10

A Variational Approach to the Fuel Optimal Control Problem for UAV Formations

Andrea L'Afflitto and Wassim M. Haddad
Georgia Institute of Technology
USA

1. Introduction

The pivotal role of unmanned aerial vehicles (UAVs) in modern aircraft technology is evidenced by the large number of civil and military applications they are employed in. For example, UAVs successfully serve as platforms carrying payloads aimed at land monitoring (Ramage et al., 2009), wildfire detection and management (Ambrosia & Hinkley, 2008), law enforcement (Haddal & Gertler, 2010), pollution monitoring (Oyekan & Huosheng, 2009), and communication broadcast relay (Majewski, 1999), to name just a few.

A formation of UAVs, defined by a set of vehicles whose states are coupled through a common control law (Scharf et al., 2003b), is often more valuable than a single aircraft because it can accomplish several tasks concurrently. In particular, UAV formations can guarantee higher flexibility and redundancy, as well as increased capability of distributed payloads (Scharf et al., 2003a). For example, an aircraft formation can successfully intercept a vehicle which is faster than its chasers (Jang & Tomlin, 2005). Alternatively, a UAV formation equipped with interferometic synthetic aperture radar (In-SAR) antennas can pursue both along-track and cross-track interferometry, which allow harvesting information that a single radar cannot detect otherwise (Lillesand et al., 2007).

Path planning is one of the main problems when designing missions involving multiple vehicles; a UAV formation typically needs to accomplish diverse tasks while meeting some assigned constraints. For example, a UAV formation may need to intercept given targets while its members maintain an assigned relative attitude. Trajectories should also be optimized with respect to some performance measure capturing minimum time or minimum fuel expenditure. In particular, trajectory optimization is critical for mini and micro UAVs (μUAVs) because they often operate independently from remote human controllers for extended periods of time (Shanmugavel et al., 2010) and also because of limited amount of available energy sources (Plnes & Bohorquez, 2006).

The scope of the present paper is to provide a rigorous and sufficiently broad formulation of the optimal path planning problem for UAV formations, modeled as a system of n 6-degrees of freedom (DoF) rigid bodies subject to a constant gravitational acceleration and aerodynamic forces and moments. Specifically, system trajectories are optimized in terms of control effort, that is, we design a control law that minimizes the forces and moments needed to operate a UAV formation, while meeting all the mission objectives. Minimizing the control effort is equivalent to minimizing the formation's fuel consumption in the case of vehicles equipped with conventional fuel-based propulsion systems (Schouwenaars et al., 2006) and is a suitable indicator of the energy consumption for vehicles powered by batteries or other power sources.

In this paper, we derive an optimal control law which is independent of the size of the formation, the system constraints, and the environmental model adopted, and hence, our framework applies to aircraft, spacecraft, autonomous marine vehicles, and robot formations. The direction and magnitude of the optimal control forces and moments is a function of the dynamics of two vectors, namely the translational and rotational primer vectors. In general, finding the dynamics of these two vectors over a given time interval is a demanding task that does not allow for an analytical closed-form solution, and hence, a numerical approach is required. Our main result involves necessary conditions for optimality of the formations' trajectories.

The contents of this paper are as follows. In Section 2, we present notation and definitions of the physical variables needed to formulate the fuel optimization problem. Section 3 gives a problem statement of the UAV path planning optimization problem, whereas Section 4 provides the necessary mathematical background for this problem. Next, in Section 5, we survey the relevant literature and highlight the advantages related to the proposed approach. Section 6 discusses results achieved by applying the theoretical framework developed in Section 4. In Section 7, we present an illustrative numerical example that highlights the efficacy of the proposed approach. Finally, in Section 8, we draw conclusions and highlight future research directions.

2. Notation and definitions

The notation used in this paper is fairly standard. When a word is defined in the text, the concept defined is *italicized* and it should be understood as an "if and only if" statement. Mathematical definitions are introduced by the symbol "$\triangleq$." The symbol $\mathbb{N}$ denotes the set of positive integers, $\mathbb{R}$ denotes the set of real numbers, $\overline{\mathbb{R}}_+$ denotes the set of nonnegative real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ column vectors on the field of real numbers, and $\mathbb{R}^{n\times m}$ denotes the set of real $n \times m$ matrices. Both natural and real numbers are denoted by lower case letters, e.g., $j \in \mathbb{N}$ and $a \in \mathbb{R}$, vectors are denoted by bold lower case letters, e.g., $\mathbf{x} \in \mathbb{R}^n$, and matrices are denoted by bold upper case letters, e.g., $\mathbf{A} \in \mathbb{R}^{n\times m}$. Subsets of $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ are denoted by italicized upper case letters, e.g., $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^{n\times m}$. The interior of the set A is denoted by $\mathrm{int}(A)$. The zero vector in $\mathbb{R}^n$ is denoted by $\mathbf{0}_n$, the zero matrix in $\mathbb{R}^{n\times m}$ is denoted by $\mathbf{0}_{n\times m}$, and the identity matrix in $\mathbb{R}^{n\times n}$ is denoted by $\mathbf{I}_n$.

For $\mathbf{x} \in \mathbb{R}^n$ we write $\mathbf{x} \geq\geq \mathbf{0}_n$ (respectively, $\mathbf{x} >> \mathbf{0}_n$) to indicate that every component of $\mathbf{x}$ is nonnegative (respectively, positive). We write $||\cdot||_p$ for the p-norm of a vector and its corresponding equi-induced matrix norm, e.g., $||\mathbf{x}||_p$ and $||\mathbf{A}||_p$. The transpose of a vector or of a matrix is denoted by the superscript $(\cdot)^T$, e.g., $\mathbf{x}^T$ and $\mathbf{A}^T$. The cross product between two vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a} \wedge \mathbf{b}$. Given $\mathbf{x} \in \mathbb{R}^3$ such that $\mathbf{x} \triangleq [x_1, x_2, x_3]^T$, we define

$$\mathbf{x}^\times \triangleq \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

The inverse of a square matrix $\mathbf{A}$ is denoted by $\mathbf{A}^{-1}$, the transpose of $\mathbf{A}^{-1}$ is denoted by $\mathbf{A}^{-\mathrm{T}}$, the determinant of $\mathbf{A}$ is denoted by $\det(\mathbf{A})$, the diagonal of $\mathbf{A}$ is denoted by $\mathrm{diag}(\mathbf{A})$, and the nullspace of a matrix $\mathbf{A}$ is denoted by $\mathcal{N}(\mathbf{A})$.

Functions are always introduced by specifying their domain and codomain, e.g., $\mathbf{h} : A_1 \times A_2 \to B$. The arguments of a function will not be indicated in the text unless necessary, e.g., $\mathbf{h}(\mathbf{x}, \mathbf{y})$ is simply denoted by $\mathbf{h}$. If a function is dependent on some unspecified variables, then its arguments will be replaced by dots, e.g., $\mathbf{h}(\cdot, \cdot)$. The same convention is used for functionals; however, their arguments are embraced by square brackets, i.e., $\mathrm{J}[\mathbf{x}, \mathbf{y}]$.

The first derivative with respect to time of a differentiable function $\mathbf{q} : [t_1, t_2] \to \mathbb{R}^{\mathrm{n}}$ is denoted by the a dot on top of the function, e.g., $\dot{\mathbf{q}}(t)$. Given $\mathbf{g} : A \to \mathbb{R}^{\mathrm{m}}$, where $A \subset \mathbb{R}^{\mathrm{n}}$ is an open set, we say that $\mathbf{g}(\cdot)$ *is of class* $\mathcal{C}^{\mathrm{k}}$, that is, $\mathbf{g}(\cdot) \in \mathcal{C}^{\mathrm{k}}(A)$, if $\mathbf{g}(\cdot)$ is continuous on A with k-continuous derivatives. If $\mathbf{g}(\cdot) \in \mathcal{C}^1(A)$, then $\mathbf{g}(\cdot)$ is *continuously differentiable*.

Throughout the paper we use two types of mathematical statements, namely, existential and universal statements. An existential statement has the form: "there exist $\mathbf{x} \in A$ such that condition Φ is satisfied." A universal statement has the form: "condition Φ is satisfied for all $\mathbf{x} \in A$." For universal statements we often omit the words "for all" and write: "condition Φ holds, $\mathbf{x} \in A$."

Time is the only independent variable used in this paper and is denoted by t. In this paper, $t \in [t_1, t_2]$, where $[t_1, t_2] \subset \mathbb{R}$ is a fixed time interval and is a priori assigned. A generic member of a formation of $\mathrm{n} \in \mathbb{N}$ UAVs is identified by the subscript i and, hence, $\mathrm{i} = 1, ..., \mathrm{n}$. We define $\mathbf{r}_{\mathrm{i}} : [t_1, t_2] \to \mathbb{R}^3$ as the *position vector* of the center of mass of the i-th vehicle in a given inertial reference frame, $\boldsymbol{\sigma}_{\mathrm{i}} : [t_1, t_2] \to \mathbb{R}^3$ as the *attitude vector* of the i-th vehicle in modified rodrigues parameters (MRPs) (Shuster, 1993), and $\mathbf{x}_{\mathrm{i}} \triangleq [\mathbf{r}_{\mathrm{i}}^{\mathrm{T}}, \boldsymbol{\sigma}_{\mathrm{i}}^{\mathrm{T}}]^{\mathrm{T}}$ as the *state vector* of the i-th vehicle. The *system's configuration* at time t is defined by $\left[\mathbf{x}_1^{\mathrm{T}}(t), ..., \mathbf{x}_{\mathrm{n}}^{\mathrm{T}}(t)\right]^{\mathrm{T}}$.

The vector $\mathbf{v}_{\mathrm{i}} : [t_1, t_2] \to \mathbb{R}^3$ denotes the *velocity* of the center of mass of the i-th vehicle, $\boldsymbol{\omega}_{\mathrm{i}} : [t_1, t_2] \to \mathbb{R}^3$ denotes the *angular velocity* of the i-th vehicle in a principal body reference frame, and $\widetilde{\mathbf{x}}_{\mathrm{i}} \triangleq \left[\mathbf{r}_{\mathrm{i}}^{\mathrm{T}}, \mathbf{v}_{\mathrm{i}}^{\mathrm{T}}, \boldsymbol{\sigma}_{\mathrm{i}}^{\mathrm{T}}, \boldsymbol{\omega}_{\mathrm{i}}^{\mathrm{T}}\right]^{\mathrm{T}}$ is the *augmented state vector* of the i-th vehicle. For all $t \in [t_1, t_2]$, $\mathbf{r}_{\mathrm{i}}(t) = \int_{t_1}^{t} \mathbf{v}_{\mathrm{i}}(\tau)\,\mathrm{d}\tau$ and $\dot{\boldsymbol{\sigma}}_{\mathrm{i}}(t) = \mathbf{R}_{\mathrm{rod}}(\boldsymbol{\sigma}_{\mathrm{i}}(t))\boldsymbol{\omega}_{\mathrm{i}}(t)$, where $\mathbf{R}_{\mathrm{rod}}(\boldsymbol{\sigma}_{\mathrm{i}}(t)) \triangleq \frac{1}{4}(1 - \boldsymbol{\sigma}_{\mathrm{i}}^{\mathrm{T}}(t)\boldsymbol{\sigma}_{\mathrm{i}}(t))\mathbf{I}_3 + \frac{1}{2}\boldsymbol{\sigma}_{\mathrm{i}}^{\times}(t) + \frac{1}{2}\boldsymbol{\sigma}_{\mathrm{i}}(t)\boldsymbol{\sigma}_{\mathrm{i}}^{\mathrm{T}}(t)$ (Neimark & Fufaev, 1972; Shuster, 1993). We assume $\left[\mathbf{x}_1^{\mathrm{T}}(t), ..., \mathbf{x}_{\mathrm{n}}^{\mathrm{T}}(t)\right]^{\mathrm{T}} \in D_{\mathrm{rel}} \subseteq \mathbb{R}^{6\mathrm{n}}$ and $\left[\widetilde{\mathbf{x}}_1^{\mathrm{T}}(t), ..., \widetilde{\mathbf{x}}_{\mathrm{n}}^{\mathrm{T}}(t)\right]^{\mathrm{T}} \in D_{\mathrm{abs}} \subseteq \mathbb{R}^{12\mathrm{n}}$, $t \in [t_1, t_2]$.

We define $\mathbf{u}_{\mathrm{i,tran}} : [t_1, t_2] \to \Gamma_{\mathrm{i,tran}}$ (respectively, $\mathbf{u}_{\mathrm{i,rot}} : [t_1, t_2] \to \Gamma_{\mathrm{i,rot}}$) as the *translational acceleration* (respectively, the *rotational acceleration*) provided by the control system of the i-th vehicle in the formation, e.g., $\mathbf{u}_{\mathrm{i,tran}}$ is the acceleration provided by the propulsion system and $\mathbf{u}_{\mathrm{i,rot}}$ is the acceleration provided by the ailerons. The vector $\mathbf{u}_{\mathrm{i,tran}}$ (respectively, $\mathbf{u}_{\mathrm{i,rot}}$) is also referred to as the *i-th translational control vector* (respectively, the *i-th rotational control vector*). For a given set of real constants $\rho_{\mathrm{i},1}$, $\rho_{\mathrm{i},2}$, $\rho_{\mathrm{i},3}$, and $\rho_{\mathrm{i},4}$ such that $0 \leq \rho_{\mathrm{i},1} < \rho_{\mathrm{i},2}$ and $0 \leq \rho_{\mathrm{i},3} < \rho_{\mathrm{i},4}$, $\Gamma_{\mathrm{i,tran}}$ and $\Gamma_{\mathrm{i,rot}}$ are defined as

$$\Gamma_{\mathrm{i,tran}} \triangleq \left\{\mathbf{a} \in \mathbb{R}^3 : \rho_{\mathrm{i},1} \leq ||\mathbf{a}||_2 \leq \rho_{\mathrm{i},2}\right\} \cup \{\mathbf{0}_3\},$$
$$\Gamma_{\mathrm{i,rot}} \triangleq \left\{\mathbf{a} \in \mathbb{R}^3 : \rho_{\mathrm{i},3} \leq ||\mathbf{a}||_2 \leq \rho_{\mathrm{i},4}\right\} \cup \{\mathbf{0}_3\}.$$

Finally, for a given set $\Gamma \subset \mathbb{R}^p$, $\mathbf{u} : [t_1, t_2] \to \Gamma$ is an *admissible control in* Γ if *i)* $\mathbf{u}(\cdot)$ is continuous at the endpoints of $[t_1, t_2]$, *ii)* $\mathbf{u}(\cdot)$ is continuous for all $t \in (t_1, t_2)$ with the exception of a finite number of times t at which $\mathbf{u}(\cdot)$ may have discontinuities of the first kind, and *iii)* $\mathbf{u}(\tau) = \lim_{t\to\tau^-} \mathbf{u}(t)$, where $\tau \in [t_1, t_2]$ is a point of discontinuity of first kind for $\mathbf{u}(t)$ (Pontryagin et al., 1962). We assume that $\mathbf{u}_{\mathrm{i,tran}}$ (respectively, $\mathbf{u}_{\mathrm{i,rot}}$) is an admissible control in $\Gamma_{\mathrm{i,tran}}$ (respectively, $\Gamma_{\mathrm{i,rot}}$) for each $\mathrm{i} \in \{1, \ldots, \mathrm{n}\}$.

3. Problem statement

3.1 Fuel consumption performance functional

A measure of the effort needed to control the i-th formation vehicle is given by the *performance functional*

$$\mathrm{J}\left[\mathbf{u}_{\mathrm{i}}(\cdot)\right] \triangleq \int_{t_1}^{t_2} ||\mathbf{u}_{\mathrm{i}}(t)||_2 \,\mathrm{d}t, \tag{1}$$

where $\mathbf{u}_{\mathrm{i}}(t) \triangleq [\mathbf{u}^{\mathrm{T}}_{\mathrm{i,tran}}(t), \mathrm{c}\mathbf{u}^{\mathrm{T}}_{\mathrm{i,rot}}(t)]^{\mathrm{T}}$ and c is a real constant with units of distance. Without loss of generality we assume that $|\mathrm{c}| = 1$. The performance functional $\int_{t_1}^{t_2} ||\mathbf{u}_{\mathrm{i,tran}}(t)||_2 \,\mathrm{d}t$ represents a measure of the fuel consumed over the time interval $[t_1, t_2]$ (Schouwenaars et al., 2006). Path planning for UAV formations is sometimes addressed by minimizing the more conservative performance functional $\int_{t_1}^{t_2} ||\mathbf{u}_{\mathrm{i,tran}}(t)||_1 \,\mathrm{d}t$ (Blackmore, 2008). It is important to note that $||\mathbf{u}_{\mathrm{i,rot}}(t)||_2$ is much smaller than $||\mathbf{u}_{\mathrm{i,tran}}(t)||_2$ for conventional aircraft and, hence, its contribution to the performance functional (1) is negligible. However, this assumption does not hold for the case of μUAVs (Bataillé et al., 2009).

The control effort for the entire formation can be captured by the performance measure

$$\mathrm{J}_{\mathrm{formation}}\left[\tilde{\mathbf{u}}(\cdot)\right] \triangleq \sum_{\mathrm{i}=1}^{\mathrm{n}} \mu_{\mathrm{i}} \mathrm{J}\left[\mathbf{u}_{\mathrm{i}}(\cdot)\right], \tag{2}$$

where $\tilde{\mathbf{u}}(t) \triangleq [\mathbf{u}^{\mathrm{T}}_1(t), \ldots, \mathbf{u}^{\mathrm{T}}_{\mathrm{n}}(t)]^{\mathrm{T}}$ and $\mu_{\mathrm{i}} \in [0, 1]$, with $\sum_{\mathrm{i}=1}^{\mathrm{n}} \mu_{\mathrm{i}} = 1$, which represents the relative importance of minimizing the control effort of the i-th vehicle with respect to the others.

3.2 Aircraft dynamic equations

Aircraft are subject to external forces and moments from the environment. Specifically, an aerial vehicle is subject to gravitational forces, aerodynamic forces, and aerodynamic moments. Accelerations induced by external forces and external moments acting on a formation vehicle are denoted by $\mathbf{a} : \mathbb{R}^{12} \to \mathbb{R}^3$ and $\mathbf{m} : \mathbb{R}^{12} \to \mathbb{R}^3$, respectively, where $\mathbf{a}\left(\widetilde{\mathbf{x}}_{\mathrm{i}}\right), \mathbf{m}\left(\widetilde{\mathbf{x}}_{\mathrm{i}}\right) \in \mathcal{C}^1\left(\mathbb{R}^{12}\right)$.

The unconstrained dynamic equations for the i-th vehicle are given by (Greenwood, 2003)

$$\frac{\mathrm{d}}{\mathrm{dt}}\widetilde{\mathbf{x}}_{\mathrm{i}}(t) = \begin{bmatrix} \mathbf{v}_{\mathrm{i}}(t) \\ \mathbf{a}\left(\widetilde{\mathbf{x}}_{\mathrm{i}}(t)\right) \\ \mathbf{R}_{\mathrm{rod}}(\boldsymbol{\sigma}_{\mathrm{i}}(t))\boldsymbol{\omega}_{\mathrm{i}}(t) \\ -\mathbf{I}^{-1}_{\mathrm{in,i}}\boldsymbol{\omega}^{\times}_{\mathrm{i}}\left(\boldsymbol{\omega}_{\mathrm{i}}(t)\right)\mathbf{I}_{\mathrm{in,i}}\boldsymbol{\omega}_{\mathrm{i}}(t) + \widetilde{\boldsymbol{\omega}}_{\mathrm{i}}\left(\widetilde{\mathbf{x}}_{\mathrm{i}}(t)\right) \end{bmatrix} + \begin{bmatrix} \mathbf{0}_3 \\ \mathbf{u}_{\mathrm{i,tran}}(t) \\ \mathbf{0}_3 \\ \mathbf{u}_{\mathrm{i,rot}}(t) \end{bmatrix}, \tag{3}$$

where $\mathbf{I}_{\mathrm{in,i}}$ is the inertia matrix of the i-th vehicle in a principal body reference frame and $\widetilde{\boldsymbol{\omega}}_{\mathrm{i}}\left(\widetilde{\mathbf{x}}_{\mathrm{i}}(t)\right) \triangleq \mathbf{I}_{\mathrm{in,i}}^{-1}\mathbf{m}\left(\widetilde{\mathbf{x}}_{\mathrm{i}}(t)\right)$, $t \in [t_1, t_2]$. The boundary conditions for (3) are given by the endpoint constraints discussed in Section 3.3.

