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Quantum Fields on the de Sitter Expanding Universe

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1. Introduction

The quantum theory of fields on curved space-times, which is of actual interest in astrophysics and cosmology, is faced with serious difficulties arising from the fact that some global properties of the theory on flat space-times become local when the background is curved. The principal impediment on curved manifolds is the absence of the Poincaré symmetry which gives rise to the principal invariants of special relativity (i.e. the mass and spin) and assures the stability of the vacuum state. This drawback encouraged many authors to avoid the principal steps of the quantum theory based on canonical quantization looking for new effective methods able to be used in theories on curved manifolds. Thus a particular attention was payed to the construction of the two-point functions of the axiomatic quantum field theory (Allen & Jacobson, 1986) or to some specific local effects as, for example, the Unruh's one (Unruh, 1976). However, in this manner some delicate problems related to the fields with spin may remain obscure.

For this reason we believe that turning back to the traditional method of canonical quantization we may open new perspectives in developing the quantum field theory on curved space-times. According to the standard interpretation of the quantum mechanics, our basic assumption is that the quantum states are *prepared* by a *global* classical apparatus which includes the natural and local frames. This means that the quantum observables must be defined globally as being *conserved* operators which commute with that of the field equation. Therefore, the quantum modes have to be determined as common eigenfunctions of several complete systems of commuting operators which include the operator of the field equation. These mode functions must be orthonormalized with respect to a suitable relativistic scalar product such that the subspaces of functions of positive and respectively negative frequencies remain orthogonal to each other in any frame. This conjecture leads to a stable vacuum state when we perform the canonical quantization which enables us to derive the propagators and especially the one-particle operators. Moreover, the theory of the interacting fields may be developed then as in the flat case using the S-matrix and the perturbation theory since the normal ordering of the operator products do make sense thanks to the stability of the vacuum state.

The theory of quantum fields with spin on curved backgrounds has a specific structure since the spin half can be defined only in orthogonal local (non-holonomic) frames. Therefore, in general, the Lagrangian theory of the matter fields has to be written in local frames assuming that this is tetrad-gauge covariant. Thus the gauge group $L^{\uparrow}_{+} \subset SO(1,3)$ (of the principal fiber bundle) and its universal covering group, $SL(2,\mathbb{C})$, (of the spin fiber bundle) become crucial since their finite-dimensional (non-unitary) representations *induce* the *covariant* representations of the universal covering group of the isometry one according to which the matter fields transform under isometries (Cotăescu, 2000). The generators of these covariant representations are the differential operators given by the Killing vectors associated to isometries according to the generalized Carter and McLenagan formula (Carter & McLenaghan, 1979). The theory of the fields with integer spin can be written either in local frames or exclusively in natural ones where we have shown how the spin must be defined in order to recover the Carter and McLenagan formula (Cotăescu, 2009). Thus the external symmetry offers us the conserved operators among them we may select different sets of commuting operators which have to define the free quantum modes. A convenient relativistic scalar product and a stable vacuum state have to complete the framework we need for performing the canonical quantization.

Our method is helpful on the de Sitter space-time where all the free field equations can be analytically solved while the $SO(1,4)$ isometries provide us with a large collection of conserved operators. A particular feature of this symmetry is that the energy and momentum operators do not commute to each other. Consequently, there are no mass-shells and the energy and momentum are diagonal in different bases, called the energy and respectively momentum bases (or representations). In spite of this new behavior, we pointed out that the principal invariants of the covariant representations can be expressed in terms of rest energy and spin as in special relativity (Cotăescu, 2011a). Moreover, we have derived the principal sets of quantum modes of the free scalar (Cotăescu et al., 2008), Dirac (Cotăescu, 2002; Cotăescu & Crucean, 2008), Proca (Cotăescu, 2010) and Maxwell (Cotăescu & Crucean, 2010) fields applying the same procedure of canonical quantization. In what follows we would like to present these results for the scalar and Dirac fields which are the typical examples of the field theory formulated exclusively either in natural or in local frames.

In the second section we briefly review our theory of external symmetry focusing on the de Sitter isometries and their invariants. The third section is devoted to the de Sitter space-time and its isometries. The next two sections are devoted to the second quantization of the scalar and Dirac fields. For these fields we derive the quantum modes in momentum representation using the free field equations in the (co)moving frames of the de Sitter expanding universe and the local frames defined by the diagonal gauge in the case of the Dirac equation. The polarization of the Dirac field is given in the helicity basis which can be easily defined in the momentum representation we use. Considering appropriate scalar products in each particular case, a special attention is paid to the orthogonality and completeness properties as well as to the choice of the vacuum state. We argue that the vacuum of Bunch-Davies type (Bunch & Davies, 1978) we define here is stable as long as the particle and antiparticle sets of mode functions remain orthogonal among themselves in any frame. Under such circumstances the method of canonical quantization is working well allowing us to obtain important pieces as propagators and one-particle operators. Finally, we present our conclusions marking out what is new in interpreting the global quantum modes studied here.

2. External symmetry in general relativity

In general relativity the space-time symmetries are the isometries of the background associated to the Killing vectors. The physical fields minimally coupled to gravity take over this symmetry transforming according to appropriate representations of the isometry group.

In the case of the scalar vector or tensor fields these representations are completely defined by the well-known rules of the general coordinate transformations since the isometries are in fact particular automorphisms. However, the theory of spinor fields is formulated in orthogonal local frames where the basis-generators of the spinor representation were discovered by Carter and McLenaghan (Carter & McLenaghan, 1979).

For this reason we proposed a new theory of external symmetry in the context of the gauge-covariant theories in local orthogonal frames (Cotăescu, 2000). We introduced there new transformations which combine isometries and gauge transformations such that the tetrad fields should remain invariant. In this way we obtained the external symmetry group and we derived the general form of the basis-generators of the representations of this group shoving that these are given by a formula which generalizes the Carter and McLenaghan one. Moreover, we pointed out that this theory can be formulated exclusively in natural frames if there are only scalar, vector and tensor fields (Cotăescu, 2009).

2.1 Tetrad gauge covariance

Let us consider the pseudo-Riemannian space-time (M, g) and a local chart (or natural frame) of coordinates x^μ (labeled by natural indices, $\mu, \nu, ... = 0, 1, 2, 3$) (Wald, 1984). Given a gauge, we denote by $e_{\hat\mu}$ the tetrad fields that define the local frames and by $\hat e^{\hat\mu}$ those of the corresponding coframes. These have the usual duality, $\hat e^{\hat\mu}_\alpha\, e^\alpha_{\hat\nu} = \delta^{\hat\mu}_{\hat\nu}$, $\hat e^{\hat\mu}_\alpha\, e^\beta_{\hat\mu} = \delta^\beta_\alpha$, and orthonormalization, $e_{\hat\mu} \cdot e_{\hat\nu} = \eta_{\hat\mu\hat\nu}$, $\hat e^{\hat\mu} \cdot \hat e^{\hat\nu} = \eta^{\hat\mu\hat\nu}$, properties. The metric tensor $g_{\mu\nu} = \eta_{\hat\alpha\hat\beta}\hat e^{\hat\alpha}_\mu \hat e^{\hat\beta}_\nu$ raises or lowers the natural indices while for the local indices ($\hat\mu, \hat\nu, ... = 0, 1, 2, 3$) we have to use the flat Minkowski metric $\eta = \mathrm{diag}(1, -1, -1, -1)$. The derivatives in local frames are the vector fields $\hat\partial_{\hat\nu} = e^\mu_{\hat\nu}\partial_\mu$ which satisfy the commutation rules $[\hat\partial_{\hat\mu}, \hat\partial_{\hat\nu}] = C_{\hat\mu\hat\nu\cdot}^{\cdot\cdot\hat\sigma}\hat\partial_{\hat\sigma}$ defining the Cartan coefficients.

The metric η remains invariant under the transformations of its gauge group, $O(1, 3)$. This has as subgroup the Lorentz group, $L^\uparrow_+$, of the transformations $\Lambda[A(\omega)]$ corresponding to the transformations $A(\omega) \in SL(2, \mathbb{C})$ through the canonical homomorphism (Tung, 1984). In the standard covariant parametrization, with the real parameters $\omega^{\hat\alpha\hat\beta} = -\omega^{\hat\beta\hat\alpha}$, the $SL(2, \mathbb{C})$ transformations $A(\omega) = \exp(-\frac{i}{2}\omega^{\hat\alpha\hat\beta}S_{\hat\alpha\hat\beta})$ are generated by the covariant basis-generators of the $sl(2, \mathbb{C})$ algebra, denoted by $S_{\hat\alpha\hat\beta}$. For small values of $\omega^{\hat\alpha\hat\beta}$ the matrix elements of the transformations Λ in the local basis can be expanded as $\Lambda^{\hat\mu\cdot}_{\cdot\hat\nu}[A(\omega)] = \delta^{\hat\mu}_{\hat\nu} + \omega^{\hat\mu\cdot}_{\cdot\hat\nu} + \cdots$.

Assuming now that (M, g) is orientable and time-orientable we can consider $G(\eta) = L^\uparrow_+$ as the gauge group of the Minkowski metric η (Wald, 1984). This is the structure group of the principal fiber bundle whose basis is M. The group $\mathrm{Spin}(\eta) = SL(2, \mathbb{C})$ is the universal covering group of $G(\eta)$ and represents the structure group of the spin fiber bundle (Lawson & Michaelson, 1989). In general, a matter field $\psi_{(\rho)} : M \to \mathcal{V}_{(\rho)}$ is locally defined over M with values in the vector space $\mathcal{V}_{(\rho)}$ of a representation ρ, generally reducible, of the group $\mathrm{Spin}(\eta)$. The covariant derivatives of the field $\psi_{(\rho)}$,

$$D^{(\rho)}_{\hat\alpha} = e^\mu_{\hat\alpha} D^{(\rho)}_\mu = \hat\partial_{\hat\alpha} + \frac{i}{2}\,\rho(S^{\hat\beta\cdot}_{\cdot\hat\gamma})\,\hat\Gamma^{\hat\gamma}_{\hat\alpha\hat\beta}\,, \tag{1}$$

depend on the connection coefficients in local frames,

$$\begin{aligned}\hat{\Gamma}^{\hat{\sigma}}_{\hat{\mu}\hat{\nu}} &= e^{\alpha}_{\hat{\mu}} e^{\beta}_{\hat{\nu}} (\hat{e}^{\hat{\sigma}}_{\gamma} \Gamma^{\gamma}_{\alpha\beta} - \hat{e}^{\hat{\sigma}}_{\beta,\alpha}) \\ &= \frac{1}{2}\eta^{\hat{\sigma}\hat{\lambda}} (C_{\hat{\mu}\hat{\nu}\hat{\lambda}} - C_{\hat{\mu}\hat{\lambda}\hat{\nu}} - C_{\hat{\nu}\hat{\lambda}\hat{\mu}}),\end{aligned} \tag{2}$$

which assure the covariance of the whole theory under tetrad gauge transformations produced by automorphisms A of the spin fiber bundle. This is the general framework of the theories involving fields with half integer spin which can not be treated in natural frames.

2.2 External symmetries in local frames

A special difficulty in local frames is that the theory is no longer covariant under isometries since these can change the tetrad fields that carry natural indices. For this reason we proposed a theory of external symmetry in which each isometry transformation is coupled to a gauge one able to correct the position of the local frames such that the whole transformation should preserve not only the metric but the tetrad gauge too (Cotăescu, 2000). Thus, for any isometry transformation $x \to x' = \phi_\xi(x) = x + \xi^a k_a + ...$, depending on the parameters ξ^a $(a, b, ... = 1, 2...N)$ of the isometry group $I(M)$, one must perform the gauge transformation A_ξ defined as

$$\Lambda^{\hat{\alpha}\cdot}_{\cdot\,\hat{\beta}}[A_\xi(x)] = \hat{e}^{\hat{\alpha}}_{\mu}[\phi_\xi(x)] \frac{\partial \phi^{\mu}_{\xi}(x)}{\partial x^{\nu}}\, e^{\nu}_{\hat{\beta}}(x) \tag{3}$$

with the supplementary condition $A_{\xi=0}(x) = 1 \in SL(2, \mathbb{C})$. Then the transformation laws of our fields are

$$(A_\xi, \phi_\xi): \qquad \begin{aligned} e(x) \to &\quad e'(x') = e[\phi_\xi(x)] \\ \hat{e}(x) \to &\quad \hat{e}'(x') = \hat{e}[\phi_\xi(x)] \\ \psi_{(\rho)}(x) \to &\ \psi'_{(\rho)}(x') = \rho[A_\xi(x)]\psi_{(\rho)}(x)\,. \end{aligned} \tag{4}$$

We have shown that the pairs (A_ξ, ϕ_ξ) constitute a well-defined Lie group we called the external symmetry group, $S(M)$, pointing out that this is just the universal covering group of $I(M)$ (Cotăescu, 2000). For small values of ξ^a, the $SL(2, \mathbb{C})$ parameters of $A_\xi(x) \equiv A[\omega_\xi(x)]$ can be expanded as $\omega^{\hat{\alpha}\hat{\beta}}_{\xi}(x) = \xi^a \Omega^{\hat{\alpha}\hat{\beta}}_a(x) + \cdots$, in terms of the functions

$$\Omega^{\hat{\alpha}\hat{\beta}}_a \equiv \frac{\partial \omega^{\hat{\alpha}\hat{\beta}}_{\xi}}{\partial \xi^a}_{|\xi=0} = \left(\hat{e}^{\hat{\alpha}}_{\mu}\, k^{\mu}_{a,\nu} + \hat{e}^{\hat{\alpha}}_{\nu,\mu} k^{\mu}_{a}\right) e^{\nu}_{\hat{\lambda}} \eta^{\hat{\lambda}\hat{\beta}} \tag{5}$$

which depend on the Killing vectors $k_a = \partial_{\xi_a}\phi_\xi|_{\xi=0}$ associated to ξ^a.

