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Quantum Gravity Insights from Smooth 4-Geometries on Trivial $\mathbb{R}^4$

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1. Introduction

The search for the theory of quantum gravity (QG) in 4-dimensions (4D) is one of the most significant challenges of temporary physics. The great effort and insights of many theoreticians and experimentalists resulted in the emergence of one of the greatest achievements of 20th century science, i.e. standard model of particles and fields (SM). SM (with its minimal extensions by massive neutrinos and after renormalization) describes and predicts, with enormous accuracy, almost all perturbative aspects and behaviour of interacting quantum fields and particles which place themselves in the realm of electromagnetic, strong and weak nuclear interactions, within the range of energies up to few TeV. However, gravity at quantum level is not covered by this pattern. The oldest, semiclassical, approach to QG relies on the quantization of metric field which is understood as the perturbation of the ground spacetime metric. This is exactly in the spirit of quantum field theory (QFT) as in SM. There should follow various correlation functions of physical processes where gravity at quantum level is present. There should, but actually they do not since the expressions are divergent and the theory is not renormalizable. Even the presence of supersymmetry does not change this substantially. On the other hand, we have a wonderful theory of general relativity (GR) which, however, is a theory of classical gravity and it prevents its quantization in 4D.

Among existing approaches to QG, superstring theory is probably the most advanced and conservative one. It attempts to follow GR and quantum mechanics as much as possible. However, superstring theory has to be formulated in 10 spacetime dimensions and on fixed, not dynamical, background. Many proposals how to reach the observed physics from 10D superstrings were worked out within the years. These are among others, compactiffication, flux stabilization, brane configuration model-buildings, brane worlds, holography or anti-de-Sitter/conformal field theory duality, i.e. AdS/CFT. There exists much ambiguity, however, with determining 4D results by these methods. Some authors estimate that there exist something about 10^{500} different backgrounds of superstring theory which all could be "good" candidates expressing 4D physics. This means that similar variety of possible models for true physics is predicted by superstring theory. To manage with such huge amount of "good" solutions, there was proposed to use the methods of statistical analysis to such *landscape* of possible backgrounds. Anyway, one could expect better prediction power from the fundamental theory which would unify gravity with other interactions at quantum level. On the other hand, superstring theory presents beautiful, fascinating and extremly rich mathematics which is still not fully comprehended.

Therefore Asselmeyer-Maluga & Król (2010) have proposed recently how to find connections of superstring theory with dimension 4 in a new way not relying on the standard techniques. Whole the approach derives from mathematics, especially low dimensional differential topology and geometry. In that approach one considers superstring theory in 10D and supersymmetry as "merely" mathematics describing directly, at least in a variety of important cases, the special smooth geometry on Euclidean, topologically trivial, manifold $\mathbb{R}^4$. These are various non-diffeomorphic different smooth structures. Smooth manifold $\mathbb{R}^4$ with such non-standard smoothness is called exotic $\mathbb{R}^4$ and as a smooth Riemannian manifold allows for a variety of metrics. This exotic geometry in turn, is regarded as underlying smoothness for 4-spacetime and is directly related to physics in dimension 4.

The way towards crystallizing such point of view on string theory was laborious and required many important steps. The breakthrough findings in differential topology, from the eighties of the previous century, showed that indeed there are different from the standard one, smoothings of the simplest Euclidean 4-space (see e.g. Asselmeyer-Maluga & Brans (2007)). Spacetime models usually are based on 4D smooth manifolds, hence they are locally described with respect to the standard smooth $\mathbb{R}^4$. Anything what happens to this fundamental building block might be important at least to classical physics formulated on such spacetime. Indeed, it was conjectured by Brans (1994a;b), and then proved by Asselmeyer (1996) and Sładkowski (2001), that exotic smooth $\mathbb{R}^4$'s can act as sources for the external gravitational field in spacetime. Even mathematics alone, strongly distinguishes these smooth open 4-manifolds: among all $\mathbb{R}^n$ only the case $n = 4$ allows for different smoothings of Euclidean $\mathbb{R}^n$. For any other $\mathbb{R}^n$, $n \neq 4$ there exists unique smooth structure. Moreover, there exists infinitely continuum many different smoothings for $\mathbb{R}^4$. However, mathematical tools suitable for the direct description of, say, metrics or functions on exotic $\mathbb{R}^4$ are mostly unknown (see however Asselmeyer-Maluga & Brans (2011)). The main obstruction which prevents progress in our understanding of exotic smoothness on $\mathbb{R}^4$ is that there is no known effective coordinate presentation. As the result, no exotic smooth function on any such $\mathbb{R}^4$ is known, even though there exist infinite continuum many different exotic $\mathbb{R}^4$. Such functions are smooth in the exotic smoothness structure, but fail to be differentiable in a standard way determined by the topological product of axes. This is also a strong limitation for the applicability of the structures to physics. Let us note that smooth structures on open 4-manifolds, like on $\mathbb{R}^4$, are of special character and require special mathematics which, in general, is not completely understood now. The case of *compact* 4-manifolds and their smooth structures is much better recognized also from the point of view of physics (see e.g. Asselmeyer-Maluga (2010); Asselmeyer-Maluga & Brans (2007); Witten (1985)). The famous exception is, however, not resolved yet, negation of the 4D Poincaré conjecture stating that there exists exotic S^4.

Bižaca (1994) constructed an infinite coordinate patch presentation by using Casson handles. Still, it seems hopeless to extract physical information from that. The proposition by Król (2004a;b; 2005) indicated that one should use methods of set theory, model theory and categories to grasp properly some results relevant to quantum physics. Such low level constructions modify the smoothness on $\mathbb{R}^4$ and the structures survive the modifications as a *classical* exotic $\mathbb{R}^4$. Thus, functions, although from different logic and category, approach exotic smooth ones, such that some quantum structures emerge due to the rich categorical formalism involved. Still, to apply exotic 4-smoothness in variety of situations one needs more direct relation to existing calculus.

Even though neither any explicit exotic metric nor the function on $\mathbb{R}^4$ is known, recent relative results made it possible to apply these exotic structures in a variety of contexts relevant to physics. In particular, strong connection with quantum theories and quantization was shown by Asselmeyer-Maluga & Król (2011b). First, we deal here exclusively with *small* exotic $\mathbb{R}^4$. These arise as the result of failing *h*-cobordism theorem in 4D (see e.g. Asselmeyer-Maluga & Brans (2007)). The others, so called large exotic $\mathbb{R}^4$, emerge from failing the smooth surgery in 4D. Second, the main technical ingredient of the relative approach to small 4-exotics is the relation of these with some structures defined on a 3-sphere. This S^3 should be placed as a part of the boundary of some contractible 4-submanifold of $\mathbb{R}^4$. This manifold is the Akbulut cork and its boundary is, in general, a closed 3-manifold which has the same homologies as ordinary 3-sphere – homology 3-sphere. Next, we deal with the parameterized by the radii $\rho \in \mathbb{R}$ of S^4 as a subset of R^4, a family of exotic $\mathbb{R}^4_\rho$ each of which is the open submanifold of standard $\mathbb{R}^4$. This is the *radial family* of small exotic $\mathbb{R}^4$'s or the deMichellis-Freedman family DeMichelis & Freedman (1992). Let CS be the standard Cantor set as a subset of $\mathbb{R}$, then the crucial result is:

Theorem 1 (Asselmeyer-Maluga & Król (2011b)). *Let us consider a radial family R_t of small exotic $\mathbb{R}^4_t$ with radius ρ and $t = 1 - \frac{1}{\rho} \subset CS \subset [0,1]$ induced from the non-product h-cobordism W between M and M_0 with Akbulut cork $A \subset M$ and $A \subset M_0$, respectively. Then, the radial family R_t determines a family of codimension-one foliations of ∂A with Godbillon-Vey invariant ρ^2. Furthermore, given two exotic spaces R_t and R_s, homeomorphic but non-diffeomorphic to each other (and so $t \neq s$), then the two corresponding codimension-one foliations of ∂A are non-cobordant to each other.*

This theorem gives a direct relation of small exotic $\mathbb{R}^4$'s - from the radial family, and (codimension one) foliations of some S^3 - from the boundary of the Akbulut cork. M and M_0 are compact non-cobordant 4-manifolds, resulting from the failure of the 4D *h*-cobordism theorem (see the next section). Such relativization of 4-exotics to the foliations of S^3 is the source of variety of further mathematical results and their applications in physics. One example of these is the quantization of electric charge in 4D where instead of magnetic monopoles one considers exotic smoothness in some region in spacetime Asselmeyer-Maluga & Król (2009a). More examples of this kind will be presented in the course of this Chapter.

Now we are ready to formulate two important questions as guidelines for this work:

i. What if smooth structure, with respect to which standard model of particles is defined, is not the "correct" one and it does not match with the smooth structure underlying GR and theories of quantum gravity, in 4 dimensions?

ii. What if particles and fields, as in standard model of particles, are not fundamental from the point of view of gravity in 4 dimensions? Rather, more natural are effective condensed matter states, and these states should be used in order to unify quantum matter with general relativity.

This Chapter is thought as giving the explanation for the above questions and for the existence of a fundamental connection between these, differently looking, problems. New point of view on the reconciliation of quantum field theory with general relativity in 4 physical dimensions, emerges. The exact description of quantum matter and fields coupled with QG in 4D, at least in some important cases, is presented. The task to build a final theory of QG in 4D is thus seen from different perspective where rather effective states of condensed matter are well suited for the reconciliation with QG. Such approach is also motivated by the AdS/CFT dualities where

effective matter states (without gravity) are described by dual theories with gravity. Hence gravity is inherently present in description of such condensed matter states.

In the next section we describe the relation of small exotic $\mathbb{R}^4$ with foliations of S^3 and WZW models on $SU(2)$. Then we show the connections between string theory and exotic $\mathbb{R}^4$. In particular 4-smoothness underlying spacetime emerges from superstring calculations and it modifies the spectra of charged particles in such spacetime. In Sec. 4 we discuss the Kondo state and show that it generates the same exotic 4-smoothness. Moreover, the Kondo state, when survive the high energy and relativistic limit, would couple to the gravity backgrounds of superstring theory. The backgrounds are precisely those related with exotic smooth $\mathbb{R}^4$ as in Sec. 3. We conjecture that one could encounter the experimental trace of existence of exotic $\mathbb{R}^4_p$ in the k-channel, $k > 2$, Kondo effect, where the usual fusion rules of the $SU(2)_k$ WZW model would be modified to these of $SU(2)_p$ WZW in high energies.

Next in Sec. 5 we present the connections of branes configurations in superstring theory with non-standard 4-smoothness of $\mathbb{R}^4$. Discussion and conclusions close the Chapter.