3.3 Formation constraints

Given $D_1 \subset \mathbb{R}^{\mathrm{p}}$ and $D_2 \subset \mathbb{R}^{\mathrm{m}}$, the function $\mathbf{S} : D_1 \to D_2$ is a *continuously differentiable manifold* if $\mathbf{S}(\mathbf{y}) = 0$, $\mathrm{m} < \mathrm{p}$, $\mathbf{S}(\mathbf{y}) \in \mathcal{C}^1(D_1)$, and rank $\frac{\partial \mathbf{S}(\mathbf{y})}{\partial \mathbf{y}} = \mathrm{m}$ (Pontryagin et al., 1962). Let $\mathbf{S}_1 : D_{\mathrm{abs}} \to \mathbb{R}^{\mathrm{n}_1}$ and $\mathbf{S}_2 : D_{\mathrm{abs}} \to \mathbb{R}^{\mathrm{n}_2}$ be two continuously differentiable manifolds, and define the *endpoint constraints*

$$\begin{aligned} \mathbf{S}_1\left(\left[\widetilde{\mathbf{x}}_1^{\mathrm{T}}(t_1), \ldots, \widetilde{\mathbf{x}}_{\mathrm{n}}^{\mathrm{T}}(t_1)\right]^{\mathrm{T}}\right) &= \mathbf{0}_{\mathrm{r}_1}, \\ \mathbf{S}_2\left(\left[\widetilde{\mathbf{x}}_1^{\mathrm{T}}(t_2), \ldots, \widetilde{\mathbf{x}}_{\mathrm{n}}^{\mathrm{T}}(t_2)\right]^{\mathrm{T}}\right) &= \mathbf{0}_{\mathrm{r}_2}. \end{aligned} \tag{4}$$

Endpoint constraints partly impose the formation's configuration at times t_1 and t_2, and hence, can model point-to-point or rendezvous maneuvers.

State inequality constraints are given by

$$\mathbf{f}_{\mathrm{ineq}}(\mathbf{x}_1(t), \ldots, \mathbf{x}_{\mathrm{n}}(t)) \leq\leq \mathbf{0}_{\mathrm{r}_3}, \tag{5}$$

where $\mathbf{f}_{\mathrm{ineq}} : D_{\mathrm{rel}} \to \mathbb{R}^{\mathrm{n}_3}$ and $\mathbf{f}_{\mathrm{ineq}}(\mathbf{x}_1, \ldots, \mathbf{x}_{\mathrm{n}}) \in \mathcal{C}^3(\mathrm{int}\,(D_{\mathrm{rel}}))$. *State equality constraints* are given by

$$\mathbf{f}_{\mathrm{eq}}(t, \mathbf{x}_1(t), \ldots, \mathbf{x}_{\mathrm{n}}(t)) = \mathbf{0}_{\mathrm{r}_4}, \tag{6}$$

where $\mathbf{f}_{\mathrm{eq}} : [t_1, t_2] \times D_{\mathrm{rel}} \to \mathbb{R}^{\mathrm{n}_4}$ and $\mathbf{f}_{\mathrm{eq}}(t, \mathbf{x}_1, \ldots, \mathbf{x}_{\mathrm{n}}) \in \mathcal{C}^2((t_1, t_2) \times \mathrm{int}\,(D_{\mathrm{rel}}))$. Here we assume that the constraints are *compatible*, that is, for all $t \in [t_1, t_2]$ there exists at least one set of 2n admissible controls $\{\mathbf{u}_{1,\mathrm{tran}}(t), \ldots, \mathbf{u}_{\mathrm{n,tran}}(t); \mathbf{u}_{1,\mathrm{rot}}(t), \ldots, \mathbf{u}_{\mathrm{n,rot}}(t)\}$ that satisfies (3) – (6).

State constraints given in terms of $\widetilde{\mathbf{x}}_1(t)$,..., $\widetilde{\mathbf{x}}_{\mathrm{n}}(t)$ that can be reduced to the form given by (5) and (6) are called *holonomic* constraints. In particular, for $\mathrm{n} = 2$ and $t \in [t_1, t_2]$, the constraint $\mathbf{v}_1(t) = \mathbf{v}_2(t)$ is holonomic since it can be rewritten as $\mathbf{r}_1(t) + \mathbf{r}_1(t_1) = \mathbf{r}_2(t) + \mathbf{r}_2(t_1)$, $t \in [t_1, t_2]$. It is important to note that the constraint $\boldsymbol{\omega}_1(t) \leq\leq \boldsymbol{\omega}_2(t)$, $t \in [t_1, t_2]$, is nonholonomic since $\boldsymbol{\sigma}_{\mathrm{i}}(t) \neq \int_{t_1}^{t} \boldsymbol{\omega}_{\mathrm{i}}(\tau)\,\mathrm{d}\tau + \boldsymbol{\sigma}_{\mathrm{i}}(t_1)$, $t \in [t_1, t_2]$ and $\mathrm{i} = 1, 2$ (Greenwood, 2003).

State constraints can model collision avoidance, keeping the formation far from no-fly zones, or the requirement of pointing payloads toward the same target. It is obvious that (6) is a special case of (5); however, as noted in Section 4.2, this distinction is useful in reducing computational complexity.

3.4 Path planning optimization problem

For all $\mathrm{i} = 1, \ldots, \mathrm{n}$ and $t \in [t_1, t_2]$ find the control vectors $\mathbf{u}_{\mathrm{i,tran}}(t)$ and $\mathbf{u}_{\mathrm{i,rot}}(t)$ among all admissible controls in $\Gamma_{\mathrm{i,tran}}$ and $\Gamma_{\mathrm{i,tran}}$ such that the performance measure (2) is minimized and $\widetilde{\mathbf{x}}_{\mathrm{i}}(t)$ satisfies (3) – (6).

4. Mathematical background

4.1 Slack variables

Inequality constraints (5) can be reduced to equality constraints by introducing $\mathbf{s} : [t_1, t_2] \to \mathbb{R}^{n_3}$ such that $\mathbf{s}(t) \in \mathcal{C}^2(t_1, t_2)$ and $\mathbf{f}_{\rm ineq}(t, \mathbf{x}_1(t), ..., \mathbf{x}_{\rm n}(t)) + \frac{1}{2}{\rm diag}\left(\mathbf{s}\mathbf{s}^{\rm T}\right) = \mathbf{0}_{\rm r_3}$. The components of $\mathbf{s}$ are called *slack variables*. Thus, (5) can be rewritten as (Valentine, 1937)

$$\widetilde{\mathbf{f}}_{\rm ineq}(\mathbf{s}(t), \mathbf{x}_1(t), ..., \mathbf{x}_{\rm n}(t)) = \mathbf{0}_{\rm r_3}, \tag{7}$$

where $\widetilde{\mathbf{f}}_{\rm ineq}\left(\mathbf{s}(t), \mathbf{x}_1(t), ..., \mathbf{x}_{\rm n}(t)\right) \triangleq \mathbf{f}_{\rm ineq}(\mathbf{x}_1(t), ..., \mathbf{x}_{\rm n}(t)) + \frac{1}{2}{\rm diag}\left(\mathbf{s}\mathbf{s}^{\rm T}\right)$.

4.2 Lagrange coordinates

The following theorem is needed for the main results of this paper.

Theorem 4.1. *(Pars, 1965) Let* $D_{\rm q} \subseteq \mathbb{R}^{6n-n_4}$ *be an open connected set and let* $\mathbf{q} : [t_1, t_2] \times \mathbb{R}^{n_3} \times D_{\rm rel} \to D_q$ *be such that* $\mathbf{q}\left(t, \mathbf{s}(t), \mathbf{x}_1(t), \dots, \mathbf{x}_{\rm n}(t)\right) \in \mathcal{C}^2\left((t_1, t_2) \times \mathbb{R}^{n_3} \times {\rm int}\left(D_{\rm rel}\right)\right)$. *Assume that*

$$\det\left(\frac{\partial\left[\widetilde{\mathbf{f}}^{\rm T}_{\rm ineq}\left(\mathbf{s}, \mathbf{x}_1, ..., \mathbf{x}_{\rm n}\right) \mathbf{f}^{\rm T}_{\rm eq}\left(t, \mathbf{s}, \mathbf{x}_1, ..., \mathbf{x}_{\rm n}\right) \mathbf{q}^{\rm T}\left(t, \mathbf{s},\ \mathbf{x}_1,\ ...,\ \mathbf{x}_{\rm n}\right)\right]^{\rm T}}{\partial\left[\mathbf{s}^{\rm T},\ \mathbf{x}_1^{\rm T},\ ...,\ \mathbf{x}_{\rm n}^{\rm T}\right]^{\rm T}}\right) \neq 0 \tag{8}$$

for all $(t, \mathbf{s}, \mathbf{x}_1 ..., \mathbf{x}_{\rm n}) \in \mathcal{I} \times \Delta$, *where* $\mathcal{I} \subset (t_1, t_2)$ *and* $\Delta \subset \mathbb{R}^{n_3} \times D_{\rm rel}$ *are open connected sets. Then* $\mathbf{q}$ *can be rewritten as a function of t, that is,* $\mathbf{q} : \mathcal{I} \to D_{\rm q}$, *and* $\mathbf{s}$, $\mathbf{x}_1$, ...,$\mathbf{x}_{\rm n}$, $\widetilde{\mathbf{x}}_1$,..., $\widetilde{\mathbf{x}}_{\rm n}$ *can be rewritten as unique functions of t and* $\mathbf{q}$, *that is,* $\mathbf{s} : \mathcal{I} \times D_{\rm q} \to \mathbb{R}^{n_3}$, $\mathbf{x}_{\rm i} : \mathcal{I} \times D_{\rm q} \to \mathbb{R}^6$, *and* $\widetilde{\mathbf{x}}_{\rm i} : \mathcal{I} \times D_{\rm q} \to \mathbb{R}^{12}$ *for all* ${\rm i} = 1, ..., {\rm n}$ *and* $(t, \mathbf{s}, \mathbf{x}_1 ..., \mathbf{x}_{\rm n}) \in \mathcal{I} \times \Delta$. *Furthermore, the components of* $\mathbf{q}$ *are independent and uniquely characterize the system's configuration.*

Under the hypothesis of Theorem 4.1, the components of $\mathbf{q}(t)$ are called *Lagrange coordinates*. As will be shown in Section 4.3, the key advantage of using Lagrange coordinates is that the constraints (5) – (7) are automatically accounted for when rewriting the formation's dynamic equations in terms of t and $\mathbf{q}(t)$ (Pars, 1965). In this paper, we assume that $\mathbf{s}$, $\mathbf{x}_1, \dots, \mathbf{x}_{\rm n}$, $\widetilde{\mathbf{x}}_1, \dots, \widetilde{\mathbf{x}}_{\rm n}$ are explicit functions of $\mathbf{q}$ only and not t, which occurs in most practical applications (Pars, 1965). In practice, given constraints in the form of (6) and (7), $\mathbf{q}$ is chosen such that Theorem 4.1 holds. As will be further discussed in Section 4.3, we select $\mathbf{q}\left(t, \mathbf{s}(t), \mathbf{x}_1(t), \dots, \mathbf{x}_{\rm n}(t)\right)$ as an explicit function of $\left(\mathbf{s}(t), \mathbf{x}_1(t), \dots, \mathbf{x}_{\rm n}(t)\right)$.

Given $\mathbf{q}\left(t, \mathbf{s}(t), \mathbf{x}_1(t), \dots, \mathbf{x}_{\rm n}(t)\right)$, $\dot{\mathbf{q}}$ is a function of $\mathbf{s}(t)$, $\mathbf{r}_{\rm i}(t)$, $\boldsymbol{\sigma}_{\rm i}(t)$, $i = 1, ..., n$, and their first time derivatives. In practice, however, we measure $\boldsymbol{\omega}_{\rm i}(t)$ rather than $\boldsymbol{\sigma}_{\rm i}(t)$, and hence, if the assumptions of Theorem 4.1 hold, we define the *kinematic equation*

$$\mathbf{q}_{\rm dot}(t) \triangleq \boldsymbol{\Psi}\left(\mathbf{q}(t)\right)\dot{\mathbf{q}}(t) + \psi\left(\mathbf{q}(t)\right), \tag{9}$$

where $\boldsymbol{\omega}_{\rm i}(t)$, ${\rm i} = 1, 2, \dots, {\rm n}$, explicitly appears in $\mathbf{q}_{\rm dot}(t)$, $\boldsymbol{\Psi} : D_{\rm q} \to \mathbb{R}^{(6n-n_4)\times(6n-n_4)}$ is an invertible continuously differentiable matrix function, and $\psi : D_{\rm q} \to \mathbb{R}^{6n-n_4}$ is continuously differentiable. Consequently, $\mathbf{s}$, $\mathbf{x}_1$, ..., $\mathbf{x}_{\rm n}$, $\widetilde{\mathbf{x}}_1$, ..., $\widetilde{\mathbf{x}}_{\rm n}$ can be rewritten as unique functions of $\mathbf{q}$ and $\mathbf{q}_{\rm dot}$, that is, $\mathbf{s} : D_{\rm q} \times \mathbb{R}^{6n-n_4} \to \mathbb{R}^{n_3}$, $\mathbf{x}_{\rm i} : D_{\rm q} \times \mathbb{R}^{6n-n_4} \to \mathbb{R}^6$, and $\widetilde{\mathbf{x}}_{\rm i} : D_{\rm q} \times \mathbb{R}^{6n-n_4} \to \mathbb{R}^{12}$, $(t, \mathbf{s}, \mathbf{x}_1 ..., \mathbf{x}_{\rm n}) \in \mathcal{I} \times \Delta$ (Greenwood, 2003). Here, we assume that $\mathbf{q}_{\rm dot}$ satisfies (23) below.

In the following we assume that the path planning optimization problem can be solved over the time interval $\left[t_1^*, t_2^*\right] \supset [t_1, t_2]$ and that the given set of Lagrange coordinates can be defined on the open connected set $\widetilde{\mathcal{I}}$, where $[t_1, t_2] \subset \widetilde{\mathcal{I}} \subset \left(t_1^*, t_2^*\right)$. Thus, (4) can be rewritten as

$$\mathbf{S}_1\left(\left[\widetilde{\mathbf{x}}_1^{\mathrm{T}}\left(\mathbf{q}(t_1), \mathbf{q}_{\mathrm{dot}}(\mathbf{q}(t_1))\right), ..., \widetilde{\mathbf{x}}_{\mathrm{n}}^{\mathrm{T}}\left(\mathbf{q}(t_1), \mathbf{q}_{\mathrm{dot}}(\mathbf{q}(t_1))\right)\right]^{\mathrm{T}}\right) = \mathbf{0}_{\mathrm{r}_1}, \tag{10}$$

$$\mathbf{S}_2\left(\left[\widetilde{\mathbf{x}}_1^{\mathrm{T}}\left(\mathbf{q}(t_2), \mathbf{q}_{\mathrm{dot}}(\mathbf{q}(t_2))\right), ..., \widetilde{\mathbf{x}}_{\mathrm{n}}^{\mathrm{T}}\left(\mathbf{q}(t_2), \mathbf{q}_{\mathrm{dot}}(\mathbf{q}(t_2))\right)\right]^{\mathrm{T}}\right) = \mathbf{0}_{\mathrm{r}_2}. \tag{11}$$

Example 4.1. Consider a UAV formation with two vehicles so that $\mathrm{n} = 2$. Assume that

$$\mathbf{f}_{\mathrm{ineq}}\left(\mathbf{x}_1(t), \mathbf{x}_2(t)\right) = \begin{bmatrix} ||\mathbf{r}_1(t) - \mathbf{r}_2(t)||_2^2 - \mathrm{r}_{\max} \\ \mathrm{r}_{\min} - ||\mathbf{r}_1(t) - \mathbf{r}_2(t)||_2^2 \end{bmatrix} \leq\leq \mathbf{0}_2, \tag{12}$$

$$\mathbf{f}_{\mathrm{eq}}\left(t, \mathbf{x}_1(t), \mathbf{x}_2(t)\right) = \boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t) = \mathbf{0}_3, \tag{13}$$

$$\mathbf{S}_1\left(\left[\widetilde{\mathbf{x}}_1^{\mathrm{T}}(t_1)\, \widetilde{\mathbf{x}}_2^{\mathrm{T}}(t_1)\right]^{\mathrm{T}}\right) = \begin{bmatrix} ||\mathbf{r}_1(t_1) - \mathbf{r}_2(t_1)||_2^2 - \left(\frac{\mathrm{r}_{\max}+\mathrm{r}_{\min}}{2}\right) \\ \boldsymbol{\sigma}_1(t_1) - \boldsymbol{\sigma}_2(t_1) \end{bmatrix} = \mathbf{0}_4, \tag{14}$$

$$\mathbf{S}_2\left(\left[\widetilde{\mathbf{x}}_1^{\mathrm{T}}(t_2)\, \widetilde{\mathbf{x}}_2^{\mathrm{T}}(t_2)\right]^{\mathrm{T}}\right) = \begin{bmatrix} ||\mathbf{r}_1(t_2) - \mathbf{r}_2(t_2)||_2^2 - \frac{2(\mathrm{r}_{\max}-\mathrm{r}_{\min})}{3} \\ \boldsymbol{\sigma}_1(t_2) - \boldsymbol{\sigma}_2(t_2) \end{bmatrix} = \mathbf{0}_4, \tag{15}$$

where $\mathrm{r}_{\min}$ and $\mathrm{r}_{\max}$ are real constants such that $0 < \mathrm{r}_{\min} < \mathrm{r}_{\max}$. Equation (12) ensures that $\mathrm{r}_{\min} \leq ||\mathbf{r}_1(t) - \mathbf{r}_2(t)||_2^2 \leq \mathrm{r}_{\max}$ and (13) ensures that both vehicles always have the same attitude: $D_{\mathrm{rel}} = \left\{\left[\mathbf{x}_1^{\mathrm{T}}(t)\, \mathbf{x}_2^{\mathrm{T}}(t)\right]^{\mathrm{T}} : \mathrm{r}_{\min} \leq ||\mathbf{r}_1(t) - \mathbf{r}_2(t)||_2^2 \leq \mathrm{r}_{\max}, \boldsymbol{\sigma}_1(t) = \boldsymbol{\sigma}_2(t), t \in [t_1, t_2]\right\}$.

Introducing the slack variables $\mathrm{s}_1 : [t_1, t_2] \to \mathbb{R}$ and $\mathrm{s}_2 : [t_1, t_2] \to \mathbb{R}$, (12) becomes

$$\widetilde{\mathbf{f}}_{\mathrm{ineq}}(\mathbf{s}(t), \mathbf{x}_1(t), \mathbf{x}_2(t)) = \begin{bmatrix} ||\mathbf{r}_1(t) - \mathbf{r}_2(t)||_2^2 - \mathrm{r}_{\max} + \frac{1}{2}\mathrm{s}_1^2(t) \\ \mathrm{r}_{\min} - ||\mathbf{r}_1(t) - \mathbf{r}_2(t)||_2^2 + \frac{1}{2}\mathrm{s}_2^2(t) \end{bmatrix} = \mathbf{0}_2. \tag{16}$$

As noted in Section 3.3, the equality constraint (13) can be embedded into (12) to give

$$\widetilde{\mathbf{f}}_{\mathrm{ineq}}(\mathbf{s}(t), \mathbf{x}_1(t), \mathbf{x}_2(t)) = \begin{bmatrix} ||\mathbf{r}_1(t) - \mathbf{r}_2(t)||_2^2 - \mathrm{r}_{\max} + \frac{1}{2}\mathrm{s}_1^2(t) \\ \mathrm{r}_{\min} - ||\mathbf{r}_1(t) - \mathbf{r}_2(t)||_2^2 + \frac{1}{2}\mathrm{s}_2^2(t) \\ \boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t) + \frac{1}{2}\mathrm{diag}(\mathbf{s}_3\mathbf{s}_3^{\mathrm{T}}) \\ \boldsymbol{\sigma}_2(t) - \boldsymbol{\sigma}_1(t) + \frac{1}{2}\mathrm{diag}(\mathbf{s}_4\mathbf{s}_4^{\mathrm{T}}) \end{bmatrix} = \mathbf{0}_8,$$

where $\mathbf{s}_{\mathrm{j}} : [t_1, t_2] \to \mathbb{R}^3$, $\mathrm{j} = 3, 4$. Note that in this case, the dimension of $\widetilde{\mathbf{f}}_{\mathrm{ineq}}$ is increased since six additional slack variables have been introduced, which increases computational complexity.