The last of Eqs. (4) defines the operator-valued representations $T^{(\rho)} : (A_\xi, \phi_\xi) \to T^{(\rho)}_{\xi}$ of the group $S(M)$ which are called the *covariant* representations (CR) *induced* by the finite-dimensional representations ρ of the group $SL(2, \mathbb{C})$. The covariant transformations,

$$(T^{(\rho)}_{\xi}\psi_{(\rho)})[\phi_\xi(x)] = \rho[A_\xi(x)]\psi_{(\rho)}(x)\,, \tag{6}$$

leave the field equation invariant since their basis-generators (Cotăescu, 2000),

$$
\begin{aligned}
X_a^{(\rho)} &= i\partial_{\xi^a} T_{\xi}^{(\rho)}{}_{|\xi=0} \\
&= -ik_a^{\mu}\partial_{\mu} + \frac{1}{2}\,\Omega_a^{\hat{\alpha}\hat{\beta}}\rho(S_{\hat{\alpha}\hat{\beta}})\,,
\end{aligned}
\tag{7}
$$

commute with the operator of the field equation and satisfy the commutation rules $[X_a^{(\rho)}, X_b^{(\rho)}] = ic_{abc}X_c^{(\rho)}$ determined by the structure constants, c_{abc}, of the algebras $s(M) \sim i(M)$. In other words, the operators (7) are the basis-generators of a CR of the $s(M)$ algebra *induced* by the representation ρ of the $sl(2,\mathbb{C})$ algebra. These generators can be put in the covariant form (Cotăescu, 2000),

$$
X_a^{(\rho)} = -ik_a^{\mu}D_{\mu}^{(\rho)} + \frac{1}{2}\,k_{a\,\mu;\nu}\,e^{\mu}_{\hat{\alpha}}\,e^{\nu}_{\hat{\beta}}\,\rho(S^{\hat{\alpha}\hat{\beta}})\,,
\tag{8}
$$

which represents the generalization to any representation ρ of the famous formula given by Carter and McLenaghan (Carter & McLenaghan, 1979) for the spinor representation $\rho_s = (\frac{1}{2},0)\oplus(0,\frac{1}{2})$.

A specific feature of the CRs is that their generators have, in general, point-dependent spin terms which do not commute with the orbital parts. However, there are tetrad-gauges in which at least the generators of a subgroup $G \subset I(M)$ may have point-independent spin terms commuting with the orbital parts. Then we say that the restriction to G of the CR $T^{(\rho)}$ is *manifest* covariant. Obviously, if $G = I(M)$ then the whole representation $T^{(\rho)}$ is manifest covariant (Cotăescu, 2000).

2.3 Isometries in natural frames

Whenever there are no spinors, the matter fields are vectors or tensors of different ranks and the whole theory is independent on the tetrad fields dealing with the natural frames only. In general, any tensor field, Θ, transforms under isometries as $\Theta \to \Theta' = T_{\xi}\Theta$, according to a tensor representation of the group $S(M)$ defined by the well-known rule in natural frames

$$
\left[\frac{\partial\phi_{\xi}^{\alpha}(x)}{\partial x_{\mu}}\frac{\partial\phi_{\xi}^{\beta}(x)}{\partial x_{\nu}}\cdots\right]\left(T_{\xi}\Theta\right)_{\alpha\beta\ldots}[\phi(x)] = \Theta_{\mu\nu\ldots}(x)\,.
\tag{9}
$$

Hereby one derives the basis-generators of the tensor representation, $X_a = i\,\partial_{\xi_a}T_{\xi}|_{\xi=0}$, whose action reads

$$
(X_a\,\Theta)_{\alpha\beta\ldots} = -i(k_a{}^{\nu}\Theta_{\alpha\beta\ldots;\nu} + k_a{}^{\nu}{}_{;\alpha}\Theta_{\nu\beta\ldots} + k_a{}^{\nu}{}_{;\beta}\Theta_{\alpha\nu\ldots}\ldots)\,.
\tag{10}
$$

Our purpose is to show that the operators X_a can be written in a form which is equivalent to equation (8) of Carter and McLenaghan.

In order to accomplish this we start with the vector representation $\rho_v = (\frac{1}{2},\frac{1}{2})$ of the $SL(2,\mathbb{C})$ group, generated by the spin matrices $\rho_v(S^{\hat{\alpha}\hat{\beta}})$ which have the well-known matrix elements

$$
[\rho_v(S^{\hat{\alpha}\hat{\beta}})]^{\hat{\mu}\cdot}_{\cdot\hat{\nu}} = i(\eta^{\hat{\alpha}\hat{\mu}}\delta^{\hat{\beta}}_{\hat{\nu}} - \eta^{\hat{\beta}\hat{\mu}}\delta^{\hat{\alpha}}_{\hat{\nu}})\,,
\tag{11}
$$

in local bases. Furthermore, we define the point-dependent spin matrices in natural frames whose matrix elements in the natural basis read

$$(\tilde{S}^{\mu\nu})^{\sigma\cdot}_{\cdot\,\tau} = e^{\mu}_{\hat{\alpha}} e^{\nu}_{\hat{\beta}} e^{\sigma}_{\hat{\gamma}} [\rho_v(S^{\hat{\alpha}\hat{\beta}})]^{\hat{\gamma}\cdot}_{\cdot\,\hat{\delta}} \hat{e}^{\hat{\delta}}_{\tau} = i(g^{\mu\sigma}\delta^{\nu}_{\tau} - g^{\nu\sigma}\delta^{\mu}_{\tau})\,. \tag{12}$$

These matrices represent the spin operators of the vector representation in natural frames. We observe that these are the basis-generators of the groups $G[g(x)] \sim G(\eta)$ which leave the metric tensor $g(x)$ invariant in each point x. Since the representations of these groups are point-wise equivalent with those of $G(\eta)$, one can show that in each point x the basis-generators $\tilde{S}^{\mu\nu}(x)$ satisfy the standard commutation rules of the vector representation ρ_v (but with $g(x)$ instead of η).

In general, the spin matrices of a tensor Θ of any rank, n, are the basis-generators of the representation $\rho_n = \rho_v^1 \otimes \rho_v^2 \otimes \rho_v^3 \otimes .... \otimes \rho_v^n$ which read $\rho_n(\tilde{S}) = \tilde{S}^1 \otimes I^2 \otimes I^3 ... + I^1 \otimes \tilde{S}^2 \otimes I^3 ... + ...$ Using these spin matrices a straightforward calculation shows that equation (8) can be rewritten in natural frames as (Cotăescu, 2009),

$$X^n_a = -ik^{\mu}_a \nabla_{\mu} + \frac{1}{2}\, k_{a\,\mu;\,\nu}\, \rho_n(\tilde{S}^{\mu\nu})\,, \tag{13}$$

where ∇_{μ} are the usual covariant derivatives. It is not difficult to verify that the action of these operators is just that given by equation (10) which means that X^n_a are the basis-generators of a tensor representation of rank n of the group $S(M)$.

Thus, it is clear that the tensor representations are *equivalent* with the CRs defined in local frames while the equation (13) represents the generalization to natural frames of the Carter and McLenaghan formula. We stress that this result is not trivial since it can not be seen as a simple basis transformation like in the usual tensor theory of the linear algebra. The principal conclusion here is that the Carter and McLenaghan formula is universal since it holds not only in local frames but in natural frames too (Cotăescu, 2009).

3. The de Sitter expanding universe

Our approach is helpful on the four-dimensional de Sitter space-time where all the usual free field equations can be analytically solved while the isometries give rise to the rich $so(1,4)$ algebra. Hereby we selected various sets of commuting operators determining the quantum modes of the scalar, vector and Dirac fields.

3.1 Natural and local frames

Let us consider (M, g) be the de Sitter space-time defined as the hyperboloid of radius $1/\omega$ (where ω denotes the Hubble de Sitter constant) embedded in the five-dimensional flat space-time (M^5, η^5) with Cartesian coordinates z^A (labeled by the indices $A, B, ... = 0, 1, 2, 3, 4$) and the metric $\eta^5 = \mathrm{diag}(1, -1, -1, -1, -1)$ (Birrel & Davies, 1982). The local charts (or natural frames) of coordinates $\{x\}$ can be easily introduced on (M, g) defining the sets of functions $z^A(x)$ able to solve the hyperboloid equation, $\eta^5_{AB} z^A(x) z^B(x) = -\omega^{-2}$. In a given chart the line element $ds^2 = \eta^5_{AB} dz^A dz^B = g_{\mu\nu}(x) dx^{\mu} dx^{\nu}$ defines the metric tensor of (M, g).

In what follows we restrict ourselves to consider only the *(co)moving* charts, $\{t, \vec{x}\}$ and $\{t_c, \vec{x}\}$ which have the same Cartesian space coordinates, x^i ($i, j, k... = 1, 2, 3$), but different time coordinates. The first chart is equipped with the *proper* time $t \in (-\infty, \infty)$ while

$t_c = -\omega^{-1} e^{-\omega t} \in (-\infty, 0]$ is the *conformal* time. These charts are defined by the functions (Birrel & Davies, 1982),

$$z^0 = \frac{e^{\omega t}}{2\omega}\left(1+\omega^2\vec{x}^{\,2} - e^{-2\omega t}\right) = -\frac{1}{2\omega^2 t_c}\left[1-\omega^2({t_c}^2-\vec{x}^{\,2})\right] \tag{14}$$

$$z^4 = \frac{e^{\omega t}}{2\omega}\left(1-\omega^2\vec{x}^{\,2} + e^{-2\omega t}\right) = -\frac{1}{2\omega^2 t_c}\left[1+\omega^2({t_c}^2-\vec{x}^{\,2})\right] \tag{15}$$

$$z^i = e^{\omega t}x^i = -\frac{1}{\omega t_c}x^i \tag{16}$$

and have the line elements

$$ds^2 = dt^2 - e^{2\omega t}\, d\vec{x}\cdot d\vec{x} = \frac{1}{\omega^2 {t_c}^2}\left({dt_c}^2 - d\vec{x}\cdot d\vec{x}\right). \tag{17}$$

We remind the reader that these charts cover only the *expanding* portion of (M, g) known as the de Sitter expanding universe.

The theory of external symmetry in local frame depends on the choice of the tetrad-gauge. The simplest gauge in the chart $\{t, \vec{x}\}$ is the diagonal one in which the non-vanishing components of the tetrad fields are

$$\hat{e}^0_0 = e^0_0 = 1\,, \quad \hat{e}^i_j = e^{\omega t}\delta^i_j\,, \quad e^i_j = e^{-\omega t}\delta^i_j\,. \tag{18}$$

The corresponding gauge in the chart $\{t_c, \vec{x}\}$ is given by the non-vanishing tetrad components,

$$e^0_0 = -\omega t_c\,, \quad e^i_j = -\delta^i_j\,\omega t_c\,, \quad \hat{e}^0_0 = -\frac{1}{\omega t_c}\,, \quad \hat{e}^i_j = -\delta^i_j\,\frac{1}{\omega t_c}\,. \tag{19}$$

3.2 The Killing vectors of the $SO(1,4)$ isometries

The de Sitter manifold (M, g) is defined as a homogeneous space of the pseudo-orthogonal group $SO(1,4)$ which is in the same time the gauge group of the metric η^5 and the isometry group, $I(M)$, of the de Sitter space-time . The group of the external symmetry, $S(M) = \mathrm{Spin}(\eta^5) = Sp(2,2)$, has the Lie algebra $s(M) = sp(2,2) \sim so(1,4)$ for which we use the covariant real parameters $\xi^{AB} = -\xi^{BA}$. In this parametrization, the Killing vectors corresponding to the $SO(1,4)$ isometries can be derived considering the natural representation carried by the space of the scalar functions over M^5. The basis-generators of this representation are the genuine orbital operators

$$L^5_{AB} = i\left[\eta^5_{AC}z^C\partial_B - \eta^5_{BC}z^C\partial_A\right] = -iK^C_{(AB)}\partial_C \tag{20}$$

which define the components of the Killing vectors $K_{(AB)}$ on (M^5, η^5). With their help we can derive the corresponding Killing vectors of (M, g), denoted by $k_{(AB)}$, using the obvious identities $k_{(AB)\mu}dx^\mu = K_{(AB)C}dz^C$.