2. Foliations, WZW σ-models and exotic $\mathbb{R}^4$

An exotic $\mathbb{R}^4$ is a topological space with $\mathbb{R}^4-$topology but with a smooth structure different (i.e. non-diffeomorphic) from the standard $\mathbb{R}^4_{std}$ obtaining its differential structure from the product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. The exotic $\mathbb{R}^4$ is the only Euclidean space $\mathbb{R}^n$ with an exotic smoothness structure. The exotic $\mathbb{R}^4$ can be constructed in two ways: by the failure to split arbitrarily a smooth 4-manifold into pieces (large exotic $\mathbb{R}^4$) and by the failure of the so-called smooth h-cobordism theorem (small exotic $\mathbb{R}^4$). Here, we deal with the later kind of exotics. We refer the reader to Asselmeyer-Maluga & Brans (2007) for general presentation of various topological and geometrical constructions and their physical perspective. Another useful mathematical books are Gompf & Stipsicz (1999); Scorpan (2005). The reader can find further results in original scientific papers.

Even though there are known, and by now rather widely discussed (see the Introduction), difficulties with making use of different differential structures on $\mathbb{R}^4$ (and on other open 4-manifolds) in explicit coordinate-like way (see e.g. Asselmeyer-Maluga & Brans (2007)), it was, however, established, in a series of recent papers, the way how to relate these 4-exotics with some structures on S^3 (see e.g. Asselmeyer-Maluga & Król (2009a;b; 2011b)). This S^3 is supposed to fulfil specific topological conditions: it has to lie in ambient $\mathbb{R}^4$ such that it is a part of the boundary of some compact 4-submanifold with boundary, i.e. Akbulut cork. If so, one can prove that exotic smoothness of the $\mathbb{R}^4$ is tightly related to codimension-one foliations of this S^3, hence, with the 3-rd real cohomology classes of S^3. Reformulating Theorem 1 we have [Asselmeyer-Maluga & Król (2009a)]:

The exotic $\mathbb{R}^4$'s, from the radial family of exotic $\mathbb{R}^4$'s embedded in standard $\mathbb{R}^4$, are determined by the codimension-1 foliations, $\mathcal{F}$'s, with non-vanishing Godbillon-Vey (GV) class in $H^3(S^3, \mathbb{R})$ of a 3-sphere lying at the boundary of the Akbulut corks of $\mathbb{R}^4$'s. The radius in the family, ρ, and value of GV are related by $GV = \rho^2$. We maintain: the exoticness is localized at a 3-sphere inside the small exotic $\mathbb{R}^4$ (seen as a submanifold of $\mathbb{R}^4$).

Let us explain briefly, following Asselmeyer-Maluga & Król (2011d), how the codimension-1 foliations of S^3 emerges from the structure of exotic $\mathbb{R}^4$. The complete construction and proof can be found in Asselmeyer-Maluga & Król (2009a).

Small exotic $\mathbb{R}^4$ is determined by the compact 4-manifold A with boundary ∂A which is homology 3-sphere, and attached several Casson handles CH's. A is the Akbulut cork and CH is built from many stages towers of immersed 2-disks. These 2-disks cannot be embedded and the intersection points can be placed in general position in 4D in separated double points. Every CH has infinite many stages of intersecting disks. However, as Freedman proved, CH is topologically the same as (homeomorphic to) open 2-handle, i.e. $D^2 \times \mathbb{R}^2$. Now if one replaces CH's, from the above description of small exotic $\mathbb{R}^4$, by ordinary open 2-handles (with suitable linking numbers in the attaching regions) the resulting object is standard $\mathbb{R}^4$. The reason is the existence of infinite (continuum) many diffeomorphism classes of CH, even though all are topologically the same.

Consider the following situation: one has two topologically equivalent (i.e. homeomorphic), simple-connected, smooth 4-manifolds M, M', which are not diffeomorphic. There are two ways to compare them. First, one calculates differential-topological invariants like Donaldson polynomials Donaldson & Kronheimer (1990) or Seiberg-Witten invariants Akbulut (1996). But there is yet another possibility – one can change a manifold M to M' by using a series of operations called surgeries. This procedure can be visualized by a 5-manifold W, the cobordism. The cobordism W is a 5-manifold having the boundary $\partial W = M \sqcup M'$. If the embedding of both manifolds M, M' into W induces a homotopy-equivalence then W is called an h-cobordism. Moreover, we assume that both manifolds M, M' are compact, closed (without boundary) and simply-connected. Freedman (1982) showed that every h-cobordism implies a homeomorphism, hence h-cobordisms and homeomorphisms are equivalent in that case. Furthermore, the following structure theorem for such h-cobordisms holds true [Curtis & Stong (1997)]:

Let W be a h-cobordism between M, M'. Then there are contractable submanifolds $A \subset M, A' \subset M'$ together with a sub-cobordism $V \subset W$ with $\partial V = A \sqcup A'$ (the disjoint oriented sum), so that the h-cobordism $W \setminus V$ induces a diffeomorphism between $M \setminus A$ and $M' \setminus A'$.

Thus, the smoothness of M is completely determined (see also Akbulut & Yasui (2008; 2009)) by the contractible submanifold A (Akbulut cork) and its embedding $A \hookrightarrow M$ determined by a map $\tau : \partial A \to \partial A$ with $\tau \circ \tau = id_{\partial A}$ and $\tau \neq \pm id_{\partial A} (\tau$ is an involution). Again, according to Freedman (1982), the boundary of every contractible 4-manifold is a homology 3-sphere. This h-cobordism theorem is employed to construct an exotic $\mathbb{R}^4$. First, one considers a neighborhood (tubular) of the sub-cobordism V between A and A'. The interior of V, $int(V)$, (as open manifold) is homeomorphic to $\mathbb{R}^4$. However, if (and only if) M and M' are not diffeomorphic (still being homeomorphic), then $int(V) \cap M$ is an exotic $\mathbb{R}^4$.

Next, Bižaca (1994) and Bižaca & Gompf (1996) showed how to construct an explicit handle decomposition of the exotic $\mathbb{R}^4$ by using $int(V)$. The details of the construction can be found in their papers or in the book Gompf & Stipsicz (1999). The idea is simply to use the cork A and add some Casson handle to it. The interior of this resulting structure is an exotic $\mathbb{R}^4$. The key feature here is the appearance of CH. Briefly, a Casson handle CH is the result of attempts to embed a disk D^2 into a 4-manifold. In most cases this attempt fails and Casson (1986) searched for a possible substitute, which is just what we now call a Casson handle. Freedman (1982) showed that every Casson handle CH is homeomorphic to the open 2-handle $D^2 \times \mathbb{R}^2$ but in nearly all cases it is not diffeomorphic to the standard handle, Gompf (1984; 1989). The Casson handle is built by iteration, starting from an immersed disk in some 4-manifold M, i.e. a map $D^2 \to M$ which has injective differential. Every immersion $D^2 \to M$ is an embedding except on a countable set of points, the double points. One can "kill" one double point by immersing

another disk into that point. These disks form the first stage of the Casson handle. By iteration one can produce the other stages. Finally, we consider a tubular neighborhood $D^2 \times D^2$ of this immersed disk, called a kinky handle, on each stage. The union of all neighborhoods of all stages is the Casson handle. So, there are two input data involved with the construction of a CH: the number of double points in each stage and their orientation $\pm$. Thus, we can visualize the Casson handle CH by a tree: the root is the immersion $D^2 \to M$ with k double points, the first stage forms the next level of the tree with k vertices connected with the root by edges etc. The edges are evaluated using the orientation $\pm$. Every Casson handle can be represented by such an infinite tree. The structure of CH as immersed many-layers 2-disks will be important in Sec. 4 where we will assign fermion fields to CH's.

Next, we turn again to the radial family of small exotic $\mathbb{R}^4$, i.e. a continuous family of exotic $\{\mathbb{R}^4_\rho\}_{\rho \in [0,+\infty]}$ with parameter ρ so that $\mathbb{R}^4_\rho$ and $\mathbb{R}^4_{\rho'}$ are non-diffeomorphic for $\rho \neq \rho'$. The point is that this radial family has a natural foliation (see Theorem 3.2 in DeMichelis & Freedman (1992)) which can be induced by a polygon P in the two-dimensional hyperbolic space $\mathbb{H}^2$. The area of P is a well-known invariant, the Godbillon-Vey class as the element in $H^3(S^3, \mathbb{R})$. Every GV class determines a codimension-one foliation on the 3-sphere (firstly constructed by Thurston (1972); see also the book Tamura (1992) chapter VIII for the details). This 3-sphere is a part of the boundary ∂A of the Akbulut cork A (there is an embedding $S^3 \to \partial A$). Furthermore, one can show that the codimension-one foliation of the 3-sphere induces a codimension-one foliation of ∂A so that the area of the corresponding polygons (and therefore the foliation invariants) agree. The Godbillon-Vey invariant $[GV] \in H^3(S^3, \mathbb{R})$ of the foliation is related to the parameter of the radial family by $\langle GV, [S^3] \rangle = \rho^2$ using the pairing between cohomology and homology (the fundamental homology class $[S^3] \in H_3(S^3)$).

Thus, the relation between an exotic $\mathbb{R}^4$ (of Bizaca as constructed from the failure of the smooth h-cobordism theorem) and codimension-one foliation of the S^3 emerges. Two non-diffeomorphic exotic $\mathbb{R}^4$ imply non-cobordant codimension-one foliations of the 3-sphere described by the Godbillon-Vey class in $H^3(S^3, \mathbb{R})$ (proportional to the surface of the polygon). This relation is very strict, i.e. if we change the Casson handle, then we must change the polygon. But that changes the foliation and vice verse. Finally, we obtain the result:

The exotic $\mathbb{R}^4$ (of Bizaca) is determined by the codimension-1 foliations with non-vanishing Godbillon-Vey class in $H^3(S^3, \mathbb{R}^3)$ of a 3-sphere seen as submanifold $S^3 \subset \mathbb{R}^4$. We say: the exoticness is localized at a 3-sphere inside the small exotic $\mathbb{R}^4$.

