta_n - \beta'_{n-1}) = 0,$$

.
.
.

$$d^{\,2}_{\,jj}(\beta'_1)d^2_{jj}(\beta'_2 - \beta'_1)\dots d^2_{j,-j}(\beta_n - \beta'_{n-1}) = 0, \tag{82}$$

$$d^{\,2}_{\,jj}(\beta'_1)d^2_{jj}(\beta'_2 - \beta'_1)\dots d^2_{j,j}(\beta'_n - \beta'_{n-1}) = p. \tag{83}$$

From Eq (80), at least one of the factors must be 0. So

$$d^2_{jj}(\beta_1) = 0 \implies \beta_1 = \pi$$

$$\text{or}$$

$$d^2_{jj}(\beta_2 - \beta_1) \implies |\beta_2 - \beta_1| = \pi$$

$$\text{or}$$

.
.
.

$$d^2_{jj}(\beta_n - \beta_{n-1}) \implies |\beta_n - \beta_{n-1}| = \pi$$

To satisfy all equations (81)-(82), we have the following conditions:

$$(\beta_1 = 0) \text{ or } (\beta'_2 = \beta_1)$$
$$\text{and}$$
$$(\beta_2 = \beta'_1) \text{ or } (\beta_2 = \beta'_3)$$

.
.
.

$$(\beta_l = \beta'_{l-1}) \text{ or } (\beta_l = \beta'_{l+1}) \tag{84}$$

.
.
.

$$\text{and}$$
$$(\beta_n = \beta'_{n-1}).$$

Now, we can calculate the maximum value p by using these conditions. For example, if we select $\beta_1 = \pi$, so we must have $\beta'_2 = \beta_1 = \pi$. In this case,

$$p = d^2_{jj}(\beta'_1)d^2_{jj}(\pi - \beta'_1)d^2_{jj}(\beta'_3 - \pi)\dots d^2_{j,j}(\beta'_n - \beta'_{n-1}). \tag{85}$$

By substituting $d^j_{jj}(\beta) = \cos^{2j}(\beta/2)$, we have

$$p = \cos^{4j}(\frac{\beta'_1}{2}) \cos^{4j}(\frac{\pi - \beta'_1}{2}) \cos^{4j}(\frac{\beta'_3 - \pi}{2}) \ldots$$
$$\cos^{4j}(\frac{\beta'_n - \beta'_{n-1}}{2}). \qquad (86)$$

By selecting $\beta'_n = \beta'_{n-1} = \ldots = \beta'_3 = \pi$ and $\beta'_1 = \pi$,

$$p \leq (\frac{1}{2})^{4j}. \qquad (87)$$

We can obtain this result in the general case. We choose $|\beta_l - \beta_{l-1}| = \pi$, where $2 \leq l \leq n$. Without loss of generality, we select $\beta_l = \pi$ and $\beta_{l-1} = 0$. From the results obtained with Eq.(84), $\beta'_{l+1} = \beta_l = \pi$ or $\beta'_{l-1} = \beta_l = \pi$. Exactly for (l-1)th in Eq.(84), we have $\beta'_l = \beta_{l-1} = 0$ or $\beta'_{l-2} = \beta_{l-1} = 0$. So we have four cases: (i) $\beta'_{l-1} = \pi$ and $\beta'_{l-2} = 0$ (ii)$\beta'_{l-1} = \pi$ and $\beta'_l = 0$ (iii)$\beta'_{l+1} = \pi$ and $\beta'_l = 0$ and (iv)$\beta'_{l+1} = \pi$ and $\beta'_{l-2} = 0$. It is easy see that for the first three cases , the maximum value of p is 0, but in the forth case, by selecting $\beta'_1 = \beta'_2 = \ldots = \beta'_{l-1} = 0$ and $\beta'_{l+2} = \beta'_{l+3} = \ldots = \beta'_n = \pi$, we get

$$p = \cos^{4j}(\frac{\beta'_l}{2}) \sin^{4j}(\frac{\beta'_l}{2}) \leq (\frac{1}{2})^{4j}. \qquad (88)$$

We see that $p > 0$ for all spins, and also, the maximum probability of success of Hardy's non-time locality is independent of the number of successive measurements and decreases with s.

