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Quantum Theory of Multi-Local Particle

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1. Introduction

In those days before the development of the gauge field theories there were many attempts to construct multi-local field theories of hadrons. The motivation for that was in the existence of wide variety of hadrons which may be categorized by quantum numbers indebted to the presumed internal structures. The success of QFD and/or QCD, however, impressed us the power of the local field theories of quarks and leptons, and swept away almost all alternative attempts describing the low energy physics. On the other hand the concept of the multi-locality (or non-locality) was promoted to the string model which is now regarded as one of candidates of the quantum gravity.

Although the realm of validity of the local field theory may be extended to the Planck scale the conceptual gap between the string and the local field theory is so large that we cannot treat them on an equal footing. Is there no room for the multi-local field theory in describing the phenomena near the Planck scale?

We have sought the theoretical possibility of the multi-local objects, consisting of N particles, which stay in an intermediate position between the local particle and the string. In the papers (Hori, 1992)−(Hori, 2009) we have constructed the models with $N = 2$, which have resembling properties as the string, though extremely simple in structure. The simplest model with $N = 2$ is a system of two relativistic particles with specific interaction among them. We called the object as a bilocal particle. We have found a hidden gauge symmetry in the bilocal model (Hori, 1992), which reveals $SL(2, \mathbb{R})$ in the canonical theory. This causes the pathological property that the amount of the gauge invariance does not match with the number of the first class constraints in the canonical theory. This means breakdown of Dirac's conjecture (Dirac, 1950).

The BRST analysis of the bilocal model shows the existence of spacetime critical dimensions, $D = 2$ or $D = 4$ (Hori, 1996). But the quantum theory of the model can not be treated in the similar way as the ordinary gauge theories, since the ghost numbers of the physical states are not zero. Because the reason of the difficulty is in the constraint structure we have constructed an improved version of $N = 2$ model based on the object called complex particle (Hori, 2009).

The coordinates of a complex particle are complex numbers and depend on the internal time. In the lagrangian formulation the gauge degrees of freedom is two in the ordinary sense. This causes the breakdown of Dirac's conjecture as in the bilocal model. We argued that a modification of the definition of the physical equivalence remedies the situation, and the system has all of the three gauge freedom of $SL(2,\mathbb{R})$. The constraint structure is different from that of the bilocal model in such a way that two of three constraints are hermitian conjugate to each other, and a natural quantization scheme can be applied similar to the string theory. The physical state conditions are fulfilled in the ghost number zero sector, and the requirement that the momentum eigenstate should be physical restricts the dimension of the spacetime to be two or four.

In the present paper we achieve the complete first quantization of the complex particle, supplementing the results obtained in ref.(Hori, 2009). We also propose the field theory action of the Chern-Simons type. The action is shown to be invariant under gauge transformations in the field theoretical sense only if $D = 4$.

Finally we extend the previous results to $N \geq 3$ particle system. Especially we define an open N-particle and closed N-particle systems. We restrict ourselves, however, to the open cases because the constraint structure in the canonical theory is much complicated in the closed cases compared with the open cases.

2. Preliminary remark

The notion of gauge invariance or physical equivalence in the models considered in the present paper is so subtle that one may easily fall into confusion. Therefore let us consider first the ordinary relativistic particle, and count the number of gauge as well as physical degrees of freedom. The spacetime coordinates of the particles $x^\mu, (\mu = 0, 1, 2, .., D-1)$ are functions of internal time τ, where D is the dimension of the spacetime. The action is written as [1]

$$I_0 = \int d\tau \, \frac{\dot{x}^\mu \dot{x}_\mu}{2g}, \tag{1}$$

where g is the einbein needed for the reparametrization of the internal time. The action is invariant under the transformations

$$\delta x^\mu = \epsilon \dot{x}^\mu + \epsilon^{\mu\nu} x_\nu + a^\mu, \qquad \delta g = \frac{d}{d\tau}(\epsilon g), \qquad (\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}) \tag{2}$$

where infinitesimal constant parameters $\epsilon^{\mu\nu}, a^\mu$ are those of Lorenz transformations and translations, respectively, and the parameter ϵ depends on τ, corresponding to the reparametrization of τ.

Now what is the gauge freedom, by which the τ development of variables is not determined uniquely? The existence of the invariance of the action, with τ dependent parameter, ϵ, leads to redundant variables, because of which the Euler-Lagrange(EL) equations have not unique solutions even if one chooses suitable initial conditions.

We get the answer by first choosing gauge fixing conditions and by ascertaining consistency of the solutions to EL equations. The first integral of EL equations is

$$\dot{x}^\mu = c^\mu g, \qquad \dot{x}^\mu \dot{x}_\mu = 0, \tag{3}$$

[1] The metric convension is $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, ..1)$.

where c's are arbitrary constants. By using the freedom $\epsilon(\tau)$ we can fix the gauge as

$$x^0(\tau) = \tau. \tag{4}$$

The remaining freedom (of finite degrees) is $\epsilon^{0i}, \epsilon^{ij}, a^i$, which is counted $\frac{1}{2}(D-1)(D+2)$. Setting the initial conditions as

$$x^i(0) = x_0^i, \qquad \dot{x}^i(0) = c^i g_0, \qquad g(0) = g_0, \qquad (c^i)^2 = 1/g_0^2, \tag{5}$$

we get the unique solution

$$x^i(\tau) = c^i g_0 \tau + x_0^i, \qquad g(\tau) = g_0. \tag{6}$$

That is, if the spacial coordinates and the spacial direction of the particle both at $\tau = 0$ and the value $g(0)$ are given, the whole orbit of the particle moving with velocity of light is determined. The number of the physical degrees of freedom must be the number of degrees of freedom to put the independent initial condition, *i.e.*, $2(D-1)$. On the other hand, among the $\frac{1}{2}(D-1)(D+2)$ degrees of freedom of the remaining symmetry in the gauge (4) the number of freedom which does not move c^i is $\frac{1}{2}(D-1)(D-2)$. Hence the net degrees of freedom for changing the initial condition is $\frac{1}{2}(D-1)(D+2) - \frac{1}{2}(D-1)(D-2) = 2(D-1)$. This coincidence implies that the gauge freedom corresponds to the transformation with the parameter $\epsilon(\tau)$, by which one can fix one variable for all τ.

Presumably, the above coincidence may be due to Dirac's conjecture in the canonical theory, which claims that every first class constraint should generate gauge transformations. In the subsequent sections we will encounter the situations where a naive counting leads to mismatch of degrees of freedom in the lagrangian form.

3. $N = 2$ model

3.1 Classical action

The simplest example of the multi-local particle is the two particle system with some bilinear *interactions*. We call it bilocal particle (Hori, 1992). Let us denote the coordinates of the two particles as $x_a^\mu, (a = 1,2;\ \mu = 0,1,2,..,D-1)$, which are functions of internal time, τ. We introduce the einbeins, $g_a, (a = 1,2)$, for the sake of the reparametrization invariance along the world lines, which are auxiliary variables and their equations of motion make the trajectories of the particles put on the light-cones. The proposed action of the bilocal particle is written as

$$I = \int d\tau L, \qquad L = \frac{\dot{x}_1^2}{2g_1} + \frac{\dot{x}_2^2}{2g_2} + \kappa(\dot{x}_1 x_2 - \dot{x}_2 x_1), \tag{7}$$

where κ is a constant with dimension of mass squared. (Here and hereafter we suppress the spacetime indices μ, if no confusions occur.) The first two terms in the action are separately invariant under

$$\delta x_a = \epsilon_a \dot{x}_a, \qquad \delta g_a = \frac{d}{d\tau}(\epsilon_a g_a), \qquad (a = 1,2) \tag{8}$$

where $\epsilon_a, (a = 1,2)$ are infinitesimal parameters depending on τ. This is the well known reparametrization gauge invariance of the relativistic particle. Apparently the third term in the action would violate this invariance with independent ϵ_1 and ϵ_2, but we found larger invariance under the following transformations (Hori, 1992),

$$\delta x_1 = \epsilon_1 \dot{x}_1 + \frac{\epsilon_0}{g_2}\dot{x}_2, \qquad \delta x_2 = \epsilon_2 \dot{x}_2 + \frac{\epsilon_0}{g_1}\dot{x}_1, \tag{9}$$

$$\delta g_1 = \frac{d}{d\tau}(\epsilon_1 g_1) + 4\kappa\epsilon_0 g_1, \qquad \delta g_2 = \frac{d}{d\tau}(\epsilon_2 g_2) - 4\kappa\epsilon_0 g_2, \tag{10}$$

where the infinitesimal parameters $\epsilon_1, \epsilon_2, \epsilon_0$ are functions of τ, two of which are arbitrary, while another is subjected to the constraint

$$\dot{\epsilon}_0 + 2\kappa g_1 g_2(\epsilon_2 - \epsilon_1) = 0. \tag{11}$$

In fact the variation of the lagrangian under (9) and (10) is

$$\delta L = \frac{d}{d\tau}\left[\epsilon_0\left(\frac{\dot{x}_1\dot{x}_2}{g_1 g_2} + \kappa\left(\frac{\dot{x}_2 x_2}{g_2} - \frac{\dot{x}_1 x_1}{g_1}\right)\right) + \sum_{a,b=1}^{2}\epsilon_a\left(\kappa\epsilon_{ab}x_a\dot{x}_b + \delta_{ab}\frac{\dot{x}_a\dot{x}_b}{g_b}\right)\right]$$
$$+\left[\dot{\epsilon}_0 + 2\kappa g_1 g_2(\epsilon_2 - \epsilon_1)\right]\frac{\dot{x}_1\dot{x}_2}{g_1 g_2}, \qquad (\epsilon_{12} = -\epsilon_{21} = 1,\ \epsilon_{11} = \epsilon_{22} = 0). \tag{12}$$