In the particular case of integral $H^3(S^3, \mathbb{Z})$ one yields the relation of exotic $\mathbb{R}^4_k$, $k[\ ] \in H^3(S^3, \mathbb{Z})$, $k \in \mathbb{Z}$ with the WZ term of the k WZW model on $SU(2)$. This is because the integer classes in $H^3(S^3, \mathbb{Z})$ are of special character. Topologically, this case refers to flat $PSL(2, \mathbb{R})-$bundles over the space $(S^2 \setminus \{k \text{ punctures}\}) \times S^1$ where the gluing of k solid tori produces a 3-sphere (so-called Heegard decomposition). Then, one obtains the relation [Asselmeyer-Maluga & Król (2009a)]:

$$\frac{1}{(4\pi)^2} \langle GV(\mathcal{F}), [S^3] \rangle = \frac{1}{(4\pi)^2} \int_{S^3} GV(\mathcal{F}) = \pm(2 - k) \tag{1}$$

in dependence on the orientation of the fundamental class $[S^3]$. We can interpret the Godbillon-Vey invariant as WZ term. For that purpose, we use the group structure $SU(2) = S^3$ of the 3-sphere S^3 and identify $SU(2) = S^3$. Let $g \in SU(2)$ be a unitary matrix with

det $g = -1$. The left invariant 1-form $g^{-1}dg$ generates locally the cotangent space connected to the unit. The forms $\omega_k = Tr((g^{-1}dg)^k)$ are complex $k-$forms generating the deRham cohomology of the Lie group. The cohomology classes of the forms ω_1, ω_2 vanish and $\omega_3 \in H^3(SU(2), \mathbb{R})$ generates the cohomology group. Then, we obtain as the value for the integral of the generator

$$\frac{1}{8\pi^2} \int\limits_{S^3 = SU(2)} \omega_3 = 1 \, .$$

This integral can be interpreted as winding number of g. Now, we consider a smooth map $G : S^3 \to SU(2)$ with 3-form $\Omega_3 = Tr((G^{-1}dG)^3)$ so that the integral

$$\frac{1}{8\pi^2} \int\limits_{S^3 = SU(2)} \Omega_3 = \frac{1}{8\pi^2} \int\limits_{S^3} Tr((G^{-1}dG)^3) \in \mathbb{Z}$$

is the winding number of G. Every Godbillon-Vey class with integer value like (1) is generated by a 3-form Ω_3. Therefore, the Godbillon-Vey class is the WZ term of the $SU(2)_k$. Thus, we obtain the relation:

The structure of exotic $\mathbb{R}^4_k$'s, $k \in \mathbb{Z}$ from the radial family determines the WZ term of the k WZW model on $SU(2)$.

This WZ term enables one for the cancellation of the quantum anomaly due to the conformal invariance of the classical σ-model on $SU(2)$. Thus, we have a method of including this cancellation term from smooth 4-geometry: when a smoothness of the ambient 4-space, in which S^3 is placed as a part of the boundary of the cork, is this of exotic $\mathbb{R}^4_k$, then the WZ term of the classical σ-model with target $S^3 = SU(2)$, i.e. $SU(2)_k$ WZW, is precisely generated by this 4-smoothness. As the conclusion, we have the important correlation:

The change of smoothnes of exotic $\mathbb{R}^4_k$ to exotic $\mathbb{R}^4_l$, k, $l \in \mathbb{Z}$ both from the radial family, corresponds to the change of the level k of the WZW model on $SU(2)$, i.e. k WZW $\to l$ WZW.

Let us consider now the end of the exotic $\mathbb{R}^4_k$ i.e. $S^3 \times \mathbb{R}$. This end cannot be standard smooth and it is in fact fake smooth $S^3 \times_{\Theta_k} \mathbb{R}$, Freedman (1979). Given the connection of $\mathbb{R}^4_k$ with the WZ term as above, we have determined the "quantized" geometry of $SU(2)_k \times \mathbb{R}$ as corresponding to the exotic geometry of the end of $\mathbb{R}^4_k$. The appearance of the $SU(2)_k \times \mathbb{R}$ is a source for various further constructions. In particular, we will see that gravitational effects of $\mathbb{R}^4_k$ on the quantum level are determined via string theory where one replaces consistently flat $\mathbb{R}^4$ part of the background by curved 4D $SU(2)_k \times \mathbb{R}$.

3. 10d string theory and 4d-smoothness

Let us, following Asselmeyer-Maluga & Król (2011a) (see also Król (2011a;b)), begin with a charged quantum particle, say e, moving through non-flat gravitational background, i.e. smooth 4-spacetime manifold. The amount of gravity due to the curvature of this background affects the particle trajectory as predicted by GR. There should exist, however, a high energy limit where gravity contained in this geometrical background becomes quantum rather than classical and the particle may not be described by perturbative field theory any longer. This rather natural, from the point of view of physics, scenario requires, however, quantum gravity calculations which is not in reach in dimension 4. Moreover, mathematics underlying classical
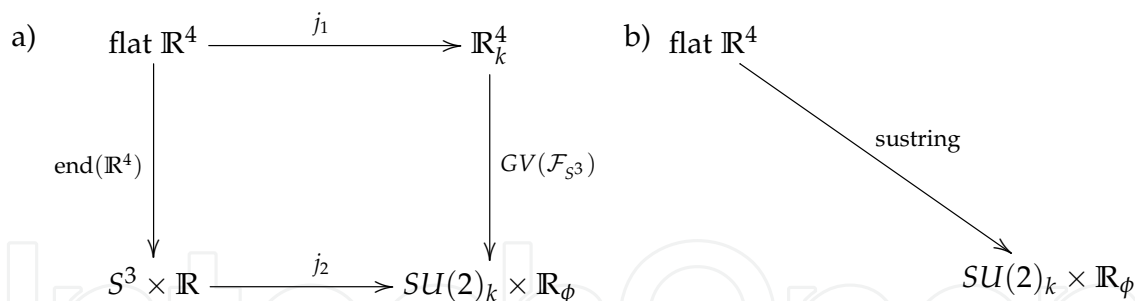
Fig. 1. a) j_1 is the change of the standard smooth $\mathbb{R}^4$ to the exotic $\mathbb{R}^4_k$, end($\mathbb{R}^4$) assigns the standard end to $\mathbb{R}^4$, $GV(\mathcal{F}_{S^3})$ generates the WZ$_k$-term from exotic $\mathbb{R}^4_k$ via GV invariant of the codim.-1 foliation of S^3. b) The change of string backgrounds s.t. flat $\mathbb{R}^4$ part is replaced by the linear dilaton background $SU(2)_k \times \mathbb{R}_\phi$

gravity is of (pseudo-)Riemannian smooth geometry and should change to a new 'geometry, or to, unknown at all, mathematics, when the transformation of GR to QG is performed.

As discussed already in the Introduction, it was proposed in Asselmeyer-Maluga & Król (2010; 2011c;d; 2012); Król (2011a;b) that 4-dimensional effects of string theory should be seen via its connections with exotic smoothness of topologically trivial $\mathbb{R}^4$. Several results were derived as if superstring theory were formulated on backgrounds which contain 4-dimensional part which is exotic $\mathbb{R}^4$ rather than standard smooth $\mathbb{R}^4$. This serves as a new window to 4-dimensional physics. The argumentation dealt with exact string backgrounds in any order of α'. The existence of such backgrounds is rather exceptional in superstring theory (see e.g. Orlando (2006; 2007)) and this always indicates important and exactly calculable effects of the theory. This is precisely the tool which we want to apply to the above stated problems, i.e. the description of both, 4D QG effects due to gravity present in background spacetime, and mathematics behind the shift GR $\rightarrow$ QG in 4D.

Exotic $\mathbb{R}^4_k$ is a smooth Riemannian manifold, however, its structure essentially deals with non-commutative geometry and quantization Asselmeyer-Maluga & Król (2011b). The connection with string exact backgrounds was also recognized in Asselmeyer-Maluga & Król (2010; 2011c). Thus, under the topological assumptions discussed in Sec. 2, the following correspondence emerges:

> The change of the smoothness from the standard $\mathbb{R}^4$ to exotic $\mathbb{R}^4_k$, corresponds to the change of exact string backgrounds from $\mathbb{R}^4 \times K^6$ to $SU(2)_k \times \mathbb{R}_\phi \times K^6$.

Let us note that only because of the *exotic* smooth structure of $\mathbb{R}^4$, the link to string backgrounds exists. If smoothness of $\mathbb{R}^4$ were standard, only separated regimes of 4-geometry (GR) and superstrings (QG) would appear. In superstring theory one understands fairy well how to change the exact background containing flat $\mathbb{R}^4$ to this with curved 4-dimensional part: $\mathbb{R}^4 \times K^6 \rightarrow SU(2)_k \times \mathbb{R}_\phi \times K^6$. This requires supersymmetry in 10 dimensions. The presence of supersymmetry is, however, just a technical mean allowing for the consistent shifts between the backgrounds and performing the QG calculations effectively.

3.1 The magnetic deformation of 4D part of the string background

To be specific let us consider the $SO(3)_{k/2} \times \mathbb{R}_\phi$ as the 4D part of the 10D string background which replaces the flat $\mathbb{R}^4$ part. This $SO(3)_{k/2} \times \mathbb{R}_\phi$ geometry is the result of the projection

from $SU(2)_k \times \mathbb{R}_\phi$, so k is even. Here $SU(2)_k$ is the affine Kac-Moody algebra at level k and $\mathbb{R}_\phi$ the linear dilaton, both appearing in the exact string background realized by the superconformal 2D field theory (see, e.g. Kiritsis & Kounnas (1995b)). The deformation of such curved 4D part of the background will be performed in heterotic superstring theory in the language of σ-model. The deformations will correspond to the introducing almost constant magnetic field H and its gravitational backreaction on the 4D curved part of the background. First let us describe undeformed theory. The action for heterotic σ-model in this $SO(3)_{k/2} \times \mathbb{R}_\phi$ background is:

$$S_4 = \frac{k}{4} \mathrm{I}_{SO(3)}(\alpha, \beta, \gamma) + \frac{1}{2\pi} \int d^2z \left[\partial x^0 \bar{\partial} x^0 + \psi^0 \partial \psi^0 + \sum_{a=1}^{3} \psi^a \partial \psi^a \right] + \frac{Q}{4\pi} \int \sqrt{g} R^{(2)} x^0 \quad (2)$$

where $\mathrm{I}_{SO(3)}(\alpha, \beta, \gamma) = \frac{1}{2\pi} \int d^2z \left[\partial\alpha\partial\bar{\alpha} + \partial\beta\partial\bar{\beta} + \partial\gamma\partial\bar{\gamma} + 2cos\beta\partial\alpha\partial\gamma \right]$ in Euler angles of $SU(2) = S^3$, $R^{(2)}$ is the 2D worldsheet curvature, g is the determinant of the target metric and Q is the dilaton charge with x^0 the coordinate of $\mathbb{R}_\phi$. The bosonic σ-model action reads in general:

$$S = \frac{1}{2\pi} \int d^2z (G_{\mu\nu} + B_{\mu\nu}) \partial x^\mu \bar{\partial} x^\nu + \frac{1}{4\pi} \int \sqrt{g} R^{(2)} \Phi(x) \quad (3)$$

so comparing with (2) gives the non-zero background fields as:

$$G_{00} = 1, \; G_{\alpha\alpha} = G_{\beta\beta} = G_{\gamma\gamma} = \frac{k}{4}, \; G_{\alpha\gamma} = \frac{k}{4} \cos \beta$$
$$B_{\alpha\gamma} = \frac{k}{4} \cos \beta, \; \Phi = Q x^0 = \frac{x^0}{\sqrt{k+2}} \, . \quad (4)$$

One can decompose (see e.g. Prezas & Sfetsos (2008)) the supersymmetric WZW model into the bosonic $SU(2)_{k-2}$ with affine currents J^i and three free fermions ψ^a, $a = 1,2,3$ in the adjoint representation of $SU(2)$. As the result the supersymmetric $\mathcal{N} = 1$ affine currents are $\mathcal{J}^a = J^a - \frac{i}{2}\epsilon^{abc}\psi^b\psi^c$. After introducing the complex fermions combination $\psi^\pm = \frac{1}{\sqrt{2}}(\psi^1 \pm i\psi^2)$ and the corresponding change of the affine bosonic currents $J^\pm = J^1 \pm iJ^2$, the supersymmetric affine currents read:

$$\mathcal{J}^3 = J^3 + \psi^+\psi^-, \; \mathcal{J}^\pm = J^\pm \pm \sqrt{2}\psi^3\psi^\pm \quad (5)$$

Let us redefine the indices in the fermion fields as: $+ \rightarrow 1, - \rightarrow 2$, then $\mathcal{J}^3 = J^3 + \psi^1\psi^2$.