7. Summary and comments

Entanglement in space displays one of the most interesting features of quantum mechanics, often called quantum non locality. Locality in space and realism impose constraints -Bell's inequalities- on certain combinations of correlations for measurements of spatially separated systems, which are violated by quantum mechanics. Non locality is one of the strangest properties of quantum mechanics, and understanding this notion remains an important problem.

Entanglement in time is not introduced in quantum mechanics because of different roles time and space play in quantum theory. The meaning of locality in time is that the results of measurement at time t_2 are independent of any measurement performed at some earlier time t_1 or later time t_3. The temporal Bell's inequalities are derived from the realistic hidden variable theory.

In this chapter we have considered a hidden variable theory of successive measurements on a single spin-s system. In all the previous scenarios comparing HVT and QM the principal hypothesis being tested was that, in a given state (having spatial correlation), HVT implies the existence of a joint probability distribution for all observables even if some of them are not compatible. QM is shown to contradict the consequence of this requirement as it does not assign joint probabilities to the values of incompatible observables. The particular implication that is tested is whether the marginal of the observable A in the joint distribution of the compatible observables A and B is the same as the marginal for A in joint distribution

for the observables A and C even if B and C are not compatible. In other words, HVT implies noncontextuality for which QM can be tested. The celebrated theorem of Bell and Kochen-Specker showed that QM is contextual (Bell, 1966),(Kochen & Specker, 1967). In our scenario, the set of measured observables have a well defined joint probability distribution as each of them acts on a different state. Note that the Bell-type inequalities we have derived follow from equation (4) which says that, for a given value of stochastic hidden variable λ, the joint probability for the outcomes of successive measurements must be statistically independent. In other words the hidden variable λ completely decides the probabilities of individual measurement outcomes independent of other measurements. We show that QM is not consistent with this requirement of HVT. A Bell-type inequality (for single particle), testing contextuality of QM was proposed by Basu et al. (Basu et al., 2001) and was shown that it could be empirically tested. However, the approach given in the present chapter furnishes a test for realistic nature of QM independent of contextuality. We have compared QM with HVT for different values of spin and for different number of successive measurements. The dependence of the deviation of QM from HVT on the spin value and on the number of successive measurements opens up new possibilities for comparison of these models, and may lead to a sharper understanding of QM.

In the following, I bring some of the key surprising results obtained in this chapter.

1- We obtained temporal Mermin-Klyshko inequality (MKI) and svetlinchi inequality (SI) for n successive measurements by using realism and non locality in time. We showed quantum correlations violate temporal MKI and satisfy temporal SI.

2- It was interesting that, for a spin-s particle, maximum deviation of quantum mechanics from realism was obtained for all convex combinations of $\alpha_0 = \pm 1$ states (the case when input state is a mixed state whose eigenstates coincide with those of $\vec{S} \cdot \hat{a}_0$ for some $\hat{a}_0$ whose eigenvalues we denote by $\alpha_0 \in \{-s, \ldots, s\}$). This is surprising as one would expect pure states to be more 'quantum' than the mixed ones thus breaking Bell inequalities by larger amount.

3- All spin 1/2 states maximally break Mermin-Klyshko inequalities for n successive measurements ($\eta_n = \sqrt{2}$) as against only the entangled states break it in multipartite case. Interestingly that for $s = \frac{1}{2}$ the random mixture (maximum noisy state) also breaks BI. This indicates that the notion of "classicality", compatible with the usual local HVT, is different in nature from the notion of classicality that would arise from the non-violation of BI here.

4- We saw that for all spins, BI and MKI is violated in two and three successive measurements ($\eta_2 > 1, \eta_3 > 1$) and the value of violation MKI in three successive measurements is a little more than the value of violation BI in two successive measurements $\eta_3 > \eta_2$ except $s = \frac{1}{2}$, while $\eta_3 = \eta_2 = \sqrt{2}$ for spin $s = \frac{1}{2}$. Also η_3 and η_2 decrease monotonically with increase in the value of s. It is interesting, in the case of two and three successive measurements of spin s prepared in a pure state, that the maximum violation of BI and MKI falls off as the spin of the particle increases, but tends to a constant for arbitrary large s, $\eta_2(s \longrightarrow \infty) = 1.143$ and $\eta_3(s \longrightarrow \infty) = 1.153$. It is thus seen that large quantum numbers do not guarantee classical behavior.