We can regard ϵ_1 and ϵ_2 as the independent gauge parameters, and ϵ_0 as dependent one determined by (11) except its *constant mode*, a relic of the global invariance of the action. The existence of the above unexpected gauge invariance is the origin of some curious properties of our model such that the $SL(2,\mathbb{R})$ gauge symmetry in the canonical theory and the existence of the critical dimension as is shown later.

In the case $\kappa \neq 0$, the first integral to the equations of motion derived by the action (7) is

$$\frac{\dot{x}_a}{g_a} + 2\kappa\sum_b \epsilon_{ab}(x_b - c_b) = 0, \qquad (a = 1,2), \tag{13}$$

where $c_a, (a = 1,2)$ are constants. Since the variations of g_a give $\dot{x}_a^2 = 0$, we have

$$(x_a - c_a)^2 = 0, \qquad (a = 1,2). \tag{14}$$

Thus the two particles are put on the light-cone with tops of arbitrary spacetime points, and moving with velocity of light.

For the sake of the reparametrization invariance we can fix the gauge as

$$x_1^0(\tau) = x_2^0(\tau) = \tau, \tag{15}$$

then from (13) we have

$$\frac{1}{g_1} = -2\kappa\left(\tau - c_2^0\right), \qquad \frac{1}{g_2} = 2\kappa\left(\tau - c_1^0\right). \tag{16}$$

Substituting them back into (13), and dividing by κ, we obtain

$$(\tau - c_2^0)\dot{x}_1^i = (x_2 - c_2)^i, \qquad (\tau - c_1^0)\dot{x}_2^i = (x_1 - c_1)^i. \qquad (i = 1, 2, .., D-1). \tag{17}$$

Note that eqs.(17) do *not* depend on κ. (The case $\kappa = 0$ should be treated separately, since in that case we must set $\dot{x}_a/g_a = c'_a$ instead of (13) with other constants c'_a, leading to $\dot{x}_a = c'_a$ in the above gauge. We assume $\kappa \neq 0$ henceforth.) Differentiating (17) with respect to τ we have

$$(\tau - c_1^0)(\tau - c_2^0)\ddot{x}_a^i + (\tau - c_a^0)\dot{x}_a^i - (x_a - c_a)^i = 0, \qquad (i = 1, 2, .., D-1;\ a = 1, 2). \tag{18}$$

These are the ordinary linear differential equations of second rank with regular singularities, and are solved by Frobenius's method. The general solutions are written as

$$x_a^i(\tau) = c_a^i + (\tau - c_a^0)v_a^i + f_a(\tau)w_a^i, \qquad (i = 1, 2, .., D-1;\ a = 1, 2), \tag{19}$$

where v_a^i, w_a^i, $(a = 1, 2)$ are arbitrary constants, and $f_a(\tau)$ are the solutions to the equations $(\tau - c_2^0)\dot{f}_1 = f_2$ and $(\tau - c_1^0)\dot{f}_2 = f_1$, with vanishing asymptotic values, the concrete form of which are written as

$$f_a(\tau) = c_1^0 - c_2^0 + (\tau - c_a^0)\ln\left|\frac{\tau - c_1^0}{\tau - c_2^0}\right|, \qquad (a = 1, 2). \tag{20}$$

Substituting (19) once again into (17), we see $v_1^i = v_2^i \equiv v^i$, $w_1^i = w_2^i \equiv w^i$. Furthermore using (14), we see $\boldsymbol{v}^2 = 1$, $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w}^2 = 0$. Since we assume the Euclidean signature for the spacial part of the metric, w^i must vanish. Then we obtain

$$x_a^i(\tau) = c_a^i + (\tau - c_a^0)v^i, \qquad \boldsymbol{v}^2 = 1, \qquad (i = 1, 2, .., D-1;\ a = 1, 2). \tag{21}$$

Thus we see that the relative coordinates $x_1^i - x_2^i$ do not depend on τ, and each particle moves with velocity of light. In other words the bilocal particle is the two end points of a *rigid stick* with arbitrary length, which moves with velocity of light. Since this result is independent of κ, the system does *not* transfered to that of two free particles in the limit $\kappa \to 0$.

In the gauge choice (15), the einbeins are determined by (16) for arbitrary τ. The independent parameters determining the initial condition are $x_a^i(0), (a = 1, 2; i = 1, 2, .., D-1)$ and $v^i, (i = 1, 2, .., D-2)$. The number of them, $3D - 4$, should be the number of the physical degrees of freedom. On the other hand the number of constant parameters corresponding to the τ-independent symmetry of the action, including ϵ_0 as well as Lorenz and translations, which survives after gauge fixing is $\frac{1}{2}(D-1)(D+4) + 1$. Among them the number of freedom which fixes v^i is $\frac{1}{2}(D-1)(D-2)$. The net freedom to move the initial condition counts $3D - 2$. The discrepancy, $(3D-2) - (3D-4) = 2$, suggests existence of extra gauge degrees of freedom in the case $\kappa \neq 0$, which is not explicit in the lagrangian formulation.

3.2 Canonical theory

The canonical conjugate variables corresponding to x_1 and x_2 are $p_1 = \dot{x}_1/g_1 + \kappa x_2$, and $p_2 = \dot{x}_2/g_2 - \kappa x_1$, respectively, while those corresponding to g_a, denote π_a, are subjected to primary constraint $\pi_a \sim 0$. The total hamiltonian is

$$H_T = \frac{1}{2}(g_1\chi_1 + g_2\chi_{-1}) + v_1\pi_1 + v_2\pi_2, \tag{22}$$

where

$$\chi_1 = \frac{1}{2}(p_1 - \kappa x_2)^2, \qquad \chi_{-1} = \frac{1}{2}(p_2 + \kappa x_2)^2, \tag{23}$$

and v_1, v_2 are the Dirac variables which are unphysical.

Preservation of the primary constraints $\pi_a \sim 0$ along time development gives the secondary constraints $\chi_{\pm 1} \sim 0$, while the preservation of them gives tertiary constraints

$$\chi_0 = \frac{1}{2}(p_1 - \kappa x_2)(p_2 + \kappa x_2) \sim 0. \tag{24}$$

There are no other constraints in our model. These constraint functions satisfy the $SL(2,\mathbb{R})$ algebra on account of the Poisson brackets:

$$\{\chi_n, \chi_m\} = -2\kappa(n-m)\chi_{n+m}, \qquad (n, m = 0, \pm 1). \tag{25}$$

The whole first class constraints of our model are $\chi_0 \sim \chi_{\pm 1} \sim \pi_1 \sim \pi_2 \sim 0$. The coefficients of the first class constraints in the hamiltonian are all unphysical variables, and the values of which can be arbitrarily fixed for the sake of the gauge freedom. Two hamiltonians with different coefficients are called gauge equivalent.

According to Dirac (Dirac, 1950)(Dirac, 1964), one may say that two points in the phase space are physically equivalent if there exists another point in the phase space which develops to the two points through equations of motion determined by respective guage equivalent hamiltonians. Transformations from a point in the phase space to the physically equivalent point are called gauge transformation. Dirac conjectured (Dirac, 1950) that all first class constraints generate gauge transformations. (On the validity of Dirac's conjecture it has been argued by some authors, see, *e.g.*, (Sugano & Kamo, 1982), (Frenkel, 1982).)

