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# Topological Singularity of Fermion Current in Abelian Gauge Theory

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## 1. Introduction

Since calculations in a quantized theory are often plagued by divergencies, we have to impose a regularization scheme in order to eliminate the singularity from ill-field functions such as the infrared-divergence problem in quantum electrodynamics[1,2]. Among these divergence phenomena, a type of divergence appears in product of quantum operators taken at the same time. The product of the local operators is often singular, which can destroy symmetries of conservation currents and classical equations of motions. For instance, in derivation of the conservation equation for the axial vector current, the differential equation of motion for fermion fields can not be used to reduce fermion-photon vertex in quantum electrodynamics in the most straightforward way[3]. Through standard perturbative treatments of singularity in the operator product of fermion current, a closer look at the manipulations reveals some subtleties. These unavoidable divergences reflect the feature that the symmetry of the classical theory may be ruined in quantum theory by quantum anomaly. Usually, we deal with the divergence of the anomaly by examining the "triangle graph", which consist of an internal fermion loop connected to two vector fields and to one axial vector field. These triangle graphs give rise to anomaly in orders of perturbation theory, which arise from momentum-space integrals. In consequence, the anomaly is affected by interactions of gauge bosons attached to the fermion loop[4,5].

From the path-integral viewpoint, the anomaly term associated with the axial-vector current can be understood in the path-integral formulation of quantum gauge theory as a consequence of the fact that the functional measure is not invariant with respect to the relevant local group transformations on the fields[6]. The chiral anomaly responds to local group transformations but vanishes for a class of space-independent transformations. One discovers that the anomaly has a topological character given by the index of a Dirac operator in a gauge background[1]. The mathematical explanation of such a anomaly is directly related to Atiyah-Singer index theorem. The theorem relates the number of chirality zero modes of Dirac operator to the topological charge of the gauge field. Furthermore, a number of papers have addressed the relationships among chiral anomalies, the geometry of the space of vector potentials and families of Dirac operators[7,8,9]. The authors showed that the first characteristic class of the index bundle for the Dirac operator is related to anomalies. In particular, in the gravitation case, the

Abelian and non-Abelian chiral anomalies in  $2n$  dimensional spacetime is determined by a variation of gauged effective action in differential geometric form which satisfies the Wess-Zumino consistency condition. Obviously, the use of the family's index theorem in the study of gravitational anomalies has the advantage over using Feynman diagram methods, which enables us to obtain the anomalies without having to calculate Feynman diagrams[10]. This implies that singularity in fermion currents connects to the non-perturbative effects of anomalies in the presences of topologically non-trivial field configurations. This sort of problem also implies the index of Dirac operator relates to quantization of the space integral of the anomaly, and the supertrace of the kernel function of the square of a Dirac operator is the quantization of the topological character of the corresponding connection on a manifold[11]. From this point of view, it is natural how to find a means to distinguish the divergence of various fermion currents coupling with gauge field becomes extremely important. In this paper we present an exposition that the topological singularity of operator product of various fermion current in gauge theory can be described in terms of Atiyah-Segal-Singer index theorem for Dirac operator.

## 2. Transformation property of fermion current

Now we are in position to specify a transformation of operator product of fermion current in Abelian quantum field to determine the property of the current. According to the form of fermion current coupling with gauge field  $B_\mu(x)$  in Dirac equation, the fermion current coupling with gauge field is defined as

$$I^{[\mu\nu]} = \bar{\psi}(x) \gamma^\mu \Gamma^{[\nu]} \psi(x) \quad (1)$$

where  $\Gamma^{[\nu]}$  indicates the corresponding Dirac matrix, which is also an element of Dirac algebra,  $\psi(x)$  and  $\bar{\psi}(x)$  is Dirac spinors.

One easily verifies that these fermion currents possess bilinear covariant property under Lorentz transformation. Some of the "products of currents" itself have quantum field theoretical meaning, which is subject to infinitesimal change of variables along the symmetry direction. To do this, in the broad sense, a set of the changing variables of matter field in quantum theory can be transformed as [3]

$$\begin{aligned} \psi'_\alpha(x) &= \psi_\alpha(x) + \delta\psi_\alpha(x) \\ \delta\psi_\alpha(x) &= \theta_{[\nu]}(x) H_\alpha^{[\nu]}(\psi_\beta, \psi_{\beta\lambda}) + \partial_\mu \theta_{[\nu]}(x) h_\alpha^{\mu[\nu]}(\psi_\beta, \psi_{\beta\lambda}) \end{aligned} \quad (2)$$

where the matrices  $H_\alpha^{[\nu]}, h_\alpha^{\mu[\nu]}$  are the functions composed of both Dirac matrix and field variables, the group parameter  $\theta_{[\nu]}(x)$  is a real function.