In the chart $\{t, \vec{x}\}$ the Killing vectors of the de Sitter symmetry have the components,

$$k^0_{(0i)} = k^0_{(4i)} = x^i\,, \qquad k^j_{(0i)} = k^j_{(4i)} - \frac{1}{\omega}\,\delta^j_i = \omega x^i x^j - \delta^j_i\,\vartheta\,, \tag{21}$$

$$k^0_{(ij)} = 0\,, \quad k^l_{(ij)} = \delta^l_j x^i - \delta^l_i x^j\,; \qquad k^0_{(04)} = -\frac{1}{\omega}\,, \quad k^i_{(04)} = x^i\,. \tag{22}$$

while in the other moving chart, $\{t_c, \vec{x}\}$, these components become

$$k^0_{(0i)} = k^0_{(4i)} = \omega t_c x^i\,, \qquad k^j_{(0i)} = k^j_{(4i)} - \frac{1}{\omega}\,\delta^j_i = \omega x^i x^j - \delta^j_i \vartheta\,, \tag{23}$$

$$k^0_{(ij)} = 0\,, \quad k^l_{(ij)} = \delta^l_j x^i - \delta^l_i x^j\,; \qquad k^\mu_{(04)} = x^\mu\,, \tag{24}$$

where the function ϑ is defined as

$$\vartheta = \frac{1}{2\omega}(1 + \omega^2 \vec{x}^{\,2} - e^{-2\omega t}) = \frac{1}{2\omega}[1 + \omega^2(\vec{x}^{\,2} - {t_c}^2)]\,, \tag{25}$$

3.3 The $so(1,4)$ generators of covariant representations

According to our general theory, the generators of the CRs $T^{(\rho)}$ of the group $S(M) = Sp(2,2)$, induced by the representations ρ of the $SL(2,\mathbb{C})$ group, constitute CRs of the $sp(2,2)$ algebra induced by the representations ρ of the $sl(2,\mathbb{C})$ algebra. Therefore, their commutation relations are determined by the structure constant of the group $Sp(2,2)$ and the principal invariants are the Casimir operators of the CRs which can be derived as those of the algebras $sp(2,2) \sim so(1,4)$.

In the covariant parametrization of the $sp(2,2)$ algebra adopted here, the generators $X^{(\rho)}_{(AB)}$ corresponding to the Killing vectors $k_{(AB)}$ result from equation (7) and the functions (5) with the new labels $a \to (AB)$. Using then the Killing vectors (21) and (22) and the tetrad-gauge (18) of the chart $\{t, \vec{x}\}$, after a little calculation, we find first the $sl(2,\mathbb{C})$ generators. These are the total angular momentum,

$$J^{(\rho)}_i \equiv \frac{1}{2}\,\varepsilon_{ijk} X^{(\rho)}_{(jk)} = -i\varepsilon_{ijk} x^j \partial_k + S^{(\rho)}_i\,, \tag{26}$$

and the generators of the Lorentz boosts

$$K^{(\rho)}_i \equiv X^{(\rho)}_{(0i)} = i x^i \partial_t + i\vartheta(x)\partial_i - i\omega x^i\, x^j \partial_j + e^{-\omega t} S^{(\rho)}_{0i} + \omega S^{(\rho)}_{ij} x^j\,, \tag{27}$$

where ϑ is defined by Eq. (25). In addition, there are three generators,

$$R^{(\rho)}_i \equiv X^{(\rho)}_{(i4)} = -K^{(\rho)}_i + \frac{1}{\omega}\, i\partial_i\,, \tag{28}$$

which play the role of a Runge-Lenz vector, in the sense that $\{J_i, R_i\}$ generate a $so(4)$ subalgebra. The energy (or Hamiltonian) operator,

$$H \equiv \omega X^{(\rho)}_{(04)} = i\partial_t - i\omega x^i \partial_i\,, \tag{29}$$

is given by the Killing vector $k_{(04)}$ which is time-like only for $\omega|\vec{x}|e^{\omega t} \le 1$. Fortunately, this condition is accomplished everywhere inside the light-cone of an observer at rest in $\vec{x} = 0$. Therefore, the operator H is correctly defined.

The generators introduced above form the basis $\{H, J^{(\rho)}_i, K^{(\rho)}_i, R^{(\rho)}_i\}$ of the covariant representation of the $sp(2,2)$ algebra with the following commutation rules (Cotăescu, 2011a):

$$\left[J^{(\rho)}_i, J^{(\rho)}_j\right] = i\varepsilon_{ijk} J^{(\rho)}_k\,, \quad \left[J^{(\rho)}_i, R^{(\rho)}_j\right] = i\varepsilon_{ijk} R^{(\rho)}_k\,, \quad \left[K^{(\rho)}_i, K^{(\rho)}_j\right] = -i\varepsilon_{ijk} J^{(\rho)}_k \tag{30}$$

$$\left[J^{(\rho)}_i, K^{(\rho)}_j\right] = i\varepsilon_{ijk} K^{(\rho)}_k\,, \quad \left[R^{(\rho)}_i, R^{(\rho)}_j\right] = i\varepsilon_{ijk} J^{(\rho)}_k\,, \quad \left[R^{(\rho)}_i, K^{(\rho)}_j\right] = \frac{i}{\omega}\,\delta_{ij} H\,, \tag{31}$$

and

$$\left[H, J_i^{(\rho)}\right] = 0\,, \quad \left[H, K_i^{(\rho)}\right] = i\omega R_i^{(\rho)}\,, \quad \left[H, R_i^{(\rho)}\right] = i\omega K_i^{(\rho)}\,. \tag{32}$$

As mentioned in the previous section, it is useful to replace the operators $\vec{K}^{(\rho)}$ and $\vec{R}^{(\rho)}$ by the momentum operator $\vec{P}$ and its dual, $\vec{Q}^{(\rho)}$, whose components are defined as

$$P_i = \omega(R_i^{(\rho)} + K_i^{(\rho)}) = i\partial_i\,, \quad Q_i^{(\rho)} = \omega(R_i^{(\rho)} - K_i^{(\rho)})\,. \tag{33}$$

which have the remarkable properties (Cotăescu, 2011a)

$$[H, P_i] = i\omega P_i\,, \quad \left[H, Q_i^{(\rho)}\right] = -i\omega Q_i^{(\rho)}\,, \quad \left[Q_i^{(\rho)}, Q_j^{(\rho)}\right] = [P_i, P_j] = 0\,. \tag{34}$$

We obtain thus the basis $\{H, P_i, Q_i^{(\rho)}, J_i^{(\rho)}\}$ and the basis of the Poincaré type formed by $\{H, P_i, J_i^{(\rho)}, K_i^{(\rho)}\}$.

The last two bases bring together the conserved energy (29) and momentum (33a) which are the only genuine orbital operators, independent on ρ. What is specific for the de Sitter symmetry is that these operators can not be put simultaneously in diagonal form since they do not commute to each other,as it results from Eq. (34a). Therefore, there are no mass-shells.

4. The massive Klein-Gordon field

The quantum modes of the scalar field in moving frames of de Sitter manifolds are well-known from long time (Birrel & Davies, 1982) paying attention to the scalar propagators, known as two-point functions (Candelas & Raine, 1975; Chernikov & Tagiriv, 1968), we recover here using the canonical quantization (Cotăescu et al., 2008).

4.1 Scalar quantum mechanics

In what follows we study the scalar field minimally coupled to the de Sitter gravity using our recently proposed new quantum mechanics on spatially flat Robertson-Walker space-times in which we defined different time evolution pictures (Cotăescu, 2007).

4.1.1 Lagrangian theory

In an arbitrary chart $\{x\}$ of a curved manifold the action of a charged scalar field ϕ of mass m, minimally coupled to gravity, reads (Birrel & Davies, 1982),

$$\mathcal{S}[\phi, \phi^*] = \int d^4x\sqrt{g}\,\mathcal{L} = \int d^4x\sqrt{g}\left(\partial^\mu\phi^*\partial_\mu\phi - m^2\phi^*\phi\right)\,, \tag{35}$$

where $g = |\det(g_{\mu\nu})|$. This action gives rise to the Klein-Gordon equation

$$\frac{1}{\sqrt{g}}\,\partial_\mu\left[\sqrt{g}\,g^{\mu\nu}\partial_\nu\phi\right] + m^2\phi = 0\,. \tag{36}$$

The conserved quantities predicted by the Noether theorem can be calculated with the help of the stress-energy tensor

$$T_{\mu\nu} = \partial_\mu\phi^*\partial_\nu\phi + \partial_\nu\phi^*\partial_\mu\phi - g_{\mu\nu}\mathcal{L}\,. \tag{37}$$

Thus, for each isometry corresponding to a Killing vector $k_{(AB)}$ there exists the conserved current $\Theta^{\mu}[k_{(AB)}] = -T^{\mu\,\cdot}_{\cdot\,\nu} k^{\nu}_{(AB)}$ which satisfies $\Theta^{\mu}[k_{(AB)}]_{;\mu} = 0$ producing the conserved quantity

$$C[k_{(AB)}] = \int_{\Sigma} d\sigma_{\mu}\sqrt{g}\;\Theta^{\mu}[k_{(AB)}]\,, \tag{38}$$

on a given hypersurface $\Sigma \subset M$. Moreover, generalizing the form of the conserved electric charge due to the internal $U(1)$ symmetry one defines the relativistic scalar product of two scalar fields as

$$\langle \phi, \phi' \rangle = i \int_{\Sigma} d\sigma^{\mu}\sqrt{g}\,\phi^{*} \overleftrightarrow{\partial_{\mu}}\, \phi'\,, \tag{39}$$

using the notation $f \overleftrightarrow{\partial} h = f(\partial h) - h(\partial f)$. With this definition one obtains the following identities

$$C[k_{(AB)}] = \langle \phi, X_{(AB)}\phi \rangle \tag{40}$$

which can be proved for any Killing vector using the field equation (36) and the Green's theorem. These identities will be useful in quantization, giving directly the conserved one-particle operators of the quantum field theory (Cotăescu et al., 2008).

4.1.2 Time-evolution pictures on de Sitter space-time

Let us consider now the de Sitter expanding universe (M, g) and the chart $\{t, \vec{x}\}$ with FRW line element. We say that the *natural* time-evolution picture (NP) is the genuine quantum theory in this chart where the time evolution of the massive scalar field is governed by the Klein-Gordon equation

$$\left(\partial_t^2 - e^{-2\omega t}\Delta + 3\omega\partial_t + m^2\right)\phi(x) = 0\,. \tag{41}$$

The solutions of this equation may be square integrable functions or tempered distributions with respect to the scalar product (39) that in NP and for $\Sigma = \mathbb{R}^3$ takes the form

$$\langle \phi, \phi' \rangle = i \int d^3x\, e^{3\omega t}\, \phi^{*}(x) \overleftrightarrow{\partial_t}\, \phi'(x)\,. \tag{42}$$

since in this chart $\sqrt{g} = e^{3\omega t}$.

The principal operators of NP are the isometry generators given by Eqs. (26)-(29) but calculated for the scalar representation $(0, 0)$ whose generators vanish. Thus, these operators are just the genuine orbital generators of the natural representation which is equivalent to the scalar CR. In addition, we consider the coordinate operator, $\vec{X}$, defined as $(X^i\phi)(x) = x^i\phi(x)$, that obeys,

$$[P_i, X^j] = i\delta_{ij} I\,, \quad [H, X^i] = -i\omega X^i\,, \tag{43}$$

where I is the identity operator.

The NP can be changed using point-dependent operators which could be even non-unitary operators since the relativistic scalar product does not have a direct physical meaning as that of the non-relativistic quantum mechanics . We exploit this opportunity for introducing the new time-evolution picture, called the Schrödinger picture (SP), with the help of the transformation $\phi(x) \to \phi_S(x) = W(x)\phi(x)W^{-1}(x)$ produced by the operator of time dependent *dilatations* (Cotăescu, 2007),

$$W(x) = \exp\left[-\omega t(x^i\partial_i)\right], \quad W^+(x) = e^{3\omega t}\, W^{-1}(x), \tag{44}$$

which has the following convenient actions

$$W(x)F(x^i)W^{-1}(x) = F\left(e^{-\omega t}x^i\right), \quad W(x)G(\partial_i)W^{-1}(x) = G\left(e^{\omega t}\partial_i\right), \tag{45}$$

upon any analytical functions F and G. This transformation leads to the Klein-Gordon equation of the SP

$$\left[\left(\partial_t + \omega x^i\partial_i\right)^2 - \Delta + 3\omega(\partial_t + \omega x^i\partial_i) + m^2\right]\phi_S(x) = 0, \tag{46}$$

and allows us to define the scalar product of this picture,

$$\langle\phi_S, \phi'_S\rangle \equiv \langle\phi, \phi'\rangle = i\int d^3x\left[\phi_S^* \overleftrightarrow{\partial_t}\, \phi'_S + \omega x^i(\phi_S^* \overleftrightarrow{\partial_i}\, \phi'_S)\right], \tag{47}$$

as it results from Eqs. (44).