Next, define $\mathrm{r}_{\mathrm{i,j}} : [t_1, t_2] \to \mathbb{R}$ (respectively, $\sigma_{\mathrm{i,j}} : [t_1, t_2] \to \mathbb{R}$) as the j-th component of $\mathbf{r}_{\mathrm{i}}(t)$ (respectively, $\boldsymbol{\sigma}_{\mathrm{i}}(t)$). If $\mathbf{q}(t) = \left[s_1(t), s_2(t), \mathbf{r}_1^{\mathrm{T}}(t), \boldsymbol{\sigma}_1^{\mathrm{T}}(t), r_{2,1}(t)\right]^{\mathrm{T}}$, then (8) gives

$$\det\left(\frac{\partial\left[\widetilde{\mathbf{f}}_{\mathrm{ineq}}^{\mathrm{T}}\left(\mathbf{s}, \mathbf{x}_1, \mathbf{x}_2\right), \mathbf{f}_{\mathrm{eq}}^{\mathrm{T}}\left(t, \mathbf{s}, \mathbf{x}_1, \mathbf{x}_2\right), \mathbf{q}^{\mathrm{T}}\left(t, \mathbf{s}, \mathbf{x}_1, \mathbf{x}_2\right)\right]^{\mathrm{T}}}{\partial\left[\mathbf{s}^{\mathrm{T}}, \mathbf{x}_1^{\mathrm{T}}, \mathbf{x}_2^{\mathrm{T}}\right]^{\mathrm{T}}}\right) = 0.$$

Thus, by Theorem 4.1, the components of $\mathbf{q}$ are not Lagrange coordinates.

Alternatively, if $\mathbf{q}(t) = \left[\mathrm{s}_1(t),\, \mathrm{r}_{1,1}(t),\, \mathrm{r}_{1,2}(t),\, \boldsymbol{\sigma}_1^{\mathrm{T}}(t),\, \mathbf{r}_2^{\mathrm{T}}(t)\right]^{\mathrm{T}}$, then

$$\det\left(\frac{\partial\left[\widetilde{\mathbf{f}}_{\mathrm{ineq}}^{\mathrm{T}}\left(\mathbf{s},\mathbf{x}_1,\mathbf{x}_2\right),\, \mathbf{f}_{\mathrm{eq}}^{\mathrm{T}}\left(t,\mathbf{s},\mathbf{x}_1,\mathbf{x}_2\right),\, \mathbf{q}^{\mathrm{T}}\left(t,\mathbf{s},\mathbf{x}_1,\mathbf{x}_2\right)\right]^{\mathrm{T}}}{\partial\left[\mathbf{s}^{\mathrm{T}},\, \mathbf{x}_1^{\mathrm{T}},\, \mathbf{x}_2^{\mathrm{T}}\right]^{\mathrm{T}}}\right) = -2\mathrm{s}_2(t)\left(\mathrm{r}_{1,3}(t) - \mathrm{r}_{2,3}(t)\right),$$

for all $(t,\mathbf{s},\mathbf{x}_1,\mathbf{x}_2) \in (t_1,t_2) \times \mathbb{R}^2 \times \operatorname{int}(D_{\mathrm{rel}})$ such that $\mathrm{r}_{1,3}(t) \neq \mathrm{r}_{2,3}(t)$, and hence, the components of $\mathbf{q}$ are suitable Lagrange coordinates if $\mathrm{r}_{\min} < ||\mathbf{r}_1(t) - \mathbf{r}_2(t)||_2^2$ and $\mathrm{r}_{1,3}(t) \neq \mathrm{r}_{2,3}(t)$. In this case, (9) gives

$$\mathbf{q}_{\mathrm{dot}}(t) = \begin{bmatrix} \dot{s}_1(t) \\ \mathrm{v}_{1,1}(t) \\ \mathrm{v}_{1,2}(t) \\ \boldsymbol{\omega}_1(t) \\ \mathbf{v}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{R}_{\mathrm{rod}}^{-1}(\boldsymbol{\sigma}_1(t)) & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \dot{s}_1(t) \\ \mathrm{v}_{1,1}(t) \\ \mathrm{v}_{1,2}(t) \\ \dot{\boldsymbol{\sigma}}_1(t) \\ \mathbf{v}_2(t) \end{bmatrix}, \tag{17}$$

where $\mathrm{v}_{1,\mathrm{j}} : [t_1,t_2] \to \mathbb{R}$ is the j-th component of $\mathbf{v}_1(t)$.

A more suitable choice of Lagrange coordinates is given by $\mathbf{q}(t) = \left[\mathbf{x}_1^{\mathrm{T}}(t),\, \mathbf{r}_2^{\mathrm{T}}(t)\right]^{\mathrm{T}}$ since

$$\det\left(\frac{\partial\left[\widetilde{\mathbf{f}}_{\mathrm{ineq}}^{\mathrm{T}}\left(\mathbf{s},\mathbf{x}_1,\mathbf{x}_2\right),\, \mathbf{f}_{\mathrm{eq}}^{\mathrm{T}}\left(t,\mathbf{s},\mathbf{x}_1,\mathbf{x}_2\right),\, \mathbf{q}^{\mathrm{T}}\left(t,\mathbf{s},\mathbf{x}_1,\mathbf{x}_2\right)\right]^{\mathrm{T}}}{\partial\left[\mathbf{s}^{\mathrm{T}}\ \mathbf{x}_1^{\mathrm{T}},\, \mathbf{x}_2^{\mathrm{T}}\right]^{\mathrm{T}}}\right) = s_1(t)s_2(t)$$

for all $(t,\mathbf{s},\mathbf{x}_1,\mathbf{x}_2) \in (t_1,t_2) \times \mathbb{R}^2 \times \operatorname{int}(D_{\mathrm{rel}})$, and hence, the components of $\mathbf{q}$ are suitable Lagrange coordinates if $\mathrm{r}_{\min} < ||\mathbf{r}_1(t) - \mathbf{r}_2(t)||_2^2 < \mathrm{r}_{\max}$. In this case, (9) gives

$$\mathbf{q}_{\mathrm{dot}}(t) = \begin{bmatrix} \mathbf{v}_1(t) \\ \boldsymbol{\omega}_1(t) \\ \mathbf{v}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{R}_{\mathrm{rod}}^{-1}(\boldsymbol{\sigma}_1(t)) & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1(t) \\ \dot{\boldsymbol{\sigma}}_1(t) \\ \mathbf{v}_2(t) \end{bmatrix}. \tag{18}$$

Since we use this example throughout the paper, we define $\mathbf{q}_{\mathrm{dot},1} \triangleq [\mathbf{v}_1^{\mathrm{T}},\, \boldsymbol{\omega}_1^{\mathrm{T}}]^{\mathrm{T}}$, $\mathbf{q}_{\mathrm{dot},2} \triangleq \mathbf{v}_2$, and

$$\boldsymbol{\Psi}_1(\mathbf{x}_1(t)) \triangleq \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{R}_{\mathrm{rod}}^{-1}(\boldsymbol{\sigma}_1(t)) \end{bmatrix}.$$

Finally, note that if $\mathrm{r}_{\min} < ||\mathbf{r}_1(t) - \mathbf{r}_2(t)||_2^2 < \mathrm{r}_{\max}$ for $t \in \left(t_1^*, t_2^*\right) \supset [t_1,t_2]$, then (14) and (15) reduce to

$$||\mathbf{r}_1(t_1) - \mathbf{r}_2(t_1)||_2^2 - \left(\frac{\mathrm{r}_{\max} + \mathrm{r}_{\min}}{2}\right) = 0, \tag{19}$$

$$||\mathbf{r}_1(t_2) - \mathbf{r}_2(t_2)||_2^2 - \frac{2\left(\mathrm{r}_{\max} - \mathrm{r}_{\min}\right)}{3} = 0. \tag{20}$$

4.3 Constrained formation dynamic equations

The formation's kinetic energy is given by *König's theorem* (Pars, 1965) and for our problem takes the form

$$
\begin{aligned}
\mathrm{k}\left(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)\right) = {} & \frac{1}{2}\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{m}_{\mathrm{i}} \mathbf{v}_{\mathrm{i}}^{\mathrm{T}}\left(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)\right) \mathbf{v}_{\mathrm{i}}\left(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)\right) \\
& + \frac{1}{2}\sum_{\mathrm{i}=1}^{\mathrm{n}} \boldsymbol{\omega}_{\mathrm{i}}^{\mathrm{T}}\left(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)\right) \mathbf{I}_{\mathrm{in,i}} \boldsymbol{\omega}_{\mathrm{i}}\left(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)\right),
\end{aligned} \tag{21}
$$

where $\mathrm{m_i}$ is the mass of the i-th vehicle, which is assumed to be constant. The dynamic equations of the constrained formation can be written in terms of Lagrange coordinates by applying the *Boltzmann-Hammel equation* (Greenwood, 2003) to give

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathrm{k}\left(\mathbf{q}, \mathbf{q}_{\mathrm{dot}}\right)}{\partial \mathbf{q}_{\mathrm{dot}}}\right) = {} & \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{m}_{\mathrm{i}} \mathbf{v}_{\mathrm{i}}^{\mathrm{T}}\left(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)\right) \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathbf{v}_{\mathrm{i}}\left(\mathbf{q}, \mathbf{q}_{\mathrm{dot}}\right)}{\partial \mathbf{q}_{\mathrm{dot}}} \\
& + \sum_{\mathrm{i}=1}^{\mathrm{n}} \boldsymbol{\omega}_{\mathrm{i}}^{\mathrm{T}}\left(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)\right) \mathbf{I}_{\mathrm{in,i}} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \boldsymbol{\omega}_{\mathrm{i}}\left(\mathbf{q}, \mathbf{q}_{\mathrm{dot}}\right)}{\partial \mathbf{q}_{\mathrm{dot}}} \\
& + \sum_{\mathrm{i}=1}^{\mathrm{n}} \left(\mathbf{a}\left(\tilde{\mathbf{x}}_{\mathrm{i}}\left(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)\right)\right) + \mathbf{u}_{\mathrm{i,tran}}(t)\right) \frac{\partial \mathbf{v}_{\mathrm{i}}\left(\mathbf{q}, \mathbf{q}_{\mathrm{dot}}\right)}{\partial \mathbf{q}_{\mathrm{dot}}} \\
& + \sum_{\mathrm{i}=1}^{\mathrm{n}} \left(\mathbf{m}\left(\tilde{\mathbf{x}}_{\mathrm{i}}\left(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)\right)\right) + \mathbf{u}_{\mathrm{i,rot}}(t)\right) \frac{\partial \boldsymbol{\omega}_{\mathrm{i}}\left(\mathbf{q}, \mathbf{q}_{\mathrm{dot}}\right)}{\partial \mathbf{q}_{\mathrm{dot}}}.
\end{aligned} \tag{22}
$$

Equations (10) and (11) are the boundary conditions for (22). It is important to note that the dynamic equation (22) is written in terms of Lagrange coordinates, and hence, accounts for (5) and (6).

Analytical optimization techniques such as Pontryagin's minimum principle, Bellman's theorem, and calculus of variations require the dynamic equations to be written as a first-order ordinary differential equation in explicit form. Therefore, using the hypothesis on $\mathbf{q}_{\mathrm{dot}}$, the second-order ordinary differential equation (22) needs to be written in a first-order form

$$
\dot{\mathbf{q}}_{\mathrm{dot}}(t) = \mathbf{f}_{\mathrm{dyn}}(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t), \tilde{\mathbf{u}}(t)), \tag{23}
$$

where $\tilde{\mathbf{u}}(t) \triangleq [\mathbf{u}_1^{\mathrm{T}}(t), \ldots, \mathbf{u}_{\mathrm{n}}^{\mathrm{T}}(t)]^{\mathrm{T}}$ and $\mathbf{f}_{\mathrm{dyn}} : D_{\mathrm{q}} \times \mathbb{R}^{6\mathrm{n}-\mathrm{n}_4} \times \mathbb{R}^{12\mathrm{n}} \to \mathbb{R}^{6\mathrm{n}-\mathrm{n}_4}$. In order to isolate the contribution of $\tilde{\mathbf{u}}$ in (24), we define $\hat{\mathbf{f}}_{\mathrm{dyn}}(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)) \triangleq \mathbf{f}_{\mathrm{dyn}}(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t), \tilde{\mathbf{u}}(t)) - \mathbf{u}_{\mathrm{i,tran}}(t)\frac{\partial \mathbf{v}_{\mathrm{i}}(\mathbf{q}, \mathbf{q}_{\mathrm{dot}})}{\partial \mathbf{q}_{\mathrm{dot}}} - \mathbf{u}_{\mathrm{i,rot}}(t)\frac{\partial \boldsymbol{\omega}_{\mathrm{i}}(\mathbf{q}, \mathbf{q}_{\mathrm{dot}})}{\partial \mathbf{q}_{\mathrm{dot}}}$.

Equation (22) or, equivalently, (23) gives a set of $6\mathrm{n} - \mathrm{n}_4$ equations in $2(6\mathrm{n} - \mathrm{n}_4)$ unknowns, which are $\mathbf{q}$ and $\mathbf{q}_{\mathrm{dot}}$. Thus, (22) needs to be solved together with (9) (Greenwood, 2003) to give

$$
\begin{bmatrix} \mathbf{q}_{\mathrm{dot}}(t) \\ \dot{\mathbf{q}}_{\mathrm{dot}}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Psi}\left(\mathbf{q}(t)\right)\dot{\mathbf{q}}(t) + \boldsymbol{\psi}\left(\mathbf{q}(t)\right) \\ \mathbf{f}_{\mathrm{dyn}}(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t), \tilde{\mathbf{u}}(t)) \end{bmatrix}. \tag{24}
$$

From (21) it follows that the formation's kinetic energy k is not an explicit function of $\mathbf{s}$, and hence, if $\mathbf{q}$ is chosen as an explicit function of p components of $\mathbf{s}(t) \in \mathbb{R}^{n_3}$, then p of the $6n - n_4$ equations in (22) cannot be straightforwardly recast in the explicit form given by (23). In this case, assume, without loss of generality, that $\mathbf{q}$ explicitly depends on the first p components of $\mathbf{s}$ and substitute the corresponding p equations in (24) with

$$s_j(t)\ddot{s}_j(t) = -\dot{s}_j^2(t) - \frac{d^2}{dt^2} f_{ineq,j}(\mathbf{x}_1(\mathbf{q}(t), \mathbf{q}_{dot}(t)), \ldots, \mathbf{x}_n(\mathbf{q}(t), \mathbf{q}_{dot}(t))), \tag{25}$$

which is obtained by differentiating (7). In this case, the boundary conditions are given by

$$\widetilde{\mathbf{f}}_{ineq}(\mathbf{s}(\mathbf{q}(t_1), \mathbf{q}_{dot}(\mathbf{q}(t_1))), \mathbf{x}_1(\mathbf{q}(t_1), \mathbf{q}_{dot}(\mathbf{q}(t_1))), ..., \mathbf{x}_n(\mathbf{q}(t_1), \mathbf{q}_{dot}(\mathbf{q}(t_1)))) = \mathbf{0}_{r_3}, \tag{26}$$

$$\widetilde{\mathbf{f}}_{ineq}(\mathbf{s}(\mathbf{q}(t_2), \mathbf{q}_{dot}(\mathbf{q}(t_2))), \mathbf{x}_1(\mathbf{q}(t_2), \mathbf{q}_{dot}(\mathbf{q}(t_2))), ..., \mathbf{x}_n(\mathbf{q}(t_2), \mathbf{q}_{dot}(\mathbf{q}(t_2)))) = \mathbf{0}_{r_3}, \tag{27}$$

where $f_{ineq,j} : \mathbb{R}^{n_3} \times D_{rel} \to \mathbb{R}$ is the j-th component of $\mathbf{f}_{ineq}(\mathbf{s}(t), \mathbf{x}_1(t), ..., \mathbf{x}_n(t))$ (Jacobson & Lele, 1969), for $j = 1, \ldots, p$. If $s_j(t^*) = 0$ for some $t^* \in [t_1, t_2]$, then (25) can be replaced by

$$3\dot{s}_j(t)\ddot{s}_j(t) + s_j(t)\frac{d^3 s_j(t)}{dt^3} = -\frac{d^3}{dt^3} f_{ineq,j}(\mathbf{x}_1(\mathbf{q}(t), \mathbf{q}_{dot}(t)), \ldots, \mathbf{x}_n(\mathbf{q}(t), \mathbf{q}_{dot}(t))), \tag{28}$$

where $\mathbf{s} \in \mathcal{C}^3(t_1, t_2)$. In general, (7) must be differentiated so that $\ddot{s}_j(t)$, or one of its higher-order derivatives, explicitly appears and is multiplied by a term that is non-zero for all $t \in [t_1, t_2]$. In this case, the differentiability assumptions on $\mathbf{s}$ and $\mathbf{f}_{ineq}$ must be modified accordingly.