From the point of view of the σ-model, the vertex for the magnetic field H on 4-dimensional $\mathbb{R}_\phi \times SU(2)_k$ part of the background is the exact marginal operator given by $V_m = H(J^3 + \psi^1\psi^2)\bar{J}^a$. Similarly, the vertex for the corresponding gravitational part is $V_{gr} = \mathcal{R}(J^3 + \psi^1\psi^2)\bar{J}^3$, and represents truly marginal deformations too.

The shape of these operators follow from the fact that, in general, the marginal deformations of the WZW model can be constructed as bilinears in the currents $J, \bar{J}$ of the model [Orlando (2007)]:

$$\mathcal{O}(z,\bar{z}) = \sum_{i,j} c_{ij} J^i(z) \overline{J^j}(\bar{z}) \quad (6)$$

where $J^i, \overline{J^j}$ are left and right-moving affine currents respectively, Orlando (2007).

Here, following Kiritsis & Kounnas (1995b), we consider covariantly constant magnetic field $H_i^a = \epsilon^{ijk}F_{jk}^a$ and constant curvature $\mathcal{R}^{il} = \epsilon^{ijk}\epsilon^{lmn}\mathcal{R}_{jmkn}$ in the 4-dimensional background as

above of closed superstring theory. When this chromo-magnetic field is in the $\mu = 3$ direction the following deformation is proportional to $(J^3 + \psi^1\psi^2)\bar{J}$ and the right moving current $\bar{J}$ is normalized as $< \bar{J}(1)\bar{J}(0) > = k_g/2$. Rewriting the currents in the Euler angles, i.e. $J^3 = k(\partial\gamma + \cos\beta\partial\alpha)$, $\bar{J}^3 = k(\bar{\partial}\alpha + \cos\beta\bar{\partial}\gamma)$, we obtain for the perturbation of the (heterotic) action in (2), the following expression:

$$\delta S_4 = \frac{\sqrt{kk_g}H}{2\pi} \int d^2z(\partial\gamma + \cos\beta\partial\alpha)\bar{J} . \tag{7}$$

The new σ-model with the action $S_4 + \delta S_4$ is again conformally invariant with all orders in α' since:

$S_4 + \delta S_4 = \frac{k}{4}I_{SO(3)}(\alpha, \beta, \gamma) + \delta S_4 + \frac{k_g}{4\pi} \int d^2z\partial\phi\bar{\partial}\phi = \frac{k}{4}I_{SO(3)}(\alpha, \beta, \gamma + 2\sqrt{\frac{k_g}{k}}H\phi) + \frac{k_g(1-2H^2)}{4\pi} \int d^2z\partial\phi\bar{\partial}\phi$. This shows that, in fact the magnetic deformation is exactly marginal. Here we have chosen for the currents J and $\bar{J}$, $\partial\phi$ and $\bar{\partial}\phi$ correspondingly, as their bosonizations.

The background corresponding to the perturbation (7) is determined by background fields, i.e. a graviton $G_{\mu\nu}$, gauge fields $F^a_{\mu\nu}$, an antisymmetric field (three form) $H_{\mu\nu\rho}$ and a dilaton Φ, which, in turn, are solutions to the following equations of motion:

$$\begin{aligned}
\frac{3}{2}\left[4(\nabla\Phi)^2 - \frac{10}{3}\Box\Phi - \frac{2}{3}R + \frac{1}{12g^2}F^a_{\mu\nu}F^{a,\mu\nu}\right] &= C \\
R_{\mu\nu} - \frac{1}{4}H^2_{\mu\nu} - \frac{1}{2g^2}F^a_{\mu\rho}F^{a\rho}_\nu + 2\nabla_\mu\nabla_\nu\Phi &= 0 \\
\nabla^\mu\left[e^{-2\Phi}H_{\mu\nu\rho}\right] &= 0 \\
\nabla^\nu\left[e^{-2\Phi}F^a_{\mu\nu}\right] - \frac{1}{2}F^{a,\nu\rho}H_{\mu\nu\rho}e^{-2\Phi} &= 0
\end{aligned} \tag{8}$$

These are derived from the variations of the following effective 4-dimensional gauge theory action:

$$S = \int d^4x\sqrt{G}e^{-2\Phi}\left[R + 4(\nabla\Phi)^2 - \frac{1}{12}H^2 - \frac{1}{4g^2}F^a_{\mu\nu}F^{a,\mu\nu} + \frac{C}{3}\right] \tag{9}$$

where C is the l.h.s. of the first equation in (8). Here $g_{str} = 1$, the gauge coupling $g^2 = 2/k_g$, $F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + f^{abc}A^b_\mu A^c_\nu$, $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2g^2}\left[A^a_\mu F^a_{\nu\rho} - \frac{1}{3}f^{abc}A^a_\mu A^b_\nu A^c_\rho\right] +$ permutations. f^{abc} are structure constants of the gauge group and A^a_μ is the effective gauge field. One can observe that the term in the square bracket in $H_{\mu\nu\rho}$ is the Chern-Simons term for the gauge potential A^a_μ.

Now, the background complying with these equations and which respects the deformation (7), reads:

$$\begin{aligned}
G_{00} = 1, \quad G_{\beta\beta} = \frac{k}{4}, \quad G_{\alpha\gamma} = \frac{k}{4}(1 - 2H^2)\cos\beta \\
G_{\alpha\alpha} = \frac{k}{4}(1 - 2H^2\cos^2\beta), \quad G_{\gamma\gamma} = \frac{k}{4}(1 - 2H^2), \quad B_{\alpha\gamma} = \frac{k}{4}\cos\beta \\
A_\alpha = g\sqrt{k}H\cos\beta, \quad A_\gamma = g\sqrt{k}H, \quad \Phi = \frac{x^0}{\sqrt{k+2}} .
\end{aligned} \tag{10}$$

where H is the magnetic field as in (7).

Similarly, when gravitational marginal deformations as in the vertex $V_{gr} = \mathcal{R}(J^3 + \psi^1\psi^2)\bar{J}^3$ are included, where $\mathcal{R}$ is the curvature parameter of the deformation, one can derive

corresponding exact background of string theory via σ-model calculations, Hassan & Sen (1993); Kiritsis & Kounnas (1995b). Again, the fields in this background which solve the effective field theory equations (8), are [Kiritsis & Kounnas (1995b)]:

$$G_{00} = 1, \; G_{\beta\beta} = \tfrac{k}{4}$$

$$G_{\alpha\alpha} = \frac{k}{4} \frac{(\lambda^2+1)^2 - (8H^2\lambda^2 + (\lambda^2-1)^2)\cos^2\beta)}{(\lambda^2+1+(\lambda^2-1)\cos\beta)^2}$$

$$G_{\gamma\gamma} = \frac{k}{4} \frac{(\lambda^2+1)^2 - (8H^2\lambda^2 - (\lambda^2-1)^2)\cos^2\beta)}{(\lambda^2+1+(\lambda^2-1)\cos\beta)^2}$$

$$G_{\alpha\gamma} = \frac{k}{4} \frac{4\lambda^2(1-2H^2)\cos\beta + (\lambda^4-1)\sin^2\beta}{(\lambda^2+1+(\lambda^2-1)\cos\beta)^2}$$

$$B_{\alpha\gamma} = \frac{k}{4} \frac{\lambda^2-1+(\lambda^2+1)\cos\beta}{(\lambda^2+1+(\lambda^2-1)\cos\beta)^2}$$

$$A_\alpha = 2g\sqrt{k} \frac{H\lambda\cos\beta}{(\lambda^2+1+(\lambda^2-1)\cos\beta)^2}$$

$$A_\gamma = 2g\sqrt{k} \frac{H\lambda}{(\lambda^2+1+(\lambda^2-1)\cos\beta)^2}$$

$$\Phi = \frac{t}{\sqrt{k+2}} - \tfrac{1}{2}\log\left[\lambda + \tfrac{1}{\lambda} + (\lambda - \tfrac{1}{\lambda})\cos\beta\right]$$

$$(11)$$

The dependence on λ shows the existence of gravitational backreaction which was absent in the purely magnetic deformed background (10).

3.2 Field theory vs. string theory spectra of charged particles in standard 4-space

In the case of field theory in 4 dimensions we introduce the magnetic field on S^3 which agrees with the magnetic part of the string background (10) as:

$$A_\alpha = H\cos\beta, \; A_\beta = 0, \; A_\gamma = H. \tag{12}$$

The Hamiltonian for a particle with electric charge e moving on S^3, is

$$\overline{\mathbf{H}} = \frac{1}{\sqrt{\det G}}(\partial_\mu - ieA_\mu)\sqrt{\det G}\, G^{\mu\nu}(\partial_\nu - ieA_\nu). \tag{13}$$

where we assume at the beginning that $G_{\mu\nu}$ is standard metric on S^3.

The energy spectrum for $\overline{\mathbf{H}}$ is then given by:

$$\Delta E_{j,m} = \frac{1}{R^2}\left[j(j+1) - m^2 + (eH - m)^2\right] \tag{14}$$

where R is the radius of S^3, $j \in \mathbb{Z}$ and $-j \le m \le j$, as is the case for $SO(3)$. In the flat limit we retrieve the Landau spectrum in 3-dimensional space of spinless particles:

$$\Delta E_{n,p_3} = e\tilde{H}(2n+1) + p_3^2 + \mathcal{O}(R^{-1}) \tag{15}$$

where magnetic field is pointing into 3-rd direction and the re-scaling of eH is performed as $eH = e\tilde{H} + \kappa R + \mathcal{O}(1)$, $m = e\tilde{H}R^2 + (p_3 + \kappa) + \mathcal{O}(1)$. This follows from rewriting the spectrum (14) as $\Delta E_{n,m} = \frac{1}{R^2}\left[n(n+1) + |m|(2n+1)\right] + \left(\frac{eH-m^2}{R}\right)^2$ by introducing new parameter n: $j = |m| + n$ for $|m|, n \in \mathbb{N}$.