Now let us examine whether the first class constraints of our model, $\chi_0, \chi_{\pm 1}, \pi_1, \pi_2$, generate the gauge transformations. Consider the transformations of a canonical variable q, generated by the constraint functions $\chi_{\pm 1}, \chi_0$ and π_1, π_2;

$$\delta q = \{q, Q\}, \qquad Q = \sum_{a=0,\pm 1} \epsilon_a \chi_a + \eta_1\pi_1 + \eta_2\pi_2, \tag{26}$$

where transformation parameters $\epsilon, \eta_{1,2}$ are time dependent with $\epsilon_a(0) = \eta_{1,2}(0) = 0$. If time development of q is generated by the total hamiltonian, *i.e.*, $\dot{q} = \{q, H_T\}$, then it turns out, using the Jacobi identity, that $q' = q + \{q, Q\}$ develops as

$$\dot{q}' = \{q, H'_T\}\big|_{q=q'} + O(\epsilon^2), \tag{27}$$

$$H'_T = H_T + \tilde{Q} + \dot{\eta}_1\pi_1 + \dot{\eta}_2\pi_2, \qquad \tilde{Q} = \sum_{a=0,\pm 1} \dot{\epsilon}_a\chi_a + \{Q, H_T\}. \tag{28}$$

We get the point $q'(\tau)$ from the initial point $q(0)$ through the "hamiltonian" H'_T. The point $q(\tau)$ is also developed from the same initial point but through the hamiltonian H_T. Thus if H'_T and H_T are gauge equivalent, then the two points $q'(t)$ and $q(t)$ are physically equivalent in Dirac's sense. But this is *not* the case, since H_T does not contain the tertiary constraint χ_0 but H'_T does. Even if we set $\epsilon_0 = 0$ the situation does not change, so we see that not only χ_0 but $\chi_{\pm 1}$ do not generate gauge transformations. This indicates the breakdown of Dirac's conjecture in our model.

The above fact, however, does not contradict with the gauge invariance in the lagrangian formulation. If we restrict ourselves to the transformation parameters so that $\tilde{Q} = 0$, then we see H_T is gauge equivalent to H'_T. Thus $\chi_0, \chi_{\pm 1}$ generate the gauge transformations with the restricted parameters. These transformations coincide with those of the lagrangian form, (9),(10), as is shown bellow. Rewriting the parameters ϵ_a in (26) as ϵ'_a, the condition $\tilde{Q} = 0$ gives

$$\eta_1 = \dot{\epsilon}'_1 + 2\kappa\epsilon'_0 g_1, \qquad \eta_2 = \dot{\epsilon}'_{-1} - 2\kappa\epsilon'_0 g_1, \qquad \dot{\epsilon}'_0 + 4\kappa(\epsilon'_2 g_1 - \epsilon'_{-1} g_2) = 0. \tag{29}$$

Therefore we have

$$\delta x_1 = \epsilon'_1(p_1 - \kappa x_2) - \frac{1}{2}\epsilon'_0(p_2 + \kappa x_1), \qquad \delta x_2 = \epsilon'_{-1}(p_2 + \kappa x_1) - \frac{1}{2}\epsilon'_0(p_1 - \kappa x_2), \tag{30}$$

$$\delta g_1 = \dot{\epsilon}'_1 + 2\kappa\epsilon'_0 g_1, \qquad \delta g_1 = \dot{\epsilon}'_1 - 2\kappa\epsilon'_0 g_2. \tag{31}$$

Substituting the definition of momenta, $p_1 = \dot{x}_1/g_1 + \kappa x_2$, $p_2 = \dot{x}_2/g_2 - \kappa x_1$, into above equations, we have

$$\delta x_1 = \epsilon'_1 \frac{\dot{x}_1}{g_1} + \frac{1}{2}\epsilon'_0 \frac{\dot{x}_2}{g_2}, \qquad \delta x_2 = \epsilon'_1 \frac{\dot{x}_2}{g_2} + \frac{1}{2}\epsilon'_0 \frac{\dot{x}_1}{g_1}. \tag{32}$$

Finally redefining the parameter as $\epsilon'_1 = g_1\epsilon_1$, $\epsilon'_{-1} = g_2\epsilon_2$, $\epsilon'_0 = 2\epsilon_0$, we get (9),(10). The last condition in (29) is the same as (11) if the redefined parameters are used.

The definition of the physical equivalence in the phase space owing to Dirac and the concept of gauge transformations based on it may be cumbersome at least in the present model. In ref.(Hori, 2009) we proposed another definition of physical equivalence, which seems natural both in the lagrangian and the canonical theories, and in accordance with Dirac's conjecture. The basic observation is that every conserved quantities have the same values along the gauge invariant orbits of canonical variables. Therefore any two physically equivalent points in Dirac's sense have the same values of all conserved quantities. Our claim is that the concept of the physical equivalence should be relaxed so that the reverse proposition holds. That is, we define that if all of the conserved variables at two points in the physical phase space (lied in the constrained subspace) coincide then the two points are called physically equivalent.

In order to determine the conserved quantities in our model let us examine the global symmetries. Since the deviations of the lagrangian under the global translations $\delta x_a = E_a$ is $\delta L = \kappa\frac{d}{d\tau}(x_1 E_2 - x_2 E_1)$, the corresponding conserved charges are

$$\tilde{p}_1^\mu = \frac{\dot{x}_1^\mu}{g_1} + 2\kappa x_2^\mu, \qquad \tilde{p}_2^\mu = \frac{\dot{x}_2^\mu}{g_2} - 2\kappa x_1^\mu, \tag{33}$$

which, in terms of the canonical variables, are written as

$$\tilde{p}_1^\mu = p_1^\mu + \kappa x_2^\mu, \qquad \tilde{p}_2^\mu = p_2^\mu - \kappa x_1^\mu. \tag{34}$$

Similarly the conserved charges corresponding to the Lorenz invariance are

$$M_{\mu\nu} = x_{1[\mu} p_{1\nu]} + x_{2[\mu} p_{2\nu]}. \tag{35}$$

All of the Poisson brackets between the charges $\tilde{p}_a^\mu$, $M_{\mu\nu}$ and the constraints $\chi_0, \chi_{\pm 1}$ vanish. There is another global symmetry of the lagrangian, which is seen by setting ϵ_0 to a constant and $\epsilon_1 = \epsilon_2 = 0$ in (9) and (10). The corresponding conserved charge turnes out to be χ_0 which is vanishing in the physically admissible orbits. In our model the maximal set of conserved quantities are $\tilde{p}_a^\mu, M_{\mu\nu}$ and χ_0. Since these variables are invariant (up to the first class constraints) under the transformations generated by all of the first class constraints, Dirac's conjecture holds.

In the canonical theory the gauge transformations are generated by five constraints $\chi_{0,\pm 1}, \pi_{1,2}$. If one fixes the gauge by five subsidiary conditions, then the equations of motion determine unique solutions. These ten conditions eliminate ten variables among $4(D+1)$ canonical variables x_a, p_a, g_a, π_a, and the remaining $2(2D-3)$ canonical variables, *i.e.*, $2D-3$ canonical pairs become the physical variables.

3.3 Quantization

In this subsection we present the quantum theory, assuming that our model is a constrained hamiltonian system with gauge symmetries generated by $\chi_{0,\pm 1}$. This point of view is consistent with the reduction of the classical degrees of freedom mentioned in **3.1**.

Let us represent the dynamical variables as linear operators on the space of differentiable and square integrable functions of $x_{1,2}$. The momentum observables of the two particles are defined by

$$\tilde{p}_1 = -i\partial_1 + \kappa x_2, \qquad \tilde{p}_2 = -i\partial_2 - \kappa x_1. \tag{36}$$

They satisfy the commutation relation

$$[\tilde{p}_1^\mu, \tilde{p}_2^\nu] = 2\kappa i \eta^{\mu\nu}. \tag{37}$$

That is, the momenta of the two particles do not have simultaneous eigenvalues. This is the reason why we call our system a bilocal particle instead of two particles. This is the fundamental uncertainty relation of the model.

The classical constraint functions are replaced by the following operators:

$$L_1 = \frac{i}{4\kappa}(-i\partial_1 - \kappa x_2)^2, \tag{38}$$

$$L_{-1} = \frac{i}{4\kappa}(-i\partial_2 + \kappa x_1)^2, \tag{39}$$

$$L_0 = \frac{i}{4\kappa}(-i\partial_1 - \kappa x_2)(-i\partial_2 + \kappa x_1) - \alpha, \tag{40}$$

where $\partial_a = \partial/\partial x_a$, and the constant α represents the ambiguity due to the operator ordering. The above operators constitute a basis of $SL(2,\mathbb{R})$ with central term as

$$[L_n, L_m] = (n-m)\left(L_{n+m} + \left(\alpha - \frac{D}{4}\right)\delta_{n+m}\right), \qquad (n, m = 0, \pm 1), \tag{41}$$

where D is the dimension of spacetime.

According to the gauge algebra (41) the BRST charge is defined by

$$Q = \sum_{n=0,\pm 1} c_n L_n - \frac{1}{2}\sum_{n,m=0,\pm 1}(n-m)c_n c_m \frac{\partial}{\partial c_{n+m}}, \tag{42}$$

where $c_a, (a = 0, \pm 1)$ are the BRST ghost variables. The square of the BRS charge is

$$Q^2 = 2\left(\alpha - \frac{D}{4}\right)c_1 c_{-1}. \tag{43}$$

As in the ordinary gauge theory we require the nilpotency of Q so that the ordering ambiguity is fixes as $\alpha = D/4$, which also eliminates the central term in (41).