In the light of Eq.(2), the infinitesimal local transformation of fermion fields in Abelian gauge theory is taken tentatively as [12]

$$\begin{aligned} \psi'(x) &= e^{-i\theta_{[\nu]}(x)\Gamma^{[\nu]}} \psi(x) \\ \bar{\psi}'(x) &= \bar{\psi}(x) \gamma^0 e^{i\theta_{[\nu]}(x)\Gamma^{[\nu]+}} \gamma^0 \end{aligned} \quad (3)$$

In fact, there is no contradiction between Eq.(2) and Eq.(3). In the case of QED, the origin of the Wald-Takahashi identities relative to fermion currents lies in the gauge invariance of the generating functional of QED. We know that the generating functional  $Z(B_\mu(x), \bar{\eta}, \eta)$  remains the same. This is because changing variables in an integral never affects its value. In this sense, we can consider a gauge transformation to be more general type of field redefinition. This is the transformation Eq.(2) as a guiding rule.

Thus, to discuss the property of the fermion current, we need to choose a comparatively simple transformation on the fields for fermion currents. This is transformation Eq.(3), in which the term coupling with the spacetime derivative of  $\theta_{[v]}(x)$  is not included.

To expand the fermion current carefully, let us introduce a complete set of eigenstates  $\phi_n$  of the Dirac operator  $iD = i\gamma^\mu(\partial_\mu - igB_\mu(x))$  as follows

$$iD\phi_n(x) = \lambda_n\phi_n(x) \tag{4}$$

here real  $\lambda_n$  is the discrete eigenvalues. This means the fermion fields can be decomposed in this set of eigenstates

$$\begin{aligned} \psi'(x) &= \sum_n c'_n \phi_n(x) \\ \bar{\psi}'(x) &= \sum_n \bar{c}'_n \phi_n(x) \\ \delta_{nm} &= \int d^4x \phi_n^+ \phi_m \quad (1 \leq n, m \leq d) \end{aligned} \tag{5}$$

where  $d$  stands for the eigenstate space dimension. It is true that the anticommuting coefficients  $c'_n, \bar{c}'_n$  of Dirac spins  $\psi'(x)$  and  $\bar{\psi}'(x)$  span a Grassmann algebra.

Thus under the transformation Eq.(3), the fermion current Eq.(1) is changed into

$$\begin{aligned} I^{[\mu\nu]} &= \bar{\psi}'(x) \gamma^\mu \Gamma^{[\nu]} \psi'(x) \\ &= \sum_{nm} \bar{c}'_n c'_m \phi_n^+ \gamma^\mu \Gamma^{[\nu]} \phi_m \end{aligned} \tag{6}$$

To find the relation between the fermion current and the corresponding integral measure, let's construct the power function  $I^{[\mu\nu]d}$  of the fermion current over  $\psi(x)$  and  $\bar{\psi}(x)$ :

$$\begin{aligned} I^{[\mu\nu]d} &= \left[ \bar{\psi}'(x) \gamma^\mu \Gamma^{[\nu]} \psi'(x) \right]^d \\ &= \sum_{n_1 m_1} \bar{c}'_{n_1} c'_{m_1} \phi_{n_1}^+ \gamma^\mu \Gamma^{[\nu]} \phi_{m_1} \sum_{n_2 m_2} \bar{c}'_{n_2} c'_{m_2} \phi_{n_2}^+ \gamma^\mu \Gamma^{[\nu]} \phi_{m_2} \cdots \sum_{n_d m_d} \bar{c}'_{n_d} c'_{m_d} \phi_{n_d}^+ \gamma^\mu \Gamma^{[\nu]} \phi_{m_d} \end{aligned} \tag{7}$$