The specific operators of the SP, denoted by H_S, P^i_S and X^i_S, are defined as

$$(H_S\phi_S)(x) = i\partial_t\phi_S(x), \quad (P^i_S\phi_S)(x) = -i\partial_i\phi_S(x), \quad (X^i_S\phi_S)(x) = x^i\phi_S(x), \tag{48}$$

The meaning of these operators can be understood in the NP. Indeed, performing the inverse transformation we recover the conserved energy operator $H = W^{-1}(x)\, H_S\, W(x)$ and we find the new interesting time-dependent operators of the NP,

$$X^i(t) = W^{-1}(x)\, X^i_S\, W(x) = e^{\omega t}X^i, \tag{49}$$

$$P^i(t) = W^{-1}(x)\, P^i_S\, W(x) = e^{-\omega t}P^i, \tag{50}$$

which satisfy the canonical commutation rules (Cotăescu, 2007),

$$[X^i(t), P^j(t)] = i\delta_{ij}I, \quad [H, X^i(t)] = [H, P^i(t)] = 0. \tag{51}$$

The angular momentum has the same expression in both these pictures since it commutes with $W(x)$. We note that even if $X^i(t)$ and $P^i(t)$ commute with H they can not be considered conserved operators since they do not commute with the Klein-Gordon operator.

In NP picture the eigenvalues problem $Hf_E(t,\vec{x}) = Ef_E(t,\vec{x})$ of the energy operator leads to energy eigenfunctions of the form

$$f_E(t,\vec{x}) = F[e^{\omega t}\vec{x}]e^{-iEt} \tag{52}$$

where F is an arbitrary function. This explains why in this picture one can not find energy eigenfunctions separating variables. However, in our SP these eigenfunctions become the new functions

$$f^S_E(t,\vec{x}) = W(x)f_E(t,\vec{x})W^{-1}(x) = F(\vec{x})e^{-iEt} \tag{53}$$

which have separated variables. This means that in SP new quantum modes could be derived using the method of separating variables in coordinates or even in momentum representation.

4.2 Scalar plane waves

As mentioned before, the specific feature of the quantum mechanics on M is that the conserved energy and momentum can not be measured simultaneously with desired accuracy. Consequently, there are no particular solutions of the Klein-Gordon equation with well-determined energy and momentum, being forced to consider different plane waves solutions depending either on momentum or on energy and momentum direction. Thus we shall work with two bases of fundamental solutions we call here the momentum and energy bases . The momentum basis is well-known (Birrel & Davies, 1982; Chernikov & Tagiriv, 1968) but the energy one is a new basis derived using our SP (Cotăescu et al., 2008).

4.2.1 The momentum basis

It is known that the Klein-Gordon equation (41) of NP can be analytically solved in terms of Bessel functions (Birrel & Davies, 1982). There are fundamental solutions determined as eigenfunctions of the set of commuting operators $\{P^i\}$ of NP whose eigenvalues p^i are the components of the momentum $\vec{p}$. Among different versions of solutions which are currently used we prefer the normalized solutions of positive frequencies that read

$$f_{\vec{p}}(x) = \frac{1}{2}\sqrt{\frac{\pi}{\omega}}\frac{1}{(2\pi)^{3/2}}\, e^{-3\omega t/2} Z_k\left(\frac{p}{\omega}\, e^{-\omega t}\right) e^{i\vec{p}\cdot\vec{x}}\,, \quad k = \sqrt{\mu^2 - \tfrac{9}{4}}\,, \tag{54}$$

where $p = |\vec{p}|$, the functions Z_k are defined in the Appendix A and we denote $\mu = \frac{m}{\omega}$. Obviously, the fundamental solutions of negative frequencies are $f^*_{\vec{p}}(x)$.

All these solutions satisfy the orthonormalization relations

$$\langle f_{\vec{p}}, f_{\vec{p}\,'}\rangle = -\langle f^*_{\vec{p}}, f^*_{\vec{p}\,'}\rangle = \delta^3(\vec{p}-\vec{p}\,')\,, \quad \langle f_{\vec{p}}, f^*_{\vec{p}\,'}\rangle = 0\,, \tag{55}$$

and the completeness condition

$$i\int d^3p\, f^*_{\vec{p}}(t,\vec{x})\, \overset{\leftrightarrow}{\partial_t}\, f_{\vec{p}}(t,\vec{x}\,') = e^{-3\omega t}\delta^3(\vec{x}-\vec{x}\,')\,. \tag{56}$$

For this reason we say that the set $\{f_{\vec{p}}\,|\vec{p}\in\mathbb{R}^3_p\}$ forms the complete system of fundamental solutions of positive frequencies of the *momentum* basis of the Hilbert space $\mathbf{H}^{(+)}_{KG}$ of particle states. The solutions of negative frequencies, $\{f^*_{\vec{p}}\,|\vec{p}\in\mathbb{R}^3_p\}$, span an orthogonal Hilbert space, $\mathbf{H}^{(-)}_{KG}$, associated to the antiparticle states. We must stress that this separation of the positive and negative frequencies defines the Bunch-Davies vacuum which is known to be stable.

In this basis, the Klein-Gordon field can expanded in terms of plane waves of positive and negative frequencies in usual manner as

$$\begin{aligned}\phi(x) &= \phi^{(+)}(x) + \phi^{(-)}(x)\\ &= \int d^3p\left[f_{\vec{p}}(x)a(\vec{p}) + f^*_{\vec{p}}(x)b^*(\vec{p})\right]\end{aligned} \tag{57}$$

where a and b are the particle and respectively antiparticle wave functions of the momentum representation. These can be calculated using the inversion formulas $a(\vec{p}) = \langle f_{\vec{p}}, \phi\rangle$ and $b(\vec{p}) = \langle f_{\vec{p}}, \phi^*\rangle$.

4.2.2 The energy basis

The plane waves of given energy have to be derived in the SP (Cotăescu, 2007) where the Klein-Gordon equation has the suitable form (46). We assume that in this picture the scalar field can be expanded as

$$\phi_S(x) = \phi_S^{(+)}(x) + \phi_S^{(-)}(x) = \int_0^\infty dE \int d^3q \left[\hat{\phi}_S^{(+)}(E,\vec{q})e^{-iEt+i\vec{q}\cdot\vec{x}} + \hat{\phi}_S^{(-)}(E,\vec{q})e^{iEt-i\vec{q}\cdot\vec{x}}\right] \quad (58)$$

where $\hat{\phi}_S^{(\pm)}$ behave as tempered distributions on the domain $\mathbb{R}^3_q$ such that the Green theorem may be used. Then we can replace the momentum operators P_S^i by q^i and the coordinate operators X_S^i by $i\partial_{q_i}$ obtaining the Klein-Gordon equation of the SP in momentum representation,

$$\left\{\left[\pm iE + \omega\left(q^i\partial_{q_i} + 3\right)\right]^2 - 3\omega\left[\pm iE + \omega\left(q^i\partial_{q_i} + 3\right)\right] + \vec{q}^2 + m^2\right\}\hat{\phi}_S^{(\pm)}(E,\vec{q}) = 0\,, \quad (59)$$

where E is the energy defined as the eigenvalue of H_S. We remind the reader that the operators P_S^i and X_S^i become in NP the time dependent operators (49) and respectively (50) while H_S is related to the conserved energy operator H. This means the energy E is a conserved quantity but the momentum $\vec{q}$ does not have this property. More specific, only the scalar momentum $q = |\vec{q}|$ is not conserved while the momentum direction is conserved since the operator (50) is parallel with the conserved momentum $\vec{P}$. For this reason we denote $\vec{q} = q\,\vec{n}$ observing that the differential operator of Eq. (59) is of radial type and reads $q^i\partial_{q_i} = q\,\partial_q$. Consequently, this operator acts only on the functions depending on q while the functions which depend on the momentum direction $\vec{n}$ behave as constants. Therefore, we have to look for solutions of the form

$$\hat{\phi}_S^{(+)}(E,\vec{q}) = h_S(E,q)\,a(E,\vec{n})\,, \quad (60)$$

$$\hat{\phi}_S^{(-)}(E,\vec{q}) = [h_S(E,q)]^*\,b^*(E,\vec{n})\,, \quad (61)$$

where the function h_S satisfies an equation derived from Eq. (59) that can be written simply using the new variable $s = \frac{q}{\omega}$ and the notation $\epsilon = \frac{E}{\omega}$. This equation,

$$\left[\frac{d^2}{ds^2} + \frac{2i\epsilon + 4}{s}\frac{d}{ds} + \frac{\mu^2 - \epsilon^2 + 3i\epsilon}{s^2} + 1\right] h_S(\epsilon, s) = 0\,, \quad (62)$$

is of the Bessel type having solutions of the form $h_S(\epsilon,s) = \text{const}\; s^{-i\epsilon-3/2}\,Z_k(s)$. Collecting all the above results we derive the final expression of the Klein-Gordon field (58) as

$$\phi_S(x) = \int_0^\infty dE \int_{S^2} d\Omega_n\, \left\{f^S_{E,\vec{n}}(x)a(E,\vec{n}) + [f^S_{E,\vec{n}}(x)]^* b^*(E,\vec{n})\right\}\,, \quad (63)$$

where the integration covers the sphere $S^2 \subset \mathbb{R}^3_p$. The fundamental solutions $f^S_{E,\vec{n}}$ of positive frequencies, with energy E and momentum direction $\vec{n}$ result to have the integral representation

$$f^S_{E,\vec{n}}(x) = N_0 e^{-iEt}\int_0^\infty ds\,\sqrt{s}\;Z_k(s)\,e^{i\omega s\vec{n}\cdot\vec{x} - i\epsilon\ln s}\,, \quad (64)$$

where N_0 is a normalization constant.

For understanding the physical meaning of this result we must turn back to NP where the scalar field

$$\phi(x) = \int_0^\infty dE \int_{S^2} d\Omega_n \; \{f_{E,\vec{n}}(x) a(E,\vec{n}) + [f_{E,\vec{n}}(x)]^* b^*(E,\vec{n})\} \,, \tag{65}$$

is expressed in terms of the solutions of NP that can be put in the form

$$\begin{aligned} f_{E,\vec{n}}(t,\vec{x}) &= W^{-1}(x) f^S_{E,\vec{n}}(t,\vec{x}) W(x) = f^S_{E,\vec{n}}(t, e^{\omega t}\vec{x}) \\ &= N_0 e^{-\frac{3}{2}\omega t} \int_0^\infty ds\, \sqrt{s}\; Z_k\left(s e^{-\omega t}\right) e^{i\omega s \vec{n}\cdot\vec{x} - i\epsilon \ln s} \,, \end{aligned} \tag{66}$$

changing the integration variable $e^{\omega t} s \to s$ in the integral (64). Using then the scalar product (108) and the method of the Appendix B we can show that the normalization constant

$$N_0 = \frac{1}{2}\sqrt{\frac{\omega}{2}} \frac{1}{(2\pi)^{3/2}} \,, \tag{67}$$

assures the desired orthonormalization relations

$$\langle f_{E,\vec{n}}, f_{E',\vec{n}\,'}\rangle = -\langle f^*_{E,\vec{n}}, f^*_{E',\vec{n}\,'}\rangle = \delta(E-E')\,\delta^2(\vec{n}-\vec{n}\,') \,, \quad \langle f_{E,\vec{n}}, f^*_{E',\vec{n}\,'}\rangle = 0 \,, \tag{68}$$

and the completeness condition

$$i\int_0^\infty dE \int_{S^2} d\Omega_n \left\{ [f_{E,\vec{n}}(t,\vec{x})]^* \stackrel{\leftrightarrow}{\partial_t} f_{E,\vec{n}}(t,\vec{x}\,') \right\} = e^{-3\omega t} \delta^3(\vec{x}-\vec{x}\,') \,. \tag{69}$$

This means that the set of functions $\{f_{E,\vec{n}} | E \in \mathbb{R}^+, \vec{n} \in S^2\}$ constitutes the complete system of fundamental solutions of the *energy* basis of $\mathbf{H}^{(+)}_{KG}$. The set $\{f^*_{E,\vec{n}} | E \in \mathbb{R}^+, \vec{n} \in S^2\}$ forms the energy basis of $\mathbf{H}^{(-)}_{KG}$.

The last step is to calculate the transition coefficients between the momentum and energy bases of the NP that read (Cotăescu et al., 2008)

$$\langle f_{\vec{p}}, f_{E,\vec{n}}\rangle = \langle f_{E,\vec{n}}, f_{\vec{p}}\rangle^* = \frac{p^{-\frac{3}{2}}}{\sqrt{2\pi\omega}}\, \delta^2(\vec{n}-\vec{n}_p)\, e^{-i\frac{E}{\omega}\ln\frac{p}{\omega}} \,, \tag{70}$$

where $\vec{n}_p = \vec{p}/p$. With their help we deduce the transformations

$$a(\vec{p}) = \int_0^\infty dE \int_{S^2} d\Omega_n \langle f_{\vec{p}}, f_{E,\vec{n}}\rangle a(E,\vec{n}) = \frac{p^{-\frac{3}{2}}}{\sqrt{2\pi\omega}} \int_0^\infty dE\, e^{-i\frac{E}{\omega}\ln\frac{p}{\omega}}\, a(E,\vec{n}_p) \,, \tag{71}$$

$$a(E,\vec{n}) = \int d^3p\, \langle f_{E,\vec{n}}, f_{\vec{p}}\rangle a(\vec{p}) = \frac{1}{\sqrt{2\pi\omega}} \int_0^\infty dp\, \sqrt{p}\; e^{i\frac{E}{\omega}\ln\frac{p}{\omega}}\, a(p\,\vec{n}) \,, \tag{72}$$

and similarly for the wave functions b. These transformations do not mix the particle and antiparticle states such that we can conclude that the Bunch-Davies vacuum defined here is stable with respect to the basis transformations.