Example 4.2. Consider Example 4.1 with $\mathbf{q}(t) = \left[s_1(t), r_{1,1}(t), r_{1,2}(t), \boldsymbol{\sigma}_1^T(t), \mathbf{r}_2^T(t)\right]^T$. In this case, the formation's kinetic energy is given by

$$k(\mathbf{q}(t), \mathbf{q}_{dot}(t)) = \frac{1}{2} m_1 \mathbf{v}_1^T(\mathbf{q}(t), \mathbf{q}_{dot}(t))\, \mathbf{v}_1(\mathbf{q}(t), \mathbf{q}_{dot}(t)) + \frac{1}{2} m_2 \mathbf{v}_2^T(t)\mathbf{v}_2(t) + \frac{1}{2}\sum_{i=1}^{2} \boldsymbol{\omega}_1^T(t)\mathbf{I}_{in,i}\boldsymbol{\omega}_1(t).$$

The dynamic equations can now be found by applying (22) and accounting for (17) giving

$$v_{1,j}(t) = \frac{dr_{1,j}(t)}{dt}, \quad \boldsymbol{\omega}_1(t) = \mathbf{R}_{rod}^{-1}(\boldsymbol{\sigma}_1(t))\dot{\boldsymbol{\sigma}}_1(t), \quad \mathbf{v}_2(t) = \frac{d\mathbf{r}_2(t)}{dt}, \tag{29}$$

$$m_1 \frac{dv_{1,1}(t)}{dt} = m_1 a_1(\widetilde{\mathbf{x}}_1(\mathbf{q}(t), \mathbf{q}_{dot}(t))) + m_1 u_{1,tran,1}(t), \tag{30}$$

$$m_1 \frac{dv_{1,2}(t)}{dt} = m_1 a_2(\widetilde{\mathbf{x}}_1(\mathbf{q}(t), \mathbf{q}_{dot}(t))) + m_1 u_{1,tran,2}(t), \tag{31}$$

$$\mathbf{I}_{in,1}\frac{d\omega_1(t)}{dt} = -\boldsymbol{\omega}_1^{\times}(\boldsymbol{\omega}_1(t))\,\mathbf{I}_{in,1}\boldsymbol{\omega}_1(t) + \mathbf{m}(\widetilde{\mathbf{x}}_1(\mathbf{q}(t), \mathbf{q}_{dot}(t, \mathbf{q}(t)))) + \mathbf{I}_{in,1}\mathbf{u}_{1,rot}(t), \tag{32}$$

$$m_2 \frac{d\mathbf{v}_2(t)}{dt} = m_2 \mathbf{a}(\widetilde{\mathbf{x}}_2(\mathbf{q}(t), \mathbf{q}_{dot}(t))) + m_2 \mathbf{u}_{2,tran}(t), \tag{33}$$

where, for $j = 1, 2$, $u_{1,\mathrm{tran},j} : [t_1, t_2] \to \mathbb{R}$ (respectively, $a_j : \mathbb{R}^{12} \to \mathbb{R}$) is the j-th component of $\mathbf{u}_{1,\mathrm{tran}}(t)$ (respectively, $\mathbf{a}\left(\tilde{\mathbf{x}}_1\left(t, \mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t, \mathbf{q}(t))\right)\right)$). Instead of deducing the dynamics of $s_1(t)$ from

$$m_1 \frac{dv_{1,3}(t)}{dt} = m_1 a_3\left(\tilde{\mathbf{x}}_1\left(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)\right)\right) + m_1 u_{1,\mathrm{tran},3}(t),$$

we use (16) and (25) to obtain

$$\begin{aligned} \dot{s}_1^2(t) + s_1(t)\ddot{s}_1(t) = &- 2||\mathbf{v}_1(t) - \mathbf{v}_2(t)||_2^2 \\ &+ (\mathbf{r}_1(t) - \mathbf{r}_2(t))^{\mathrm{T}}\left(\mathbf{a}\left(\tilde{\mathbf{x}}_1\left(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)\right)\right) + \mathbf{u}_{1,\mathrm{tran}}(t)\right) \\ &- (\mathbf{r}_1(t) - \mathbf{r}_2(t))^{\mathrm{T}}\left(\mathbf{a}\left(\tilde{\mathbf{x}}_2\left(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)\right)\right) + \mathbf{u}_{2,\mathrm{tran}}(t)\right), \end{aligned} \tag{34}$$

which can be solved for $s_1(t)$ if $||\mathbf{r}_1(t) - \mathbf{r}_2(t)||_2^2 < r_{\max}$, $t \in [t_1, t_2]$. In this case , the boundary conditions to (34) are given by

$$\begin{bmatrix} ||\mathbf{r}_1(t_1) - \mathbf{r}_2(t_1)||_2^2 - r_{\max} + \frac{1}{2}s_1^2(t_1) \\ r_{\min} - ||\mathbf{r}_1(t_1) - \mathbf{r}_2(t_1)||_2^2 + \frac{1}{2}s_2^2(t_1) \end{bmatrix} = \mathbf{0}_2,$$

$$\begin{bmatrix} ||\mathbf{r}_1(t_2) - \mathbf{r}_2(t_2)||_2^2 - r_{\max} + \frac{1}{2}s_1^2(t_2) \\ r_{\min} - ||\mathbf{r}_1(t_2) - \mathbf{r}_2(t_2)||_2^2 + \frac{1}{2}s_2^2(t_2) \end{bmatrix} = \mathbf{0}_2.$$

If, alternatively, $\mathbf{q}(t) = \left[\mathbf{x}_1^{\mathrm{T}}(t), \mathbf{r}_2^{\mathrm{T}}(t)\right]^{\mathrm{T}}$, then the formation's kinetic energy is given by

$$k\left(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)\right) = \frac{1}{2}\sum_{i=1}^{2} m_i \mathbf{v}_i^{\mathrm{T}}(t)\mathbf{v}_i(t) + \frac{1}{2}\sum_{i=1}^{2} \boldsymbol{\omega}_1^{\mathrm{T}}(t)\mathbf{I}_{\mathrm{in},i}\boldsymbol{\omega}_1(t) \tag{35}$$

and the dynamic equations, obtained by applying (22) and (18), are given by

$$\mathbf{v}_1(t) = \frac{d\mathbf{r}_1(t)}{dt}, \quad \boldsymbol{\omega}_1(t) = \mathbf{R}_{\mathrm{rod}}^{-1}(\boldsymbol{\sigma}_1(t))\dot{\boldsymbol{\sigma}}_1(t), \quad \mathbf{v}_2(t) = \frac{d\mathbf{r}_2(t)}{dt}, \tag{36}$$

$$m_1 \frac{d}{dt}\mathbf{v}_1(t) = m_1 \mathbf{a}\left(\tilde{\mathbf{x}}_1(t)\right) + m_1 \mathbf{u}_{1,\mathrm{tran}}(t), \tag{37}$$

$$\mathbf{I}_{\mathrm{in},1}\frac{d}{dt}\boldsymbol{\omega}_1(t) = -\boldsymbol{\omega}_1^{\times}\left(\boldsymbol{\omega}_1(t)\right)\mathbf{I}_{\mathrm{in},1}\boldsymbol{\omega}_1(t) + \mathbf{m}\left(\tilde{\mathbf{x}}_1(t)\right) + \mathbf{I}_{\mathrm{in},1}\mathbf{u}_{1,rot}(t), \tag{38}$$

$$m_2 \frac{d}{dt}\mathbf{v}_2(t) = m_2 \mathbf{a}\left(\left[\mathbf{r}_2^{\mathrm{T}}(t), \mathbf{v}_2^{\mathrm{T}}(t), \boldsymbol{\sigma}_1^{\mathrm{T}}(t), \boldsymbol{\omega}_1^{\mathrm{T}}(t)\right]^{\mathrm{T}}\right) + m_2 \mathbf{u}_{2,\mathrm{tran}}(t). \tag{39}$$

The Lagrange coordinates chosen imply that the first vehicle can be considered as unconstrained, that is, subject to (3), (14), and (15) only, and therefore, the dynamic equations (36) – (38) can be directly deduced from (3). Similarly, the translational dynamics of the second vehicle can be considered as unconstrained. Thus, (39) can be directly obtained from (3). Recall from Example 4.1 that the components of $\mathbf{q}$ are suitable Lagrange coordinates if $r_{\min} < ||\mathbf{r}_1(t) - \mathbf{r}_2(t)||_2^2 < r_{\max}$, whereas (29) – (33) hold if $r_{\min} < ||\mathbf{r}_1(t) - \mathbf{r}_2(t)||_2^2 < r_{\max}$ and $r_{1,3}(t) \neq r_{2,3}(t)$. Thus, $\mathbf{q}(t) = \left[\mathbf{x}_1^{\mathrm{T}}(t), \mathbf{r}_2^{\mathrm{T}}(t)\right]^{\mathrm{T}}$ is a more convenient choice of Lagrange coordinates than $\mathbf{q}(t) = \left[s_1(t), r_{1,1}(t), r_{1,2}(t), \boldsymbol{\sigma}_1^{\mathrm{T}}(t), \mathbf{r}_2^{\mathrm{T}}(t)\right]^{\mathrm{T}}$.

This example will be further elaborated on in Section 6 for $\mathbf{q}(t) = \left[\mathbf{x}_1^{\mathrm{T}}(t),\ \mathbf{r}_2^{\mathrm{T}}(t)\right]^{\mathrm{T}}$, and hence, for notational convenience define $\mathbf{f}_{\mathrm{dyn},2}\left(\widetilde{\mathbf{x}}_2(t), \mathbf{u}_{2,\mathrm{tran}}(t)\right) \triangleq \mathbf{a}\left(\widetilde{\mathbf{x}}_2(t)\right) + \mathbf{u}_{2,\mathrm{tran}}(t)$ and

$$\mathbf{f}_{\mathrm{dyn},1}(\mathbf{x}_1, \mathbf{q}_{\mathrm{dot},1}(\mathbf{x}_1), \mathbf{u}_1) \triangleq \begin{bmatrix} \mathbf{a}\left(\widetilde{\mathbf{x}}_1(t)\right) + \mathbf{u}_{1,\mathrm{tran}}(t) \\ -\mathbf{I}_{\mathrm{in},1}^{-1}\boldsymbol{\omega}_1^{\times}\left(\boldsymbol{\omega}_1(t)\right)\mathbf{I}_{\mathrm{in},1}\boldsymbol{\omega}_1(t) + \widetilde{\boldsymbol{\omega}}_{\mathrm{i}}\left(\widetilde{\mathbf{x}}_1(t)\right) + \mathbf{u}_{1,\mathrm{rot}}(t) \end{bmatrix}.$$

4.4 Path planning optimization problem revisited

The trajectory optimization problem defined in Section 3.4 can be reformulated as follows. For all $\mathrm{i} = 1, ..., \mathrm{n}$ and $t \in [t_1, t_2]$, find $\mathbf{u}_{\mathrm{i,tran}}(t)$ (respectively, $\mathbf{u}_{\mathrm{i,rot}}(t)$) among all admissible controls in $\Gamma_{\mathrm{i,tran}}$ (respectively, $\Gamma_{\mathrm{i,rot}}$) such that the performance measure (2) is minimized and $\mathbf{q}(t)$ satisfies (24), (10), and (11).

By comparing this problem statement to the problem statement given in Section 3.4, it is clear that (5) and (6) are not explicitly accounted for in the above reformulation of the optimization problem. Hence, the constrained optimization problem has been reduced to an unconstrained optimization problem by the introduction of slack variables and Lagrange coordinates.

4.5 Transversality condition

Let $\mathbf{S} : D_1 \to D_2$, where $D_1 \subset \mathbb{R}^{\mathrm{p}}$ and $D_2 \subset \mathbb{R}^{\mathrm{m}}$, be a a continuously differentiable manifold and let the *manifold tangent* to $\mathbf{S}$ at $\mathbf{y}_0$ be given by

$$\left.\frac{\partial \mathbf{S}(\mathbf{y})}{\partial \mathbf{y}}\right|_{\mathbf{y}=\mathbf{y}_0} (\mathbf{y} - \mathbf{y}_0) = \mathbf{0}_{\mathrm{m}}. \tag{40}$$

Every vector $\mathbf{v} \in \mathbb{R}^{\mathrm{p}}$ that is normal to the manifold tangent to $\mathbf{S}$ at $\mathbf{y}_0$, that is, $\mathbf{v}^{\mathrm{T}}\mathbf{y} = 0$ for all $\mathbf{y} \in \mathbb{R}^{\mathrm{p}}$ such that (40) holds, is said to verify the *transversality condition* for $\mathbf{S}$ at $\mathbf{y}_0$.

4.6 Pontryagin's minimum principle

Assume that a set of Lagrange coordinates has been found and that the formation's dynamic equations can be written in the form given by (24). Define the *costate vectors* $\boldsymbol{\lambda}_{\mathrm{dot}} : [t_1, t_2] \to \mathbb{R}^{6\mathrm{n}-\mathrm{n}_4}$ and $\boldsymbol{\lambda}_{\mathrm{dyn}} : [t_1, t_2] \to \mathbb{R}^{6\mathrm{n}-\mathrm{n}_4}$ so that the *costate equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \boldsymbol{\lambda}_{\mathrm{dot}}(t) \\ \boldsymbol{\lambda}_{\mathrm{dyn}}(t) \end{bmatrix} = -\left(\frac{\partial}{\partial[\mathbf{q}^{\mathrm{T}},\ \mathbf{q}_{\mathrm{dot}}^{\mathrm{T}}]^{\mathrm{T}}}\begin{bmatrix} \boldsymbol{\Psi}\left(\mathbf{q}(t)\right)\dot{\mathbf{q}}(t) + \boldsymbol{\psi}\left(\mathbf{q}(t)\right) \\ \mathbf{f}_{\mathrm{dyn}}(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t), \tilde{\mathbf{u}}(t)) \end{bmatrix}\right)^{\mathrm{T}}\begin{bmatrix} \boldsymbol{\lambda}_{\mathrm{dot}}(t) \\ \boldsymbol{\lambda}_{\mathrm{dyn}}(t) \end{bmatrix} \tag{41}$$

holds. The boundary conditions for (41) are given in Theorem 4.2 below. Given $\lambda_0 \in \mathbb{R}$, define the *Hamiltonian function*

$$\begin{aligned} \mathfrak{h}\left(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t), \tilde{\mathbf{u}}(t), \boldsymbol{\lambda}_{\mathrm{dyn}}(t), \boldsymbol{\lambda}_{\mathrm{dot}}(t)\right) \triangleq\ & \lambda_0 \sum_{\mathrm{i}=1}^{\mathrm{n}} \mu_{\mathrm{i}}||\mathbf{u}_{\mathrm{i}}(t)||_2 + \boldsymbol{\lambda}_{\mathrm{dot}}^{\mathrm{T}}(t)\mathbf{q}_{\mathrm{dot}}(t) \\ & + \boldsymbol{\lambda}_{\mathrm{dyn}}^{\mathrm{T}}(t)\mathbf{f}_{\mathrm{dyn}}(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t), \tilde{\mathbf{u}}(t)). \end{aligned} \tag{42}$$

Finally, define

$$\mathfrak{m}\left(\mathbf{q}(t),\mathbf{q}_{\mathrm{dot}}(t),\boldsymbol{\lambda}_{\mathrm{dyn}}(t),\boldsymbol{\lambda}_{\mathrm{dot}}(t)\right)\triangleq\min_{\tilde{\mathbf{u}}\in\prod_{\mathrm{i}=1}^{\mathrm{n}}(\Gamma_{\mathrm{i,tran}}\times\Gamma_{\mathrm{i,rot}})}\mathfrak{h}\left(\mathbf{q}(t),\mathbf{q}_{\mathrm{dot}}(t),\tilde{\mathbf{u}}(t),\boldsymbol{\lambda}_{\mathrm{dyn}}(t),\boldsymbol{\lambda}_{\mathrm{dot}}(t)\right). \tag{43}$$

The following theorem is known as the *Pontryagin minimum principle*. For details on this theorem and its numerous applications to optimal control, see Pontryagin et al. (1962).

Theorem 4.2. *(Pontryagin et al., 1962) For all* i = 1, ..., n, *let* $\mathbf{u}^*_{\mathrm{i,tran}}(t)$ *and* $\mathbf{u}^*_{\mathrm{i,rot}}(t)$, $t\in[t_1,t_2]$, *be admissible controls in* $\Gamma_{\mathrm{i,tran}}$ *and* $\Gamma_{\mathrm{i,rot}}$*, respectively, such that* $\mathbf{q}^*(t)$ *satisfies* (24)*,* (10)*, and* (11)*. If* $\mathbf{u}^*_{\mathrm{i,tran}}(t)$ *and* $\mathbf{u}^*_{\mathrm{i,rot}}(t)$ *solve the trajectory optimization problem stated in Section 4.4, then there exist* $\lambda^*_0\in\overline{\mathbb{R}}_+$, $\boldsymbol{\lambda}^*_{\mathrm{dyn}}(t)$*, and* $\boldsymbol{\lambda}^*_{\mathrm{dot}}(t)$ *such that i)* $|\lambda^*_0|+||\boldsymbol{\lambda}^*_{\mathrm{dyn}}(t)||_2+||\boldsymbol{\lambda}^*_{\mathrm{dot}}(t)||_2\neq 0$, $t\in[t_1,t_2]$*, ii)* (41) *holds, iii)* $\mathfrak{h}\left(\mathbf{q}^*(t),\tilde{\mathbf{u}}^*(t),\boldsymbol{\lambda}^*_{\mathrm{dyn}}(t),\boldsymbol{\lambda}^*_{\mathrm{dot}}(t)\right)$ *attains its minimum almost everywhere on* $[t_1,t_2]$ *except on a finite number of points, and iv)* $\boldsymbol{\lambda}^*_{\mathrm{dyn}}(t_1)$ *and* $\boldsymbol{\lambda}^*_{\mathrm{dot}}(t_1)$ *(respectively,* $\boldsymbol{\lambda}^*_{\mathrm{dyn}}(t_2)$ *and* $\boldsymbol{\lambda}^*_{\mathrm{dot}}(t_2)$*) satisfy the transversality condition for* $\mathbf{S}_1$ *(respectively,* $\mathbf{S}_2$*) at* $\mathbf{q}^*(t_1)$ *(respectively,* $\mathbf{q}^*(t_2)$*).*

Pontryagin minimum principle is a necessary condition for optimality, and hence, it provides *candidate* optimal control vectors. Sufficient conditions for optimality that are currently available in the literature do not apply to the optimization problem discussed herein.

It is worth noting that, instead of introducing the Lagrange coordinates, the equality constraints (7) and (5) can be accounted for by introducing Lagrange multipliers. This approach requires modifying the assigned performance measure and introducing additional costate vectors (Giaquinta & Hildebrandt, 1996; Lee & Markus, 1968). The dynamics of the costate vectors are characterized by ordinary differential equations known as costate equations, which need to be integrated numerically together with the dynamic equations of the state vector. Therefore, the computational complexity of finding optimal trajectories for large formations increases drastically when Lagrange multipliers are employed (L'Afflitto & Sultan, 2010). Alternatively, finding a suitable set of Lagrange coordinates can be a demanding task and in some cases the Lagrange coordinates may not have physical meaning (Pars, 1965); however, this reduces the dimension of the costate equation and consequently reduces the computational complexity.

Finally, we say the optimization problem is *normal* if $\lambda_0\neq 0$, otherwise the optimization problem is *abnormal*. Normality can be shown by using the *Euler necessary condition*

$$\left.\frac{\partial\mathfrak{h}\left(\mathbf{q}(t),\mathbf{q}_{\mathrm{dot}}(t),\tilde{\mathbf{u}}(t),\boldsymbol{\lambda}_{\mathrm{dyn}}(t),\boldsymbol{\lambda}_{\mathrm{dot}}(t)\right)}{\partial\tilde{\mathbf{u}}}\right|_{\tilde{\mathbf{u}}=\tilde{\mathbf{u}}^*}=\mathbf{0}^{\mathrm{T}}_{6\mathrm{n}}, \tag{44}$$

where $\tilde{\mathbf{u}}^*(t)\triangleq\left[[\mathbf{u}^{*T}_{1,\mathrm{tran}}(t),\mathbf{u}^{*T}_{1,\mathrm{rot}}(t)]^{\mathrm{T}},...,[\mathbf{u}^{*T}_{\mathrm{n,tran}}(t),\mathbf{u}^{*T}_{\mathrm{n,rot}}(t)]^{\mathrm{T}}\right]^{\mathrm{T}}\in\mathrm{int}\,(\prod_{\mathrm{i}=1}^{\mathrm{n}}(\Gamma_{\mathrm{i,tran}}\times\Gamma_{\mathrm{i,rot}}))$. In particular, assume, *ad absurdum*, that $\lambda_0=0$. Now, if (41) and (44) imply that $\boldsymbol{\lambda}_{\mathrm{dot}}(t)=\mathbf{0}_{6\mathrm{n}-\mathrm{n}_4}$ and $\boldsymbol{\lambda}_{\mathrm{dyn}}(t)=\mathbf{0}_{6\mathrm{n}-\mathrm{n}_4}$ for some $t\in[t_1,t_2]$, then assertion *i)* of Theorem 4.2 is contradicted. Therefore, $\lambda_0\neq 0$, and hence, the optimization problem is normal. In this case, we assume without loss of generality that $\lambda_0=1$.

5. Analytical and numerical approaches to the optimal path planning problem

Finding minimizers to (2) subject to the constraints (3) – (6) can be formulated as a Lagrange optimization problem (Ewing, 1969), which has been extensively studied both analytically and numerically in the literature. Analytical methods rely on either Lagrange's variational approach using calculus of variations or on the direct approach. In the classical variational approach, candidate minimizers for a given performance functional can be found by applying the Euler necessary condition. In order to find the minimizers, candidate optimal solutions need to be further tested by applying the Clebsh necessary condition, Jacobi necessary condition, Weierstrass necessary condition, as well as the associated sufficient conditions (Ewing, 1969; Giaquinta & Hildebrandt, 1996).