5- We showed that for $s = \frac{1}{2}$, the correlation between the outputs of measurements from last k out of n successive measurements ($k < n$) depend on the measurement prior to $(n - k)$, when k is even, while for odd k, these correlations are independent of the outputs of measurements prior to $n - k$.

6- Interestingly if the initial state is the random mixture or first Stern-Gerloch measurements for the qubit component along the directions a_1 perpendicular to initial state $\hat{a}_0 \perp \hat{a}_1$ so that, $\langle \alpha_1 \rangle_{QM} = 0$, then always quantum averages for all odd number of successive measurements

are zero.

7- We proved that the correlation function between first and third measurement (t_1 and t_3) on spin-s particle for a given measurement performed at t_2 can not violate the temporal Bell inequality. Therefore, any measurement performed at time t_2 disentangles events at time t_1 and t_3 if $t_1 < t_2 < t_3$.

8- Three successive measurements on spin-s particles do not break Svetlinchi Inequality. But it is proved that three successive measurements on qubit violate Scarani-Gisin inequality. Thus, although there are no genuine three-fold temporal correlations, a specific combination of two-fold correlations can have values that are not achievable with correlations in space for any three-qubit system.

9- Also we showed that three successive measurements violate two types of Bell inequalities involving two and three successive measurements. So two successive measurement correlations are relevant to those of three successive measurements. This behavior is analogous to three particle W-state.

10- Quantum correlations between two successive measurements on a qubit violates chained Bell inequality which is obtained by providing more than two alternative experiments in every step.

11- Also, we have studied Hardy's argument for the correlations between the outputs of n successive measurements for all s-spin measurements. We have shown that the maximum probability of success of Hardy's argument for n successive measurements is $\left(\frac{1}{2}\right)^{4s}$, which is independent of the number of successive measurements of spin (n) and decreases with increase of s. This can be compared with the correlations corresponding to measurement of spin observables in a spacelike separated two-particles scenario where only the non-maximally entangled states of any spin-s bipartite system respond to Hardy's nonlocality test.

8. Appendix A

We evaluate $\langle \alpha_1 \rangle$, $\langle \alpha_1 \alpha_2 \rangle$ and $\langle \alpha_1 \alpha_2 \alpha_3 \rangle$ in the state ρ_0 given in(4.1).
$(|\vec{S} \cdot \hat{a}_0, \alpha_0\rangle \equiv |\hat{a}_0, \alpha_0\rangle)$

$$\langle \alpha_1 \rangle = \sum_{\alpha_1 = -s}^{s} \alpha_1 p(\alpha_1) = \langle \hat{a}_0, \alpha_0 | \vec{S} \cdot \hat{a}_1 | \hat{a}_0, \alpha_0 \rangle = \langle \hat{a}_1, \alpha_0 | e^{i\vec{S} \cdot \hat{n}\theta_1} (\vec{S} \cdot \hat{a}_1) e^{-i\vec{S} \cdot \hat{n}\theta_1} | \hat{a}_1, \alpha_0 \rangle \qquad \text{(A.1)}$$

where θ_1 is the angle between $\hat{a}_0$ and $\hat{a}_1$ and $\hat{n}$ is the unit vector along the direction defined by $\hat{n} = \hat{a}_0 \times \hat{a}_1$. By using Baker- Hausdorff Lemma

$$e^{iG\lambda} A e^{-iG\lambda} = A + i\lambda[G, A] + \left(\frac{i^2\lambda^2}{2!}\right) [G, [G, A]] + \cdots \qquad \text{(A.2)}$$

we get,

$$\langle \alpha_1 \rangle = \langle \hat{a}_1, \alpha_0 | \vec{S} \cdot \hat{a}_1 | \hat{a}_1, \alpha_0 \rangle + \frac{i\theta_1}{1!} \langle \hat{a}_1, \alpha_0 | [\vec{S} \cdot \hat{n}, \vec{S} \cdot \hat{a}_1] | \hat{a}_1, \alpha_0 \rangle$$

$$+ \frac{i^2\theta_1^2}{2!} \langle \hat{a}_1, \alpha_0 | [\vec{S} \cdot \hat{n}, [\vec{S} \cdot \hat{n}, \vec{S} \cdot \hat{a}_1]] | \hat{a}_1 \alpha_0 \rangle + \cdots \qquad \text{(A.3)}$$