4.3 Quantization and one-particle operators

The quantization can be done in canonical manner considering that the wave functions a and b of the fields (57) and (65) become field operators (such that $b^* \to b^\dagger$). We assume that the particle (a, $a^\dagger$) and antiparticle (b, $b^\dagger$) operators fulfill the standard commutation relations in the momentum basis, from which the non-vanishing ones are

$$[a(\vec{p}), a^\dagger(\vec{p}\,')] = [b(\vec{p}), b^\dagger(\vec{p}\,')] = \delta^3(\vec{p} - \vec{p}\,')\,. \tag{73}$$

Then, from Eq. (71) it results that the field operators of the energy basis satisfy

$$[a(E,\vec{n}), a^\dagger(E',\vec{n}\,')] = [b(E,\vec{n}), b^\dagger(E',\vec{n}\,')] = \delta(E - E')\delta^2(\vec{n} - \vec{n}\,')\,, \tag{74}$$

and

$$[a(\vec{p}), a^\dagger(E,\vec{n})] = [b(\vec{p}), b^\dagger(E,\vec{n})] = \langle f_{\vec{p}}, f_{E,\vec{n}}\rangle\,, \tag{75}$$

while other commutators are vanishing. In this way the field ϕ is correctly quantized according to the *canonical* rule (Drell & Bjorken, 1965),

$$[\phi(t,\vec{x}), \pi(t,\vec{x}\,')] = e^{3\omega t}\,[\phi(t,\vec{x}), \partial_t\phi^\dagger(t,\vec{x}\,')] = i\,\delta^3(\vec{x} - \vec{x}\,')\,, \tag{76}$$

where $\pi = \sqrt{g}\,\partial_t\phi^\dagger$ is the momentum density derived from the action (35). All these operators act on the Fock space which has the unique Bunch-Davies vacuum state $|0\rangle$ accomplishing

$$a(\vec{p})|0\rangle = b(\vec{p})|0\rangle = 0\,, \quad \langle 0|a^\dagger(\vec{p}) = \langle 0|b^\dagger(\vec{p}) = 0\,, \tag{77}$$

and similarly for the energy basis. The sectors with a given number of particles have to be constructed using the standard methods, obtaining thus the generalized bases of momentum or energy.

The one-particle operators corresponding to the conserved operators can be calculated bearing in mind that for any self-adjoint generator X of the scalar representation of the group $I(M)$ there exists a *conserved* one-particle operator of the quantum field theory which can be calculated simply as

$$\mathcal{X} =: \langle \phi, X\phi\rangle : \tag{78}$$

respecting the normal ordering of the operator products. Hereby we recover the standard algebraic properties

$$[\mathcal{A}, \phi(x)] = -A\phi(x)\,, \quad [\mathcal{A}, \mathcal{B}\,] =: \langle \phi, [A,B]\,\phi\rangle : \tag{79}$$

due to the canonical quantization adopted here. In other respects, the electric charge operator corresponding to the $U(1)$ internal symmetry (of Abelian gauge transformations $\phi \to e^{i\alpha I}\phi$) results from the Noether theorem to be $\mathcal{Q} =: \langle \phi, I\phi\rangle := :\langle \phi, \phi\rangle :$.

However, there are many other conserved operators which do not have corresponding differential operators at the level of quantum mechanics. The simplest examples are the operators of number of particles,

$$\mathcal{N}_{pa} = \int d^3p\, a^\dagger(\vec{p})a(\vec{p}) = \int_0^\infty dE \int_{S^2} d\Omega_n a^\dagger(E,\vec{n})a(E,\vec{n})\,, \tag{80}$$

and that of antiparticles, $\mathcal{N}_{ap}$ (depending on b and $b^\dagger$), giving rise to the charge operator $\mathcal{Q} = \mathcal{N}_{pa} - \mathcal{N}_{ap}$ and that of the total number of particles, $\mathcal{N} = \mathcal{N}_{pa} + \mathcal{N}_{ap}$.

In what follows we focus on the conserved one-particle operators determining the momentum and energy bases. The diagonal operators of the momentum basis the are $\mathcal{Q}$ and the components of momentum operator,

$$\mathcal{P}^i =: \langle \phi, P^i \phi \rangle := \int d^3p\, p^i \left[a^{\dagger}(\vec{p}) a(\vec{p}) + b^{\dagger}(\vec{p}) b(\vec{p}) \right] . \tag{81}$$

In other words, the momentum basis is determined by the set of commuting operators $\{\mathcal{Q}, \mathcal{P}^i\}$. The energy basis is formed by the common eigenvectors of the set of commuting operators $\{\mathcal{Q}, \mathcal{H}, \tilde{\mathcal{P}}^i\}$, i.e. the charge, energy and momentum direction operators. The energy operator can be easily calculated since the solutions (66) are eigenfunctions of the operator H. In this way we find

$$\mathcal{H} =: \langle \phi, H\phi \rangle := \int_0^{\infty} dE\, E \int_{S^2} d\Omega_n \left[a^{\dagger}(E, \vec{n}) a(E, \vec{n}) + b^{\dagger}(E, \vec{n}) b(E, \vec{n}) \right] . \tag{82}$$

More interesting are the operators $\tilde{\mathcal{P}}^i$ of the momentum direction since they do not come from differential operators and, therefore, must be defined directly as

$$\tilde{\mathcal{P}}^i = \int_0^{\infty} dE \int_{S^2} d\Omega_n\, n^i \left[a^{\dagger}(E, \vec{n}) a(E, \vec{n}) + b^{\dagger}(E, \vec{n}) b(E, \vec{n}) \right] . \tag{83}$$

The above operators which satisfy simple commutation relations,

$$[\mathcal{H}, \mathcal{P}^i] = i\omega \mathcal{P}^i\,, \quad [\mathcal{H}, \tilde{\mathcal{P}}^i] = 0\,, \quad [\mathcal{Q}, \mathcal{H}] = [\mathcal{Q}, \mathcal{P}^i] = [\mathcal{Q}, \tilde{\mathcal{P}}^i] = 0\,, \tag{84}$$

are enough for defining the bases considered hare.

Our approach offers the opportunity to deduce mode expansions of conserved one-particle operators but in bases where these are not diagonal. For example, we can calculate the mode expansion of the energy operator in the momentum basis either starting with the identity

$$(Hf_{\vec{p}})(x) = -i\omega \left(p^i \partial_{p_i} + \frac{3}{2} \right) f_{\vec{p}}(x) \tag{85}$$

or using Eq. (72). The final result (Cotăescu et al., 2008),

$$\mathcal{H} = \frac{i\omega}{2} \int d^3p\, p^i \left[a^{\dagger}(\vec{p}) \overleftrightarrow{\partial}_{p_i} a(\vec{p}) + b^{\dagger}(\vec{p}) \overleftrightarrow{\partial}_{p_i} b(\vec{p}) \right] , \tag{86}$$

is similar to those obtained for other fields as we shall see later. This expansion has a remarkable property namely, the change of the phase factors,

$$f_{\vec{p}}(x) \to f_{\vec{p}}(x) e^{i\chi(\vec{p})}\,, \quad a(\vec{p}) \to e^{-i\chi(\vec{p})} a(\vec{p})\,, \quad b(\vec{p}) \to e^{-i\chi(\vec{p})} b(\vec{p})\,, \tag{87}$$

using a real phase function $\chi(\vec{p})$, preserve the form of the operators ϕ, $\mathcal{Q}$ and $\mathcal{P}^i$ but transforms the Hamiltonian operator as

$$\mathcal{H} \to \mathcal{H} + \omega \int d^3p\, [p^i \partial_{p^i} \chi(\vec{p})] \left[a^{\dagger}(\vec{p}) a(\vec{p}) + b^{\dagger}(\vec{p}) b(\vec{p}) \right] . \tag{88}$$

Our preliminary investigations indicate that this property may be helpful for avoid some mathematical difficulties related to the flat limit $\omega \sim 0$ (Cotăescu, 2011b).

We note that beside the above conserved operators we can introduce other one-particle operators extending the definition (78) to the non-conserved operators of our quantum mechanics. However, these operators will depend explicitly on time, their expressions being complicated and without an intuitive physical meaning.

4.4 Commutator and Green functions

In the quantum theory of fields the Green functions are related to the partial commutator functions (of positive or negative frequencies) defined as

$$D^{(\pm)}(x,x') = i[\phi^{(\pm)}(x), \phi^{(\pm)\dagger}(x')] \tag{89}$$

and the total one, $D = D^{(+)} + D^{(-)}$. These function are solutions of the Klein-Gordon equation in both the sets of variables and obey $[D^{(\pm)}(x,x')]^* = D^{(\mp)}(x,x')$ such that D results to be a real function. This property suggests us to restrict ourselves to study only the functions of positive frequencies,

$$D^{(+)}(x,x') = i\int d^3p\, f_{\vec{p}}(x) f_{\vec{p}}(x')^* = i\int_0^\infty dE \int_{S^2} d\Omega_n\, f_{E,\vec{n}}(x) f_{E,\vec{n}}(x')^*, \tag{90}$$

resulted from Eqs. (57) and (65). Both these versions lead to the final expression

$$D^{(+)}(x,x') = \frac{\pi}{4\omega}\frac{i}{(2\pi)^3}\, e^{-\frac{3}{2}\omega(t+t')}\int d^3p\, Z_k\left(\frac{p}{\omega}e^{-\omega t}\right) Z_k^*\left(\frac{p}{\omega}e^{-\omega t'}\right) e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} \tag{91}$$

from which we understand that $D^{(+)}(x,x') = D^{(+)}(t,t',\vec{x}-\vec{x}\,')$ and may deduce what happens at equal time. First we observe that for $t'=t$ the values of the function $D^{(+)}(t,t,\vec{x}-\vec{x}\,')$ are c-numbers which means that $D(t,t,\vec{x}-\vec{x}\,')=0$. Moreover, from Eqs. (56) or (69) we find

$$(\partial_t - \partial_{t'})D^{(+)}(t,t',\vec{x}-\vec{x}\,')\Big|_{t'=t} = e^{-3\omega t}\delta^3(\vec{x}-\vec{x}\,') \tag{92}$$

and similarly for $D^{(-)}$.

The commutator functions can be written in analytical forms since the mode integral (91) may be solved in terms of Gauss hypergeometric functions (Chernikov & Tagiriv, 1968). Indeed, in the chart $\{t_c, \vec{x}\}$ this integral becomes

$$D^{(+)}(t_c, t_c{}', \vec{x}-\vec{x}\,') = \frac{i\pi\omega^2}{4}\frac{e^{-\pi k}}{(2\pi)^3}(t_c t_c{}')^{\frac{3}{2}}\int d^3p\, e^{i(\vec{x}-\vec{x}\,')\cdot\vec{p}} H^{(1)}_{ik}(-pt_c) H^{(1)}_{ik}(-pt_c{}')^* \tag{93}$$

and can be solved as,

$$D^{(+)}(t_c, t_c{}', \vec{x}-\vec{x}') = \frac{im^2}{16\pi}\, e^{-\pi k}\mathrm{sech}(\pi k)\, {}_2F_1\left(\frac{3}{2}+ik, \frac{3}{2}-ik; 2; 1+\frac{y}{4}\right), \tag{94}$$

where the quantity

$$y(x,x') = \frac{(t_c - t_c{}' - i\epsilon)^2 - (\vec{x}-\vec{x}\,')^2}{t_c t_c{}'} \tag{95}$$

is related to the geodesic length between x and x' (Birrel & Davies, 1982).

In general, $G(x,x') = G(t,t',\vec{x}-\vec{x}\,')$ is a Green function of the Klein-Gordon equation if this obeys

$$\left[E_{KG}(x) - m^2\right] G(x,x') = -e^{-3\omega t}\delta^4(x-x'). \tag{96}$$

The properties of the commutator functions allow us to construct the Green function just as in the scalar theory on Minkowski space-time. We assume that the retarded, D_R, and advanced, D_A, Green functions read

$$D_R(t,t',\vec{x}-\vec{x}\,') = \theta(t-t')D(t,t',\vec{x}-\vec{x}\,')\,, \tag{97}$$

$$D_A(t,t',\vec{x}-\vec{x}\,') = -\,\theta(t'-t)D(t,t',\vec{x}-\vec{x}\,')\,, \tag{98}$$

while the Feynman propagator,

$$\begin{aligned} D_F(t,t',\vec{x}-\vec{x}\,') &= i\langle 0|T[\phi(x)\phi^{\dagger}(x')]\,|0\rangle \\ &= \theta(t-t')D^{(+)}(t,t',\vec{x}-\vec{x}\,') - \theta(t'-t)D^{(-)}(t,t',\vec{x}-\vec{x}\,')\,, \end{aligned} \tag{99}$$

is defined as a causal Green function. It is not difficult to verify that all these functions satisfy Eq. (96) if one uses the identity $\partial_t^2[\theta(t)f(t)] = \delta(t)\partial_t f(t)$, the artifice $\partial_t f(t-t') = \frac{1}{2}(\partial_t - \partial_{t'})f(t-t')$ and Eq. (92).