In ref.(Hori, 1996) we have calculated the BRST cohomology classes in the bilocal model in order to get the physical Hilbert space. We found there that there exists non-trivial physical states only in the dimensions $D = 2$ or $D = 4$. In the case $D = 2$ there exists vector states, while in the case $D = 4$ only scalar states are permitted. However, the analysis is very complicated and it seems difficult to obtain simple scheme for calculations of quantum phenomena.

The reason for the difficulty is in the fact that one can not define such an inner product in the Hilbert space that L_1 is hermitian conjugate to L_{-1}. To obtain physical states represented by functions of spacetime coordinates we are forced to solve the over determined system $L_{\pm 1}|\mathrm{pys}\rangle = L_0|\mathrm{pys}\rangle = 0$, which has no solution.

A field theory, however, has been constructed (Hori, 1993) by using the Chern-Simons action whose exterior derivative is replaced by the BRST operator as Witten has done (Witten, 1986) in a string field theory. But the formulation is formal and a concrete calculation of physical processes has not been achieved due to lack of connections to the first quantized theory.

This situation has been partially overcome by a modification of the model, where two particles in the bilocal model are replaced by the real and the imaginary parts of one complex particle (Hori, 2009). The model is illustrated in the next section.

4. Complex particle

4.1 Action and invariance

The improved version of the $N = 2$ model is defined as follows. Let us consider the spacetime with complex coordinates $z^\mu, (\mu = 0, 1, 2, ..D-1)$, and a particle moving in the spacetime, the complex coordinates of which are functions of the internal time τ. The einvein g is also complex valued function of τ. The proposed action is

$$I_C = \int d\tau\, L_C, \qquad L_C = \frac{\dot{z}^2}{2g} + i\kappa\dot{z}\bar{z} + c.c. \tag{44}$$

The action describes dynamics of two real coordinates corresponding to the real and the imaginary parts of $z = x + ia$. We call the object defined above as *complex particle* (Hori, 2009).

The action is invariant under the transformations

$$\delta z = \epsilon\dot{z} + \frac{\epsilon_0}{\bar{g}}\dot{\bar{z}}, \qquad \delta g = \frac{d}{d\tau}(\epsilon g) + 4i\kappa\epsilon_0 g, \tag{45}$$

where ϵ and ϵ_0 depend on τ. While ϵ has arbitrary complex value, ϵ_0 is real and subjected to the constraint,

$$\dot{\epsilon}_0 - i\kappa g\bar{g}(\epsilon - \bar{\epsilon}) = 0. \tag{46}$$

Classical solutions, the constraint structure and so force are analyzed in the similar way as those of the bilocal model. Thus we recapitulate the results. In the gauge choice

$$g^{-1} = 2\kappa|\tau - \tau_0|, \tag{47}$$

it turns out that in the kinetic terms of the action x(real part) and a(imaginary part) have correct and wrong signs, respectively. Thus x's are physical variables, while a's are ghosts. The solution for $z = x + ia$ to the equations of motion is

$$x(\tau) = x_0 + (\tau - \tau_0)k + \frac{e}{\tau - \tau_0}, \tag{48}$$

$$a(\tau) = a_0 + s(\tau - \tau_0)\left((\tau - \tau_0)k + \frac{e}{\tau - \tau_0}\right), \tag{49}$$

where $s(\tau)$ is the step function, and k and e are D-dimensional light-like vectors with real valued components, which are mutually orthogonal.

The canonical momenta of z and $\bar{z}$ are

$$\begin{aligned} p &= \frac{\dot{z}}{g} + i\kappa\bar{z}, \\ \bar{p} &= \frac{\dot{\bar{z}}}{\bar{g}} - i\kappa z, \end{aligned} \tag{50}$$

while those of g and $\bar{g}$, denoting π and $\bar{\pi}$, respectively, vanish. Note that the momenta (50) are *not* conserved quantities with respect to τ, and the conserved momenta, denoted $\tilde{p}$ and $\tilde{\bar{p}}$, are

$$\begin{aligned} \tilde{p} &= p + i\kappa\bar{z}, \\ \tilde{\bar{p}} &= \bar{p} - i\kappa z, \end{aligned} \tag{51}$$

while the generators of Lorentz transformations defined by $M_{\mu\nu} = z_{[\mu}p_{\nu]} + \bar{z}_{[\mu}\bar{p}_{\nu]}$ are conserved. The deviation of p's and $\tilde{p}$'s (and their c.c.) comes from the fact that under global translations the action is invariant but the lagrangian varies by total derivatives. For an arbitrary variation δz (and $\delta\bar{z}$), the identity

$$\int d\tau \left[[\mathrm{EL}] + \frac{d}{d\tau}\left(\tilde{p}\delta z + \tilde{\bar{p}}\delta\bar{z}\right)\right] = 0 \tag{52}$$

holds, where

$$[\mathrm{EL}] = \left(\frac{\partial L}{\partial z} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{z}}\right)\delta z + \text{c.c.} \tag{53}$$

vanishes if the Euler-Lagrange equations are satisfied. Since in the translations, δz and $\delta\bar{z}$ are constants, $\tilde{p}$ and $\tilde{\bar{p}}$ are conserved. From the invariance under the Lorenz transformations we get $M_{\mu\nu}$ as conserved quantities.

Now the total Hamiltonian generating τ development is

$$H_T = g\chi_1 + \bar{g}\chi_{-1} + v\pi + \bar{v}\bar{\pi}, \tag{54}$$

where

$$\begin{aligned} \chi_1 &= \frac{1}{2}(p - i\kappa\bar{z})^2, \\ \chi_{-1} &= \frac{1}{2}(\bar{p} + i\kappa z)^2, \end{aligned} \tag{55}$$

and v and $\bar{v}$ are the Dirac variables corresponding to the primary constraints, $\pi \sim \bar{\pi} \sim 0$. The preservation of the primary constraints requires the secondary constraints, $\chi_1 \sim \chi_{-1} \sim 0$, and the preservation of them requires the tertiary constraint

$$\chi_0 = \frac{1}{2}(p - i\kappa\bar{z})(\bar{p} + i\kappa z) \sim 0. \tag{56}$$

These constraint functions form a $SL(2,\mathbb{R})$ algebra with regard to Poisson brackets:

$$\{\chi_n, \chi_m\} = -2i\kappa(n-m)\chi_{n+m}, \qquad (n, m = 0, \pm 1), \tag{57}$$

and generate gauge transformations as argued in the bilocal model.

4.2 1st quantization

Now let us proceed to the quantum theory. We represent the canonical variables as operators on the Hilbert space of differentiable and square integrable functions of z and $\bar{z}$. The state vectors are functions in the Hilbert space. The inner product of two states ϕ_1, ϕ_2 is defined by

$$\langle\phi_1|\phi_2\rangle = \int d^D z d^D \bar{z}\, \phi_1^*(z,\bar{z})\phi_2(z,\bar{z}). \tag{58}$$

A dynamical variable q is replaced by the differential operator $-i\partial/\partial q$. The classical constraint functions are replaced by

$$L_1 = \frac{1}{4\kappa}(-i\partial - i\kappa\bar{z})^2, \tag{59}$$

$$L_{-1} = \frac{1}{4\kappa}(-i\bar{\partial} + i\kappa z)^2, \tag{60}$$

$$L_0 = \frac{1}{4\kappa}(-i\bar{\partial} + i\kappa z)(-i\partial - i\kappa\bar{z}) + \alpha, \tag{61}$$

where $\partial = \partial/\partial z$, and the constant α represents the ambiguity due to the operator ordering. $L_{0,\pm1}$ satisfy the algebra,

$$[L_n, L_m] = (n-m)\left(L_{n+m} - \left(\alpha - \frac{D}{4}\right)\delta_{n+m}\right), \qquad (n, m = 0, \pm1). \tag{62}$$

The expression for the BRST operator is the same as eq.(42), and the requirement of the nilpotency of it is guaranteed by $\alpha = D/4$.

In the classical theory the constraints, $\chi_n = 0, (n = 0, \pm1)$, are imposed for guaranteeing the equivalence of the lagrangian and the hamiltonian formulations [2]. These constraints define the physical subspace of whole phase space. In the quantum theory we cannot regard them neither as operator equations nor as the equations to physical states, $L_{0,\pm1}|\text{phys}\rangle = 0$, since they have no solution. Hence the conditions are relaxed so that a product of the constraint operators has vanishing matrix elements between any physical states, $|\varphi\rangle$ and $|\phi\rangle$:

$$\langle\varphi|L_{n_1}\cdots L_{n_N}|\phi\rangle = 0. \tag{63}$$

This is realized by requiring

$$L_1|\phi\rangle = L_0|\phi\rangle = 0, \tag{64}$$

for physical state $|\phi\rangle$, since we have $\langle\phi|L_{-1} = 0$ by virtue of the Hermiticity, $L_1^\dagger = L_{-1}$, the property lacking in the original bilocal model. The above conditions for physical states are analogous to those of string model, and seems most natural ones.