This classical analytical approach is not practical since applying the Euler necessary condition involves solving a differential-algebraic boundary value problem, whose analytical solutions are impossible to find for many practical problems of interest. Moreover, numerical solutions to this boundary value problem are affected by a strong sensitivity to the boundary conditions (Bryson, 1975). Furthermore, verifying the Jacobi necessary condition or the Weierstrass necessary condition can be a dauting task (L'Afflitto & Sultan, 2010).

A variational approach to the optimal path planning problem for a single vehicle, known as *primer vector theory*, was addressed by Lawden (1963). Lawden's problem was formulated using the assumptions that the acceleration vector $\mathbf{a}$ induced by external forces due to the environment is function of only the position vector, the vehicle is a 3 DoF point mass, and the state and control are only subject to equality constraints (Lawden, 1963). Primer vector theory is successfully employed in spacecraft trajectory optimization (Jamison & Coverstone, 2010), orbit transfers (Petropoulos & Russell, 2008), and optimal rendezvous problems (Zaitri et al., 2010), however, vehicles are often assumed to be point masses subject to only gravitational acceleration. Among the few studies on primer vector theory applied to vehicle formations, it is worth noting the work of Mailhe & Guzman (2004), where the formation initialization problem is addressed. Applications of primer vector theory to 6 DoF single vehicles have been employed to optimize the descent on Mars (Topcu et al., 2007). These studies, however, assume that the spacecraft is subject to a constant gravity acceleration, the control variables are the translational acceleration and the angular rates, and the translational acceleration can be pointed in any direction by rotating the vehicle.

Pontryagin's minimum principle is a variational method that is equivalent to the Weierstrass necessary condition with the advantage of addressing constraints on the control more effectively than applying the classical variational approach. State constraints need to be addressed by applying an optimal switching condition on the costate equation (Pontryagin et al., 1962), which generally increases the complexity of the problem. In the present formulation, the constraints on the formation are addressed by employing Lagrange coordinates, which does not introduce further conditions on the costate vector dynamics.

The direct approach in the calculus of variations, which is more recent than the variational approach, is based on defining a minimizing sequence of control functions $\mathbf{u}_{\mathrm{n}}(t)$ in some set Γ such that $\lim_{\mathrm{n}\to+\infty} \mathbf{u}_{\mathrm{n}}(t) = \mathbf{u}(t)$ is a minimizer of the performance measure $J[\mathbf{u}(\cdot)]$. To this end, the following conditions should be met. *i*) Compactness of Γ, so that a minimizing sequence contains a convergent subsequence, *ii*) closedness of Γ, so that the limit of such a subsequence is contained in Γ, and *iii*) lower semicontinuity of the sequence

$\{\mathbf{u}_n(\cdot)\}_{n=0}^{\infty}$, that is, if $\lim_{n\to+\infty} \mathbf{u}_n(t) = \mathbf{u}(t)$, then $J[\mathbf{u}(t)] \leq \liminf_{n\to+\infty} J[\mathbf{u}_n(t)]$, $\mathbf{u}_n \in \Gamma$. Finally, it is also worth noting that approximate analytical methods can be used to solve the optimal path planning problem such as shape-based approximation methods (Petropoulos & Longuski, 2004), which are generally less effective due to the arbitrary parameterization of the minimizers (Wall, 2008).

Most of the results on the fuel consumption optimization employ numerical methods (Betts, 1998), which can be categorized as indirect or direct. Indirect numerical methods, which mimic the variational approach, suffer from high computational complexity since adjoint variables must be introduced. Alternatively, direct numerical methods are computationally more efficient, however, they require casting the given problem into a parameter optimization problem (Herman & Conway, 1987). Among the numerical methods commonly in use, it is worth mentioning genetic algorithms (Seereram et al., 2000) and particle swarm optimizers (Hassan et al., 2005).

One of the contributions of the present paper is that it extends Lawden's results on primer vector theory to formations of vehicles modeled as 6 DoF rigid bodies subject to generic environmental forces and moments by applying Pontryagin's minimum principle. As in all classical variational methods, Pontryagin's minimum principle is not suitable for numerically computing the optimal trajectory of a formation. However, Pontryagin's minimum principle allows us to draw analytical conclusions since it provides a generalization of the necessary conditions used by Lawden (1963), allows us to formally implement bounded integrable functions as admissible controls, and allows us to account for control constraints. Prussing (2010) and Marec (1979) have used Pontryagin's minimum principle to address primer vector theory using the same assumptions as Lawden (1963). In contrast, the present work provides additional analytical results for generic mission scenarios and complex environmental conditions for which numerical results can be verified. Furthermore, this paper exploits some properties of the costate space and consequently provides further insight into the formation system dynamics problem.

6. Necessary conditions for optimality of UAV formation trajectories

The following propositions are needed to develop the necessary conditions for optimality of the UAV formation problem.

Proposition 6.1. *Consider the performance measure* $J_{\text{formation}}\,[\tilde{\mathbf{u}}(\cdot)]$ *given by* (2)*. Then, there exists at least one* $\tilde{\mathbf{u}}^*$ *such that* $J_{\text{formation}}\,[\tilde{\mathbf{u}}^*(\cdot)] \leq J_{\text{formation}}\,[\tilde{\mathbf{u}}(\cdot)]$ *for all* $\tilde{\mathbf{u}} \in \prod_{i=1}^{n}(\Gamma_{i,\text{tran}} \times \Gamma_{i,\text{rot}})$.

Proof. Since the integrand of the performance measure (1) is a continuous function defined on the compact set $\Gamma_{i,\text{tran}} \times \Gamma_{i,\text{rot}}$, it follows from Weierstrass' theorem that (1) has a global minimizer on $\Gamma_{i,\text{tran}} \times \Gamma_{i,\text{rot}}$. Now, since $\mu_i \in [0,1]$ with $\sum_{i=1}^{n} \mu_i = 1$, the result is immediate. $\square$

Proposition 6.2. *Assume that the hypothesis of Theorem 4.1 hold. If* $\boldsymbol{\lambda}^*_{\text{dot}}(t) \in \mathcal{N}\left(\left.\frac{\partial \boldsymbol{\Psi}(\mathbf{q})}{\partial \mathbf{q}}\right|_{\mathbf{q}=\mathbf{q}^*} \dot{\mathbf{q}}^*(t) + \left.\frac{\partial \psi(\mathbf{q})}{\partial \mathbf{q}}\right|_{\mathbf{q}=\mathbf{q}^*}\right)$*, then the path planning problem is normal.*

Proof. First, note that the Hamiltonian function (42) can be rewritten as

$$\begin{aligned}\mathfrak{h}\left(\mathbf{q}(t),\mathbf{q}_{\rm dot}(t),\tilde{\mathbf{u}}(t),\boldsymbol{\lambda}_{\rm dyn}(t),\boldsymbol{\lambda}_{\rm dot}(t)\right) =&\lambda_0\sum_{\rm i=1}^{\rm n}\mu_{\rm i}||\mathbf{u}_{\rm i}(t)||_2\\ &+\sum_{\rm i=1}^{\rm n}\mathbf{u}_{\rm i,tran}(t)\frac{\partial\mathbf{v}_{\rm i}\left(\mathbf{q},\mathbf{q}_{\rm dot}\right)}{\partial\mathbf{q}_{\rm dot}}\boldsymbol{\lambda}_{\rm dyn}(t)\\ &+\sum_{\rm i=1}^{\rm n}\mathbf{u}_{\rm i,rot}(t)\frac{\partial\boldsymbol{\omega}_{\rm i}\left(\mathbf{q},\mathbf{q}_{\rm dot}\right)}{\partial\mathbf{q}_{\rm dot}}\boldsymbol{\lambda}_{\rm dyn}(t)\\ &+\boldsymbol{\lambda}_{\rm dyn}^{\rm T}(t)\hat{\mathbf{f}}_{\rm dyn}(\mathbf{q}(t),\mathbf{q}_{\rm dot}(t))+\boldsymbol{\lambda}_{\rm dot}^{\rm T}(t)\mathbf{q}_{\rm dot}(t).\end{aligned}\quad(45)$$

Furthemore, note that (44) implies that

$$\lambda_0^*\sum_{\rm i=1}^{\rm n}\mu_{\rm i}\frac{\mathbf{u}_{\rm i}^{*T}(t)}{||\mathbf{u}_{\rm i}^*(t)||_2}=-\sum_{\rm i=1}^{\rm n}\left[\left.\frac{\partial\mathbf{v}_{\rm i}\left(\mathbf{q},\mathbf{q}_{\rm dot}\right)}{\partial\mathbf{q}_{\rm dot}}\right|_{(\mathbf{q}^*,\mathbf{q}_{\rm dot}^*)}\boldsymbol{\lambda}_{\rm dyn}^*(t),\quad\left.\frac{\partial\boldsymbol{\omega}_{\rm i}\left(\mathbf{q},\mathbf{q}_{\rm dot}\right)}{\partial\mathbf{q}_{\rm dot}}\right|_{(\mathbf{q}^*,\mathbf{q}_{\rm dot}^*)}\boldsymbol{\lambda}_{\rm dyn}^*(t)\right],$$

where $\tilde{\mathbf{u}}^*\in int\left(\Pi_{\rm i=1}^{\rm n}(\Gamma_{\rm i,tran}\times\Gamma_{\rm i,rot})\right)$ and where we use the subscript $(\mathbf{q}^*,\mathbf{q}_{\rm dot}^*)$ for $(\mathbf{q},\mathbf{q}_{\rm dot})=(\mathbf{q}^*,\mathbf{q}_{\rm dot}^*)$. Now, assume, *ad absurdum*, that $\lambda_0^*=0$ and note that $\frac{\partial\mathbf{v}_{\rm i}(\mathbf{q},\mathbf{q}_{\rm dot})}{\partial\mathbf{q}_{\rm dot}}=\frac{\partial\mathbf{v}_{\rm i}(\mathbf{q},\mathbf{q}_{\rm dot})}{\partial\mathbf{q}}\frac{\partial\mathbf{q}}{\partial\mathbf{q}_{\rm dot}}$ and $\frac{\partial\boldsymbol{\omega}_{\rm i}(\mathbf{q},\mathbf{q}_{\rm dot})}{\partial\mathbf{q}_{\rm dot}}=\frac{\partial\boldsymbol{\omega}_{\rm i}(\mathbf{q},\mathbf{q}_{\rm dot})}{\partial\mathbf{q}}\frac{\partial\mathbf{q}}{\partial\mathbf{q}_{\rm dot}}$. Since $\boldsymbol{\Psi}\left(\mathbf{q}\right)$ is diffeomorphic and Theorem 4.1 holds, it follows that $\boldsymbol{\lambda}_{\rm dyn}^*(t)=\mathbf{0}_{\rm 6n-n_4}$. In this case, (41) can be explicitly written as

$$\frac{\rm d}{{\rm d}t}\begin{bmatrix}\boldsymbol{\lambda}_{\rm dot}^*(t)\\ \boldsymbol{\lambda}_{\rm dyn}^*(t)\end{bmatrix}=-\left.\begin{bmatrix}\frac{\partial\boldsymbol{\Psi}(\mathbf{q})}{\partial\mathbf{q}}\dot{\mathbf{q}}(t)+\frac{\partial\boldsymbol{\psi}(\mathbf{q})}{\partial\mathbf{q}} & \mathbf{0}_{(\rm 6n-n_4)\times(6n-n_4)}\\ \frac{\partial\mathbf{f}_{\rm dyn}(\mathbf{q},\mathbf{q}_{\rm dot},\tilde{\mathbf{u}})}{\partial\mathbf{q}} & \frac{\partial\mathbf{f}_{\rm dyn}(\mathbf{q},\mathbf{q}_{\rm dot},\tilde{\mathbf{u}})}{\partial\mathbf{q}_{\rm dot}}\end{bmatrix}^{\rm T}\right|_{(\mathbf{q}^*,\mathbf{q}_{\rm dot}^*)}\begin{bmatrix}\boldsymbol{\lambda}_{\rm dot}^*(t)\\ \boldsymbol{\lambda}_{\rm dyn}^*(t)\end{bmatrix},\quad(46)$$

and hence, $\boldsymbol{\lambda}_{\rm dot}^*(t)=\mathbf{0}_{\rm 6n-n_4}$, which contradicts *i*) of Theorem 4.2. □

If follows from Proposition 6.2 that the path planning optimization problem for a constrained formation is abnormal. Example 6.1 below, however, shows that this problem is normal for unconstrained 3 DoF vehicles, which is a well known result in the literature (Lawden, 1963).

Theorem 6.1. *Consider the path planning optimization problem. If* $\sum_{\rm i=1}^{\rm n}\mathbf{u}_{\rm i,tran}^*(t)\left.\frac{\partial\mathbf{v}_{\rm i}(\mathbf{q},\mathbf{q}_{\rm dot})}{\partial\mathbf{q}_{\rm dot}}\right|_{(\mathbf{q}^*,\mathbf{q}_{\rm dot}^*)}+\sum_{\rm i=1}^{\rm n}\mathbf{u}_{\rm i,rot}^*(t)\left.\frac{\partial\boldsymbol{\omega}_{\rm i}(\mathbf{q},\mathbf{q}_{\rm dot})}{\partial\mathbf{q}_{\rm dot}}\right|_{(\mathbf{q}^*,\mathbf{q}_{\rm dot}^*)}$ *and* $-\boldsymbol{\lambda}_{\rm dyn}^*(t)$ *are parallel, then the performance measure* (2) *is minimized. Moreover, for all* $\rm i=1,\ldots,n$, *the following conditions hold.*

i) *If* $\lambda_0^*\mu_{\rm i}>\left|\left|\left.\frac{\partial\mathbf{v}_{\rm i}(\mathbf{q},\mathbf{q}_{\rm dot})}{\partial\mathbf{q}_{\rm dot}}\right|_{(\mathbf{q}^*,\mathbf{q}_{\rm dot}^*)}\boldsymbol{\lambda}_{\rm dyn}^*(t)\right|\right|_2$, *then* $\mathbf{u}_{\rm i,tran}^*(t)=\mathbf{0}_3$.

ii) *If* $\lambda_0^*\mu_{\rm i}>\left|\left|\left.\frac{\partial\boldsymbol{\omega}_{\rm i}(\mathbf{q},\mathbf{q}_{\rm dot})}{\partial\mathbf{q}_{\rm dot}}\right|_{(\mathbf{q}^*,\mathbf{q}_{\rm dot}^*)}\boldsymbol{\lambda}_{\rm dyn}^*(t)\right|\right|_2$, *then* $\mathbf{u}_{\rm i,rot}^*(t)=\mathbf{0}_3$.

iii) *If* $\lambda_0^*\mu_\mathrm{i} < \left|\left|\frac{\partial \mathbf{v}_\mathrm{i}(\mathbf{q},\mathbf{q}_\mathrm{dot})}{\partial \mathbf{q}_\mathrm{dot}}\right|_{(\mathbf{q}^*,\mathbf{q}^*_\mathrm{dot})} \boldsymbol{\lambda}^*_\mathrm{dyn}(t)\right|\right|_2$*, then* $\mathbf{u}^*_\mathrm{i,tran}(t) = \rho_\mathrm{i,2}$.

iv) *If* $\lambda_0^*\mu_\mathrm{i} < \left|\left|\frac{\partial \boldsymbol{\omega}_\mathrm{i}(\mathbf{q},\mathbf{q}_\mathrm{dot})}{\partial \mathbf{q}_\mathrm{dot}}\right|_{(\mathbf{q}^*,\mathbf{q}^*_\mathrm{dot})} \boldsymbol{\lambda}^*_\mathrm{dyn}(t)\right|\right|_2$*, then* $\mathbf{u}^*_\mathrm{i,rot}(t) = \rho_\mathrm{i,4}$.

v) *If* $\lambda_0^*\mu_\mathrm{i} = \left|\left|\frac{\partial \mathbf{v}_\mathrm{i}(\mathbf{q},\mathbf{q}_\mathrm{dot})}{\partial \mathbf{q}_\mathrm{dot}}\right|_{(\mathbf{q}^*,\mathbf{q}^*_\mathrm{dot})} \boldsymbol{\lambda}^*_\mathrm{dyn}(t)\right|\right|_2$*, then* $\mathbf{u}^*_\mathrm{i,tran}(t)$ *is unspecified.*

vi) *If* $\lambda_0^*\mu_\mathrm{i} = \left|\left|\frac{\partial \boldsymbol{\omega}_\mathrm{i}(\mathbf{q},\mathbf{q}_\mathrm{dot})}{\partial \mathbf{q}_\mathrm{dot}}\right|_{(\mathbf{q}^*,\mathbf{q}^*_\mathrm{dot})} \boldsymbol{\lambda}^*_\mathrm{dyn}(t)\right|\right|_2$*, then* $\mathbf{u}^*_\mathrm{i,rot}(t)$ *is unspecified.*

Proof. It follows from (45) that $\mathfrak{h}\left(\mathbf{q}(t), \tilde{\mathbf{u}}(t), \boldsymbol{\lambda}_\mathrm{dyn}(t), \boldsymbol{\lambda}_\mathrm{dot}(t)\right)$ is minimized if, for all $\mathrm{i} = 1, \ldots, \mathrm{n}$, $-\frac{\partial \mathbf{v}_\mathrm{i}(\mathbf{q},\mathbf{q}_\mathrm{dot})}{\partial \mathbf{q}_\mathrm{dot}}\Big|_{(\mathbf{q}^*,\mathbf{q}^*_\mathrm{dot})} \boldsymbol{\lambda}^*_\mathrm{dyn}(t)$ is parallel to $\mathbf{u}^*_\mathrm{i,tran}(t)$ and if $-\frac{\partial \boldsymbol{\omega}_\mathrm{i}(\mathbf{q},\mathbf{q}_\mathrm{dot})}{\partial \mathbf{q}_\mathrm{dot}}\Big|_{(\mathbf{q}^*,\mathbf{q}^*_\mathrm{dot})} \boldsymbol{\lambda}^*_\mathrm{dyn}(t)$ is parallel to $\mathbf{u}^*_\mathrm{i,rot}(t)$. Thus, using the triangular inequality, it follows that

$$\begin{aligned}
&\mathfrak{h}\left(\mathbf{q}^*(t), \tilde{\mathbf{u}}^*(t), \boldsymbol{\lambda}^*_\mathrm{dyn}(t), \boldsymbol{\lambda}^*_\mathrm{dot}(t)\right) - \boldsymbol{\lambda}^{*\mathrm{T}}_\mathrm{dot}(t)\mathbf{q}_\mathrm{dot}(\mathbf{q}^*(t)) - \boldsymbol{\lambda}^\mathrm{T}_\mathrm{dyn}(t)\hat{\mathbf{f}}_\mathrm{dyn}(\mathbf{q}^*(t), \mathbf{q}_\mathrm{dot}(\mathbf{q}^*(t)))\\
&\leq \sum_{\mathrm{i}=1}^{\mathrm{n}} \left[\left(\lambda_0^*\mu_\mathrm{i} - \left|\left|\frac{\partial \mathbf{v}_\mathrm{i}\left(\mathbf{q}, \mathbf{q}_\mathrm{dot}\right)}{\partial \mathbf{q}_\mathrm{dot}}\right|_{(\mathbf{q}^*,\mathbf{q}^*_\mathrm{dot})} \boldsymbol{\lambda}^*_\mathrm{dyn}(t)\right|\right|_2\right) ||\mathbf{u}^*_\mathrm{i,tran}(t)||_2\right]\\
&\quad + \sum_{\mathrm{i}=1}^{\mathrm{n}} \left[\left(\lambda_0^*\mu_\mathrm{i} - \left|\left|\frac{\partial \boldsymbol{\omega}_\mathrm{i}\left(\mathbf{q}, \mathbf{q}_\mathrm{dot}\right)}{\partial \mathbf{q}_\mathrm{dot}}\right|_{(\mathbf{q}^*,\mathbf{q}^*_\mathrm{dot})} \boldsymbol{\lambda}^*_\mathrm{dyn}(t)\right|\right|_2\right) ||\mathbf{u}^*_\mathrm{i,rot}(t)||_2\right],
\end{aligned} \tag{47}$$

which proves i) – iv). Next, if $\lambda_0^*\mu_\mathrm{i} = \left|\left|\frac{\partial \mathbf{v}_\mathrm{i}(\mathbf{q},\mathbf{q}_\mathrm{dot})}{\partial \mathbf{q}_\mathrm{dot}}\right|_{(\mathbf{q}^*,\mathbf{q}^*_\mathrm{dot})} \boldsymbol{\lambda}^*_\mathrm{dyn}(t)\right|\right|_2$ (respectively, $\lambda_0^*\mu_\mathrm{i} = \left|\left|\frac{\partial \boldsymbol{\omega}_\mathrm{i}(\mathbf{q},\mathbf{q}_\mathrm{dot})}{\partial \mathbf{q}_\mathrm{dot}}\right|_{(\mathbf{q}^*,\mathbf{q}^*_\mathrm{dot})} \boldsymbol{\lambda}^*_\mathrm{dyn}(t)\right|\right|_2$), then Pontryagin's minimum principle does not provide any information about the optimal control, and hence, v) and vi) hold. □

Analogous to Lawden's (Lawden, 1963) primer vector theory, $\frac{\partial \mathbf{v}_\mathrm{i}(\mathbf{q},\mathbf{q}_\mathrm{dot})}{\partial \mathbf{q}_\mathrm{dot}}\Big|_{(\mathbf{q}^*,\mathbf{q}^*_\mathrm{dot})} \boldsymbol{\lambda}^*_\mathrm{dyn}(t)$ and $\frac{\partial \boldsymbol{\omega}_\mathrm{i}(\mathbf{q},\mathbf{q}_\mathrm{dot})}{\partial \mathbf{q}_\mathrm{dot}}\Big|_{(\mathbf{q}^*,\mathbf{q}^*_\mathrm{dot})} \boldsymbol{\lambda}^*_\mathrm{dyn}(t)$ determine the magnitude and the direction of the control forces, and hence, we denote them as the *translational primer vector* and the *rotational primer vector*, respectively. Moreover, the trajectory given by each of the cases in Theorem 6.1 are called *arcs*. For each $\mathrm{i} = 1, \ldots, \mathrm{n}$, the arcs corresponding to i) (respectively, ii)) are called *maximum translational* (respectively, *rotational*) *thrust arcs*. Similarly, arcs corresponding to iii) (respectively, iv)) are called *null translational* (respectively, *rotational*) *thrust arcs*. Finally, arcs corresponding to v) (respectively, vi)) are called *singular translational* (respectively, *rotational*) *thrust arcs*. The optimal translational and rotational control vectors for v) and vi) in Theorem

6.1 need to be deduced by applying the generalized Legendre-Clebsch condition (Giaquinta & Hildebrandt, 1996).