Let us, again following Kiritsis & Kounnas (1995b), calculate the spectrum in the case of full exact string background (10) as our starting point. One takes the metric components from the background (10) and derive the eigenvalues of the Hamiltonian (14). The result is [Kiritsis & Kounnas (1995b)]:

$$\Delta E_{j,m} = \frac{1}{R^2} \left[j(j+1) - m^2 + \frac{(eHR - m)^2}{(1 - 2H^2)} \right] .$$ (16)

Again, introducing $n \in \mathbb{N}$ by $j = |m| + n$, $|m| = 0, 1/2, 1, ...$ we can rewrite the spectrum (16) as:

$$\Delta E_{n,m} = \frac{1}{R^2}[n(n+1) + |m|(2n+1)] + \left(\frac{eHR - m}{R\sqrt{1 - 2H^2}} \right)^2$$ (17)

which is the energy spectrum containing the corrections due to H field appearing in the string exact background (10), but the Hamiltonian (13) is field theoretic 4-dimensional one.

One can also calculate the exact string spectrum of energy in this exact background (see Kiritsis & Kounnas (1995a;b)) and when compared with (17) gives rise to the following dictionary rules enabling passing between the spectra:

$$R^2 \to k + 2 \, , \ m \to \mathcal{Q} + J^3 \, , \ e \to \sqrt{\frac{2}{k_g}}\overline{\mathcal{Q}}$$

$$H \to \frac{F}{\sqrt{2(1+\sqrt{1+F^2})}} = \frac{1}{2\sqrt{2}} \left[F - \frac{F^3}{4} + \mathcal{O}(F^5) \right] .$$ (18)

Here $F^2 = \left\langle F_{\mu\nu}^a F_a^{\mu\nu} \right\rangle$ is the integrated (square of) field strength where $H_i^a = \epsilon^{ijk} F_{jk}^a$ as before.

For a particle with spin S setting $S = Q$ the following modification of the spectrum appear due to the above rules [Kiritsis & Kounnas (1995b)]:

$$\Delta E_{j,m,S} = \frac{1}{k+2} \left[j(j+1) - (m+S)^2 + \frac{(eHR - m - S)^2}{(1 - 2H^2)} \right] .$$ (19)

Next step is the inclusion of gravitational backreactions. One begins with the string background (11) and compute again the eigenvalues of (13). The result for scalar particles is [Kiritsis & Kounnas (1995b)]:

$$\Delta E_{j,m,\overline{m}} = \frac{1}{R^2} \left[j(j+1) - m^2 + \frac{(2ReH - (\lambda + \frac{1}{\lambda})m - (\lambda - \frac{1}{\lambda})\overline{m})^2}{4(1 - 2H^2)} \right]$$ (20)

where now $-j \leq m, \overline{m} \leq j$. Again, comparing with exact string spectra for even k we have the corresponding dictionary rules in the case where gravity backreactions are included:

$$R^2 \to k + 2 \, , \ m \to \mathcal{Q} + J^3 \, , \ e \to \sqrt{\frac{2}{k_g}}\overline{\mathcal{P}} \, , \ \overline{m} \to \overline{J}^3$$

$$H^2 \to \frac{1}{2} \frac{F^2}{F^2 + 2(1+\sqrt{1+F^2+\mathcal{R}^2})} \, , \ \lambda^2 = \frac{1+\sqrt{1+F^2+\mathcal{R}^2}+\mathcal{R}}{1+\sqrt{1+F^2+\mathcal{R}^2}-\mathcal{R}}$$ (21)

where $\mathcal{R}^2 = \left\langle R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right\rangle$ is the integrated squared scalar curvature and $R_{\mu\nu\rho\sigma}$ is the Riemann tensor of the „squashed" $SU(2) = S^3$ in the deformed background.

3.3 The exotic 4D interpretation of string calculations

Given the dictionary (21), we can rewrite (20) in a way where the dependance on the even level k is written explicitly:

$$\Delta E^k_{j,m,\overline{m}} = \frac{1}{k+2}[j(j+1) - m^2] + \frac{(2\sqrt{k+2}eH - (\lambda + \frac{1}{\lambda})m - (\lambda - \frac{1}{\lambda})\sqrt{(1+2/k)}\overline{m})^2}{4(k+2)(1-2H^2)} \ . \quad (22)$$

Thus this is the 4D spectrum of a scalar particle with charge e which is modified by the magnetic field H and its gravitational backreaction λ (20). The spectrum depends on $k = 2p$ which indicates the relevance of the stringy regime. One can interpret this dependance on k as the result of exotic $\mathbb{R}^4_k$ geometry of a 4-region where the particle travells. However, this 4-geometry, in the QG limit of string theory, generates the quantum gravity effects in 4D.

In deriving the spectrum (22) we commence with the flat standard smooth $\mathbb{R}^4$ which is a part of the exact string background. Then we switched to another exact string background where the 4D part is now $SU(2)_k \times \mathbb{R}_\phi$. This new 4D part ceases to be flat. Its curvature has defined gravitational meaning in superstring theory such that the QG calculations are possible. The effects are derived in the regime of QG, i.e. heterotic string theory. The same deformed spectrum could be obtained, in principle, via including magnetic field $\tilde{H}$ and its gravitational backreaction on exotic smooth $\mathbb{R}^4_k$ where modified metric $\tilde{G}_{\mu\nu}$ emerges. These fields, however, are not explicitly specified but still the effects in QG regime are derived from string theory as above. Such an approach serves as a way of quantization of gravity while on exotic $\mathbb{R}^4$. The relations between various ingredients appearing here are presented in Fig. 2.
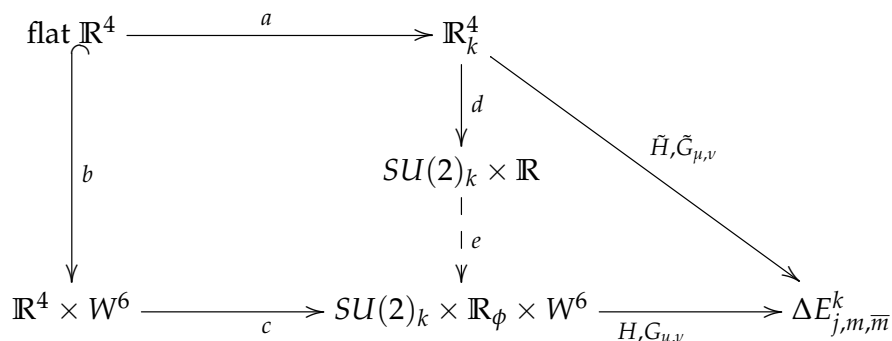

Fig. 2. a is the change of smoothness on $\mathbb{R}^4$ from standard one to exotic $\mathbb{R}^4_k$; b is the embedding of flat smooth $\mathbb{R}^4$ into the string background; c is the change of the string backgrounds; d assigns $\mathbb{R}^4_k$ $SU(2)_k \times \mathbb{R}$ the end of exotic $\mathbb{R}^4_k$, via GV invariant; e is the embedding of $SU(2)_k \times \mathbb{R}$ into the string background; $H, G_{\mu\nu}$ is the deformation of the CFT background resulting in the deformed spectrum $\Delta E^k_{j,m,\overline{m}}$; the same spectrum is obtained when $\tilde{H}, \tilde{G}_{\mu\nu}$ are on exotic $\mathbb{R}^4_k$

Let us turn to the appearance of the mass gap in the spectrum when the theory is formulated on exotic $\mathbb{R}^4_k$ rather than standard smooth $\mathbb{R}^4$. In field theory a dilaton Φ couples to a massless bosonic field T in a universal fashion:

$$S[\Phi, T] = \int e^{-2\Phi}\partial_M T \partial^M T.$$

One can introduce a new field $U = e^{-\Phi}T$ hence the above action becomes:

$$S[\Phi, U] = \int \partial_M U \partial^M U + [\partial^2 \Phi - \partial_M \Phi \partial^M \Phi]U.$$

Thus, for a linear dilaton $\Phi = q_M X^M$ the field U gets a mass square $M^2 = q_M q^M$ for q_M spacelike. This way the massless boson T is mapped to the boson U with the mass M. However, this mechanism does not work in the case of massless free fermions. In four dimensional spacetime the chiral fermion ψ can be coupled to an antisymmetric tensor $H_{\mu\nu\rho}$ as follows:

$$S[\psi, H] = \int \overline{\psi}\gamma^\mu \left[\overleftrightarrow{\partial_\mu} + H_\mu\right]\psi$$

where $H_\mu = \varepsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma}$ is the dual of the antisymmetric tensor $H_{\nu\rho\sigma}$. If one can embed this system into a string background with the fields: Φ and H_{MNP}, then using one-loop string equations:

$$R_{MN} = -2\nabla_M \nabla_N \Phi + \tfrac{1}{4}H_{MPR}H_N{}^{PR},$$
$$\nabla_L \left(e^{-2\Phi}H_{MN}^L\right) = 0, \tag{23}$$
$$\nabla^2\Phi - 2\left(\nabla\Phi\right)^2 = -\tfrac{1}{12}H^2,$$

one gets for the linear dilaton $\Phi = q_M X^M$ the following relation:

$$q_M q^M = \frac{1}{6}H^2$$

and the scalar curvature R is:

$$R = \frac{3}{2}q_M q^M.$$

If non-vanishing components of q_M and H_{MNP} are in four dimensional space, one obtains that:

$$q^\mu \sim \varepsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma}.$$

Thus, the Dirac operator acquires a mass gap proportional to $q_\mu q^\mu$.

The problem of embedding a four dimensional fermion system in the exact string background was considered in Kiritsis & Kounnas (1995c). In the case when four dimensional space is represented by the $\mathbb{R}_\phi \times SU(2)_k$ part of the string background, than the linear dilaton is $\Phi = QX^0$ and Q is given by the level k of the WZW model on $SU(2)$ as $Q = (k+2)^{-1/2}$ so that the CFT has the same central charge as flat space. Hence, the massless bosons acquire the mass gap $\Delta M^2 = \mu^2 = (k+2)^{-1}$.

That way we arrive at the important feature of the theory when on exotic $\mathbb{R}_k^4$:

The theory predicting the energy spectra of charged particles as in (22) in the flat smooth 4D limit, $k \to \infty$ does not show the existence of the mass gap in the energy spectra. However, in the exotic $\mathbb{R}_k^4$ limit the theory acquires the mass gap $\mu^2 \sim \frac{1}{(k+2)}$.

The mass gap which appears here in 4D theory is the result of QG computations i.e. those on linear dilaton background in superstring theory. Such overlapping QG with field theory in 4D is a special new feature of the approach via exotic open 4-smooth spaces which bridges 10D superstring and 4D matter fields. We will see in the next section that there are also bottom–up arguments where exotic $\mathbb{R}^4$'s emerge from the regime of low energy effective states of condensed matter. The latter means that gravity is present in the description of the effective entangled matter, since $\mathbb{R}_k^4$ is not flat and Einstein equations can be written on these 4-manifolds.

4. Quantum effective spin matter and exotic $\mathbb{R}^4$ – the Kondo effect

Gravitational interaction is very exceptional among all interactions in Nature. On the one hand gravity is the geometry of spacetime on which fields propagate and interactions take place. On the other hand, gravity couples with any kind of energy and matter. Further, it is the only interaction which restrains quantization.