In order that our model is physically meaningful there should exist the eigenstates of momentum. As is shown shortly this requirement gives rise to restriction on the space time dimension. The conserved quantities derived by the invariance under the space time translations are

$$\begin{aligned} \tilde{p} &= -i\partial + i\kappa\bar{z}, \\ \tilde{\bar{p}} &= -i\bar{\partial} - i\kappa z. \end{aligned} \tag{65}$$

Thus the momentum should be combinations of these quantities. From the reality of eigenvalues, it should have the form $P = \beta\tilde{p} + \bar{\beta}\tilde{\bar{p}}$, with arbitrary complex constant β. Any

[2] Strictly speaking, only the primary constraints are involved for the equivalence, and the secondary and tertiary constraints are imposed on the *initial conditions* so that one stays on the subspace defined by the primary constraints in later τ.

two of the eigenstates of P would be taken as independent momentum eigenstates. However, we regard one of them as the physical momentum state, since these two operators are not mutually commuting and have not simultaneous eigenvalues. Any choice of β is physically equivalent because it changes by global rotations. Here we choose the momentum of the real part of z as the physical momentum, which corresponds to $\beta = 1$ (see eq.(52)).

Now let us solve the following equations:

$$L_1|k\rangle = L_0|k\rangle = 0, \tag{66}$$

$$P|k\rangle = k|k\rangle. \tag{67}$$

If one puts

$$|k\rangle = e^{-\kappa z\bar{z}} f(z,\bar{z}), \tag{68}$$

the condition $L_1|k\rangle = 0$ reduces to $\partial\partial f = 0$, *i.e.*, $f(z,\bar{z})$ is an harmonic function with respect to z. The eigenvalue equation $P|k\rangle = k|k\rangle$ reduces to

$$(\partial + \bar{\partial} - 2\kappa\bar{z} - ik) f(z,\bar{z}) = 0. \tag{69}$$

This equation is of the form with separate variables, and has the solution of the form $g_1(z)g_2(\bar{z})$. The solution is written as

$$f(z,\bar{z}) = e^{ik_1 z + i(k-k_1)\bar{z} + \kappa\bar{z}^2}, \tag{70}$$

with arbitrary separation constant k_1. Multiplying arbitrary function $a(k_1)$ to (70), and integrating over k_1, we obtain the general solution to (69) as

$$f(z,\bar{z}) = e^{ik\bar{z} + \kappa\bar{z}^2} g(y), \qquad y = i(\bar{z} - z), \tag{71}$$

where $g(y)$ is an arbitrary differentiable function of real arguments y's, which can be Fourier expanded. Since $f(z,\bar{z})$ is harmonic with respect to z, $g(y)$ must be an harmonic function.

Finally, the condition $L_0|k\rangle = 0$ reduces to

$$(\bar{\partial}\partial - 2\kappa z\partial - 4\kappa\alpha) f(z,\bar{z}) = 0. \tag{72}$$

Substituting (71) into this, we get

$$\left[\left(y^\mu - \frac{k^\mu}{2\kappa} \right) \frac{\partial}{\partial y^\mu} + 2\alpha \right] g(y) = 0. \tag{73}$$

If we put

$$g(y) = \left[\left(y - \frac{k}{2\kappa} \right)^2 \right]^{-\alpha} h\left(y - \frac{k}{2\kappa} \right), \tag{74}$$

eq.(73) and the harmonicity of g are reduced to

$$u^{\mu}\frac{\partial}{\partial u^{\mu}}h(u) = 0, \tag{75}$$

$$\left(\Box_u - \frac{K}{u^2}\right)h(u) = 0, \qquad K = 2\alpha(D-2(\alpha+1)) = \frac{1}{4}D(D-4), \tag{76}$$

where $\Box_u = \partial^2/\partial u^{\mu}\partial u_{\mu}$ and u's are D-dimensional real coordinates. These equations are solved by the pseudo-harmonic analysis in D-dimensions. Transforming to the pseudo-polar coordinates [3], $(r, \theta_1, \theta_2, ..., \theta_{D-1})$, we see that from eq.(75) h does not depend on r. Since the d'Alembertian $\Box_u$ is written as $r\partial/\partial r + (1/u^2)\Delta$, where Δ is the Laplace-Beltrami operator on $S^{1,D-2}$, we see from (76) that $\Delta h = Kh$. It is well known in the theory of spherical functions (Takeuchi, 1975) that if the Laplace-Beltrami operator on S^{D-1} have single valued bounded eigenfunctions, then the eigenvalues must be of the form $K = -\ell(\ell + D - 2)$ with non negative integer ℓ. Hence if we take the Eucledian signature for the metric we get $D = 4 - 2\ell$, *i.e.*, $D = 2$ or $D = 4$. In the present case, however, the signature of the metric is Minkowskian, and the base space is $S^{1,D-2}$ which is non-compact. The theory of pseudo-spherical functions on non-compact space (Raczka et al., 1966) (Limi'c et al., 1966) (Limi'c et al., 1967) (Strichartz, 1973) shows variety of series of eigenvalues, including continuous as well as discrete ones. An explicit form of the eigenfunctions are recently obtained for $D = 3$ (Kowalski et al., 2011). The real eigenvalues of single valued eigenfunctions on the non-compact base space are of the same form as those of the compact space except some supplementary continuous series. Here we restrict ourselves to the former cases.

We have assumed here that the eigenfunctions are single valued. If one permits double valued eigenfunctions a half integer value of K should be taken into account. The double values might come from rotations around y^0 axis. Since physical meaning of the rotations around the time axis is not clear, we simply do not consider the effects.

The eigenfunctions are expressed by Gegenbauer's polynomials for general ℓ, but are constants for $\ell = 0$. In the case $D = 2$, eqs.(75) and (76) are directly solved [4], and we get $h(u) = (u^0 \pm u^1)(u^2)^{-1/2}$.

The physical eigenstates of the momentum in four and two dimensions is written as

$$|k\rangle \propto \frac{e^{ik\bar{z}+\kappa\bar{z}(\bar{z}-z)}}{\left(z - \bar{z} - \frac{ik}{2\kappa}\right)^2} \times \begin{cases} 1 & (\text{for } D = 4) \\ z^0 - \bar{z}^0 - \frac{ik^0}{2\kappa} \pm \left(z^1 - \bar{z}^1 - \frac{ik^1}{2\kappa}\right) & (\text{for } D = 2). \end{cases} \tag{77}$$

There are *spurious* states defined by $L_{-1}^n|k\rangle, (n = 1, 2, ..)$, which are orthogonal to all physical states and have zero norm. In the string theory there are many spurious states which are physical and have zero norm, especially in the critical dimension. Existence of these states in the string theory suggests some underlying gauge invariance, since they must be decoupled from physical S-matrix. In the present model, however, spurious states are all unphysical by

[3] According to the metric $\eta_{\mu\nu} = (-1, 1, 1, .., 1)$, the pseudo-polar coordinates are defined by $y^0 = r\sinh\theta_1$, $y^1 = r\cosh\theta_1 \sin\theta_2 \cdots \sin\theta_{D-1}$, $y^2 = r\cosh\theta_1 \sin\theta_2 \cdots \cos\theta_{D-1}$, ..., $y^{D-1} = r\cosh\theta_1\cos\theta_2$.

[4] The solution $D = 2$ was overlooked in ref.(Hori, 2009).

virtue of the constraint algebra without central term, and do not enter in physical S-matrix from the outset.

4.3 Toward a field theory

A field theory based on the complex particle might have some gauge symmetries in the field theoretic sense, which may have some connections with the $SL(2,\mathbb{R})$ in the first quantized theory. The most likely candidate for the action of the field theory may be the Chern-Simons form written, for example, as (Hori, 1993)

$$\mathcal{I} = \int d^3c d^D z d^D \bar{z}\, V(z) \left(A_i \star QA^i - \frac{g}{3}\, \epsilon_{ijk} A^i \star A^j \star A^k \right), \tag{78}$$

where Q is the BRST charge and $\star$ is some associative binary operator like a convolution. $V(z)$ is a possible measure factor. The fields $A_i, (i = 1,2,3)$ are fermionic and may be written as $A^i = \sum_n c_n \Psi^i_n$ with ghost variables c_n.