Theorem 6.2. *Consider the path planning optimization problem. Then, there exists $c^* \in \mathbb{R}$ such that*

$$\mathfrak{m}\left(\mathbf{q}^*(t), \mathbf{q}^*_{\rm dot}(t), \boldsymbol{\lambda}^*_{\rm dyn}(t), \boldsymbol{\lambda}^*_{\rm dot}(t)\right) = c^*. \tag{48}$$

Proof. It follows from the Weierstrass - Erdmann condition (Giaquinta & Hildebrandt, 1996) that on an optimal trajectory,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{h}\left(\mathbf{q}^*(t), \mathbf{q}^*_{\rm dot}(t), \tilde{\mathbf{u}}^*(t), \boldsymbol{\lambda}^*_{\rm dyn}(t), \boldsymbol{\lambda}^*_{\rm dot}(t)\right) = \frac{\partial}{\partial t}\mathfrak{h}\left(\mathbf{q}(t), \mathbf{q}^*_{\rm dot}(t), \tilde{\mathbf{u}}^*(t), \boldsymbol{\lambda}^*_{\rm dyn}(t), \boldsymbol{\lambda}^*_{\rm dot}(t)\right)$$

holds for all $t \in (t_1, t_2)$. Now, since $\mathfrak{h}$ does not explicitly depend on t, it follows that there exists $c^* \in \mathbb{R}$ such that $\mathfrak{h}\left(\mathbf{q}^*(t), \mathbf{q}^*_{\rm dot}(t), \mathbf{0}_{6n}, \boldsymbol{\lambda}^*_{\rm dyn}(t), \boldsymbol{\lambda}^*_{\rm dot}(t)\right) = c^*$, which proves (48). □

Proposition 6.3. *Consider the costate dynamics given by* (46)*. Then, the dynamics of $\boldsymbol{\lambda}^*_{\rm dyn}(t)$ are decoupled from the dynamics of $\boldsymbol{\lambda}^*_{\rm dot}(t)$.*

Proof. The result is immediate from the form of (46). □

It follows from Proposition 6.3 that the translational primer vector and the rotational primer vector dynamics are independent of the choice of $\mathbf{q}_{\rm dot}$. Moreover, in solving for $\boldsymbol{\lambda}^*_{\rm dyn}(t)$ we need not integrate a system of $2(6n - n_4)$ ordinary differential equations as in (41), but rather a system of $(6n - n_4)$ ordinary differential equations, which is very advantageous for large formations.

Proposition 6.4. *The translational primer vector and the rotational primer vector are continuously differentiable functions.*

Proof. First, note that $\boldsymbol{\lambda}^*_{\rm dyn}(\cdot)$ and $\boldsymbol{\lambda}^*_{\rm dot}(\cdot)$ are continuous with continuous derivatives almost everywhere on $t \in (t_1, t_2)$ except for a finite number of points (Pontryagin et al., 1962). Next, the differentiability assumption on the environmental model for $\mathbf{a}(\cdot)$ and $\mathbf{m}(\cdot)$ implies that the matrix on the right-hand side of (41) is of class $C^1(\mathbb{R}^{6n-n_4} \times \mathbb{R}^{6n-n_4} \times \mathbb{R}^{12n})$. Hence, $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\lambda}^*_{\rm dyn}(\cdot)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\lambda}^*_{\rm dot}(\cdot)$ are continuous on (t_1, t_2). □

In order to elucidate the translational primer vector and rotational primer vector dynamics for a vehicle formation problem, we focus on specific formation configurations and on a specific environmental model. Hence, in the reminder of the paper we concentrate on the case where n_v components of $\mathbf{v}_i$ and n_ω components of $\boldsymbol{\omega}_i$ are also components of $\mathbf{q}_{\rm dot}$. A justification for this model is as follows. Assume that the i-th formation vehicle behaves as unconstrained, e.g., the first vehicle in Examples 4.1 and 4.2, or the dynamics of the i-th vehicle can be addressed as partly unconstrained, e.g., the second formation vehicle in the aforementioned examples. In either of these cases, it is natural to choose the unconstrained components of $\mathbf{v}_i$ and $\boldsymbol{\omega}_i$ as some of the components of $\mathbf{q}_{\rm dot}$. This model includes the classical formation configuration known as the *leader-follower* model, whose trajectories are computed as a function of the leader's path (Wang, 1991).

To simplify the environmental model assume that

$$\mathbf{a}\left(\widetilde{\mathbf{x}}_{\mathrm{i}}\left(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)\right)\right) = \mathbf{a}\left(\left[\mathbf{0}_3^{\mathrm{T}}, \mathbf{v}_{\mathrm{i}}^{\mathrm{T}}(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)), \mathbf{0}_3^{\mathrm{T}}, \mathbf{0}_3^{\mathrm{T}}\right]^{\mathrm{T}}\right), \tag{49}$$

$$\widetilde{\boldsymbol{\omega}}_{\mathrm{i}}\left(\widetilde{\mathbf{x}}_{\mathrm{i}}\left(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)\right)\right) = \widetilde{\boldsymbol{\omega}}_{\mathrm{i}}\left(\left[\mathbf{0}_3^{\mathrm{T}}, \mathbf{v}_{\mathrm{i}}^{\mathrm{T}}(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)), \mathbf{0}_3^{\mathrm{T}}, \boldsymbol{\omega}_{\mathrm{i}}^{\mathrm{T}}(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t))\right]^{\mathrm{T}}\right). \tag{50}$$

For notational convenience, we will refer to (49) and (50) as $\mathbf{a}(\mathbf{v}_{\mathrm{i}}(t))$ and $\widetilde{\boldsymbol{\omega}}_{\mathrm{i}}(\mathbf{v}_{\mathrm{i}}(t), \boldsymbol{\omega}_{\mathrm{i}}(t))$, respectively. This assumption on the accelerations induced by external forces and external moments is justified by a common environmental model given by (Anderson, 2001)

$$\mathbf{a}\left(\widetilde{\mathbf{x}}_{\mathrm{i}}\left(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)\right)\right) = \mathbf{g} + ||\mathbf{v}_{\mathrm{i}}(t)||_2^2\left(-k_{\mathrm{i,D}}\widehat{\mathbf{v}}_{\mathrm{i}}(t) + k_{\mathrm{i,L}}\widehat{\mathbf{v}}_{\mathrm{i}}^{\mathrm{L}}(t) - k_{\mathrm{i,S}}\widehat{\mathbf{v}}_{\mathrm{i}}^{\mathrm{S}}(t)\right), \tag{51}$$

$$\mathbf{m}\left(\widetilde{\mathbf{x}}_{\mathrm{i}}\left(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t)\right)\right) = ||\mathbf{v}_{\mathrm{i}}(t)||_2^2\left(k_{\mathrm{i,R}}\widehat{\boldsymbol{\omega}}_{\mathrm{i}}^{\mathrm{R}}(t) + k_{\mathrm{i,P}}\widehat{\boldsymbol{\omega}}_{\mathrm{i}}^{\mathrm{P}}(t) + k_{\mathrm{i,Y}}\widehat{\boldsymbol{\omega}}_{\mathrm{i}}^{\mathrm{Y}}(t)\right), \tag{52}$$

where $\mathbf{g}$ is the constant gravitational acceleration, $\widehat{\mathbf{v}}_{\mathrm{i}} \triangleq \mathbf{v}_{\mathrm{i}}/||\mathbf{v}_{\mathrm{i}}||_2$, $\widehat{\mathbf{v}}_{\mathrm{i}}^{\mathrm{L}} : [t_1, t_2] \to \mathbb{R}^3$ (respectively, $\widehat{\mathbf{v}}_{\mathrm{i}}^{\mathrm{S}} : [t_1, t_2] \to \mathbb{R}^3$) is the unit vector in the direction of the aerodynamic lift (respectively, in the direction opposite to the aerodynamic side force), $\widehat{\boldsymbol{\omega}}_{\mathrm{i}}^{\mathrm{R}} : [t_1, t_2] \to \mathbb{R}^3$ (respectively, $\widehat{\boldsymbol{\omega}}_{\mathrm{i}}^{\mathrm{P}} : [t_1, t_2] \to \mathbb{R}^3$ and $\widehat{\boldsymbol{\omega}}_{\mathrm{i}}^{\mathrm{Y}} : [t_1, t_2] \to \mathbb{R}^3$) is the unit vector in the direction of roll (respectively, pitch and yaw), and $k_{\mathrm{i,D}}$, $k_{\mathrm{i,L}}$, $k_{\mathrm{i,S}}$, $k_{\mathrm{i,R}}$, $k_{\mathrm{i,P}}$, and $k_{\mathrm{i,Y}}$, are the drag, lift, side force, roll, pitch, and yaw coefficients, respectively.

Using the above assumptions, it follows from (22) that

$$\dot{\hat{\mathbf{v}}}_{\mathrm{i}}(t) = \hat{\mathbf{a}}(\mathbf{v}_{\mathrm{i}}(t)) + \hat{\mathbf{u}}_{\mathrm{i,tran}}(t), \tag{53}$$

$$\dot{\hat{\boldsymbol{\omega}}}_{\mathrm{i}}(t) = \hat{\widetilde{\boldsymbol{\omega}}}_{\mathrm{i}}(\mathbf{v}_{\mathrm{i}}(t), \boldsymbol{\omega}_{\mathrm{i}}(t)) + \hat{\mathbf{u}}_{\mathrm{i,rot}}(t), \tag{54}$$

where $\hat{\mathbf{v}}_{\mathrm{i}} : [t_1, t_2] \to \mathbb{R}^{\mathrm{n_v}}$ (respectively, $\hat{\boldsymbol{\omega}}_{\mathrm{i}} : [t_1, t_2] \to \mathbb{R}^{\mathrm{n}_\omega}$) represents the components of $\mathbf{v}_{\mathrm{i}}(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t))$ (respectively, $\boldsymbol{\omega}_{\mathrm{i}}(\mathbf{q}(t), \mathbf{q}_{\mathrm{dot}}(t))$) that are also components of $\mathbf{q}_{\mathrm{dot}}(t)$, and $\hat{\mathbf{a}} : \mathbb{R}^3 \to \mathbb{R}^{\mathrm{n_v}}$ and $\hat{\mathbf{u}}_{\mathrm{i,tran}} : [t_1, t_2] \to \mathbb{R}^{\mathrm{n_v}}$ (respectively, $\hat{\widetilde{\boldsymbol{\omega}}}_{\mathrm{i}} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^{\mathrm{n}_\omega}$ and $\hat{\mathbf{u}}_{\mathrm{i,rot}} : [t_1, t_2] \to \mathbb{R}^{\mathrm{n}_\omega}$) are the corresponding components of $\mathbf{a}(\mathbf{v}_{\mathrm{i}}(t))$ and $\mathbf{u}_{\mathrm{i,tran}}(t)$ (respectively, $\widetilde{\boldsymbol{\omega}}_{\mathrm{i}}(\mathbf{v}_{\mathrm{i}}(t), \boldsymbol{\omega}_{\mathrm{i}}(t))$ and $\mathbf{u}_{\mathrm{i,rot}}(t)$).

Next, it follows from (46), (53), and (54) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\boldsymbol{\lambda}^*_{\mathrm{dyn,i},\hat{v}}(t) \\ \boldsymbol{\lambda}^*_{\mathrm{dyn,i},\hat{\omega}}(t)\end{bmatrix} = -\begin{bmatrix}\left(\frac{\partial \hat{\mathbf{a}}(\mathbf{v}_{\mathrm{i}})}{\partial \hat{\mathbf{v}}_{\mathrm{i}}}\right)^{\mathrm{T}} & \left(\frac{\partial \hat{\widetilde{\boldsymbol{\omega}}}_{\mathrm{i}}(\mathbf{v}_{\mathrm{i}}, \boldsymbol{\omega}_{\mathrm{i}})}{\partial \hat{\mathbf{v}}_{\mathrm{i}}}\right)^{\mathrm{T}} \\ \mathbf{0}_{\mathrm{n}_\omega \times \mathrm{n_v}} & \left(\frac{\partial \hat{\widetilde{\boldsymbol{\omega}}}_{\mathrm{i}}(\mathbf{v}_{\mathrm{i}}, \boldsymbol{\omega}_{\mathrm{i}})}{\partial \hat{\boldsymbol{\omega}}_{\mathrm{i}}}\right)^{\mathrm{T}}\end{bmatrix}_{(\hat{\mathbf{v}}_{\mathrm{i}}^*, \hat{\boldsymbol{\omega}}_{\mathrm{i}}^*)}\begin{bmatrix}\boldsymbol{\lambda}^*_{\mathrm{dyn,i},\hat{v}}(t) \\ \boldsymbol{\lambda}^*_{\mathrm{dyn,i},\hat{\omega}(t)}\end{bmatrix}, \tag{55}$$

where $\boldsymbol{\lambda}_{\mathrm{dyn,i},\hat{v}} : [t_1, t_2] \to \mathbb{R}^{\mathrm{n_v}}$ and $\boldsymbol{\lambda}_{\mathrm{dyn,i},\hat{\omega}(t)} : [t_1, t_2] \to \mathbb{R}^{\mathrm{n}_\omega}$ are the $\mathrm{n_v}$ and n_ω components of $\boldsymbol{\lambda}^*_{\mathrm{dyn,i}}(t)$ corresponding to the $\mathrm{n_v}$ and n_ω components of $\dot{\mathbf{v}}_{\mathrm{i}}(t)$ and $\dot{\boldsymbol{\omega}}_{\mathrm{i}}(t)$, respectively.

Theorem 6.3. *Assume that* $||\hat{\mathbf{u}}^*_{\rm i,tran}(t)||_2 = \hat{\rho}_{\rm i,tran}$, $||\hat{\mathbf{u}}^*_{\rm i,rot}(t)||_2 = \hat{\rho}_{\rm i,rot}$, $||\mathbf{u}^*_{\rm i,tran}(t)||_2 = \rho_{\rm i,tran}$, $||\mathbf{u}^*_{\rm i,rot}(t)||_2 = \rho_{\rm i,rot}$, *where* $\hat{\rho}_{\rm i,tran}$ *and* $\rho_{\rm i,tran} \in (\rho_{\rm i,1}, \rho_{\rm i,2})$, $\rho_{\rm i,rot}$ *and* $\rho_{\rm i,rot} \in (\rho_{\rm i,3}, \rho_{\rm i,4})$, *and* $\left[\frac{\partial\hat{\hat{\boldsymbol{\omega}}}_{\rm i}(\mathbf{v}_{\rm i},\boldsymbol{\omega}_{\rm i})}{\partial\hat{\boldsymbol{\omega}}_{\rm i}}\right]_{(\mathbf{v}^*_{\rm i},\boldsymbol{\omega}^*_{\rm i})}$ *is invertible. Then,*

$$\left|\left| \left[\frac{\partial\hat{\hat{\boldsymbol{\omega}}}_{\rm i}\left(\mathbf{v}_{\rm i},\boldsymbol{\omega}_{\rm i}\right)}{\partial\hat{\boldsymbol{\omega}}_{\rm i}}\right]^{-\rm T}_{(\mathbf{v}^*_{\rm i},\boldsymbol{\omega}^*_{\rm i})} \left(\mathbf{I}_3 + \left[\frac{\partial\hat{\mathbf{a}}_{\rm i}\left(\mathbf{v}_{\rm i}\right)}{\partial\hat{\mathbf{v}}_{\rm i}}\right]^T_{\mathbf{v}^*_{\rm i}}\right)\mathbf{u}^*_{\rm i,tran}(t)\right|\right|_2 \leq \sqrt{\rho^2_{\rm i,tran} + \rho^2_{\rm i,rot}}, \tag{56}$$

$$\left|\left| \left[\frac{\partial\hat{\hat{\boldsymbol{\omega}}}_{\rm i}\left(\mathbf{v}_{\rm i},\boldsymbol{\omega}_{\rm i}\right)}{\partial\hat{\boldsymbol{\omega}}_{\rm i}}\right]^{-\rm T}_{(\mathbf{v}^*_{\rm i},\boldsymbol{\omega}^*_{\rm i})} \dot{\mathbf{u}}^*_{\rm i,rot}(t)\right|\right|_2 \leq \sqrt{\rho^2_{\rm i,tran} + \rho^2_{\rm i,rot}}. \tag{57}$$

Proof. It follows from (44), (42), (53), and (54) that

$$\begin{aligned} \lambda^*_0\mu_{\rm i}\frac{\hat{\mathbf{u}}^*_{\rm i,tran}(t)}{||\tilde{\mathbf{u}}^*(t)||_2} &= -\boldsymbol{\lambda}^*_{{\rm dyn,i},\hat{\rm v}}(t), \\ \lambda^*_0\mu_{\rm i}\frac{\hat{\mathbf{u}}^*_{\rm i,rot}(t)}{||\tilde{\mathbf{u}}^*(t)||_2} &= -\boldsymbol{\lambda}^*_{{\rm rot,i},\hat{\omega}(t)}(t). \end{aligned} \tag{58}$$