where  $d$  denotes power of the operator function  $I^{[\mu\nu]}(x)$ , its value is equal to the dimension of the eigenspace of the regulation operator(see Eq.(4)). In like manner, due to the transformation Eq.(3), the power function  $I^{[\mu\nu]d}(x)$  over  $\psi'(x)$  and  $\bar{\psi}'(x)$  can be expressed as

$$\begin{aligned}
 I^{[\mu\nu]d}(x) &= \left[ \bar{\psi}'(x) \gamma^\mu \Gamma^{[\nu]} \psi'(x) \right]^d \\
 &= \sum_{n_1 m_1} \bar{c}'_{n_1} c'_{m_1} \phi_{n_1}^+ \gamma^\mu \Gamma^{[\nu]} \phi_{m_1} \sum_{n_2 m_2} \bar{c}'_{n_2} c'_{m_2} \phi_{n_2}^+ \gamma^\mu \Gamma^{[\nu]} \phi_{m_2} \cdots \sum_{n_d m_d} \bar{c}'_{n_d} c'_{m_d} \phi_{n_d}^+ \gamma^\mu \Gamma^{[\nu]} \phi_{m_d}
 \end{aligned} \tag{8}$$

Obviously in the light of the property of the Grassmann algebra, the expression (8) implies that  $I^{[\mu\nu]d}(x)$  links with the operator product of the expansion coefficients  $c'_n, \bar{c}'_n$ .

In order to understand the meaning of the operator product composed of  $c'_n$  and  $\bar{c}'_n$  in the expression  $I^{[\mu\nu]d}(x)$ , we now recall the algebraic property of functional measure in the path integral formulation. Under the transformation Eq.(3), the corresponding functional measure  $d\mu'_{[\nu]}$  over  $\psi'(x)$  and  $\bar{\psi}'(x)$  has the transformation property

$$\begin{aligned}
 d\mu'_{[\nu]} &= \prod_n d\bar{c}'_n \prod_m dc'_m \\
 &= J_{[\nu]} d\mu
 \end{aligned} \tag{9}$$

where  $d\mu'$  is the functional measure over  $\psi'(x)$  and  $\bar{\psi}'(x)$ ,  $J_{[\nu]}$  is the Jacobian determinant of the corresponding transformation of the fermion variables, which is a key quantity for anomaly current. Then we see that the Jacobian of differential operator  $d\mu$  connects with the transformation of the operator function  $\mu$ , which is defined as the operator product of the anticommuting coefficient  $c_n, \bar{c}_m$  of  $\psi(x)$  and  $\bar{\psi}(x)$

$$\mu = \prod_n \bar{c}_n \prod_m c_m \tag{10}$$

After the transformation Eq.(3), the expression of the Grassmann operator function  $\mu$  is changed explicitly into

$$\begin{aligned}
 \mu'_{[\nu]} &= \prod_n \bar{c}'_n \prod_m c'_m \\
 &= \prod_n \sum_m \int d^4x \phi_n^+(x) e^{-i\theta_{[\nu]}(x)\Gamma^{[\nu]}} \phi_m \bar{c}_m \prod_l \sum_s \int d^4x \phi_l^+ e^{i\theta_{[\nu]}(x)\Gamma^{[\nu]}} \phi_s c_s \\
 &= \tilde{J}_{[\nu]} \mu \quad (1 \leq n, m, l, s \leq d)
 \end{aligned} \tag{11}$$

where  $\tilde{J}_{[\nu]}$  is the Jacobian determinant of the transformations Eq.(3).

From the properties of the Grassmann algebra we find the identity

$$\tilde{J}_{[\nu]} = J_{[\nu]}^{-1} \tag{12}$$

It shows that the Jacobian  $\tilde{J}_{[\nu]}$  corresponding to the operator product  $\mu$  connects with the inverse of the Jacobian of the corresponding transformation of the integral measure.