By using:

$$\langle \hat{a}_1, \alpha_0 | \vec{S} \cdot \hat{a}_1 | \hat{a}_1, \alpha_0 \rangle = \alpha_0, \qquad \text{(A.4)}$$

$$\langle \hat{a}_1, \alpha_0 | [\vec{S} \cdot \hat{n}, \vec{S} \cdot \hat{a}_1] | \hat{a}_1, \alpha_0 \rangle = \langle \hat{a}_1, \alpha_0 | (i\vec{S} \cdot (\hat{n} \times \hat{a}_1)) | \hat{a}_1, \alpha_0 \rangle = 0, \tag{A.5}$$

and

$$\langle \hat{a}_1, \alpha_0 | [\vec{S} \cdot \hat{n}, [\vec{S} \cdot \hat{n}, \vec{S} \cdot \hat{a}_1]] | \hat{a}_1, \alpha_0 \rangle = \langle \hat{a}_1, \alpha_0 | \vec{S} \cdot \hat{a}_1 | \hat{a}_1, \alpha_0 \rangle = \alpha_0. \tag{A.6}$$

Terms with odd powers of θ_1 vanish

$$\langle \alpha_1 \rangle = \alpha_0 - \frac{\theta_1^2}{2!} \alpha_0 + \frac{\theta_1^4}{4!} \alpha_0 - \cdots = \alpha_0 \cos \theta_1. \tag{A.7}$$

If the initial state is mixed state(4.1):

$$\langle \alpha_1 \rangle = \sum_{\alpha_0 = -s}^{+s} p_{\alpha_0} \alpha_0 \cos \theta_1. \tag{A.8}$$

Further we compute

$$\langle \alpha_1 \alpha_2 \rangle = \sum_{\alpha_1} \alpha_1 |\langle \hat{a}_0, \alpha_0 | \hat{a}_1, \alpha_1 \rangle|^2 \sum_{\alpha_2} \alpha_2 |\langle \hat{a}_1, \alpha_1 | \hat{a}_2, \alpha_2 \rangle|^2.$$

By using (A.7)

$$\langle \alpha_1 \alpha_2 \rangle = \cos \theta_{12} \sum_{\alpha_1} \alpha_1^2 |\langle \hat{a}_0, \alpha_0 | \hat{a}_1, \alpha_1 \rangle|^2 = \cos \theta_{12} \langle \hat{a}_0, \alpha_0 | (\vec{S} \cdot \hat{a}_1)^2 | \hat{a}_0, \alpha_0 \rangle$$

$$= \cos \theta_{12} \langle \hat{a}_1, \alpha_0 | e^{i\vec{S} \cdot \hat{n} \theta_1} (\vec{S} \cdot \hat{a}_1)^2 e^{-i\vec{S} \cdot \hat{n} \theta_1} | \hat{a}_1, \alpha_0 \rangle \tag{A.9}$$

Using the Baker-Hausdorff Lemma, and using

$$\langle \hat{a}_1, \alpha_0 | [\vec{S} \cdot \hat{n}, [\vec{S} \cdot \hat{n}, [\vec{S} \cdot \hat{n}, \cdots [\vec{S} \cdot \hat{n}, (\vec{S} \cdot \hat{a}_1)^2]] \cdots]] | \hat{a}_1, \alpha_0 \rangle \tag{A.10}$$

$$= \begin{cases} 0 & \text{if } \vec{S} \cdot \hat{n} \text{ occurs odd number of times} \\ 3\alpha_0^2 - s^2 - s & \text{if } \vec{S} \cdot \hat{n} \text{ occurs } 2p \text{ times} \end{cases}$$

we get,

$$\langle \alpha_1 \alpha_2 \rangle = \frac{1}{2} \cos \theta_{12} [(s^2 + s - \alpha_0^2) + (3\alpha_0^2 - s^2 - s) \cos^2 \theta_1]. \tag{A.11}$$