Recalling that $\left|\left|\frac{\hat{\mathbf{u}}^*_{\rm i,rot}(t)}{||\tilde{\mathbf{u}}^*(t)||_2}\right|\right|_2 \leq 1$ and using (55) and (58) we obtain

$$\begin{aligned} &\left|\left|\lambda^*_0\mu_{\rm i}\left[\frac{\partial\hat{\hat{\boldsymbol{\omega}}}_{\rm i}\left(\mathbf{v}_{\rm i},\boldsymbol{\omega}_{\rm i}\right)}{\partial\hat{\boldsymbol{\omega}}_{\rm i}}\right]^{-\rm T}_{(\mathbf{v}^*_{\rm i},\boldsymbol{\omega}^*_{\rm i})} \frac{||\tilde{\mathbf{u}}^*_{\rm i}(t)||_2\dot{\hat{\mathbf{u}}}^*_{\rm i,tran}(t) + \dot{\tilde{\mathbf{u}}}^{*T}_{\rm i}(t)\tilde{\mathbf{u}}^*_{\rm i}(t)\hat{\mathbf{u}}^*_{\rm i,tran}(t)}{||\tilde{\mathbf{u}}^*_{\rm i}(t)||^2_2}\right.\right. \\ &\qquad \left.\left. + \lambda^*_0\mu_{\rm i}\left[\frac{\partial\hat{\hat{\boldsymbol{\omega}}}_{\rm i}\left(\mathbf{v}_{\rm i},\boldsymbol{\omega}_{\rm i}\right)}{\partial\hat{\boldsymbol{\omega}}_{\rm i}}\right]^{-\rm T}_{(\mathbf{v}^*_{\rm i},\boldsymbol{\omega}^*_{\rm i})} \left[\frac{\partial\hat{\mathbf{a}}_{\rm i}\left(\mathbf{v}_{\rm i}\right)}{\partial\hat{\mathbf{v}}_{\rm i}}\right]^T_{\mathbf{v}^*_{\rm i}} \frac{\hat{\mathbf{u}}^*_{\rm i,tran}(t)}{||\tilde{\mathbf{u}}^*_{\rm i}(t)||_2}\right|\right|_2 \leq \lambda^*_0\mu_{\rm i}, \end{aligned}$$

$$\left|\left|\lambda^*_0\mu_{\rm i}\left[\frac{\partial\hat{\hat{\boldsymbol{\omega}}}_{\rm i}\left(\mathbf{v}_{\rm i},\boldsymbol{\omega}_{\rm i}\right)}{\partial\hat{\boldsymbol{\omega}}_{\rm i}}\right]^{-\rm T}_{(\mathbf{v}^*_{\rm i},\boldsymbol{\omega}^*_{\rm i})} \frac{||\tilde{\mathbf{u}}^*_{\rm i}(t)||_2\dot{\hat{\mathbf{u}}}^*_{\rm i,rot}(t) + \dot{\tilde{\mathbf{u}}}^{*T}_{\rm i}(t)\tilde{\mathbf{u}}^*_{\rm i}(t)\hat{\mathbf{u}}^*_{\rm i,rot}(t)}{||\tilde{\mathbf{u}}^*_{\rm i}(t)||^2_2}\right|\right|_2 \leq \lambda^*_0\mu_{\rm i}.$$

Now, noting that $\dot{\tilde{\mathbf{u}}}^{*T}_{\rm i}(t)\tilde{\mathbf{u}}^*_{\rm i}(t) = 0$, the result follows. □

Since Theorem 6.3 is proven using the Euler necessary condition, it follows that $(\mathbf{u}^*_{\rm i,tran}, \mathbf{u}^*_{\rm i,rot}) \in {\rm int}(\Gamma_{\rm i,tran} \times {\rm int}(\Gamma_{\rm i,rot}))$. However, the parameter bounds $\rho_{\rm i,j}$, j = 1, 2, 3, 4, are imposed by physical and not mathematical considerations, and hence, for practical applications we can assume that there exists $\epsilon > 0$ such that Theorem 6.3 holds for $\rho_{\rm i,tran} \in (\rho_{\rm i,1} - \epsilon, \rho_{\rm i,2} + \epsilon)$ and $\rho_{\rm i,rot} \in (\rho_{\rm i,3} - \epsilon, \rho_{\rm i,4} + \epsilon)$. Consequently, for engineering applications we can assume that Theorem 6.3 also holds on arcs of maximum translational and rotational thrust.

Corollary 6.1. *Assume that the hypothesis of Theorem 6.3 hold. If* $n_\omega = 0$*, then*

$$\left|\left| \left[\frac{\partial \hat{\mathbf{a}}_i(\mathbf{v}_i)}{\partial \hat{\mathbf{v}}_i}\right]_{\mathbf{v}_i^*}^{-T} \hat{\mathbf{u}}^*_{i,tran}(t) \right|\right|_2 \leq \sqrt{\rho^2_{i,tran} + \rho^2_{i,rot}}. \tag{59}$$

Alternatively, if $n_v = 0$*, then*

$$\left|\left| \left[\frac{\partial \hat{\hat{\boldsymbol{\omega}}}_i(\mathbf{v}_i, \boldsymbol{\omega}_i)}{\partial \hat{\boldsymbol{\omega}}_i}\right]_{(\mathbf{v}_i^*, \boldsymbol{\omega}_i^*)}^{-T} \hat{\mathbf{u}}^*_{i,rot}(t) \right|\right|_2 \leq \sqrt{\rho^2_{i,tran} + \rho^2_{i,rot}}. \tag{60}$$

Proof. The proof is a direct consequence of Theorem 6.3. □

Example 6.1. Consider the formation of the two vehicles addressed in Examples 4.1 and 4.2, and assume that $\mathbf{q}(t) = \left[\mathbf{x}_1^T(t), \mathbf{r}_2^T(t)\right]^T$. As shown in Example 4.2, if $r_{min} < ||\mathbf{r}_1(t) - \mathbf{r}_2(t)||_2^2 < r_{max}$, then the first vehicle and the translational dynamics of the second vehicle can be considered unconstrained. Thus, the costate equation (41) can be rewritten as two decoupled ordinary differential equations given by

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{\lambda}_{dot,1}(t) \\ \boldsymbol{\lambda}_{dyn,1}(t)\end{bmatrix} = -\begin{bmatrix} \begin{bmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \frac{\partial \boldsymbol{R}^{-1}_{rod}(\boldsymbol{\sigma}_1)}{\partial \boldsymbol{\sigma}_1}\dot{\boldsymbol{\sigma}}_1 \end{bmatrix}^T & \mathbf{0}_{6\times6} \\ \left(\frac{\partial \mathbf{f}_{dyn,1}(\mathbf{x}_1, \mathbf{q}_{dot,1}(\mathbf{x}_1), \mathbf{u}_1)}{\partial \mathbf{x}_1}\right)^T & \left(\frac{\partial \mathbf{f}_{dyn,1}(\mathbf{x}_1, \mathbf{q}_{dot,1}(\mathbf{q}), \mathbf{u}_1)}{\partial \mathbf{q}_{dot,1}}\right)^T \end{bmatrix}\begin{bmatrix}\boldsymbol{\lambda}_{dot,1}(t) \\ \boldsymbol{\lambda}_{dyn,1}(t)\end{bmatrix}, \tag{61}$$

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{\lambda}_{dot,2}(t) \\ \boldsymbol{\lambda}_{dyn,2}(t)\end{bmatrix} = -\begin{bmatrix} \mathbf{0}_{3\times3} & \frac{\partial \mathbf{f}_{dyn,2}(\tilde{\mathbf{x}}_2(t), \mathbf{u}_{2,tran}(t))}{\partial \mathbf{r}_2} \\ \mathbf{0}_{3\times3} & \frac{\partial \mathbf{f}_{dyn,2}(\tilde{\mathbf{x}}_2(t), \mathbf{u}_{2,tran}(t))}{\partial \mathbf{v}_2} \end{bmatrix}^T \begin{bmatrix}\boldsymbol{\lambda}_{dot,2}(t) \\ \boldsymbol{\lambda}_{dyn,2}(t)\end{bmatrix}, \tag{62}$$

where $\boldsymbol{\lambda}_{dyn}(t) \triangleq [\boldsymbol{\lambda}^T_{dyn,1}(t)\ \boldsymbol{\lambda}^T_{dyn,2}(t)]^T$, $\boldsymbol{\lambda}_{dyn,1} : [t_1, t_2] \to \mathbb{R}^6$, $\boldsymbol{\lambda}_{dyn,2} : [t_1, t_2] \to \mathbb{R}^3$, $\boldsymbol{\lambda}_{dot}(t) \triangleq [\boldsymbol{\lambda}^T_{dot,1}(t), \boldsymbol{\lambda}^T_{dot,2}(t)]^T$, $\boldsymbol{\lambda}_{dot,1} : [t_1, t_2] \to \mathbb{R}^6$, and $\boldsymbol{\lambda}_{dot,2} : [t_1, t_2] \to \mathbb{R}^3$.

From (61) and (62) it follows that the path planning optimization problem for the first vehicle is possibly abnormal since we cannot verify a priori whether or not

$$\boldsymbol{\lambda}^*_{dot,1}(t) \in \mathcal{N}\left(\begin{bmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \frac{\partial \boldsymbol{R}^{-1}_{rod}(\boldsymbol{\sigma}_1)}{\partial \boldsymbol{\sigma}_1}\Big|\dot{\boldsymbol{\sigma}}_1(\mathbf{q}(t)) \end{bmatrix}_{\mathbf{q}=\mathbf{q}^*}\right),$$

whereas the path planning optimization problem for the second vehicle is normal since its rotational dynamics are not expressed by (62). Normality for the second formation vehicle can also be proven by rewriting the unconstrained dynamic equations (3) for a 3 DoF vehicle. For details, see L'Afflitto & Sultan (2008).

Using (18) it follows that (45) can be written as

$$\begin{aligned} \mathfrak{h}\left(\mathbf{q}(t), \mathbf{q}_{dot}(t), \tilde{\mathbf{u}}(t), \boldsymbol{\lambda}_{dyn}(t), \boldsymbol{\lambda}_{dot}(t)\right) &= \mathfrak{h}_1\left(\mathbf{x}_1(t), \mathbf{u}_1(t), \boldsymbol{\lambda}_{dyn,1}(t), \boldsymbol{\lambda}_{dot,1}(t)\right) \\ &\quad + \mathfrak{h}_2\left(\mathbf{x}_2(t), \mathbf{u}_{2,tran}(t), \boldsymbol{\lambda}_{dyn,2}(t)\right), \end{aligned} \tag{63}$$

where

$$\mathfrak{h}_1\left(\mathbf{x}_1(t), \mathbf{u}_1(t), \boldsymbol{\lambda}_{\mathrm{dyn},1}(t), \boldsymbol{\lambda}_{\mathrm{dot},1}(t)\right) = \lambda_0\mu_1||\mathbf{u}_1(t)||_2 + \boldsymbol{\lambda}^{\mathrm{T}}_{\mathrm{dyn},1,1}(t)\mathbf{u}_{1,\mathrm{tran}}(t) + \boldsymbol{\lambda}^{\mathrm{T}}_{\mathrm{dyn},1,2}(t)\mathbf{u}_{1,\mathrm{rot}}(t) + \boldsymbol{\lambda}^{\mathrm{T}}_{\mathrm{dyn},1,1}(t)\mathbf{a}\left(\widetilde{\mathbf{x}}_1(t)\right) + \boldsymbol{\lambda}^{\mathrm{T}}_{\mathrm{dyn},1,2}(t)\left(\widetilde{\boldsymbol{\omega}}_1\left(\widetilde{\mathbf{x}}_1(t)\right) - \mathbf{I}^{-1}_{\mathrm{in},1}\boldsymbol{\omega}^{\times}_1\left(\boldsymbol{\omega}_1(t)\right)\mathbf{I}_{\mathrm{in},1}\boldsymbol{\omega}_1(t)\right) + \boldsymbol{\lambda}^{\mathrm{T}}_{\mathrm{dot},1,1}\mathbf{v}_1(t) - \boldsymbol{\lambda}^{\mathrm{T}}_{\mathrm{dot},1,2}\mathbf{R}^{-1}_{\mathrm{rod}}(\boldsymbol{\sigma}_1(t))\dot{\boldsymbol{\sigma}}_1(t), \tag{64}$$

$$\mathfrak{h}_2\left(\mathbf{x}_2(t), \mathbf{u}_{2,\mathrm{tran}}(t), \boldsymbol{\lambda}_{\mathrm{dyn},2}(t)\right) = \mu_2||\mathbf{u}_{2,\mathrm{tran}}(t)||_2 + \boldsymbol{\lambda}^{\mathrm{T}}_{\mathrm{dyn},2,1}(t)\mathbf{u}_{2,\mathrm{tran}}(t) + \boldsymbol{\lambda}^{\mathrm{T}}_{\mathrm{dyn},2,1}(t)\mathbf{a}\left(\widetilde{\mathbf{x}}_2(t)\right) + \boldsymbol{\lambda}^{\mathrm{T}}_{\mathrm{dot},2,1}\mathbf{v}_2(t), \tag{65}$$

where $\boldsymbol{\lambda}_{\mathrm{dyn},1}(t) \triangleq [\boldsymbol{\lambda}^{\mathrm{T}}_{\mathrm{dyn},1,1}(t), \boldsymbol{\lambda}^{\mathrm{T}}_{\mathrm{dyn},1,2}(t)]^{\mathrm{T}}$, $\boldsymbol{\lambda}_{\mathrm{dyn},2}(t) \triangleq [\boldsymbol{\lambda}^{\mathrm{T}}_{\mathrm{dyn},2,1}(t), \boldsymbol{\lambda}^{\mathrm{T}}_{\mathrm{dyn},2,2}(t)]^{\mathrm{T}}$, and $\boldsymbol{\lambda}_{\mathrm{dyn,j,k}} : [t_1, t_2] \to \mathbb{R}^3$, j, k $= 1, 2$. Now, using Theorem 6.3 we can construct a candidate optimal control law. Remarkably, the same candidate optimal control law can be obtained by applying Theorem 6.3 to (64) and (65) independently. The fact that the candidate optimal control law for the the first vehicle can be found independently from the second vehicle is another advantage in employing Lagrange coordinates. The minimization of $\mathfrak{h}_2$ leads to the same candidate optimal control law as given by primer vector theory with the only difference being that the arcs of maximum, null, and singular thrust are not characterized by the sign of $||\boldsymbol{\lambda}^*_{\mathrm{dyn},2,1}(t)||_2 - 1$ as in Lawden's work (Lawden, 1963) but rather by the sign of $||\boldsymbol{\lambda}^*_{\mathrm{dyn},2,1}(t)||_2 - \mu_2$.

Singular translational thrust arcs for the first vehicle occur when

$$(\lambda_0\mu_1)^2 = \boldsymbol{\lambda}^{\mathrm{T}}_{\mathrm{dyn},1,1}(t)\boldsymbol{\lambda}_{\mathrm{dyn},1,1}(t) \tag{66}$$

and, as shown in Theorem 6.3, $\mathbf{u}^*_{2,\mathrm{tran}}$ cannot be found on singular arcs by applying Pontryagin's minimum principle. However, from (44) and (64), we note that $\lambda_0\mu_1\frac{\mathbf{u}^*_{1,\mathrm{tran}}(t)}{||\mathbf{u}^*_1(t)||_2} = -\boldsymbol{\lambda}^*_{\mathrm{dyn},1,1}(t)$, and hence, (66) yields

$$||\mathbf{u}^*_1(t)||^2_2 = \mathbf{u}^{*\mathrm{T}}_{1,\mathrm{tran}}(t)\mathbf{u}^*_{1,\mathrm{tran}}(t). \tag{67}$$

Thus, on singular translational thrust arcs for the first vehicle $\mathbf{u}^*_{1,\mathrm{rot}}(t) = \mathbf{0}_3$. Similarly, it can be shown that $\mathbf{u}^*_{1,\mathrm{tran}}(t) = \mathbf{0}_3$ on singular rotational thrust arcs for the first vehicle. Finally, singular arcs for the second vehicle occur when

$$\mu_2^2 = \boldsymbol{\lambda}^{*\mathrm{T}}_{\mathrm{dyn},2,1}(t)\boldsymbol{\lambda}^*_{\mathrm{dyn},2,1}(t). \tag{68}$$

From (44) and (65) it follows that $\mu_2\frac{\mathbf{u}^*_{2,\mathrm{tran}}(t)}{||\mathbf{u}^*_{2,\mathrm{tran}}(t)||_2} = -\boldsymbol{\lambda}^*_{\mathrm{dyn},2,1}(t)$, which satisfies (68). Hence, any admissible $\mathbf{u}_{2,\mathrm{tran}}$ can be applied on singular arcs. This was first noted by Lawden (1963).

7. Illustrative numerical example

In this section, we present a numerical example to highlight the efficacy of the framework presented in the paper. In particular, we consider the two vehicles presented in Examples

4.1, 4.2, and 6.1 with masses 0.1kg and inertia matrices $0.40\mathbf{I}_3\,\mathrm{kgm}^4$ flying in an environment modeled by (51) and (52), where $\mathbf{g} = [0, 0, -9.81]^{\mathrm{T}}\frac{\mathrm{m}}{\mathrm{s}^2}$, $k_{\mathrm{i,D}} = 0.20$, $k_{\mathrm{i,L}} = 1.20$, $k_{\mathrm{i,S}} = 0.50$, $k_{\mathrm{i,R}} = 0.30$, $k_{\mathrm{i,P}} = 0.30$, and $k_{\mathrm{i,Y}} = 0.30$, for $\mathrm{i} = 1, 2$. Furthermore, we assume that $t_1 = 0.00$ s, $t_2 = 60.00$ s, $\mathbf{r}_1(t_1) = [0.00, 0.00, 0.00]^{\mathrm{T}}$ m, $\mathbf{r}_1(t_2) = [0.90, -10.00, -1.80]^{\mathrm{T}}$ m, $\boldsymbol{\sigma}_1(t_1) = [0.00, 0.00, 0.00]^{\mathrm{T}}$, and $\boldsymbol{\sigma}_1(t_2) = [0.00, 0.00, 120.00\frac{\pi}{180.00}]^{\mathrm{T}}$. For our simulation we take $\rho_{\mathrm{i},1} = 10.00\frac{\mathrm{m}}{\mathrm{s}^2}$, $\rho_{\mathrm{i},2} = 45.00\frac{\mathrm{m}}{\mathrm{s}^2}$, $\rho_{\mathrm{i},3} = 10.00\frac{1}{\mathrm{s}^2}$, and $\rho_{\mathrm{i},1} = 20.00\frac{1}{\mathrm{s}^2}$, for $\mathrm{i} = 1, 2$. The boundary conditions for the second vehicle are deduced from (14) and (15) by assuming that $\mathrm{r}_{\max} = \frac{21}{25}$m and $\mathrm{r}_{\min} = \frac{33}{50}$m. It can be easily verified that the constraints given by (12) and (13) hold for all $t \in [t_1, t_2]$. Letting $\mu_1 = \mu_2 = \frac{1}{2}$ and applying Theorem 6.1, we obtain the optimal trajectory shown in Figure 1. Figures 2 and 3 show the optimal control as a function of the norm of the translational primer vector and the rotational primer vector, as well as time, respectively. For this example $\mathrm{J}\left[\mathbf{u}_1(\cdot)\right] = 10.00\frac{\mathrm{m}}{\mathrm{s}}$ and $\mathrm{J}\left[\mathbf{u}_2(\cdot)\right] = 11.60\frac{\mathrm{m}}{\mathrm{s}}$. Since $\mathfrak{m}\left(\mathbf{q}^*(t), \mathbf{q}^*_{\mathrm{dot}}(t), \boldsymbol{\lambda}^*_{\mathrm{dyn}}(t), \boldsymbol{\lambda}^*_{\mathrm{dot}}(t)\right) = 22.30\frac{\mathrm{m}}{\mathrm{s}^2}$, Theorem 6.2 holds. Finally, Figure 4 shows the translational primer vector and the rotational primer vector of the first vehicle as a function of time.