Based on the entanglement of ideas presented so far, we want to argue that gravity is present in some states of magnetic effective quantum matter in a *nonstandard* way. The latter means that some states of spin matter, already at low temperatures, are coupled with 4D gravity via special 4-geometry *directly*, rather than, by energy-momentum tensor. This coupling can be extended over quantum regime of gravity, at least in some cases, and relates effective rather than fundamental fields and particles from SM. The coupling is understood as the presence of a non-flat 4-geometry which becomes dominating in some limits. The special 4-geometry is, again, exotic smoothness of Euclidean 4-space $\mathbb{R}^4$, thus becoming a guiding principle for presented approach to QG. The presence of gravity in the description of nonperturbative, strongly entangled states of 4D matter field is not a big surprise, as recent vital activity on the methods of AdS/CFT correspondence shows. However, our approach is different and makes use of inherently 4-dimensional new geometrical findings, which, at this stage of development, do not refer to AdS/CFT techniques (cf. Król (2005)).

In the thirties of the last century strange behaviour of conducting electrons occurring in some metallic alloys was observed. Namely the resistivity $\rho(T)$ in these alloys in the presence of magnetic spin s impurities, growth substantially when the temperature is lowering below the critical temperature T_K called the Kondo temperature. T_K is as low as a few K.

Kondo proposed in 1964 a simple phenomenological Hamiltonian Affleck (1995):

$$H = \sum_{\vec{k},\alpha} \psi_{\vec{k}}^{+\alpha} \psi_{\vec{k}\alpha} \, \epsilon(k) + \lambda \vec{S} \cdot \sum_{\vec{k},\vec{k}'} \psi_{\vec{k}}^{+} \frac{\vec{\sigma}}{2} \psi_{\vec{k}'}. \tag{24}$$

explaining the growth of the resistivity $\rho(T)$. Here ψ is the annihilation operator for the conduction electron of spin α and momentum $\vec{k}$, the antiferromagnetic interaction term is that between spin s impurity $\vec{S}$ with spins of conducting electrons, at $\vec{x} = 0$; $\vec{\sigma}$ is the vector of Pauli matrices. From this Hamiltonian one can derive, in the Born approximation, that $\rho(T) \sim \left[\lambda + \nu\lambda^2 \ln \frac{D}{T} + ...\right]^2$ where D is the 'width of the band' parameter and the second term is divergent in $T = 0$. Thus, this divergence explains the growth of the resistivity. The Hamiltonian (24) can be also derived from the more microscopic Anderson model Anderson (1961). The Kondo antiferromagnetic coupling appears as the tunnelling of electrons thus screening the spin impurity (see eg. Potok et al. (2007)).

The exact low T behavior was proposed by Affleck (1995); Affleck & Ludwig (1991; 1993; 1994) and Potok et al. (2007) by the use of boundary conformal field theory (BCFT). This insightful use of the CFT methods makes it possible to work out the connection with smooth 4-geometry.

Let us see in brief how the structure of the $SU(2)_k$ WZW model is well suited to the description of the k-channel Kondo effect. Recall that Kac-Moody algebra $SU(2)_k$ is spanned

on 3-components currents $\vec{\mathcal{J}}_n$, $n = \ldots -2, -1, 0, 1, 2, \ldots$:

$$[\mathcal{J}_n^a, \mathcal{J}_m^b]_k = i\epsilon^{abc}\mathcal{J}_{n+m}^c + \frac{1}{2}kn\delta^{ab}\delta_{n,-m}. \tag{25}$$

Next, we decompose the currents $\vec{\mathcal{J}}_n$ as $\vec{\mathcal{J}}_n = \vec{J}_n + \vec{S}$ such that $\vec{J}_n$ obey the same Kac-Moody algebra, i.e. $[J_n^a, J_m^b]_k = i\epsilon^{abc}J_{n+m}^c + \frac{1}{2}kn\delta^{ab}\delta_{n,-m}$ and usual relations for $\vec{S}$, i.e. $[S^a, S^b] = i\epsilon^{abc}S^c$, $[S^a, J_n^b] = 0$. From the point of view of field theories describing the interacting currents with spins, $\vec{\mathcal{J}}_n$ corresponds to the effective infrared fixed point of the theory of interacting spins $\vec{S}$ with $\vec{J}_n$ where the coupling constant λ is taken as $\frac{2}{3}$ for $k = 1$. The interacting Hamiltonian of the theory, for $k = 1$, reads:

$$H_s = c\left(\frac{1}{3}\sum_{-\infty}^{+\infty}\vec{J}_{-n}\cdot\vec{J}_n + \lambda\sum_{-\infty}^{+\infty}\vec{J}_n\cdot\vec{S}\right). \tag{26}$$

For $\lambda = \frac{2}{3}$, one completes the square and the algebra (25) for the currents $\vec{\mathcal{J}}_n$ follows. Then, the new Hamiltonian, where $\vec{S}$ is now effectively absent (still for $k = 1$), is given by $H = c'\sum_{-\infty}^{+\infty}\left(\vec{\mathcal{J}}_{-n}\cdot\vec{\mathcal{J}}_n - \frac{3}{4}\right)$ (c, c' are some constants).

A similar procedure holds for arbitrary integer k where the spin part of the Hamiltonian reads: $H_{s,k} = \frac{1}{2\pi(k+2)}\vec{J}^2 + \lambda\vec{J}\cdot\vec{S}\delta(x)$ and the infrared effective fixed point is now reached for $k = \frac{2}{2+k}$. The spins $\vec{S}$ reappear as the boundary conditions in the boundary CFT represented by the WZW model on $SU(2)$. This model defines the Verlinde fusion rules and is determined by these. The following *fusion rules hypothesis*, was proposed by Affleck (1995), which explains the creation and nature of the multichannel Kondo states:

The infrared fixed point, in the k-channel spin-s Kondo problem, is given by fusion with the spin-s primary for $s \leqq k/2$ or with the spin $k/2$ primary for $s > k/2$. Thus, the level k Kac-Moody algebra, as in the level k WZW $SU(2)$ model, governs the behaviour of the Kondo state in the presence of k channels of conducting electrons and magnetic impurity of spin s.

This is also the reason why, already in low temperatures, entangled magnetic matter of impurities and conduction electrons indicates the correlation with exotic 4-geometry. First, every CH generates a fermion field. Every small exotic $\mathbb{R}^4$ can be represented as handlebody where Akbulut cork has several CH's attached. The important thing is that the handlenbody has a boundary and only after removing it the interior is diffeomorphic to, say, exotic $\mathbb{R}_k^4$. Let us remove a single CH from the handlebody $\mathbb{R}_k^4$. The result is $\mathbb{R}_k^4 \setminus CH$. The boundary of it reads $\partial(\mathbb{R}_k^4 \setminus CH)$. The contribution to the Einstein action $\int_{\mathbb{R}_k^4 \setminus CH} R\sqrt{g}d^4x$ from this boundary is the suitable surface term:

$$\int_{\partial(\mathbb{R}_k^4 \setminus CH)} R\sqrt{g}d^4x + \int_{\partial(\mathbb{R}_k^4 \setminus CH)} K_{CH}\sqrt{g_\partial}d^3x$$

where K_{CH} is the trace of the 2-nd fundamental form and g_∂ the metric on the boundary Asselmeyer-Maluga & Brans (2011). But as shown in Asselmeyer-Maluga & Brans (2011) this

term is expressed by the spinor field ψ describing the immersion of D^2 into $\mathbb{R}^3$, which extends to the immersion of $D^2 \times (0,1)$ into $\mathbb{R}^4$:

$$\int_{\partial(\mathbb{R}_k^4 \backslash CH)} K_{CH} \sqrt{g_\partial} d^3 x = \int_{\partial(\mathbb{R}_k^4 \backslash CH)} \psi \gamma^\mu D_\mu \overline{\psi} \sqrt{g_\partial} d^3 x . \tag{27}$$

This can be extended to 4-dimensional Einstein-Hilbert action with the source depending on the CH, hence on exotic $\mathbb{R}_k^4$:

$$S_4^{CH}(\mathbb{R}_k^4) = \int_{\mathbb{R}_k^4 \backslash CH} (R + \psi \gamma^\mu D_\mu \overline{\psi}) \sqrt{g} d^4 x . \tag{28}$$

Again, it was shown in Asselmeyer-Maluga & Brans (2011) that the spinor field ψ extends over whole 4-manifold such that the 4D Dirac equations are fulfilled. This way we have fermion fields which are determined by CH. Moreover, this fermions plays a role of gravity sources as in 28. In fact every infinite branch of the CH determines some 4D fermion.

Second, given exotic $\mathbb{R}_p^4$ we have r Casson handles in its handlebody. These r CH's generate effective $q(r)$-many infinite branches. Each such branch generates a fermion field. Attaching the CH's to the cork results in exotic $\mathbb{R}_p^4$. Hence, p is the function of q in general, $p = p(q(r))$.

Let us assign now the simplest possible CH to every CH in the handlebody of exotic $\mathbb{R}^4$, such that replacing the original CH by this simple one does not change the exotic smoothness. This is the model handlebody we refer to in the context of the Kondo effect (see Fig. 3 for the examples of the simplest CH's).

The k-channel Kondo state, in the k-channel Kondo effect, is the entangled state of conducting electrons in k bands and the magnetic spin s impurity. The physics of resulting state is described by BCFT by the Verlinde fusion rules in $SU(2)_k$ WZW model. To have the WZ term in this WZW model one certainly needs $p = k$. This kWZ term is generated by exotic $\mathbb{R}_k^4$ as we explained in Sec. 2. The draft of the dependance of the number of infinite branches on the function of the number of CH's in the handlebody of $\mathbb{R}_k^4$, is presented in Fig. 3a. Fig. 3b shows the example of the ramified structure of CH's in the precise language of the graphical Kirby calculus (see e.g. Gompf & Stipsicz (1999)).

The general correspondence appears:

One assigns the 4-smooth geometry on $\mathbb{R}^4$ to the k-channel Kondo effect such that k corresponds to the number of infinite branches of CH's in the handlebody. This 4-geometry is $\mathbb{R}_p^4$ where $p = p(k)$, $p, k \in \mathbb{N}$. The change between the physical Kondo states, from this emerging in k_1 channel Kondo effect to this with k_2 channels, $k_1 \neq k_2$, corresponds to the change between 4-geometries, from exotic $\mathbb{R}_{p_1}^4$ to $\mathbb{R}_{p_2}^4$, $p_1 \neq p_2$, $p_1, p_2 \in \mathbb{N}$, such that $p_1 = p_1(k_1)$ and $p_2 = p_2(k_2)$ as above.