The nomenclature of "Chern-Siomons" comes from the Chern-Simons gauge theory on three-manifold, which has been investigated in connection with knot theory. The formal resemblance of our model to the C-S gauge theory is that the wedge product corresponds to the operator $\star$, which we call star product, and the exterior derivatives correspond to Q, which are both nilpotent. The star product satisfies

$$A \star B(x) = (-1)^{F(A)F(B)} B \star A(x), \tag{79}$$

where $F(A) = 1$ for fermionic A and $F(A) = 0$ otherwise.

Now the action is invariant under the *gauge* transformations:

$$\delta A^i = Q\Lambda^i + g\epsilon^{ijk} \Lambda_j \star A_k, \tag{80}$$

where Λ_i is arbitrary bosonic parameters depending on z's and c's. A necessary condition for the invariance is the Leibniz rule for Q expressed as

$$Q(A \star B) = Q(A) \star B + (-1)^{F(A)F(B)} A \star Q(B), \tag{81}$$

for arbitrary fields A and B. Then the action is invariant if the integral of *total derivative* vanishes:

$$\int d^3c d^D z d^D \bar{z}\, V(z)\, QA = 0. \tag{82}$$

Expanding the fields Ψ^i_n in powers of the imaginary parts of z, the coefficients may represent physical fields. After integrations over the imaginary parts of z's and ghost variables, the action is expressed as integral of these fields over the real parts of z's, which has some gauge invariance.

Since there is no guideline for defining the star product apart from the condition (81), let us examine the Leibniz rule in the following simple representation. Consider the representation of $sl(2,\mathbb{R})$, on the space of functions of a single variable x, defined by

$$\lambda_{-1} = x - x_0, \qquad \lambda_0 = (x - x_0)\frac{d}{dx} + a, \qquad \lambda_1 = (x - x_0)\frac{d^2}{dx^2} + 2a\frac{d}{dx}\,, \tag{83}$$

where x_0 is a constant, and a is a constant which appears due to the ordering ambiguity. λ's satisfy the algebra,

$$[\lambda_n, \lambda_m] = (n - m)\lambda_{n+m}, \qquad (n, m = 0, \pm 1). \tag{84}$$

The natural choice for the (wedge) product of two functions, which permits the Leibniz rule, may be the convolution defined by

$$A \wedge B(x) = \int_{x_0}^{x} dx'\; A(x + x_0 - x')B(x'). \tag{85}$$

The limits in the integration in the definition of the product is so chosen as it is (anti-)commuting:

$$A \wedge B(x) = (-1)^{F(A)F(B)} B \wedge A(x). \tag{86}$$

Also we see from (85),

$$A \wedge B(x_0) = 0. \tag{87}$$

This suggests that the representation space, S, should be restricted to the functions which vanish at $z = z_0$:

$$S = \{\, A \mid A \in C^2, A(x_0) = 0\}. \tag{88}$$

Now let us examine the Leibniz rule for the *exterior derivative* defined by

$$d = \sum_{n=0,\pm 1} c_n \lambda_n - \frac{1}{2} \sum_{n,m=0,\pm 1} (n - m) c_n c_m \frac{\partial}{\partial c_{n+m}}. \tag{89}$$

As in the ordinary exterior derivative, d is nilpotent. After straightforward calculations we obtain

$$\begin{aligned} d(A \wedge B) - (dA \wedge B + A \wedge dB) = c_1 \left[2(1 - a)A \wedge B' + (2a - 1)A_0 B + AB_0\right] \\ +(1 - a)c_0 A \wedge B, \end{aligned} \tag{90}$$

where $A_0 = A(x_0)$, $B_0 = B(x_0)$, $B' = dB/dx$. Thus we find that if and only if $a = 1$ *and* $A, B \in S$ then d behaves like a derivative operator.

Next let us examine the eigenstate expansions. The basis functions $u_k = (z - z_0)^k, (k = 0, 1, 2, ..)$ satisfy

$$\lambda_0 u_0 = a u_0, \qquad \lambda_1 u_0 = 0, \qquad u_k = \lambda_{-1}^k u_0. \qquad (k = 1, 2, ..) \tag{91}$$

A function in S is expanded as

$$A(x) = \sum_{k=1}^{\infty} \frac{u_k}{k!} A_k. \tag{92}$$

Note that $u_0 = 1$, representing the '*ground state*', does *not* belong to S. The wedge product of A and B in S is expanded as

$$A \wedge B(x) = \sum_{k=1}^{\infty} \frac{u_k}{k!} \sum_{m=0}^{k-1} A_{k-m-1} B_m. \tag{93}$$

The summation in (93) over m is, in fact, carried out from 1 to $k-2$ due to $A_0 = B_0 = 0$. The k-th component of the wedge product is thus

$$(A \wedge B)_k = \sum_{m=1}^{k-2} A_{k-m-1} B_m. \tag{94}$$

Finally, let examine whether an integration of dA vanishes for any A. Since the ghost derivative parts in dA are of the form $c_1 c_{-1} \partial/\partial c_0$, $c_0(c_1 \partial/\partial c_1 - c_{-1}\partial/\partial c_{-1})$, they vanish after integrations by parts. Thus it is sufficient to check only that $\int dx\ V(x)\lambda_n A = 0, (n = 0, \pm 1)$, with some measure factor V. This leads to $V(x) = \delta(x - x_0)$ and $a = 0$. Therefore it is impossible in the present representations to satisfy all the requirements.

Now go back to the complex particle model. Let us define the basis functions v_k as follows:

$$L_0 v_0 = \alpha v_0, \qquad L_1 v_0 = 0, \qquad v_k = L_{-1}^k v_0. \qquad (k = 1, 2, ..) \tag{95}$$

where

$$v_0 = e^{(ip + \kappa(\bar{z} - z))\bar{z}}, \qquad \alpha = \frac{D}{4}. \tag{96}$$

v_0 is the eigenstate of the momentum with eigenvalue p, but not a physical state, since $L_0 v_0 \neq 0$. The basis $v_k(p, z, \bar{z}), (k = 0, 1, 2, ...)$ may span a dense subset of functions which are differentiable and square integrable. We consider fields which are expanded as

$$A = \sum_{k=0}^{\infty} \frac{v_k}{k!} A_k, \tag{97}$$

where A_k are functions of the ghost variables and not depend on z's . The each component of a field A is denoted as A_k. In analogy with (94) let us define the star product as

$$(A \star B)_k = \sum_{m=0}^{k+\delta} \left(A_{k-m+\beta} B_{m+\gamma} + A_{m+\gamma} B_{k-m+\beta} \right), \tag{98}$$

where integer constants, β, γ and δ, are introduced so that the Leibniz rule might be satisfied. The star product satisfies the (anti-)symmetry, $A \star B = (-1)^{F(A)F(B)} B \star A$, and the associativity, $(A \star B) \star C = A \star (B \star C)$.

The Leibniz rule can be examined merely using the commutation relations (62) with $\alpha = D/4$, and it is sufficient to check for the first term in the BRST charge, since the second terms are in the form of derivatives. For an operator O writing as

$$\mathrm{Leib}[O; A, B] = (OA) \star B + A \star (OB) - O(A \star B), \tag{99}$$

we get

$$\mathrm{Leib}[L_0; A, B]_k = (\beta+\gamma+\alpha) \sum_{m=0}^{k+\delta} (A_{k-m+\beta} B_{m+\gamma} + A_{k-m+\gamma} B_{m+\beta}), \tag{100}$$

$$\begin{aligned} \mathrm{Leib}[L_{-1}; A, B]_k &= (\beta+\gamma+1) \sum_{m=0}^{k+\delta} (A_{k-m+\beta} B_{m+\gamma-1} + A_{m+\gamma-1} B_{k-m+\beta}) \\ &\quad -(1+\beta)(A_{k+\beta} B_{\gamma-1} + A_{\gamma-1} B_{k+\beta}), \end{aligned} \tag{101}$$

$$\begin{aligned} \mathrm{Leib}[L_1; A, B]_k &= (\beta+\gamma+2\alpha-1) \sum_{m=0}^{k+\delta} (A_{k-m+\beta} B_{m+\gamma+1} + A_{m+\gamma+1} B_{k-m+\beta}) \\ &\quad +(A_{k+\beta+1} B_\gamma + A_\gamma B_{k+\beta+1}) \\ &\quad -(\beta+2\alpha-1-\delta)(A_{-\alpha+\beta} B_{k+\alpha+\gamma+1} + A_{k+\alpha+\gamma+1} B_{-\alpha+\beta}). \end{aligned} \tag{102}$$

The bulk parts (the summations) of these quantities vanish if we put

$$\beta+\gamma+\alpha = 0, \tag{103}$$

$$\beta+\gamma+1 = 0, \tag{104}$$

$$\beta+\gamma+2\alpha-1 = 0, \tag{105}$$

which are equivalent to

$$(D/4=)\alpha = 1, \qquad \beta+\gamma = -1. \tag{106}$$

The marginal parts (single terms) vanish if

$$\beta = -1, \qquad \gamma = 0, \qquad \delta = 0, \tag{107}$$

and $A_0 = B_0 = 0$. Thus we see that the Leibniz rule for the BRST charge is valid only if $D = 4$ and restricting the function space to

$$\mathcal{S} = \{A | A_0 = 0\}. \tag{108}$$

The star product should be

$$(A \star B)_k = \sum_{m=1}^{k} (A_{k-m-1} B_m + A_m B_{k-m-1}). \tag{109}$$

Note $(A \star B)_k = 0$ for $k \leq 2$. The reason for restriction to $D = 4$ seems a technical one in building a field theory, while the restriction to $D = 2$ or $D = 4$ in the first quantized theory is intrinsic in the model.