If the initial state is mixed state (4.1),

$$\langle \alpha_1 \alpha_2 \rangle = \frac{1}{2} \cos \theta_{12} \sum_{\alpha_0 = -s}^{+s} p_{\alpha_0} [(s^2 + s - \alpha_0^2) + (3\alpha_0^2 - s^2 - s) \cos^2 \theta_1]. \tag{A.12}$$

Next we calculate,

$$\langle \alpha_1 \alpha_2 \alpha_3 \rangle = \sum_{\alpha_1} \alpha_1 |\langle \hat{a}_0, \alpha_0 | \hat{a}_1, \alpha_1 \rangle|^2 \sum_{\alpha_2} \alpha_2 |\langle \hat{a}_1, \alpha_1 | \hat{a}_2, \alpha_2 \rangle|^2 \sum_{\alpha_3} \alpha_3 |\langle \hat{a}_2, \alpha_2 | \hat{a}_3, \alpha_3 \rangle|^2. \tag{A.13}$$

By using (A.7) and (A.11) we get,

$$\langle \alpha_1 \alpha_2 \alpha_3 \rangle = \frac{1}{2} \alpha_0 \cos \theta_1 \cos \theta_{23} \sin^2 \theta_{12} s(s+1) + \frac{1}{2} \cos \theta_{23} (3 \cos^2 \theta_{12} - 1) A, \tag{A.14}$$

where,

$$A = \sum_{\alpha_1} \alpha_1^3 |\langle \hat{a}_0, \alpha_0 | \hat{a}_1, \alpha_1 \rangle|^2 = \langle \hat{a}_1, \alpha_0 | e^{i\vec{S} \cdot \hat{n}\theta_1} (\vec{S} \cdot \hat{a}_1)^3 e^{-i\vec{S} \cdot \hat{n}\theta_1} |\hat{a}_1, \alpha_0 \rangle. \tag{A.15}$$

Using Baker-Hausdorff lemma and

$$\langle \hat{a}_1, \alpha_0 | [\vec{S} \cdot \hat{n}, [\vec{S} \cdot \hat{n}, [\vec{S} \cdot \hat{n}, \cdots [\vec{S} \cdot \hat{n}, (\vec{S} \cdot \hat{a}_1)^3]] \cdots]] |\hat{a}_1, \alpha_0 \rangle$$

$$= \begin{cases} 0 & \text{if } \vec{S} \cdot \hat{n} \text{ occurs odd number of times} \\ Y(X - a_0^3) + X & \text{if } \vec{S} \cdot \hat{n} \text{ occurs } 2p \text{ times} \end{cases} \tag{A.16}$$

where,

$$X = 6\alpha_0^3 + \alpha_0(1 - 3s(s+1))$$
$$Y = 3^{2p-2} + 3^{2p-4} + \cdots + 3^2 = (\frac{9}{8})[9^{2p-2} - 1] \tag{A.17}$$

we get,

$$A = \frac{1}{8} \sum_{j=0}^{\infty} (-1)^j [(9^j - 1)X - (9^j - 9)\alpha_0^3] \, (\frac{\theta_{01}^{2j}}{2j!}).$$

This gives,

$$A = \frac{1}{8}\alpha_0 \{ [3\alpha_0^2 + 3s(s+1) - 1] \cos\theta_{01} + [5\alpha_0^2 - 3s(s+1) + 1] \cos 3\theta_{01} \} \tag{A.18}$$

After substituting (A.18) in (A.14) and simplifying,

$$\langle \alpha_1\alpha_2\alpha_3 \rangle = \frac{1}{16} \cos\theta_{23} \{ \cos\theta_1 [M \cos^2\theta_{12} + N] + R[3\cos^2\theta_{12} - 1] \} \tag{A.19}$$

where,

$$M = \alpha_0[9\alpha_0^2 + s(s+1) - 3]$$
$$N = \alpha_0[5s(s+1) - 3\alpha_0^2 + 1]$$
$$R = \alpha_0[5\alpha_0^2 - 3s(s+1) + 1].$$