5. The Dirac field

The first solutions of the free Dirac equation in the moving chart with proper time and spherical coordinates were derived in (Shishkin, 1991) and normalized in (Cotăescu et al., 2006). We derived other solutions of this equation but in moving charts with Cartesian coordinates where we considered the helicity basis in momentum representation (Cotăescu, 2002; Cotăescu & Crucean, 2008; Cotăescu, 2011b). These solutions are well-normalized, satisfy the usual completeness relations and correspond to a unique vacuum state.

5.1 Spinor quantum mechanics

Let ψ be a Dirac free field of mass m, defined on the space domain D, and $\overline{\psi} = \psi^{+}\gamma^{0}$ its Dirac adjoint. The tetrad gauge invariant action of the Dirac field minimally coupled with the gravitational field is

$$\mathcal{S}[e,\psi] = \int d^4x\sqrt{g}\left\{\frac{i}{2}\,[\overline{\psi}\gamma^{\hat{\alpha}}D_{\hat{\alpha}}\psi - (\overline{D_{\hat{\alpha}}\psi})\gamma^{\hat{\alpha}}\psi] - m\overline{\psi}\psi\right\} \tag{100}$$

where the Dirac matrices, $\gamma^{\hat{\alpha}}$, satisfy $\{\gamma^{\hat{\alpha}},\gamma^{\hat{\beta}}\} = 2\eta^{\hat{\alpha}\hat{\beta}}$. The covariant derivatives in local frames, denoted simply by $D_{\hat{\alpha}}$, are given by Eq. (1) where we consider the spinor representation $\rho_s = (\frac{1}{2},0)\oplus(0,\frac{1}{2})$ of the $SL(2,\mathbb{C})$ group whose basis-generators in covariant parametrization are $S^{\hat{\alpha}\hat{\beta}} = \frac{i}{4}\,[\gamma^{\hat{\alpha}},\gamma^{\hat{\beta}}]$. The operator of the Dirac equation $E_D\psi = m\psi$, derived from the action (100), reads $E_D = i\gamma^{\hat{\alpha}}D_{\hat{\alpha}}$. In other respects, from the conservation of the electric charge one deduces that when $e^0_i = 0$ ($i, j, ... = 1,2,3$) the time-independent relativistic scalar product of two spinors,

$$\langle\psi,\psi'\rangle = \int_D d^3x\,\mu(x)\overline{\psi}(x)\gamma^0\psi'(x)\,. \tag{101}$$

has the weight function $\mu = \sqrt{g}\,e^0_0$.

Our theory of external symmetry offers us the framework we need to calculate the conserved quantities predicted by the Noether theorem. Starting with the infinitesimal transformations of the one-parameter subgroup of $S(M)$ generated by X_a, we find that there exists the conserved current $\Theta^{\mu}[X_a]$ which satisfies $\Theta^{\mu}[X_a]_{;\mu} = 0$. For the action (100) this is

$$\Theta^{\mu}[X_a] = -\tilde{T}^{\mu\,\cdot}_{\cdot\,\nu} k^{\nu}_a + \frac{1}{4}\,\overline{\psi}\{\gamma^{\hat{\alpha}}, S^{\hat{\beta}\hat{\gamma}}\}\psi\, e^{\mu}_{\hat{\alpha}}\, \Omega_{a\,\hat{\beta}\hat{\gamma}} \tag{102}$$

where

$$\tilde{T}^{\mu\,\cdot}_{\cdot\,\nu} = \frac{i}{2}\left[\overline{\psi}\gamma^{\hat{\alpha}} e^{\mu}_{\hat{\alpha}}\partial_{\nu}\psi - (\overline{\partial_{\nu}\psi})\gamma^{\hat{\alpha}} e^{\mu}_{\hat{\alpha}}\psi\right] \tag{103}$$

is a notation for a part of the stress-energy tensor of the Dirac field. Finally, it is clear that the corresponding conserved quantity is the real number (Cotăescu, 2002),

$$\int_D d^3x\sqrt{g}\,\Theta^0[X_a] = \frac{1}{2}\left[\langle\psi, X_a\psi\rangle + \langle X_a\psi, \psi\rangle\right]. \tag{104}$$

We note that it is premature to interpret this formula as an expectation value or to speak about Hermitian conjugation of the operators X_a with respect to the scalar product (101), before specifying the boundary conditions on D. What is important here is that this result is useful in quantization giving directly the one-particle operators of the quantum field theory.

On the de Sitter expanding universe we can chose the simple Cartesian gauge (18) of the chart $\{t, \vec{x}\}$ or the corresponding gauge (19) in the chart $\{t_c, \vec{x}\}$. Then the Dirac operator takes the forms

$$E_D = -i\omega t_c\left(\gamma^0\partial_{t_c} + \gamma^i\partial_i\right) + \frac{3i\omega}{2}\gamma^0 = i\gamma^0\partial_t + ie^{-\omega t}\gamma^i\partial_i + \frac{3i\omega}{2}\gamma^0 \tag{105}$$

and the weight function of the scalar product (101) reads

$$\mu = (-\omega t_c)^{-3} = e^{3\omega t}. \tag{106}$$

This operator commutes with the isometry generators (26)-(29) whose spin parts are given now by the matrices $S_{\hat{\alpha}\hat{\beta}}$. Thus we obtained the NP of the Dirac theory.

The SP can be introduced transforming the Dirac field using the same operator (44) as in the scalar case. In this picture the Dirac field becomes $\psi_S(x) = W(x)\psi(x)W^{-1}(x)$ while the genuine orbital operators (49), (50) and H remain the same as in the scalar case. Moreover, we obtain the free Dirac equation of the SP,

$$\left[i\gamma^0\partial_t + i\gamma^i\partial_i - m + i\gamma^0\omega\left(x^i\partial_i + \frac{3}{2}\right)\right]\psi_S(x) = 0, \tag{107}$$

and the new form of the relativistic scalar product,

$$\langle\psi_S, \psi'_S\rangle = \langle\psi, \psi'\rangle = \int_D d^3x\,\bar{\psi}_S(x)\gamma^0\psi'_S(x), \tag{108}$$

calculated according to Eqs. (101) and (44b). We observe that this is no longer dependent on $\sqrt{g}$, having thus the same form as in special relativity.

5.2 Polarized plane wave solutions

In what follows we present the principal polarized plane wave solutions of the free Dirac field minimally coupled to the de Sitter gravity. The polarization is described in the helicity basis such that we have to speak about the momentum-helicity basis and the energy-halicity one (Cotăescu, 2002; Cotăescu & Crucean, 2008). In addition, we derived the modes of the momentum-spin basis (Cotăescu, 2011b) but these exceed the space of this paper.

5.2.1 The momentum-helicity basis

The plane wave solutions of the Dirac equation with $m \neq 0$ may be eigenspinors of the momentum operators P^i corresponding to the eigenvalues p^i. Therefore, we assume that, in the standard representation of the Dirac matrices, with diagonal γ^0 (Thaler, 1992), these have the form

$$\psi^{(+)}_{\vec{p}} = \left(\begin{array}{c} f^+(t_c)\,\alpha(\vec{p}) \\ g^+(t_c)\,\frac{\vec{\sigma}\cdot\vec{p}}{p}\,\alpha(\vec{p}) \end{array} \right) e^{i\vec{p}\cdot\vec{x}}\,, \quad \psi^{(-)}_{\vec{p}} = \left(\begin{array}{c} g^-(t_c)\,\frac{\vec{\sigma}\cdot\vec{p}}{p}\,\beta(\vec{p}) \\ f^-(t_c)\,\beta(\vec{p}) \end{array} \right) e^{-i\vec{p}\cdot\vec{x}} \tag{109}$$

where σ_i denotes the Pauli matrices while α and β are arbitrary Pauli spinors depending on $\vec{p}$. Replacing these spinors in the Dirac equation given by (105) and denoting $\mu = \frac{m}{\omega}$ and $\nu_\pm = \frac{1}{2} \pm i\mu$, we find equations of the form (161) whose solutions can be written in terms of Hankel functions as

$$f^+ = (-f^-)^* = c\, {t_c}^2 e^{\frac{1}{2}\pi\mu} H^{(1)}_{\nu_-}(-pt_c) \tag{110}$$

$$g^+ = (-g^-)^* = c\, {t_c}^2 e^{-\frac{1}{2}\pi\mu} H^{(1)}_{\nu_+}(-pt_c)\,. \tag{111}$$

The integration constant c will be calculated from the ortonormalization condition in the momentum scale.

The plane wave solutions are determined up to the significance of the Pauli spinors α and β. In general any pair of orthogonal spinors $\xi_\sigma(\vec{p})$ with polarizations $\sigma = \pm 1/2$ represents a good basis in the space of α-spinors. According to the standard interpretation of the negative frequency terms, the corresponding basis of the β-spinors is formed by the pair of orthogonal spinors defined as $\eta_\sigma(\vec{p}) = i\sigma_2[\xi_\sigma(\vec{p})]^*$. It remains to choose specific spinor bases, considering supplementary physical assumptions. Here we choose the *helicity* basis which is formed by the orthogonal Pauli spinors of helicity $\lambda = \pm\frac{1}{2}$ which satisfy the eigenvalues equations

$$\vec{\sigma}\cdot\vec{p}\;\xi_\lambda(\vec{p}) = 2p\lambda\,\xi_\lambda(\vec{p})\,, \quad \vec{\sigma}\cdot\vec{p}\;\eta_\lambda(\vec{p}) = -2p\lambda\,\eta_\lambda(\vec{p})\,, \tag{112}$$

and the orthonormalization condition $\xi^+_\lambda(\vec{p})\xi_{\lambda'}(\vec{p}) = \eta^+_\lambda(\vec{p})\eta_{\lambda'}(\vec{p}) = \delta_{\lambda\lambda'}$.

The desired particular solutions of the Dirac equation with $m \neq 0$ result from our starting formulas (109) where we insert the functions (110) and (111) and the spinors of the helicity basis (112). It remains to calculate the normalization constant c with respect to the scalar product (101) with the weight function (106). After a few manipulation, in the chart $\{t, \vec{x}\}$, it turns out that the Dirac field can be expanded as

$$\begin{aligned} \psi(t,\vec{x}) &= \psi^{(+)}(t,\vec{x}) + \psi^{(-)}(t,\vec{x}) \\ &= \int d^3p \sum_\lambda \left[U_{\vec{p},\lambda}(x) a(\vec{p},\lambda) + V_{\vec{p},\lambda}(x) b^*(\vec{p},\lambda) \right]\,, \end{aligned} \tag{113}$$

in terms of the particle (a) and antiparticle (b) wave functions of the momentum representation. The fundamental spinors of positive and negative frequencies with momentum $\vec{p}$ and helicity λ read (Cotăescu, 2002)

$$U_{\vec{p},\lambda}(t,\vec{x}) = iN \left(\begin{array}{c} \frac{1}{2}\, e^{\frac{1}{2}\pi\mu} H^{(1)}_{\nu_-}(qe^{-\omega t})\,\xi_\lambda(\vec{p}) \\ \lambda\, e^{-\frac{1}{2}\pi\mu} H^{(1)}_{\nu_+}(qe^{-\omega t})\,\xi_\lambda(\vec{p}) \end{array} \right) e^{i\vec{p}\cdot\vec{x}-2\omega t} \tag{114}$$

$$V_{\vec{p},\lambda}(t,\vec{x}) = iN \left(\begin{array}{c} \lambda\, e^{-\frac{1}{2}\pi\mu} H^{(2)}_{\nu_-}(qe^{-\omega t})\,\eta_\lambda(\vec{p}) \\ -\frac{1}{2}\, e^{\frac{1}{2}\pi\mu} H^{(2)}_{\nu_+}(qe^{-\omega t})\,\eta_\lambda(\vec{p}) \end{array} \right) e^{-i\vec{p}\cdot\vec{x}-2\omega t}\,, \tag{115}$$

where we introduced the new parameter $q = \frac{p}{\omega}$ and

$$N = \frac{1}{(2\pi)^{3/2}}\sqrt{\pi q}\,. \tag{116}$$

These solutions are the common eigenspinors of the complete set of commuting operators $\{E_D, \vec{S}^2, P^i, W\}$ obeying

$$P^i\, U_{\vec{p},\lambda} = p^i\, U_{\vec{p},\lambda}\,, \qquad P^i\, V_{\vec{p},\lambda} = -p^i\, V_{\vec{p},\lambda}\,, \tag{117}$$

$$W\, U_{\vec{p},\lambda} = p\lambda U_{\vec{p},\lambda}\,, \qquad W\, V_{\vec{p},\lambda} = -p\lambda V_{\vec{p},\lambda}\,, \tag{118}$$

where $W = \vec{J}\cdot\vec{P} = \vec{S}\cdot\vec{P}$ is the helicity operator. For this reason we say that these spinors form the *momentum-helicity* basis.