Finally let us examine the vanishing of the integral of the *total derivatives* of the form QA. As in the simple representation (83), it is sufficient to check $\int VL_n A = 0, (n = 0, \pm 1)$. Integrating by parts, the conditions become $K_n V = 0,\ (n = 0, \pm 1)$, where

$$K_1 = \frac{1}{4\kappa}(i\partial - i\kappa\bar{z})^2, \tag{110}$$

$$K_{-1} = \frac{1}{4\kappa}(i\bar{\partial} + i\kappa z)^2, \tag{111}$$

$$K_0 = \frac{1}{4\kappa}(i\partial - i\kappa\bar{z})(i\bar{\partial} + i\kappa z) + \alpha, \qquad \alpha = \frac{D}{4}. \tag{112}$$

Now let us find the explicit form of $V(z, \bar{z})$. Putting

$$V(z, \bar{z}) = e^{i\kappa\bar{z}y} G(y), \qquad y = i(\bar{z} - z), \tag{113}$$

we find after straightforward calculations that the conditions are

$$K_1 V(z) = -\frac{1}{4\kappa} e^{i\kappa\bar{z}y} \Box_y G(y) = 0, \tag{114}$$

$$K_{-1} V(z) = e^{i\kappa\bar{z}y}\left(y\partial_y + \frac{1}{2}(y^2 + D)\right) G(y) = 0, \tag{115}$$

$$K_0 V(z) = -\frac{1}{2} e^{i\kappa\bar{z}y}\left(y\partial_y + D - 2\alpha\right) G(y) = 0. \tag{116}$$

From the last two equations we see $y^2 G(y) = 0$, so we find $G(y) \propto (y^2)$. Thus the solution must be

$$V(z, \bar{z}) = e^{i\kappa\bar{z}y}\delta(y^2). \tag{117}$$

Substituting this back into (114)-(116), we get

$$K_1 V(z) = -\frac{1}{2\kappa} e^{i\kappa\bar{z}y}\left[(D-4)\delta'(y^2) + 2(2\delta'(y^2) + y^2\delta''(y^2))\right] \tag{118}$$

$$K_{-1} V(z) = \frac{1}{2} e^{i\kappa\bar{z}y}\left[(D-4)\delta(y^2) + y^2\delta(y^2) + 4(\delta(y^2) + y^2\delta'(y^2))\right] \tag{119}$$

$$K_0 V(z) = -\frac{1}{4} e^{i\kappa\bar{z}y}\left[(D-2-2\alpha)\delta(y^2) + 4(\delta(y^2) + y^2\delta'(y^2))\right]. \tag{120}$$

From the identity $x\delta(x) = 0$, we see $\delta(x) + x\delta'(x) = 0$ and $2\delta'(x) + x\delta''(x) = 0$. Hence we see that $K_n V = 0, (n = 0, \pm 1)$ if and only if $D = 4$. Once again $D = 4$ makes us happy!

The action of the field theory is an integral over z and z, where the imaginary part of z's are restricted on the light-cone.

5. Extension to $N \geq 3$

5.1 Actions and gauge invariance

The action of the N-extended multi-local particle is defined by

$$I_N = \int d\tau\, L_N, \qquad L_N = \sum_{a=1}^{N} \frac{1}{2g_a}\dot{x}_a^2 + \sum_{a,b} \kappa_{ab}\dot{x}_a x_b, \tag{121}$$

where x_a are the (real) coordinates of the N particles and each g_a is the einbein of the a-th world line which is parametrized by τ, and dots denote the derivatives with respect to internal time τ. The difference of this N particle system from the ordinary free particles comes from the second term in eq.(121), where κ_{ab} is an arbitrary anti-symmetric constant matrix.

The action has the hidden local symmetry generated by

$$\delta x_a = \epsilon_a \dot{x}_a + \sum_b s_{ab} \frac{\dot{x}_b}{g_b}, \tag{122}$$

$$\delta g_a = \frac{d}{d\tau}(\epsilon_a g_a) + k_a g_a, \tag{123}$$

where ϵ_a, s_{ab} and k_a are infinitesimal local parameters constrained by $s_{ab} = s_{ba}$ and

$$\dot{s}_{ab} + 2\kappa_{ab} g_a g_b (\epsilon_b - \epsilon_a) + 2\sum_c (\kappa_{ac} g_a s_{cb} + \kappa_{bc} g_b s_{ca}) = g_a k_a \delta_{ab}, \tag{124}$$

In fact the variation of the lagrangian is

$$\begin{aligned}\delta L_N = &\frac{d}{d\tau}\left[\sum_a \frac{\epsilon_a \dot{x}_a^2}{2g_a} + \sum_{a,b}(\kappa_{ab}\epsilon_a x_b \dot{x}_a + \frac{s_{ab}\dot{x}_a\dot{x}_b}{2g_a g_b}) + \sum_{a,b,c} \frac{\kappa_{ab} s_{ac} x_b \dot{x}_c}{g_c}\right] \\ &+\frac{1}{2}\sum_{ab}(s_{ab} - s_{ba})\frac{\dot{x}_a}{g_a}\frac{d}{d\tau}\left(\frac{\dot{x}_b}{g_b}\right) \\ &+\sum_{a,b}\left[\frac{1}{2}\dot{s}_{ab} + 2\kappa_{ab} g_a g_b \epsilon_b + g_a\left(2\sum_c \kappa_{ac} s_{cb} - \frac{1}{2}k_a \delta_{ab}\right)\right]\frac{\dot{x}_a \dot{x}_b}{g_a g_b}.\end{aligned} \tag{125}$$

In order to fix the model we set the non-vanishing components of the anti-symmetric parameter κ_{ab} as in the following two cases:

(i) Closed N-particle ($N \geq 3$):

$$\kappa_{aa+1} = \kappa, \quad (a = 1, ..., N-1), \qquad \kappa_{N1} = -\kappa, \qquad \kappa_{ab} = -\kappa_{ba}, \tag{126}$$

(ii) Open N-particle ($N \geq 2$):

$$\kappa_{aa+1} = \kappa, \quad (a = 1, ..., N-1), \qquad \kappa_{ab} = -\kappa_{ba}, \tag{127}$$

and other κ's are set to zero, where κ is the coupling constant. The closed N-particle system is characterized by the anti-symmetric matrix κ_{ab}, each row (or column) of which has two non vanishing elements, while in the open N-particle system this is valid except for the first (or N-th) row (or column) corresponding to the two ends of the N particles. The bilocal particle is the open 2-particle. In what follows we restrict ourselves to the open N-particle, since the constraint structures in the canonical theories of the closed N-particle are rather complicated compared with the open ones.

Now the number of the gauge degrees of freedom can be counted in the similar way as in the bilocal particle, where the degrees of freedom of the initial condition are counted in suitable gauge condition. But this procedure is rather cumbersome in the lagrangian formalism compared with the hamiltonian one.

However, a shortcut derivation of the physical degrees of freedom in the lagrangian formalism is possible. The result coincides precisely with the hamiltonian one, if Dirac's conjecture holds. The reasoning is as follows. The number of the unphysical, *i.e.*, the gauge degrees of freedom is the number of the independent parameters and their time derivatives appeared in the transformation rules, where a parameter and all of its (higher order) time derivative(s) are formally regarded as *independent*. (For a skeptical reader we recommend to check the above rule in the case of the Yang-Mills or the local Lorentz symmetries.)

The counting argument in the open N-particle system is as follows. The independent transformation parameters are extracted by solving the constraint, eq.(124). For $a = b$ in eq.(124), we obtain k_a in terms of $\dot{s}_{ab}$ and s_{ab}. For $b = a + 1$, we see $\epsilon_a(a = 2, ..., N)$ are expressed in terms of ϵ_1, $\dot{s}_{aa+1}$, $(a = 1, ..., N-1)$ and s_{ab}. Next, for $b \geq a + 2$, we see $\dot{s}_{ab}$ are expressed by s_{ab}. Thus we have the independent parameters, $s_{ab}(\frac{N(N+1)}{2}), \dot{s}_{aa}(N), \dot{s}_{aa+1}(N - 1)$ and $\epsilon_1(1)$, where the numbers of each independent parameter are written in the parentheses.