If the initial state is a mixed state (4.1),

$$M = \sum_{\alpha_0=-s}^{+s} p_{\alpha_0} \alpha_0 [9\alpha_0^2 + s(s+1) - 3]$$
$$N = \sum_{\alpha_0=-s}^{+s} p_{\alpha_0} \alpha_0 [5s(s+1) - 3\alpha_0^2 + 1]$$
$$R = \sum_{\alpha_0=-s}^{+s} p_{\alpha_0} \alpha_0 [5\alpha_0^2 - 3s(s+1) + 1].$$

Also, by using equations (A.12) and (A.13), one obtains:

$$\langle \alpha_1 \alpha_3 \rangle = \sum_{\alpha_1} \alpha_1 |\langle \hat{a}_0, \alpha_0 | \hat{a}_1, \alpha_1 \rangle|^2 \sum_{\alpha_2} |\langle \hat{a}_1, \alpha_1 | \hat{a}_2, \alpha_2 \rangle|^2 \sum_{\alpha_3} \alpha_3 |\langle \hat{a}_2, \alpha_2 | \hat{a}_3, \alpha_3 \rangle|^2$$

$$= \cos \theta_{32} \langle \alpha_1 \alpha_2 \rangle$$

$$= \frac{1}{2} \cos \theta_{32} \cos \theta_{21} [(s^2 + s - \alpha_0^2) + (3\alpha_0^2 - s^2 - s) \cos^2 \theta_1], \qquad (A.20)$$

and we can obtain:

$$\langle \alpha_2 \alpha_3 \rangle = \sum_{\alpha_1} |\langle \hat{a}_0, \alpha_0 | \hat{a}_1, \alpha_1 \rangle|^2 \sum_{\alpha_2} \alpha_2 |\langle \hat{a}_1, \alpha_1 | \hat{a}_2, \alpha_2 \rangle|^2 \sum_{\alpha_3} \alpha_3 |\langle \hat{a}_2, \alpha_2 | \hat{a}_3, \alpha_3 \rangle|^2$$

$$= \cos \theta_{32} \langle \alpha_2^2 \rangle$$

$$= \frac{1}{2} s \cos \theta_{32} \{(s+1) \sin^2 \theta_{12} + \frac{1}{2} (3 \cos^2 \theta_{12} - 1)[1 + (2s - 1) \cos^2 \theta_1]\}. \qquad (A.21)$$

9. Appendix B

Let us consider a situation where an ensemble of systems prepared in state $\rho = |s.\hat{a}_0 = \alpha_0 > < s.\hat{a}_0 = \alpha_0|$ at time $t = 0$, is subjected to a measurement of the observable $A(t_1) = s.\hat{a}_1$ at time t_1 followed by a measurement of the observable $B(t_2) = s.\hat{a}_2$ at time t_2 ($t_2 > t_1 > 0$), where we have adopted the Heisenberg picture of time evolution. Further, let us assume that both $A(t_1)$ and $B(t_2)$ have purely discrete spectra. Let $\{\alpha_1\} = \{-s, -s+1, \ldots, s\}$ and $\{\alpha_2\} = \{-s, -s+1, \ldots, s\}$ denote the eigenvalues and $P^{A(t_1)}(\alpha_1) = |s.\hat{a}_1 = \alpha_1 > < s.\hat{a}_1 = \alpha_2|$, $P^{B(t_2)}(\alpha_2) = |s.\hat{a}_2 = \alpha_2 > < s.\hat{a}_2 = \alpha_2|$ the corresponding eigenprojectors of $A(t_1)$ and $B(t_2)$ respectively. Then the joint probability that a measurement of $A(t_1)$ yields the outcome α_1 and a measurement of $B(t_2)$ yields the outcome α_2 is given by

$$Pr^{\rho}_{A(t_1),B(t_2)}(\alpha_1, \alpha_2)$$

$$= Tr \left[P^{B(t_2)}(\alpha_2) P^{A(t_1)}(\alpha_1) \rho P^{A(t_1)}(\alpha_1) P^{B(t_2)}(\alpha_2) \right]$$

$$= |\langle s.\hat{a}_0 | s.\hat{a}_1 \rangle|^2 |\langle s.\hat{a}_1 | s.\hat{a}_2 \rangle|^2. \qquad (B.1)$$