In other respects, according to Eqs. (160) and (162), it is not hard to verify that these spinors are charge-conjugated to each other,

$$V_{\vec{p},\lambda} = (U_{\vec{p},\lambda})^c = \mathcal{C}(\overline{U}_{\vec{p},\lambda})^T\,, \quad \mathcal{C} = i\gamma^2\gamma^0\,, \tag{119}$$

satisfy the ortonormalization relations,

$$\left\langle U_{\vec{p},\lambda}, U_{\vec{p}\,',\lambda'}\right\rangle = \left\langle V_{\vec{p},\lambda}, V_{\vec{p}\,',\lambda'}\right\rangle = \delta_{\lambda\lambda'}\delta^3(\vec{p}-\vec{p}\,')\,, \tag{120}$$

$$\left\langle U_{\vec{p},\lambda}, V_{\vec{p}\,',\lambda'}\right\rangle = \left\langle V_{\vec{p},\lambda}, U_{\vec{p}\,',\lambda'}\right\rangle = 0\,, \tag{121}$$

and represent a *complete* system of solutions in the sense that

$$\int d^3p \sum_{\lambda}\left[U_{\vec{p},\lambda}(t,\vec{x})U^{+}_{\vec{p},\lambda}(t,\vec{x}\,') + V_{\vec{p},\lambda}(t,\vec{x})V^{+}_{\vec{p},\lambda}(t,\vec{x}\,')\right] = e^{-3\omega t}\delta^3(\vec{x}-\vec{x}\,')\,. \tag{122}$$

Thus we can conclude that the separation of the positive and negative frequency modes performed here is point-independent and corresponds to a *stable* vacuum state which is of the Bunch-Davies type.

In the case of $m = 0$ (when $\mu = 0$) it is convenient to consider the chiral representation of the Dirac matrices (with diagonal γ^5) and the chart $\{t_c, \vec{x}\}$. We find that the fundamental solutions in momentum-helicity basis of the left-handed massless Dirac field (Cotăescu, 2002),

$$U^0_{\vec{p},\lambda}(t_c,\vec{x}) = \lim_{\mu\to 0} P_L U_{\vec{p},\lambda}(t_c,\vec{x}) = \left(\frac{-\omega t_c}{2\pi}\right)^{3/2}\begin{pmatrix}(\frac{1}{2}-\lambda)\xi_\lambda(\vec{p})\\ 0\end{pmatrix} e^{-ipt_c + i\vec{p}\cdot\vec{x}}\,, \tag{123}$$

$$V^0_{\vec{p},\lambda}(t_c,\vec{x}) = \lim_{\mu\to 0} P_L V_{\vec{p},\lambda}(t_c,\vec{x}) = \left(\frac{-\omega t_c}{2\pi}\right)^{3/2}\begin{pmatrix}(\frac{1}{2}+\lambda)\eta_\lambda(\vec{p})\\ 0\end{pmatrix} e^{ipt_c - i\vec{p}\cdot\vec{x}}\,, \tag{124}$$

where $P_L = \frac{1}{2}(1-\gamma^5)$ is the left-handed projection matrix. These solutions are non-vanishing only for positive frequency and $\lambda = -\frac{1}{2}$ or negative frequency and $\lambda = \frac{1}{2}$, as in the flat case and, moreover, they have similar properties as in (119)-(118).

5.2.2 The energy-helicity basis

The energy basis formed by eigenspinors of the energy operator, H, must be studied in the SP where Eq. (107) can be solved in momentum representation (Cotăescu & Crucean, 2008). We start assuming that the spinors of the SP may be expanded in terms of plane waves of positive and negative frequencies as,

$$\begin{aligned}\psi_S(x) &= \psi_S^{(+)}(x) + \psi_S^{(-)}(x) \\ &= \int_0^\infty dE \int_{\hat{D}} d^3p \, \left[\hat{\psi}_S^{(+)}(E,\vec{p})\, e^{-i(Et-\vec{p}\cdot\vec{x})} + \hat{\psi}_S^{(-)}(E,\vec{p})\, e^{i(Et-\vec{p}\cdot\vec{x})}\right]\end{aligned} \tag{125}$$

where $\hat{\psi}_S^{(\pm)}$ are spinors which behave as tempered distributions on the domain $\hat{D} = \mathbb{R}_p^3$ such that the Green theorem may be used. Then we can replace the momentum operators P_S^i by their eigenvalues p^i and the coordinate operators X_S^i by $i\partial_{p_i}$ obtaining the free Dirac equation of the SP in momentum representation,

$$\left[\pm E\gamma^0 \mp \gamma^i p^i - m - i\gamma^0\omega\left(p^i\partial_{p_i} + \frac{3}{2}\right)\right]\hat{\psi}_S^{(\pm)}(E,\vec{p}) = 0\,, \tag{126}$$

where E is the energy defined as the eigenvalue of H_S. Denoting $\vec{p} = p\,\vec{n}$ we observe that the differential operator of Eq. (126) is of radial type and reads $p^i\partial_{p_i} = p\,\partial_p$. Therefore, this operator acts on the functions which depend on p while the functions which depend only on the momentum direction $\vec{n}$ behave as constants.

Following the method of section 4.2.2 we derive the fundamental solutions of the *helicity* basis using the standard representation of the γ-matrices (with diagonal γ^0) (Thaler, 1992). The general solutions,

$$\hat{\psi}_S^{(+)}(E,\vec{p}) = \sum_\lambda u^S(E,\vec{p},\lambda)\, a(E,\vec{n},\lambda)\,, \tag{127}$$

$$\hat{\psi}_S^{(-)}(E,\vec{p}) = \sum_\lambda v^S(E,\vec{p},\lambda)\, b^*(E,\vec{n},\lambda)\,, \tag{128}$$

involve spinors of helicity $\lambda = \pm\frac{1}{2}$ and the particle and antiparticle wave functions, a and respectively b, which play here the role of constants since they do not depend on p. According to our previous results, the spinors of the momentum representation must have the form

$$u^S(E,\vec{p},\lambda) = \begin{pmatrix} \frac{1}{2} f_E^{(+)}(p)\,\xi_\lambda(\vec{n}) \\ \lambda g_E^{(+)}(p)\,\xi_\lambda(\vec{n}) \end{pmatrix}, \quad v^S(E,\vec{p},\lambda) = \begin{pmatrix} \lambda g_E^{(-)}(p)\,\eta_\lambda(\vec{n}) \\ -\frac{1}{2} f_E^{(-)}(p)\,\eta_\lambda(\vec{n}) \end{pmatrix}, \tag{129}$$

where $\xi_\lambda(\vec{n})$ and $\eta_\lambda(\vec{n}) = i\sigma_2[\xi_\lambda(\vec{n})]^*$ denote now the Pauli spinors of the helicity basis introduced in section 5.2.1. (which depend only on the momentum direction $\vec{n}$). Furthermore, we derive the radial functions solving the system

$$\left[i\omega\left(p\frac{d}{dp} + \frac{3}{2}\right) \mp (E-m)\right] f_E^{(\pm)}(p) = \mp p\, g_E^{(\pm)}(p)\,, \tag{130}$$

$$\left[i\omega\left(p\frac{d}{dp} + \frac{3}{2}\right) \mp (E+m)\right] g_E^{(\pm)}(p) = \mp p\, f_E^{(\pm)}(p)\,, \tag{131}$$

resulted from Eq. (126). We find the solutions

$$f_E^{(+)}(p) = [-f_E^{(-)}(p)]^* = Cp^{-1-i\epsilon}e^{\frac{1}{2}\pi\mu}H^{(1)}_{\nu_-}(\tfrac{p}{\omega})\,, \tag{132}$$
$$g_E^{(+)}(p) = [-g_E^{(-)}(p)]^* = Cp^{-1-i\epsilon}e^{-\frac{1}{2}\pi\mu}H^{(1)}_{\nu_+}(\tfrac{p}{\omega})\,. \tag{133}$$

The normalization constant C has to assure the normalization in the energy scale.

Collecting all the above results we can write down the final expression of the Dirac field (125) in SP identifying the form of the fundamental spinors of given energy. Then we turn back to the NP where the Dirac field,

$$\psi(x) = \psi_S(t, e^{\omega t}\vec{x}) = \int_0^\infty dE \int_{S^2} d\Omega_n \sum_\lambda \left[U_{E,\vec{n},\lambda}(t,\vec{x})a(E,\vec{n},\lambda)\right. \tag{134}$$
$$\left. + V_{E,\vec{n},\lambda}(t,\vec{x})b^*(E,\vec{n},\lambda)\right]\,, \tag{135}$$

depends on the solutions written in NP,

$$U_{E,\vec{n},\lambda}(t,\vec{x}) = U^S_{E,\vec{n},\lambda}(t,e^{\omega t}\vec{x})\,,\quad V_{E,\vec{n},\lambda}(t,\vec{x}) = V^S_{E,\vec{n},\lambda}(t,e^{\omega t}\vec{x})\,. \tag{136}$$

According to Eqs. (129), (132) and (133), we obtain the integral representations

$$U_{E,\vec{n},\lambda}(t,\vec{x}) = i\hat{N}e^{-2\omega t}\int_0^\infty s\,ds \begin{pmatrix} \frac{1}{2}\,e^{\frac{1}{2}\pi\mu}H^{(1)}_{\nu_-}(se^{-\omega t})\,\xi_\lambda(\vec{n}) \\ \lambda\,e^{-\frac{1}{2}\pi\mu}H^{(1)}_{\nu_+}(se^{-\omega t})\,\xi_\lambda(\vec{n})\end{pmatrix} e^{i\omega s\vec{n}\cdot\vec{x}-i\epsilon\ln s}\,, \tag{137}$$
$$V_{E,\vec{n},\lambda}(t,\vec{x}) = i\hat{N}e^{-2\omega t}\int_0^\infty s\,ds \begin{pmatrix} \lambda\,e^{-\frac{1}{2}\pi\mu}H^{(2)}_{\nu_-}(se^{-\omega t})\,\eta_\lambda(\vec{n}) \\ -\frac{1}{2}\,e^{\frac{1}{2}\pi\mu}H^{(2)}_{\nu_+}(se^{-\omega t})\,\eta_\lambda(\vec{n})\end{pmatrix} e^{-i\omega s\vec{n}\cdot\vec{x}+i\epsilon\ln s}\,, \tag{138}$$

where we denote the dimensionless integration variable by $s = \frac{p}{\omega}\,e^{\omega t}$ and take

$$\hat{N} = \frac{1}{(2\pi)^{3/2}}\,\frac{\omega}{\sqrt{2}}\,. \tag{139}$$

We derived thus the *fundamental* spinor solutions of positive and, respectively, negative frequencies, with energy E, momentum direction $\vec{n}$ and helicity λ. These spinors are charge-conjugated to each other,

$$V_{E,\vec{n},\lambda} = (U_{E,\vec{n},\lambda})^c = \mathcal{C}(\overline{U}_{E,\vec{n},\lambda})^T\,,\quad \mathcal{C} = i\gamma^2\gamma^0\,, \tag{140}$$

and satisfy the orthonormalization relations

$$\langle U_{E,\vec{n},\lambda}, U_{E,\vec{n}\,',\lambda'}\rangle = \langle V_{E,\vec{n},\lambda}, V_{E,\vec{n}\,',\lambda'}\rangle = \delta_{\lambda\lambda'}\delta(E-E')\,\delta^2(\vec{n}-\vec{n}\,')\,, \tag{141}$$
$$\langle U_{E,\vec{n},\lambda}, V_{E,\vec{n}\,',\lambda'}\rangle = \langle V_{E,\vec{n},\lambda}, U_{E,\vec{n}\,',\lambda'}\rangle = 0\,. \tag{142}$$

deduced as in the Appendix B. Moreover, the completeness relation

$$\int_0^\infty dE \int_{S^2} d\Omega_n \sum_\lambda \{U_{E,\vec{n},\lambda}(t,\vec{x})[U_{E,\vec{n},\lambda}(t,\vec{x}\,')]^+$$
$$+ V_{E,\vec{n},\lambda}(t,\vec{x})[V_{E,\vec{n},\lambda}(t,\vec{x}\,')]^+\} = e^{-3\omega t}\delta^3(\vec{x}-\vec{x}\,')\,, \tag{143}$$

indicates that this system of solutions is complete. We say that this represents the *energy-helicity* basis.

Finally, we derive the transition coefficients transforming the momentum-helicity and the energy-helicity bases among themselves. After a few manipulation we find that these coefficients,

$$\left\langle U_{\vec{p},\lambda}, U_{E,\vec{n},\lambda'}\right\rangle = \left\langle V_{\vec{p},\lambda}, V_{E,\vec{n},\lambda'}\right\rangle^{*} = \delta_{\lambda\lambda'}\,\frac{p^{-\frac{3}{2}}}{\sqrt{2\pi\omega}}\,\delta^2(\vec{n}-\vec{n}_p)\,e^{-i\frac{E}{\omega}\ln\frac{p}{\omega}}\,, \tag{144}$$

$$\left\langle U_{\vec{p},\lambda}, V_{E,\vec{n},\lambda'}\right\rangle = \left\langle V_{\vec{p},\lambda}, U_{E,\vec{n},\lambda'}\right\rangle = 0\,, \tag{145}$$

are similar to those of the scalar modes (70). Therefore, the particle wave functions $a(\vec{p},\lambda)$ and $a(E,\vec{n},\lambda)$ are related among themselves by similar unitary transformations as (71) and (72) but conserving, in addition, the helicity. These transformations preserve the vacuum state since the antiparticle wave functions have the same properties and the particle and antiparticle Hilbert spaces remain orthogonal to each other, as it results from Eq. (145).