Whether actually $p = k$ or not is the question about the level of the $SU(2)$ WZW model and the corresponding fusion rules in use. If $k = p$ the exotic geometry gives the same fusion rules as the Affleck proposed. In the case $k \neq p$ and $k < p$ in the k-channel Kondo effect the fusion rules derived from the exotic geometry are those of the $SU(2)_p$ WZW model. It would be interesting to decide experimentally, which fusion rules apply for bigger k. Probably in higher energies, if the Kondo state survives, the proper fusion rules are those derived from exotic $\mathbb{R}_p^4$. This reflects the situation that electrons in different conduction bands (channels) are generated potentially by (each infinite branch of) Casson handles from the handlebody of

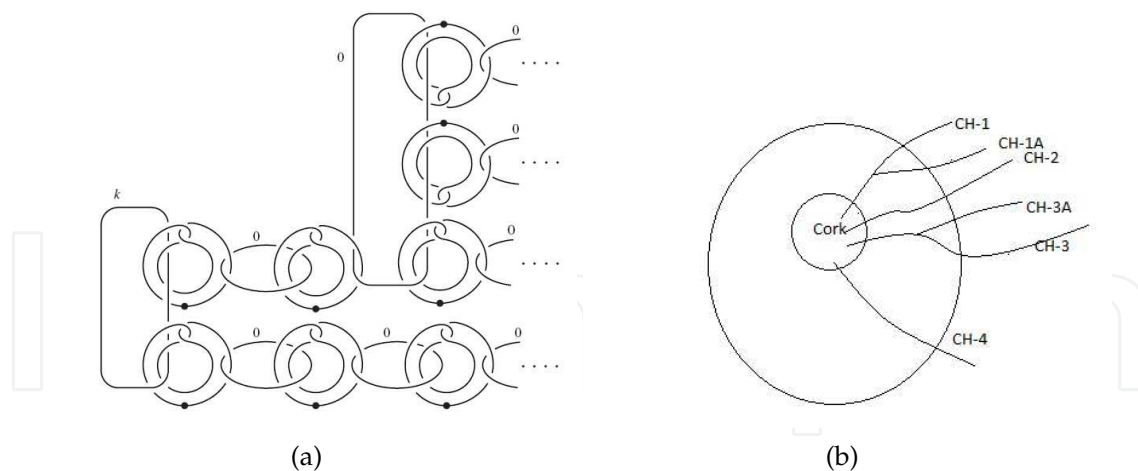<div align="center">(a)   (b)</div>

Fig. 3. (a) (Redrawn from Asselmeyer-Maluga & Brans (2011)) Two CH's, the upper one with 3 infinite branches, the lower one is the simplest CH with the single infinite labeled branch with a single intersection point at every stage. This CH appears in the simplest possible exotic smooth $\mathbb{R}^4$. (b) Schematic structure of the $r = 4$ CH's in the handlebody of exotic $\mathbb{R}_p^4$. Each infinite branch generates a fermion field hence, this exotic $\mathbb{R}_p^4$ can model the Kondo state in the $k = 6$-channels Kondo effect.

the exotic $\mathbb{R}_p^4$, though not every CH generates the actual channel contributing to the Kondo effect. The higher energy more potential CH contributes to the actual electron bands. Then, the fusion rules are given by the exotic geometry, i.e. $SU(2)_p$ WZW model. In suitable high energies 4-geometry (Casson handles) acts as anihilation or creation operator for fermions (electrons). This is the content of our *relativistic fusion rule* (RFR) hypothesis. The experimental confirmation of such discrepancy (between the levels of the WZW model) in high energies in Kondo effect for $p > 2$ channels, would serve as indication for the role of 4-exotic geometry in the relativistic limit of the Kondo state.

Let us illustrate this hypothesis and consider the simplest CH and the simplest exotic $\mathbb{R}^4$ described by Bižaca & Gompf (1996). Suppose this exotic $\mathbb{R}_1^4$ is the member of the radial family and its radius, hence GV invariant of the foliation of S^3, is equal to 1. The corresponding WZ term would be then derived from the $SU(2)_1$ WZW model. Thus, in this case, there is precisely one channel of conducting electrons in the Kondo effect. More complicated exotic $\mathbb{R}_2^4$ could have two CH's in the handlebody and the radii equal to $\sqrt{2}$. Two channels of conducting electrons give rise to the $SU(2)_2$ WZ fusion rules. However, more complicated exotic $\mathbb{R}_p^4$, $p > 2$, could spoil this 1 to 1 correspondence between number of CH's and the number of channels in the Kondo effect.

We have derived the trace of the (exotic) 4-geometry in the low energy Kondo effect. This geometry is probably not physically valid at energies of the Kondo effect (as gravity is not). However, exotic $\mathbb{R}_k^4$ in high energy (and relativistic) limit can become dominating or giving viable physical contributions. These contributions appear when geometric CH's become the actual sources for fermions in KE, thus, changing its CFT structure. In fact the appearance of non-flat $\mathbb{R}_k^4$ when describing the p-channel Kondo effect, indicates a new fundamental link between *matter, geometry* and *gravity* in dimension 4.

5. From smooth geometry of string backgrounds to quantum D-branes

One could wonder what is, if any, suitable sense assigned to geometry of spacetime in various string constructions or backgrounds. As we know the geometry of GR, hence, classical gravity, is the one of (pseudo)-Riemannian differentiable manifolds. String theory has GR (10D Einstein equations) as its classical gravitational limit; however, string theory is the theory of QG and the spacetime geometry should be modified. What is the fate of this (pseudo) Riemannian geometry when gravity is quantized? To answer this question we should find correct classical limit for some quantum string constructions. The proper way is to consider the string backgrounds. These are semi-classical solutions in string theory or supergravity, around which one develops a purturbative theory. GR is not the only ingredient of classical geometry in string theory. There are other fields which are equally fundamental. In type II we have metric $G_{\mu\nu}$, antisymmetric H-field, i.e. three-form $H_{\mu\nu\rho}$, and dilaton Φ. In heterotic strings we have additionally gauge field $F_{\mu\nu}^a$ and the calculations of Sec. 3 made use of these. The presence of B-field such that H is represented by the non-zero cohomology class (see below), is a highly non-trivial fact and indicates that the correct, semi-classical, geometry for string theory is one based on *abelian gerbes* as supplementing Riemannian geometry Król (2010a;b); Segal (2001). Small exotic $\mathbb{R}^4$'s show strong connections with abelian gerbes on S^3 Asselmeyer-Maluga & Król (2009a) which has many important consequences. Some of them are discussed in what follows.

Another crucial feature is the role assigned to D- and NS-branes. Closed string theory, as we made use of it in Sec. 3, is not complete in the sense that there are possible boundary conditions, Dirichlet (D) or Neveu-Schwarz (NS), for open strings, already appearing in closed string theories. These boundary conditions determine geometric subspaces on which open strings can end. In that sense open string theory complements the closed one and predicts the existence of D- or NS-branes. This tame picture of branes as subspaces has only very limited validity. In the quantum regime, or even in the non-zero string coupling g_s, the picture of D-branes as above fails Aspinwall (2004). Nevertheless, interesting proposals were presented recently. They are based on the ideas from non-commutative geometry and aim toward replacing D-branes and spacetime by corresponding (sub) $C^\star$-algebras Brodzki et al. (2008a;b); Szabo (2008). Surprisingly, such an $C^\star$-algebraic setting again shows deep connections with exotic $\mathbb{R}^4$'s.

The appearance of the codimension-one foliations of S^3 in the structure of small exotic $\mathbb{R}^4$, is the key for the whole spectrum of the connections of exotics, beginning with differential geometry and topology, up to non-commutative geometry. This opens very atractive possibilities for exploring both, 1) the classical limit of string geometry, as above and 2) quantum D-branes regim in string theory.

Let us comment on 1) above. The presence of non-zero B-field in a string background is crucial from the point of view of resulting geometry: in σ-model the B-field modifies metric as in (3). Moreover, supposing dilaton is constant and $F_{\mu\nu}^a$ vanishes, the second equation of (8) (the β-function), enforces the background be non-flat, unless $H = dB$ is zero. Given S^3 part of the linear dilaton background as in Sec. 3, we have non-trivial H-field on it. The topological classification of H-fields is given by 3-rd de Rham cohomology classes on background manifold M, $H^3(M, \mathbb{R})$. In order to avoid anomalies we restrict to the integral case $H^3(S^3, \mathbb{Z})$ for $M = S^3$. These classes however are equally generated by exotic $\mathbb{R}_k^4, k \in \mathbb{Z}$ (see Sec. 2). On the other hand, the classification of D-branes in string backgrounds is

governed by K-theory of the background, or in the presence of *H*-field, by, twisted by *H*, K-theory classes. This is briefly summerized in the next subsection where D and NS branes will be understood also classically as subsets in specific CFT backgrounds.

5.1 NS and D branes in type II

Let us consider again the bosonic, i.e. nonsupersymmetric, $SU(2)_k$ WZW model and follow Asselmeyer-Maluga & Król (2011c) closely. The semi-classical limit of it corresponds to taking $k \to \infty$ as in Sec. 3.3. In that limit D-branes in group manifold $SU(2)$ are determined by wrapping the conjugacy classes of $SU(2)$, i.e. are described by 2-spheres S^2's and two poles (degenarate branes) each localized at a point. Owing to the quantization conditions, there are $k + 1$ D-branes on the level k $SU(2)$ WZW model Alekseev & Schomerus (1999b); Fredenhagen & Schomerus (2001); Schomerus (2002). To grasp the dynamics of the branes one should deal with the gauge theory on the stack of N D-branes on S^3, quite similar to the flat space case where noncommutative gauge theory emerges Alekseev & Schomerus (1999a). Let J be the representation of $SU(2)_k$ i.e. $J = 0, \frac{1}{2}, 1, ..., \frac{k}{2}$. The non-commutative action for the dynamics of N branes of type J (on top of each other), in the string regim (k is finite), is then given by:

$$S_{N,J} = S_{YM} + S_{CS} = \frac{\pi^2}{k^2(2J+1)N} \left(\frac{1}{4}\text{tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{i}{2}\text{tr}(f^{\mu\nu\rho}\text{CS}_{\mu\nu\rho}) \right). \tag{29}$$