We know that $|s.a = \alpha > = \sum_{m=-s}^{+s} d_{\alpha,m}^{(s)}(\beta)|m >$ where $d_{\alpha m}^{(s)}(\beta) \equiv \left\langle s, \alpha | exp(\frac{-iS_y(\beta)}{\hbar}) | s, m \right\rangle$ and it obtains Wigner's formula (Sakurai, n.d.) and $\alpha_i \in -s, \ldots, s$ and β_i is the angle between the $\hat{a}_i$ and the z axes. In contrast,

$$|\langle s.\hat{a}_1 | s.\hat{a}_2 \rangle| = \sum_m < m | d_{\alpha_1,m}^{*(s)}(\beta_1) \sum_{m'} d_{\alpha_2,m'}^{(s)}(\beta_2) | m' >$$

$$= \sum_m d_{\alpha_1,m}^{*(s)}(\beta_1) d_{\alpha_2,m'}^{(j)}(\beta_2)$$

$$= d_{\alpha_1 \alpha_2}^{(s)}(\beta_2 - \beta_1). \qquad (B.2)$$

So, we obtain

$$pr_{QM}(\alpha_1, \alpha_2) = |d_{\alpha_0 \alpha_1}^{(s)}(\beta_1 - \beta_0)|^2 |d_{\alpha_1 \alpha_2}^{(s)}(\beta_2 - \beta_1)|^2. \qquad (B.3)$$

10. References

Alter, O. & Yamamoto, Y. (n.d.). *Quantum Measurement of a Single System*, A Wiley-interscience Publication John Wiley and sons, INC.

Anderson, E., Barnett, S. M. & Aspect, A. (2005). *Phys. Rev. A* 72: 042104.

Ballentine, L. E. (1990). *Quantum Mechanics*, prentice Hall, Englewood Cliffs, NJ.

Basu, S., Bandyopadhay, S., Kar, G. & Home, D. (2001). *Phys. Lett. A* 279: 284.

Beck, C. & Graudenz, D. (1992). *Phys. Rev. A* 46: 6265.

Belinskii, A. V. & Klyshko, D. N. (1993). *Phys. Usp.* 36: 653.

Bell, J. S. (1964). *Physics* 1: 195.

Bell, J. S. (1966). *Rev Mod. Phys.* 38: 447.

Bennett, C. H., Brassard, G., Popescu, S., Schumacher, B., Smolin, J. & Wootters, W. K. (1996). *Phys. Rev. Lett* 76: 722.

Bohm, D. (1952). *Phys. Rev.* 85: 166.

Braunstein, L. & Caves, C. M. (1990). *Ann. Phys. (NY)* 202: 22.

Brukner, C., Taylor, S., Cheung, S. & Vedral, V. (2004). *The Seventh International Conference on Quantum Communication Measurement and Computing, Glasgow, United Kingdom*, pp. 25–29.

Cabello, A. (2002a). *Phys. Rev. A* 65: 062105.

Cabello, A. (2002b). *Phys. Rev. A* 65: 032108.

Clauser, J. F., Horne, M. A., Shimony, A. & Holt, R. A. (1969). *Phys. Rev. Lett* 23: 880.

Clifton, R. & Niemann, P. (1992). *Phys. Lett. A* 166: 177.

Collins, D., Gisin, N., Linden, N., Massar, S. & Popescu., S. (2002). *Phys. Rev. Lett.* 88: 040404.

Collins, D., Gisin, N., Popescu, S., Roberts, D. & Scarani, V. (2002). *Phys. Rev. Lett* 88: 170405.

Collins, D. & Popescu, S. (2001). *J. Phys. A :Math. Gen* 34: 6821.

Einstein, A., Podoloski, B. & Rosen, N. (1935). *Phys Rev* 47.

Fine, A. (1982). *Phys. Rev. Lett.* 48: 291.

Gisin, N. & Bechmann-Pasquinucci, H. (1998). *Phys. Lett. A* 246.

Gisin, N. & Peres, A. (1992). *Phys. Lett. A* 162.