Substituting the above parameters into eqs.(122) and (123), we get the extra independent parameters, $\ddot{s}_{aa+1}(N-1)$ and $\dot{\epsilon}_1(1)$. Thus we have the total of $\frac{1}{2}N(N+1) + 3N$ independent parameters in eqs.(122) and (123). However, a short manipulation shows that s_{aa} and $\dot{s}_{aa}$ actually do not appear or be absorbed into ϵ_a by shifting $\epsilon_a \to \epsilon_a + \frac{s_{aa}}{g_a}$. Hence, finally, we see the number of the gauge degrees of freedom is $\frac{1}{2}N(N+1) + N$. Among them N degrees of freedom are used for fixing g_a, and the remaining $\frac{1}{2}N(N+1)$ are of our interest. $\frac{1}{2}N(N+1)$ constraints and the same number of gauge fixing conditions eliminate a part of the canonical variables, $x_a, p_a, (a = 1, 2, .., N)$, leaving $\frac{1}{2}N(2D - N - 1)$ canonical pairs as physical. Hence if $N \leq 2(D-1)$ there are at least one physical degrees of freedom.

In the next subsection we show that the number of the first class constraints in the hamiltonian formalism coincides precisely with the above number. This is in accordance with Dirac's conjecture, *i.e.*, all of the first class constraints generate the gauge symmetry of the system.

5.2 Canonical theory

The algebraic structure of the symmetry is clarified in the canonical formalism. Introducing the momenta p_a and π_a conjugate to x_a and g_a, respectively, and defining

$$V_{ab} = \frac{1}{2} p_a^{(-)} p_b^{(-)}, \tag{128}$$

$$p_a^{(-)} = p_a - \sum_b \kappa_{ab} x_b, \tag{129}$$

we can express the total hamiltonian as

$$H_T^{(N)} = \sum_a g_a V_{aa} + \sum_a \Lambda_a \pi_a, \tag{130}$$

where Λ's are Dirac variables which can be set to arbitrary functions of canonical variables. The Poisson brackets of V's are given by

$$\{V_{ab}, V_{cd}\} = \kappa_{c(a}V_{b)d} + \kappa_{d(a}V_{b)c}. \tag{131}$$

Now let us derive the constraints for the canonical variables in the open N-particle system. The primary constraints are $\pi_a \sim 0$, since the lagrangian does not contain $\dot{g}$'s. The stability of the primary constraints along the time development requires the secondary constraints $V_{aa} \sim 0$. The stability of the latter, in turn, requires $V_{aa+1} \sim 0, (a = 1, ..., N-1)$. In general, the stability of $V_{aa+k} \sim 0$ requires $V_{aa+k+1} \sim 0$. After all we have $\frac{1}{2}N(N+1)$ secondary and tertiary constraints, V_{ab}, which close under the Poison brackets, and form the first class constraints.

V_{ab} generate the gauge symmetry which has the form of eqs.(122) and (123) in the lagrangian formalism, and transform the hamiltonian, eq.(130), into the same form but with different coefficients of V_{aa}. This ambiguity of the coefficients is a reflection of the gauge invariance and is removed by the gauge fixing.

5.3 Quantization

In order to quantize the system we replace p_a^μ's by $-i\partial_{\mu a}$, and we denote the quantum operators obtained by this replacement by writing hats on these quantities. The generators of the gauge transformations are defined by

$$\hat{V}_{ab} = \frac{1}{4}\left(\hat{p}_a^{(-)}\hat{p}_b^{(-)} + \hat{p}_b^{(-)}\hat{p}_a^{(-)}\right). \tag{132}$$

The gauge algebra is expressed as

$$[\hat{V}_{ab}, \hat{V}_{cd}] = i\kappa_{c(a}\hat{V}_{b)d} + i\kappa_{d(a}\hat{V}_{b)c}. \tag{133}$$

The ambiguities from the operator ordering are fixed by requiring the nilpotency of the BRST operator as in the $N = 2$ theories, then the central terms in the gauge algebra also vanish.

The generators for the kinematic symmetry are as follows:

$$\text{translations}: \qquad \hat{p}_a^{(+)} = \hat{p}_a + \sum_b \kappa_{ab}x_b, \quad (a = 1, ..., N) \tag{134}$$

$$\text{Lorentz tfm.}: \qquad M_{\mu\nu} = \sum_{a=1}^{N} \hat{p}_{a[\mu}x_{a\nu]} \tag{135}$$

where $\hat{p}_{a\mu} \equiv -i\partial_{a\mu}$. $\hat{p}_{+a}^{(+)}$ generate the translation of a-th particle. They form the following algebra with a central term:

$$[M_{\mu\nu}, M_{\lambda\rho}] = i\eta_{\rho[\mu}M_{\nu]\lambda} - i\eta_{\lambda[\mu}M_{\nu]\rho}, \tag{136}$$

$$[M_{\mu\nu}, \hat{p}_{a\lambda}^{(+)}] = -i\eta_{\lambda[\mu}\hat{p}_{\nu]a}^{(+)}, \tag{137}$$

$$[\hat{p}_{a\mu}^{(+)}, \hat{p}_{b\nu}^{(+)}] = 2i\kappa_{ab}. \tag{138}$$

The algebra defined above contains the Poincaré algebra as a subalgebra. The crucial point is the uncertainty relation (138). The momentum of each particle does not have a certain value irrespective to the momentum of the neighboring particles. Another important feature is the commutativity of the kinematic generators and those of gauge generators:

$$[\hat{V}_{ab}, \hat{p}^{(+)}_{\mu a}] = [\hat{V}_{ab}, M_{\mu\nu}] = 0. \tag{139}$$

These relations assure the consistency of the gauge structure and the kinematic properties of the model.

The first quantizations of the N-extended models may be achieved in the similar way as that of the bilocal models. Chern-Simons type actions may be used in field theories. It is interesting to know whether the critical dimensions exist also in the N-extended models. However, there might be similar difficulties as in the bilocal model, and they may be overcome by improving them to those like complex particle as is done in the bilocal model. We leave these problems to future studies.

6. Summary

In the present paper we have analyzed the multi-local particle models especially emphasizing on the complex particle. At first sight the guage degrees of freedom of the multi-local particle are less than those of the canonical theory, which may lead to breakdown of Dirac's conjecture. The concept of physical equivalence is argued to be modified so that the guage transformations are extend to whole algebra, recovering Dirac's conjecture.

The constraint structure of the model of the complex particle is suited for the ordinary quantization scheme as opposed to the original bilocal model, due to the Hermiticity property of $L_{\pm 1}$. In the first quantization we see that, requiring the existence of the momentum eigenstates which satisfy the physical state conditions, the dimension of the spacetime is restricted to be two or four. The most natural action of the field theory might be of the form of Chern-Simons one, where the exterior derivative is replaced by the BRST charge. It is rather unexpected that the action has gauge invariance only in the four dimensions. This fact is caused by the Leibniz rule of the BRST charge and the vanishing of the total derivative, *i.e.*, $\int QA = 0$, which are satisfied only in four dimensions.

Although the complex particle model is favorable in many respects than the original bilocal model, the latter is more intuitive in that the classical solution is interpreted as a rigid stick. As far as we know the bilocal model is the first example of relativistically admissible rigid stick.

We extend the bilocal model to $N \geq 3$ particle system, and obtain large classes of actions. The larger the guage algebra, the less physical degrees of freedom. The models categorized into two classes, *i.e.*, open and closed types. In open N-particle system it turns out that the number of the constraints and the corresponding gauge symmetry is $\frac{1}{2}N(N+1)$. Consequently the physical degrees of freedom survives only if $N \leq 2(D-1)$.

The models proposed here have not been aimed so far to phenomenological applications but to the analysis in their theoretical aspects such as the gauge invariance or critical dimensions.

Of course we do not intend to claim that the present model is the theory of the nature and for that reason the dimension of our spacetime should be four. However, it is interesting that there exist simple models other than the string, which have critical dimensions. We hope that the future investigations along with the direction described here may open a new perspective in the area of quantum gravity where some non-locality of a fundamental object should play the central role in the Planck scale.

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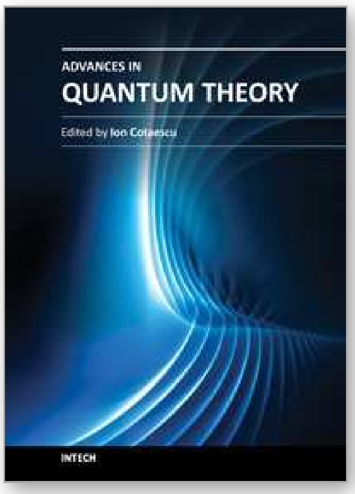

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