Here the curvature form $F_{\mu\nu}(A) = iL_\mu A_\nu - iL_\nu A_\mu + i[A_\mu, A_\nu] + f_{\mu\nu\rho}A^\rho$ and the noncommutative Chern-Simons action reads $\text{CS}_{\mu\nu\rho}(A) = L_\mu A_\nu A_\rho + \frac{1}{3}A_\mu[A_\nu, A_\rho]$. The fields A_μ, $\mu = 1, 2, 3$ are defined on a fuzzy 2-sphere S_J^2 and should be considered as $N \times N$ matrix-valued, i.e. $A_\mu = \sum_{j,a} a_{j,a}^\mu Y_a^j$ where Y_a^j are fuzzy spherical harmonics and $a_{j,a}^\mu$ are Chan-Paton matrix-valued coefficients. L_μ are generators of the rotations on fuzzy 2-spheres and they act only on fuzzy spherical harmonics Schomerus (2002). The noncommutative action S_{YM} was derived from Connes spectral triples from the noncommutative geometry, and they will be crucial in grasping quantum nature of D-branes in the next subsection. Originally the action (29) was designed to describe Maxwell theory on fuzzy spheres Carow-Watamura & Watamura (2000). The equations of motion derived from (29) read:

$$L_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0. \tag{30}$$

The solutions of (30) describe the dynamics of the branes, i.e. the condensation processes on the brane configuration (N, J) which results in another configuration (N', J'). A special class of solutions, in the semi-classical $k \to \infty$ limit, can be obtained from the $N(2J+1)$ dimensional representations of the algebra su(2). For $J = 0$ one has N branes of type $J = 0$, i.e. N point-like branes in S^3 at the identity of the group. Given another solution corresponding to $J_N = \frac{N-1}{2}$, one shows that this solution is the condensed state of N point-like branes at the identity of $SU(2)$ Schomerus (2002):

$$(N, J) = (N, 0) \to (1, \frac{N-1}{2}) = (N', J') \tag{31}$$

Turning to the finite k string regime of the $SU(2)$ WZW model one makes use of the techniques of the boundary CFT, the same as was applied to the analysis of Kondo effect in Sec. 4. It follows that there exists a continuous shift between the partition functions governed by the Verlinde fusion rules coefficients $N_{J_N j}^l$: $N\chi_j(q)$ and the sum of characters $\sum_j N_{J_N j}^l \chi_l(q)$ where

$N = 2J_N + 1$. In the case of N point-like branes one can determine the decay product of these by considering open strings ending on the branes. The result on the partition function is

$$Z_{(N,0)}(q) = N^2\chi_0(q)$$

which is continuously shifted to $N\chi_{J_N}(q)$ and next to $\sum_j N_{J_N J_N}^j \chi_j(q)$. As the result, we have the decay process:

$$Z_{(N,0)}(q) \to Z_{(1,J_N)}$$
$$(N,0) \to (1, J_N) \tag{32}$$

which extends the similar process derived at the semi-classical $k \to \infty$ limit (31), and the representations $2J_N$ are bounded now, from the above, by k.

Thus, there are $k + 1$ stable branes wrapping the conjugacy classes numbered by $J = 0, \frac{1}{2}, ..., \frac{k}{2}$. The decaying process (32) says that placing N point-like branes (each charged by the unit 1) at the pole e, they can decay to the spherical brane J_N wrapping the conjugacy class. Taking more point-like branes to the stack at e, gives the more distant S^2 branes until reaching the opposite pole $-e$, where we have single point-like brane with the opposite charge -1. Having identified $k + 1$ units of the charge with -1, we obtain the correct shape of the group of charges, as: $\mathbb{Z}_{k+2}$. More generally, the charges of branes on the background X with non-vanishing $H \in H^3(X, \mathbb{Z})$ are described by the twisted K group, $K_H^\star(X)$. In the case of $SU(2)$, we get the group of RR charges as (for $K = k + 2$):

$$K_H^\star(S^3) = \mathbb{Z}_K \tag{33}$$

Now, based on the earlier discussion from Secs. 2,3, let us place the $S^3 \simeq SU(2)$ above, at the boundary of the Akbulut cork for some exotic smooth $\mathbb{R}^4_k$. Then, we have: *Certain small exotic $\mathbb{R}^4$'s generate the group of RR charges of D-branes in the curved background of $S^3 \subset \mathbb{R}^4$.*

We have yet another important correspondence:

Theorem 2 (Asselmeyer-Maluga & Król (2011c)). *The classification of RR charges of the branes on the background given by the group manifold $SU(2)$ at the level k (hence the dynamics of D-branes in S^3 in stringy regime) is correlated with the exotic smoothness on $\mathbb{R}^4$, containing this $S^3 = SU(2)$ as the part of the boundary of the Akbulut cork.*

Turning to the linear dilaton geometry, as emerging, in the near horizon geometry, from the stack of N NS5-branes in supersymmetric model, i.e. $\mathbb{R}^{5,1} \times \mathbb{R}_\phi \times SU(2)_k$, we obtain next important relation:

Theorem 3 (Asselmeyer-Maluga & Król (2011c)). *In the geometry of the stack of NS5-branes in type II superstring theories, adding or subtracting a NS5-brane is correlated with the change of the smoothness structure on the transversal $\mathbb{R}^4$.*

5.2 Quantum and topological D-branes

The recognition of the role of exotic $\mathbb{R}^4$ in string theory, in the previous and in Sec. 3, relied on the following items:

- Standard smooth $\mathbb{R}^4$ appears as a part of an exact string background;

- The process of changing the exotic smoothness on $\mathbb{R}^4$ is capable of encoding a) the change in the configurations of specific D- and NS branes (Sec. 5.1), b) the change of the 4D part of the string background from flat to curved one in closed string theory (see Sec. 3).

- All exotic $\mathbb{R}^4$'s appearing in this setup are *small exotic* $\mathbb{R}^4$'s, i.e. those which embed smoothly in the standard smooth $\mathbb{R}^4$ as open subsets.

Given the fact that every small exotic $\mathbb{R}^4$ from the radial family (see Sec. 2) determines the codimension-1 foliation of S^3, we have natural $C^\star$-algebra assigned to this 4-exotic. Namely this is the noncommutative convolution $C^\star$-algebra of the foliation. Let us, following Brodzki et al. (2008b), represent every D-brane by suitable separable $C^\star$-algebra replacing, in the same time, spacetime by the corressponding separable $C^\star$-algebra as well. The usual semiclassical embedding of D-branes in spacetime is now reformulated in the language of morphisms between $C^\star$-algebras. In fact, taking into account the isue of stability of D-branes, we define the setup:

1. Fix the (spacetime) $C^\star$ algebra $\mathcal{A}$;

2. A * homomorphism $\phi : \mathcal{A} \to \mathbf{B}(\mathcal{H})$ (a homomorphisms of the algebras preserving their * structure), generates the embedding of the D-brane world-volume M and its noncommutative algebra $\mathcal{A}_M$ as $\mathcal{A}_M := \phi(\mathcal{A})$;

3. D-branes embedded in a spacetime $\mathcal{A}$ are represented by the spectral triple $(\mathcal{H}, \mathcal{A}_M, T)$;

4. Equivalently, a D-brane in A is given by an unbounded Fredholm module $(\mathcal{H}, \phi, T)$.

Thus, the classification of stable D-branes in $\mathcal{A}$ is given by the classification of Fredholm modules $(\mathcal{H}, \phi, T)$ where $\mathbf{B}(\mathcal{H})$ are bounded operators on the separable Hilbert space $\mathcal{H}$ and T the operators corresponding to tachyons. In general, to every foliation (V, F) one can associate its noncommutative $C^\star$ convolution algebra $C^\star(V, F)$. The interesting connection with exotic 4-smoothness then emerges:

Theorem 4. *The class of generalized stable D-branes on the $C^\star$ algebra $C^\star(S^3, F_1)$ (of the codimension 1 foliation of S^3) determines an invariant of exotic smooth $\mathbb{R}^4$,*

and

Theorem 5. *Let e be an exotic $\mathbb{R}^4$ corresponding to the codimension-1 foliation of S^3 which gives rise to the $C^\star$ algebra $\mathcal{A}_e$. The exotic smooth $\mathbb{R}^4$ embedded in e determines a generalized quantum D-brane in $\mathcal{A}_e$.*

It is interesting to note that the tame subspace interpretation of D-branes can be recovered for the special class of the topological quantum D-branes. However, the embedding is replaced now by the wild embedding into spacetime, which historically appeared in the description of the horned Alexander's spheres, known from topology.

Theorem 6. *Let $\mathbb{R}^4_H$ be some exotic $\mathbb{R}^4$ determined by element in $H^3(S^3, \mathbb{R})$, i.e. by a codimension-1 foliation of S^3. Each wild embedding $i : K^3 \to S^p$ for $p > 6$ of a 3-dimensional polyhedron determines a class in $H^n(S^n, \mathbb{R})$ which represents a wild embedding $i : K^p \to S^n$ of a p -polyhedron into S^n.*

Now, a class of *topological quantum Dp-branes* are these branes which are determined by the wild embeddings $i : K^p \to S^n$ as above and in the classical and flat limit correspond to tame embeddings. In fact, B-field on S^3 can be translated into wild embeddings of higher dimensional objects and generates quantum character of these branes.

6. Discussion and conclusions

Superstring theory (ST) appears in fact as very rich mathematics. The mathematics which is designed especially for the reconciling classical gravity, as in GR, with QFT. The richness of mathematics involved is, however, the limitation of the theory. Namely, to yield 4D physics from such huge structure is very non-unique and thus problematic. We followed the idea, proposed at the recent International Congress of Mathematician ICM 2010 [Asselmeyer-Maluga & Król (2010)], that the mathematics of ST refers to and advance understanding of the mathematics of exotic smooth $\mathbb{R}^4$. Conversely, exotic $\mathbb{R}^4$'s provide important information about the mathematics of superstrings. Exotic $\mathbb{R}^4$'s are non-flat geometries, hence contain gravity from the point of view of physics. ST is the theory of QG and gravity of exotic geometries is quantized by methods of ST. The 4-geometries also refer to effective correlated states of condensed matter as in Kondo effect. Thus, the approach presented in this Chapter indicates new fundamental link between *gravity, geometry and matter* at the quantum limit and exclusively in dimension 4. The exotic smoothness of $\mathbb{R}^4$, when underlies the 4-Minkowski spacetime, is a natural way to quantum gravity (given by superstring techniques) from the standard model of particles. On the other hand, exotic $\mathbb{R}^4$'s serve as factor reducing the ambiguity of 10D superstring theory in yielding 4D physical results. The work on these issues should be further pursued.

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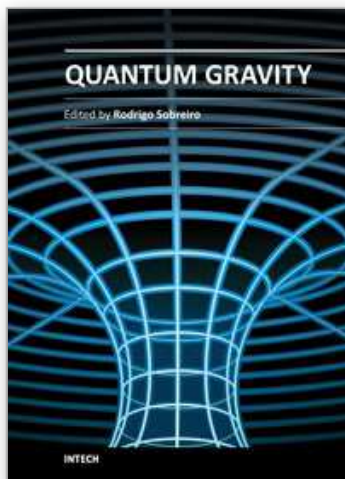
Quantum Gravity

Edited by Dr. Rodrigo Sobreiro

The unification between gravity and quantum field theory is one of the major problems in contemporary fundamental Physics. It exists for almost one century, but a final answer is yet to be found. Although string theory and loop quantum gravity have brought many answers to the quantum gravity problem, they also came with a large set of extra questions. In addition to these last two techniques, many other alternative theories have emerged along the decades. This book presents a series of selected chapters written by renowned authors. Each chapter treats gravity and its quantization through known and alternative techniques, aiming a deeper understanding on the quantum nature of gravity. Quantum Gravity is a book where the reader will find a fine collection of physical and mathematical concepts, an up to date research, about the challenging puzzle of quantum gravity.

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