Having clarified the relation between the operator product of fermion current and the integral measure, we therefore see that the power function of fermion current Eq.(8) can be rewritten as

$$\begin{aligned}
 I^{[\mu\nu]d}(x) &= \left[ \bar{\psi}'(x) \gamma^\mu \Gamma^{[\nu]} \psi'(x) \right]^d \\
 &= \sum_{n_1 m_1} \bar{c}'_{n_1} c'_{m_1} \phi_{n_1}^+ \gamma^\mu \Gamma^{[\nu]} \phi_{m_1} \sum_{n_2 m_2} \bar{c}'_{n_2} c'_{m_2} \phi_{n_2}^+ \gamma^\mu \Gamma^{[\nu]} \phi_{m_2} \cdots \sum_{n_d m_d} \bar{c}'_{n_d} c'_{m_d} \phi_{n_d}^+ \gamma^\mu \Gamma^{[\nu]} \phi_{m_d} \\
 &= \sum_{\substack{n_1 \cdots n_d \\ m_1 \cdots m_d}} \bar{c}'_{n_1} c'_{m_1} \bar{c}'_{n_2} c'_{m_2} \bar{c}'_{n_d} c'_{m_d} \phi_{n_1}^+ \gamma^\mu \Gamma^{[\nu]} \phi_{m_1} \phi_{n_2}^+ \gamma^\mu \Gamma^{[\nu]} \phi_{m_2} \cdots \phi_{n_d}^+ \gamma^\mu \Gamma^{[\nu]} \phi_{m_d} \\
 &= \mu_{[\nu]} \sum_{\substack{n_1 \cdots n_d \\ m_1 \cdots m_d}} \phi_{n_1}^+ \gamma^\mu \Gamma^{[\nu]} \phi_{m_1} \phi_{n_2}^+ \gamma^\mu \Gamma^{[\nu]} \phi_{m_2} \cdots \phi_{n_d}^+ \gamma^\mu \Gamma^{[\nu]} \phi_{m_d} \\
 &= J_{[\nu]}^{-1} \mu \sum_{\substack{n_1 \cdots n_d \\ m_1 \cdots m_d}} \phi_{n_1}^+ \gamma^\mu \Gamma^{[\nu]} \phi_{m_1} \phi_{n_2}^+ \gamma^\mu \Gamma^{[\nu]} \phi_{m_2} \cdots \phi_{n_d}^+ \gamma^\mu \Gamma^{[\nu]} \phi_{m_d} \\
 &= J_{[\nu]}^{-1} I^{[\mu\nu]d}(x)
 \end{aligned}
 \tag{13}$$

This is the desired result, which is just the transformation identity of fermion current. The expression illustrates that the property of fermion current is presented in its functional Jacobian through the operator product function  $\mu$ . In other words, the mathematical property of fermion current can be characterized by the nature of Jacobian of functional integral measure due to transformation of fermion field. For instance, the process of regularization of Jacobian in the functional measure gives rise to the anomaly for the axial vector current. Its topological property of the anomaly term is described by Atiyah-Singer index theorem for Dirac operator.

### 3. Topological property of fermion current

From the topological viewpoint, the non-perturbative effect of the Abelian anomaly associated with Ward-Takahashi type's identity for axial vector current is related to the topological character in the presence of topologically non-trivial field configuration [14,15]. The topological exposition of the quantum anomaly for the current is addressed by Atiyah-Singer index theorem in a gauge background. The Abelian anomaly term can be derived by using path integral formulation in Euclidean spacetime  $E^4$ . Of particular interest is applications of the index theorem for Dirac operator, because it can be used to discuss the topological property of fermion current.

First restricting ourselves to the axial-vector current case, we calculate the analytical index and topological index for Dirac operator. For the following discussion, think of Euclidean spacetime  $E^4$  as a four-dimensional space-time. The spacetime manifold  $E^4$  is compacted into 4-dimensional sphere manifold  $S^4$ .

If  $E$  is a vector bundle on  $S^4$ , let  $\Gamma(S^4, E)$  be the space of smooth section of  $E$ . Thus In terms of  $\gamma^5$ , the Dirac spin space  $\phi$  is decomposed into two subspace  $\phi_+(E^+)$  and  $\phi_-(E^-)$ , in which the eigenvectors of Dirac operator can be chosen eigenvectors of definite chirality with eigenvalue  $\pm 1$ . That is

$$iD\phi_n = 0, \quad \gamma^5 \phi_n = \pm \phi_n
 \tag{14}$$

Naturally the analytical index definition  $Ind_a D$  of  $D$  to be integer is defined as [11,16,17]

$$\begin{aligned} Ind_a D &= \dim(Ker(D^+)) - \dim(Ker(D^-)) \\ &= n_+ - n_- \end{aligned} \quad (15)$$

where  $D^\pm$  is the restriction of  $D$  to  $\Gamma(S^4, E^\pm)$ ,  $\dim(Ker(D^+))$  is the dimension of the kernel  $Ker(D^+)$ , here  $n_\pm$  are the numbers of zero modes of the Dirac operator.