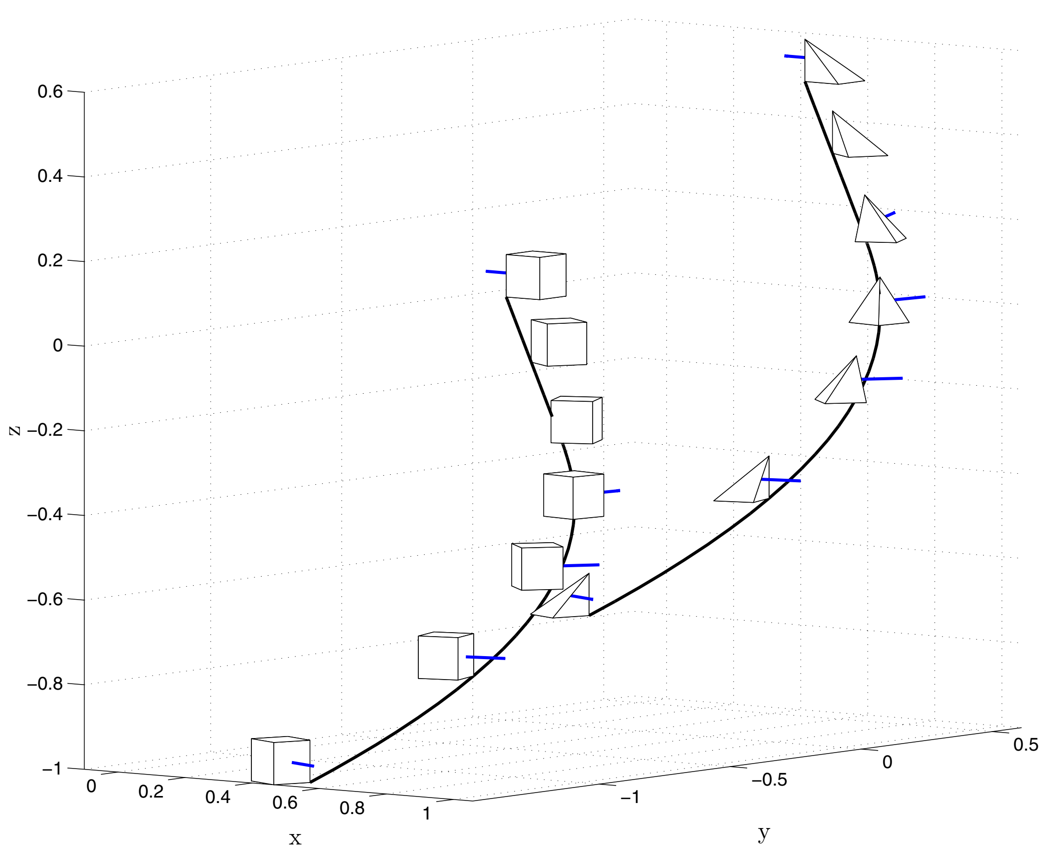


Fig. 1. Optimal trajectories for vehicles 1 and 2. The cube represents the first vehicle and the prism represents the second vehicle.

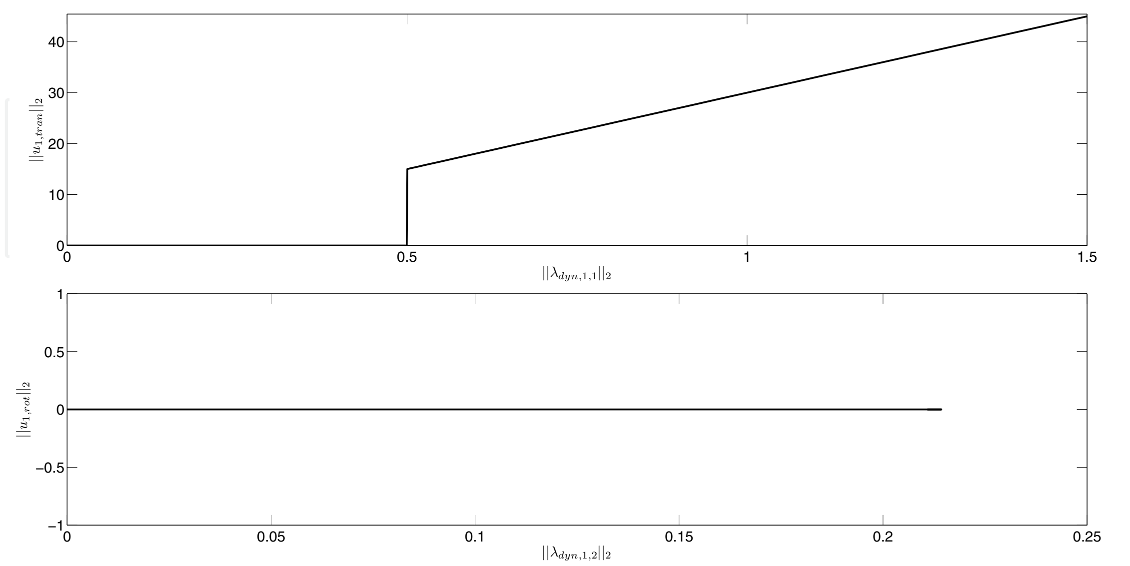


Fig. 2. Optimal control for the first vehicle as function of the norm of the translational primer vector and the rotational primer vector.

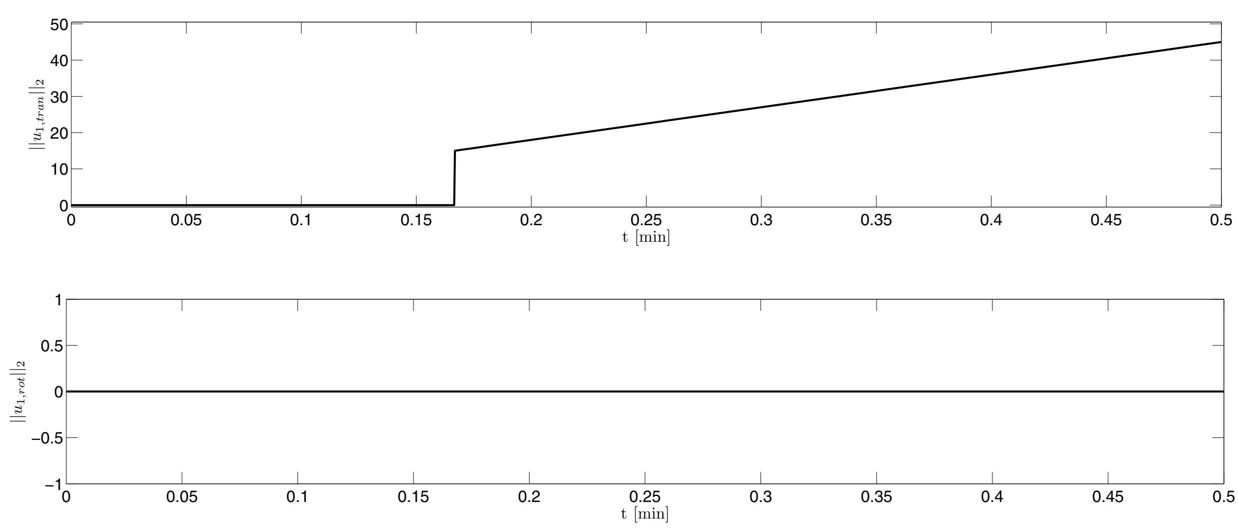


Fig. 3. Optimal control for the first vehicle as function of time.

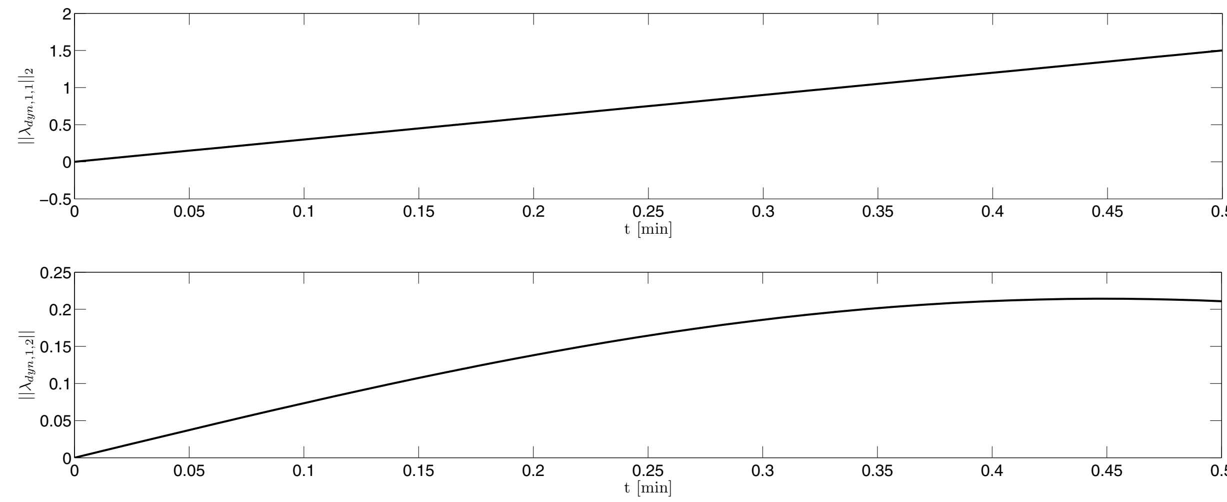


Fig. 4. Translational and rotational primer vector norms as functions of time for the first vehicle.

8. Conclusion and recommendations for future research

In this paper, we addressed the problem of minimizing the control effort needed to operate a formation of n UAVs. Specifically, a candidate optimal control law as well as necessary conditions for optimality that characterize the resulting optimal trajectories are derived and discussed assuming that the formation vehicles are 6 DoF rigid bodies flying in generic environmental conditions and subject to equality and inequality constraints. The results presented extend Lawden's seminal work (Lawden, 1963) and several papers predicated on his work.

An illustrative numerical example involving a formation of two vehicles is provided to illustrate the mathematical path planning optimization framework presented in the paper. Furthermore, we show that our framework is not restricted to UAV formations and can be applied to formations of robots, spacecraft, and underwater vehicles.

The results of the present paper can be further extended in several directions. Specifically, an analytical study of the translational primer vector and the rotational primer vector can be useful in identifying numerous properties of the formation's optimal path. In particular, the translational primer vector and the rotational primer vector can be used to measure the sensitivity of the candidate optimal control law to uncertainties in the dynamical model. In this paper, we provide a generic formulation to the optimal path planning problem in order to address a large number of formation problems. However, specializing our results to a particular formation and a particular environmental model can lead to analytical tools that can be amenable to efficient numerical methods. Additionally, nonholonomic constraints have not been accounted in our framework and can be addressed by modifying Theorem 4.1. Finally, in this paper, we penalize vehicle control effort by tuning the constants $\mu_1,...,\mu_n$ in (2). In many practical applications, however, it is preferable to trade-off the control effort in a formation of vehicles by optimizing over the free parameters $\mu_1,...,\mu_n$.

9. Acknowledgments

The first-named author would like to thank Drs. C. Sultan and E. Cliff at Virginia Polytechnic Institute and State University for several helpful discussions. This research was supported in part by the Domenica Rea d'Onofrio Fellowship Foundation and the Air Force Office of Scientific Research under Grant FA9550-09-1-0429.

10. References

Ambrosia, V. & Hinkley, E. (2008). NASA science serving society: Improving capabilities for fire characterization to effect reduction in disaster losses, *IEEE International Geoscience and Remote Sensing Symposium, 2008. IGARSS 2008.*, Vol. 4, pp. IV –628 –IV –631.

Anderson, J. D. (2001). *Fundamentals of Aerodynamics*, McGraw Hill, New York, NY.

Bataillé, B., Moschetta, J. M., Poinsot, D., Bérard, C. & Piquereau, A. (2009). Development of a VTOL mini UAV for multi-tasking missions, *The Aeronautical Journal* 13: 87–98.

Betts, J. T. (1998). Survey of numerical methods for trajectory optimization, *AIAA Journal of Guidance, Control, and Dynamics* 21: 193–207.

Blackmore, L. (2008). Robust path planning and feedback design under stochastic uncertainty, *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, AIAA, Honolulu, HI.

Bryson, A. E. (1975). *Applied Optimal Control*, Hemisphere, New York, NY.

Ewing, E. G. (1969). *Calculus of Variations with Applications*, Dover Edition, New York, NY.

Giaquinta, M. & Hildebrandt, S. (1996). *Calculus of Variations I*, Springer-Verlag, Berlin, Germany.

Greenwood, T. D. (2003). *Advanced Dynamics*, Cambridge University Press, New York, NY.

Haddal, C. C. & Gertler, J. (2010). Homeland security: Unmanned aerial vehicles and border surveillance, *Technical Report RS21698*, Congressional Research Service, Washington, D.C.

Hassan, R., Cohanim, B. & de Weck, O. (2005). A comparison of particle swarm optimization and the genetic algorithm, *Proceedings of the 46th AIAA Structures, Structural Dynamics and Materials Conference*, AIAA, Breckenridge, CO.

Herman, A. L. & Conway, B. A. (1987). Direct optimization using nonlinear programming and collocation, *AIAA Journal of Guidance, Control, and Dynamics* 10: 338–342.

Jacobson, D. & Lele, M. (1969). A transformation technique for optimal control problems with a state variable inequality constraint, *IEEE Transactions on Automatic Control* 14(5): 457–464.

Jamison, B. R. & Coverstone, V. (2010). Analytical study of the primer vector and orbit transfer switching function, *AIAA Journal of Guidance, Control, and Dynamics* 33: 235–245.

Jang, J. S. & Tomlin, C. J. (2005). Control strategies in multi-player pursuit and evasion game, *Proceeding AIAA Guidance, Navigation, and Control Conference*, AIAA, San Francisco, CA.

L'Afflitto, A. & Sultan, C. (2008). Applications of calculus of variations to aircraft and spacecraft path planning, *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, AIAA, Chicago, IL.

L'Afflitto, A. & Sultan, C. (2010). On calculus of variations in aircraft and spacecraft formation flying path planning, *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, AIAA, Toronto, Canada.

Lawden, D. F. (1963). *Optimal Trajectories for Space Navigation*, Butterworths, London, UK.

Lee, E. B. & Markus, L. (1968). *Foundations of Optimal Control Theory*, Wiley, New York, NY.

Lillesand, T., Kiefer, R. W. & Chipman, J. (2007). *Remote Sensing and Image Interpretation*, Wiley, New York, NY.

Mailhe, L. & Guzman, J. (2004). Initialization and resizing of formation flying using global and local optimization methods, *Proceedings IEEE Aerospace Conference*, Vol. 1, pp. 547–556.

Majewski, S. E. (1999). Naval command and control for future UAVs. MS Thesis, Naval Postgraduate School, Monterey, CA.

Marec, J. P. (1979). *Optimal Space Trajectories*, Elsevier, New York, NY.

Neimark, J. I. & Fufaev, N. A. (1972). *Dynamics of Nonholonimic Systems*, American Mathematical Society, New York, NY.

Oyekan, J. & Huosheng, H. (2009). Toward bacterial swarm for environmental monitoring, *IEEE International Conference on Automation and Logistics*, pp. 399 –404.

Pars, L. A. (1965). *A Treatise on Analytical Dynamics*, Wiley, New York, NY.

Petropoulos, A. E. & Longuski, J. M. (2004). Shape-based algorithm for automated design of low-thrust, gravity-assist trajectories, *AIAA Journal of Guidance, Control, and Dynamics* 32: 95–101.

Petropoulos, A. E. & Russell, R. P. (2008). Low-thrust transfers using primer vector theory and a second-order penalty method, *Proceedings of the AIAA Astrodynamics Specialist Conference*, AIAA, Honolulu, HI.

Plnes, D. & Bohorquez, F. (2006). Challenges facing future micro-air-vehicle development, *AIAA Journal of Aircraft* 43: 290–305.

Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V. & Mishchenko, E. F. (1962). *The Mathematical Theory of Optimal Processes*, Interscience Publishers, New York, NY.

Prussing, J. E. (2010). Primer vector theory and applications, *in* B. A. Conway (ed.), *Spacecraft Trajectory Optimization*, Cambridge University Press, Chicago, IL, pp. 155–188.

Ramage, J., Avalle, M., Berglund, E., Crovella, L., Frampton, R., Krogmann, U., Ravat, C., Robinson, M., Shulte, A. & Wood, S. (2009). Automation technologies and application considerations for highly integrated mission systems, *Technical Report TR-SCI-118*, North Atlantic Treaty Organisation.

Scharf, D., Hadaegh, F. & Ploen, S. (2003a). A survey of spacecraft formation flying guidance and control (part 1): Guidance, *Proceedings of the American Control Conference*, pp. 1733 – 1739.

Scharf, D., Hadaegh, F. & Ploen, S. (2003b). A survey of spacecraft formation flying guidance and control (part 2): Control, *Proceedings of the American Control Conference*, pp. 1740 – 1748.

Schouwenaars, T., Feron, E. & How, J. (2006). Multi-vehicle path planning for non-line of sight communication, *Proceedings of the American Control Conference*, pp. 5758–5762.

Seereram, S., Li, E., Ravichandran, B., Mehra, R. K., Smith, R. & Beard, R. (2000). Multispacecraft formation initialization using genetic algorithm techniques, *Proceedings of the 23rd Annual AAS Guidance and Control Conference*, AAS, Breckenridge, CO.

Shanmugavel, M., Tsourdos, A. & White, B. (2010). Collision avoidance and path planning of multiple UAVs using flyable paths in 3D, *15th International Conference on Methods and Models in Automation and Robotics*, pp. 218–222.

Shuster, M. D. (1993). Survey of attitude representations, *Journal of the Astronautical Sciences* 11: 439–517.

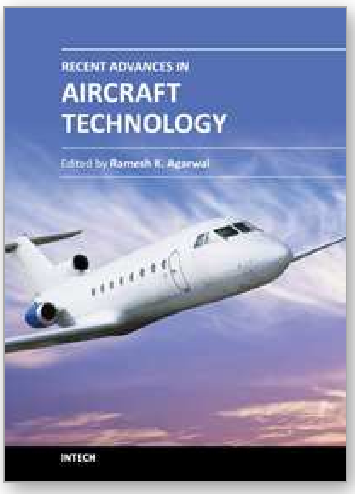
Topcu, U., Casoliva, J. & Mease, K. D. (2007). Minimum-fuel powered descent for Mars pinpoint landing, *AIAA Journal of Spacecraft and Rockets* 44(2): 324–331.

Valentine, F. A. (1937). The problem of Lagrange with differential inequalities as added side conditions, *in* G. A. Bliss (ed.), *Contributions to the Calculus of Variations*, Chicago University Press, Chicago, IL, pp. 407–448.

Wall, B. J. (2008). Shape-based approximation method for low-thrust trajectory optimization, *Proceedings of the AIAA Astrodynamics Specialist Conference*, AIAA, Honolulu, HI.

Wang, P. K. C. (1991). Navigation strategies for multiple autonomous mobile robots moving in formation, *Journal of Robotic Systems* 8: 177 – 195.

Zaitri, M. K., Arzelier, D. & Louembert, C. (2010). Mixed iterative algorithm for solving optimal impulsive time-fixed rendezvous problem, *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, AIAA, Toronto, Canada.



Recent Advances in Aircraft Technology

Edited by Dr. Ramesh Agarwal

ISBN 978-953-51-0150-5
Hard cover, 544 pages
Publisher InTech
Published online 24, February, 2012
Published in print edition February, 2012

The book describes the state of the art and latest advancements in technologies for various areas of aircraft systems. In particular it covers wide variety of topics in aircraft structures and advanced materials, control systems, electrical systems, inspection and maintenance, avionics and radar and some miscellaneous topics such as green aviation. The authors are leading experts in their fields. Both the researchers and the students should find the material useful in their work.

INTECH
open science | open minds

InTech Europe
University Campus STeP Ri
Slavka Krautzeka 83/A
51000 Rijeka, Croatia
Phone: +385 (51) 770 447
Fax: +385 (51) 686 166
www.intechopen.com

InTech China
Unit 405, Office Block, Hotel Equatorial Shanghai
No.65, Yan An Road (West), Shanghai, 200040, China
中国上海市延安西路65号上海国际贵都大饭店办公楼405单元
Phone: +86-21-62489820
Fax: +86-21-62489821