5.3 Quantization and propagators

The quantization of the Dirac field can be done easily in the helicity bases as well as in the spin one. We assume that the wave functions of the momentum-helicity basis, $a(\vec{p},\lambda)$ and $b(\vec{p},\lambda)$, become field operators (so that $b^* \to b^\dagger$) satisfying the standard anticommutation relations from which the non-vanishing ones are

$$\{a(\vec{p},\lambda), a^\dagger(\vec{p}\,',\lambda')\} = \{b(\vec{p},\lambda), b^\dagger(\vec{p}\,',\lambda')\} = \delta_{\lambda\lambda'}\delta^3(\vec{p}-\vec{p}\,')\,. \tag{146}$$

The corresponding anticommutation rules of the energy-helicity basis are

$$\begin{aligned}\{a(E,\vec{n},\lambda), a^\dagger(E',\vec{n}\,',\lambda')\} &= \{b(E,\vec{n},\lambda), b^\dagger(E',\vec{n}\,',\lambda')\}\\ &= \delta_{\lambda,\lambda'}\delta(E-E')\delta^2(\vec{n}-\vec{n}\,')\,.\end{aligned} \tag{147}$$

We say that this quantization is *canonical* since the equal-time anticommutator takes the standard form (Drell & Bjorken, 1965)

$$\{\psi(t,\vec{x}),\, \overline{\psi}(t,\vec{x}\,')\} = e^{-3\omega t}\gamma^0\delta^3(\vec{x}-\vec{x}\,')\,, \tag{148}$$

as it results from Eq. (122). In addition, we know that the mode separation we use defines a stable vacuum state. Therefore, we have to construct the Fock space canonically, applying the creation operators upon the unique vacuum state $|0\rangle$.

The one-particle operators corresponding to the isometry generators can be calculated as in the scalar case using the definition $\mathcal{X} =: \langle\psi, X\psi\rangle :$. The operators which do not come from differential operators have to be defined directly giving their mode expansions (Cotăescu, 2002). It is remarkable that all these operators have similar properties to those of the scalar field presented here or of the vector fields studied in (Cotăescu, 2010; Cotăescu & Crucean, 2010). The most interesting result is the expansion of the energy operator in the momentum-helicity basis where we may use the identity

$$H\,U_{\vec{p},\lambda}(t,\vec{x}) = -i\omega\left(p^i\partial_{p^i} + \frac{3}{2}\right)U_{\vec{p},\lambda}(t,\vec{x})\,, \tag{149}$$

and the similar one for $V_{\vec{p},\lambda}$, leading to the expansion

$$\mathcal{H} = \frac{i\omega}{2} \int d^3p\, p^i \sum_{\lambda} \left[a^{\dagger}(\vec{p},\lambda) \overset{\leftrightarrow}{\partial}_{p^i} a(\vec{p},\lambda) + b^{\dagger}(\vec{p},\lambda) \overset{\leftrightarrow}{\partial}_{p^i} b(\vec{p},\lambda) \right] \tag{150}$$

which depend on the phase factors of the field operators as in the scalar or vector cases.

The Green functions can be expressed in terms of anticommutator functions,

$$S^{(\pm)}(t,t',\vec{x}-\vec{x}\,') = i\{\psi^{(\pm)}(t,\vec{x}),\, \overline{\psi}^{(\pm)}(t',\vec{x}\,')\}\,, \tag{151}$$

and $S = S^{(+)} + S^{(-)}$ which can be written as mode integrals that can be analytically solved (Kosma & Prokopec, 2009). These functions are solutions of the Dirac equation in both their sets of coordinates and helped us to write down the Feynman propagator,

$$\begin{aligned} S_F(t,t',\vec{x}-\vec{x}\,') &= i\,\langle 0|\, T[\psi(x)\overline{\psi}(x')]\, |0\rangle \\ &= \theta(t-t')S^{(+)}(t,t',\vec{x}-\vec{x}\,') - \theta(t'-t)S^{(-)}(t,t',\vec{x}-\vec{x}\,')\,, \end{aligned} \tag{152}$$

and the retarded and advanced Green functions $S_R(t,t',\vec{x}-\vec{x}\,') = \theta(t-t')S(t,t',\vec{x}-\vec{x}\,')$ and respectively $S_A(t,t',\vec{x}-\vec{x}\,') = -\theta(t'-t)S(t,t',\vec{x}-\vec{x}\,')$, which satisfy the specific equation,

$$[E_D(x) - m]S_F(t,t',\vec{x}-\vec{x}\,') = -e^{-3\omega t}\delta^4(x-x')\,, \tag{153}$$

of the spinor Green functions on the de Sitter space-time (Cotăescu, 2002).

6. Concluding remarks

We presented here the complete quantum theory of the massive scalar and Dirac free fields minimally coupled to the gravity of the de Sitter expanding universe. Applying similar methods we succeeded to accomplish the theory of the Proca (Cotăescu, 2010) and Maxwell (Cotăescu & Crucean, 2010) fields on this background. The main points of our approach are the theory of external symmetry (Cotăescu, 2000; 2009) that provides us with the conserved operators of the fields with any spin and the Schrödinger time-evolution picture (Cotăescu, 2007) allowing us to derive new sets of fundamental solutions.

The wave functions defining quantum modes are solutions of the field equations and common eigenfunctions of suitable systems of commuting operators which represent conserved observables *globally* defined on the de Sitter manifold. All these observables form the global apparatus which *prepares* global quantum modes as it seems to be natural as long as the field equations are global too. In this manner, we obtain wave functions correctly normalized on the whole background such that the Hilbert spaces of the particle and respectively antiparticle states remain orthogonal to each other in *any* frame, assuring thus the stability of a vacuum state which is of the bunch-Davies type (Bunch & Davies, 1978).

The new energy bases introduced here completes the framework of the de Sitter quantum theory, being crucial for understanding how the energy and momentum can be measured simultaneously. We may convince that considering the simple example of a Klein-Gordon particle in the state

$$|\chi\rangle = \int d^3p\, \chi(\vec{p}) a^{\dagger}(\mathbf{p})|0\rangle\,, \quad \chi(\vec{p}) = \rho(\vec{p})e^{-i\vartheta(\vec{p})}\,, \tag{154}$$

determined by the functions $\rho, \theta : \mathbb{R}^3_p \to \mathbb{R}$. The normalization condition

$$1 = \langle \chi|\chi\rangle = \int d^3p\, |\chi(\vec{p})|^2 = \int d^3p\, |\rho(\vec{p})|^2 \tag{155}$$

shows that the function ρ must be square integrable on $\mathbb{R}^3_p$ while θ remains an arbitrary real function. Furthermore, according to Eqs. (81) and (86), we derive the expectation values of the non-commuting operators $\mathcal{P}^i$ and $\mathcal{H}$,

$$\langle \chi|\mathcal{P}^i|\chi\rangle = \int d^3p\, p^i\, |\rho(\vec{p})|^2\,, \quad \langle \chi|\mathcal{H}|\chi\rangle = \omega \int d^3p\, [p^i \partial_{p_i} \vartheta(\vec{p})]\, |\rho(\vec{p})|^2\,. \tag{156}$$

The expectation values of the momentum operators are independent on the phase θ while that the energy operator depends mainly on it. This means that we can prepare at anytime states with arbitrary desired expectation values of both these observables. Thus, in the particular case of $\theta(\vec{p}) = \epsilon \ln(p)$ we obtain $\langle \chi|\mathcal{H}|\chi\rangle = \omega\epsilon = E$ indifferent on the form of ρ if this obeys the condition (155). In other respects, we observe that the dispersion of the energy operator in this particular state,

$$\mathrm{disp}\mathcal{H} = \langle \chi|\mathcal{H}^2|\chi\rangle - \langle \chi|\mathcal{H}|\chi\rangle^2 = \omega^2 \int d^3p\, |p^i \partial_{p_i} \rho(\vec{p})|^2 - \tfrac{9}{4}\omega^2\,, \tag{157}$$

depends only on the momentum statistics given by the function ρ. Thus we can say that we meet a new quantum mechanics on the de Sitter expanding universe.

We conclude that our approach seems to be coherent at the level of the relativistic quantum mechanics where the conserved observables of different time-evolution pictures are correctly defined allowing us to derive complete sets of quantum modes. Consequently, the second quantization can be performed in canonical manner leading to quantum free fields, one-particle operators and Green functions with similar properties to those of special relativity. Under such circumstances, we believe that we constructed the appropriate framework for studying quantum effects of interacting fields on the de Sitter background. Assuming that the quantum transitions are measured by the *same* global apparatus which prepares the free quantum states we may use the perturbation theory for deriving transition amplitudes as in the flat case.

Finally we specify that our attempt to use quantum modes globally defined does not contradict the general concept of local measurements (Birrel & Davies, 1982) which is the only possible option when the isometries (or other symmetries) are absent and, consequently, the global apparatus does not work.

7. Appendix

Appendix A: Some properties of Hankel functions

Let us consider the functions Z_k depending on the Hankel functions $H^{(1,2)}_\nu(z)$ (Abramowitz & Stegun, 1964) as

$$Z_k(z) = e^{-\pi k/2} H^{(1)}_{ik}(z)\,, \quad Z^*_k(z) = e^{\pi k/2} H^{(2)}_{ik}(z)\,. \tag{158}$$

where $z, k \in \mathbb{R}$. Then, using the Wronskian W of the Bessel functions we find that

$$Z^*_k(z) \overset{\leftrightarrow}{\partial_z} Z_k(z) = W[H^{(2)}_{ik}, H^{(1)}_{ik}](z) = \frac{4i}{\pi z}\,. \tag{159}$$

A special case is of the Hankel functions $H^{(1,2)}_{\nu_\pm}(z)$ of indices $\nu_\pm = \frac{1}{2} \pm ik$ where $z, k \in \mathbb{R}$. These are related among themselves through

$$[H^{(1,2)}_{\nu_\pm}(z)]^* = H^{(2,1)}_{\nu_\mp}(z)\,, \tag{160}$$

satisfy the equations

$$\left(\frac{d}{dz} + \frac{\nu_\pm}{z}\right) H^{(1)}_{\nu_\pm}(z) = ie^{\pm\pi k} H^{(1)}_{\nu_\mp}(z)\,, \quad \left(\frac{d}{dz} + \frac{\nu_\pm}{z}\right) H^{(2)}_{\nu_\pm}(z) = -ie^{\mp\pi k} H^{(2)}_{\nu_\mp}(z) \tag{161}$$

and the identities

$$e^{\pm\pi k} H^{(1)}_{\nu_\mp}(z) H^{(2)}_{\nu_\pm}(z) + e^{\mp\pi k} H^{(1)}_{\nu_\pm}(z) H^{(2)}_{\nu_\mp}(z) = \frac{4}{\pi z}\,. \tag{162}$$

Appendix B: Normalization integrals

In spherical coordinates of the momentum space, $\vec{n} \sim (\theta_n, \phi_n)$, and the notation $\vec{q} = \omega s \vec{n}$, we have $d^3q = q^2 dq\, d\Omega_n = \omega^3\, s^2 ds\, d\Omega_n$ with $d\Omega_n = d(\cos\theta_n) d\phi_n$. Moreover, we can write

$$\delta^3(\vec{q} - \vec{q}\,') = \frac{1}{q^2}\, \delta(q - q')\delta^2(\vec{n} - \vec{n}') = \frac{1}{\omega^3 s^2}\, \delta(s - s')\delta^2(\vec{n} - \vec{n}')\,, \tag{163}$$

where we denoted $\delta^2(\vec{n} - \vec{n}') = \delta(\cos\theta_n - \cos\theta_n')\delta(\phi_n - \phi_n')$.

The normalization integrals can be calculated according to Eqs. (66) and (163), that yield

$$\langle f_{E,\vec{n}}, f_{E',\vec{n}'}\rangle = i\, \frac{N^2(2\pi)^3}{\omega^3}\, \delta^2(\vec{n} - \vec{n}\,') \int_0^\infty \frac{ds}{s} e^{i(\epsilon-\epsilon')\ln s} \left[Z_k^*(s\, e^{-\omega t})\, \overset{\leftrightarrow}{\partial_t}\, Z_k(s\, e^{-\omega t}) \right]. \tag{164}$$

The final result has to be obtained using Eq. (159) and the representation

$$\frac{1}{2\pi\omega} \int_0^\infty \frac{ds}{s}\, e^{\frac{i}{\omega}(E-E')\ln s} = \delta(E - E')\,. \tag{165}$$

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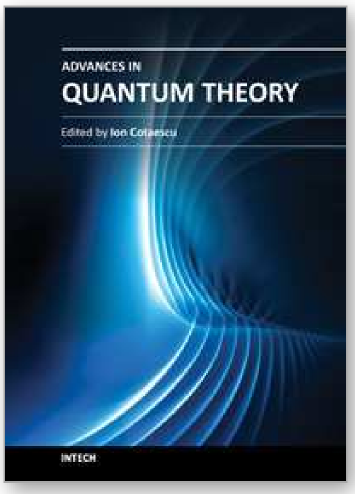

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The quantum theory is the first theoretical approach that helps one to successfully understand the atomic and sub-atomic worlds which are too far from the cognition based on the common intuition or the experience of the daily-life. This is a very coherent theory in which a good system of hypotheses and appropriate mathematical methods allow one to describe exactly the dynamics of the quantum systems whose measurements are systematically affected by objective uncertainties. Thanks to the quantum theory we are able now to use and control new quantum devices and technologies in quantum optics and lasers, quantum electronics and quantum computing or in the modern field of nano-technologies.