Goldstein, S. (1994). *Phys. Rev. Lett.* 72: 1951.

Greenberger, D. M., Horne, M. A. & Zeilinger, A. (1990). *Am. J. Phys.* 58: 1131–1143.

Hardy, L. (1992). *Phys. Rev. Lett* 68: 2981.

Hardy, L. (1993). *Phys. Rev. Lett.* 71: 1665.

Heywood, P. & Redhead, M. L. (1983). *Found. Phys.* 13: 481.

Jarrett, J. P. (1984). On the physical significance of the locality conditions in the bell arguments, *Nous* 18: 569–89.

Kaszlikowski, D., Gnacinski, P., Zukowski, M., Miklaszewski, W. & Zeilinger, A. (n.d.). *Phys.Rev. Lett.* 85.

Kochen, K. & Specker, E. (1967). *J. Math.* 17: 59.

Leggett, A. J. & Garg, A. (1985). *Phys. Rev. Lett.* 54: 857.

Mermin, N. D. (1990). *Phys. Rev. Lett* 65: 1838.

Pagonis, C. & Clifton, R. (1992). *Phys. Lett. A* 168: 100.

Parasuram, K. & Ghosh, S. (n.d.). *arXiv: quant-ph/ 0903.3020* .

Paz, J. P. & Mahler, G. (1993). *Phys. Rev. Lett.* 71: 3235.

Peres, A. (1992). *Phys. Rev. A* 46: 4413.

Peres, A. (1993). *Quantum Theory: Concepts and Methods*, Kluwer Academic Publishers.

Popescu, S. (1995). *Phys. Rev. Lett.* 74: 2619.

Redhead, M. (1987). *Incompleteness, Nonlocality, and Realism*, Clarendon Press, Oxford.

Sakurai, J. J. (n.d.). *Modern Quantum Mechanics, (revised edition)*, Addison-Wesley, 1999.

Scarani, V. & Gisin, N. (2001). *Phys. Rev. Lett.* 87: 117901.

Seevinck, M. & Svetlichny, G. (2002). *Phys. Rev. Lett.* 89: 060401.

Shimony, A. (n.d.). Historical philosophica, and physical inquiries into the foundations of quantum mechanics, *In Sixty-Two Years of Uncertainty*, Plenum, New York.

Svetlichny, G. (1987). *Phys. Rev. D* 35: 3066.

Werner, R. F. (1989). *Phys. Rev. A* 40: 4277.

Werner, R. & Wolf, M. (2000). *Phys. Rev. A.* 61: 062102.

Zuckowski, M. & Brukner, C. (2002). *Phys. Rev. Lett.* 88: 21041.
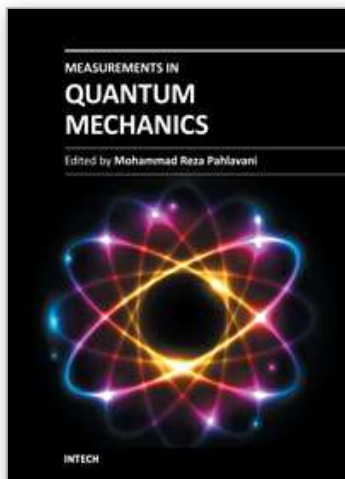
Measurements in Quantum Mechanics

Edited by Prof. Mohammad Reza Pahlavani

Perhaps quantum mechanics is viewed as the most remarkable development in 20th century physics. Each successful theory is exclusively concerned about "results of measurement". Quantum mechanics point of view is completely different from classical physics in measurement, because in microscopic world of quantum mechanics, a direct measurement as classical form is impossible. Therefore, over the years of developments of quantum mechanics, always challenging part of quantum mechanics lies in measurements. This book has been written by an international invited group of authors and it is created to clarify different interpretation about measurement in quantum mechanics.

How to reference

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Ali Ahanj (2012). Quantum Correlations in Successive Spin Measurements, Measurements in Quantum Mechanics, Prof. Mohammad Reza Pahlavani (Ed.), ISBN: 978-953-51-0058-4, InTech, Available from: http://www.intechopen.com/books/measurements-in-quantum-mechanics/successive-spin-s-measurements

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