In the mean time, in explicitly performing the regulariaztion for anomaly function, we find a local expression giving the analytical index of the Dirac operator

$$\begin{aligned} Ind_a D &= \int d^4x \sum_n \phi_n^+(x) \Gamma^{[5]} \phi_n(x) \\ &= \int d^4x A^{[5]}(x) \\ &= -\frac{1}{32\pi^2} \int d^4x \varepsilon^{\mu\nu\rho\sigma} Trace(F_{\mu\nu} F_{\rho\sigma}) \\ &= -\frac{4}{32\pi^2} \int d^4x Trace \partial_\mu (\varepsilon^{\mu\nu\rho\sigma} B_\nu \partial_\rho B_\sigma) \end{aligned} \quad (16)$$

with "trace" here denoting a trace only over indices labeling the various fermion species, which could consider as a regularization of the trace in function space.  $F_{\mu\nu}$  is the field strength, and  $A^{[5]}(x)$  is the anomaly function.

This formula shows that the index of  $D$  is given in terms of the restriction to the diagonal of the kernel of  $D^2$  at arbitrarily large masses, which we know by the asymptotic expansion of the kernel to be given by local formula in the curvature of the connection. In the above equation, the following identity is used

$$\varepsilon^{\mu\nu\rho\sigma} (F_{\mu\nu} F_{\rho\sigma}) = 4 \partial_\mu (\varepsilon^{\mu\nu\rho\sigma} B_\nu \partial_\rho B_\sigma) \quad (17)$$

The fact reveals the important information that the integral of the local anomaly function  $A^{[5]}(x)$  can not change smoothly under variations in the gauge field. Since the anomaly function in Eq.(16) can be written as a total derivative, the space integral of the anomaly function depends only the behavior of the gauge field at the boundaries. That is, there exists a number of singularities of the  $E$  vector bundle on the manifold.

Since  $D$  is an elliptic operator, the Atiyah-Singer local index theorem gives a formula for the topological index  $Ind_t D$  of  $D$  in terms of the Chern character of  $ChE$  and the  $\hat{A}$ -genus of  $E^4$  [18,19]

$$Ind_a D = Ind_t D \quad (18)$$

So that [11]

$$Ind_t D = \int_M \hat{A}(M) Ch(E) \quad (19)$$

Here the more natural definition of  $\hat{A}$ -genus form with respect the Riemannian curvature  $R$  is

$$\hat{A}(M) = \det^{1/2} \left( \frac{R/2}{\sinh(R/2)} \right) \quad (20)$$

and the Chern character form is

$$Ch(E) = Str \left( e^{-\nabla^2} \right) \quad (21)$$

Substituting this and Eq.(20) into Eq.(19) yields

$$Ind_t D = -\frac{4}{32\pi^2} \int d^4x \text{Trace} \partial_\mu \left( \varepsilon^{\mu\nu\rho\sigma} B_\nu \partial_\rho B_\sigma \right) \quad (22)$$

Obviously, the quantity on the right of the Eq.(22) is known as Chern-Pontrjagin term.

Since the anomaly function  $A^{[5]}(x)$  does not vanish in the Jacobian determinant  $J$ , we have the explicit identity

$$Ind_a D = \int d^4x \frac{1}{2i} \frac{\delta J}{\delta \theta} \Big|_{\theta=0} \quad (23)$$

The gauge group parameter  $\theta_{[v]}(x)$  is independent of the topological index.

The Abelian anomaly arisen from axial vector current is a local quantity, because it is a consequence of short distance singularities, which does violate the chiral symmetry. That is, the topological singularity of operator product of fermion current Eq.(13) is presented by Jacobian of the measure due to the transformation of fermion fields.

Now we turn to calculations of anomaly function in the transformation of measure for other fermion currents. Due to the Eq.(3), the anomaly function  $A^{[v]}(x)$  corresponding to the defined fermion currents Eq.(1) is given by the formula (see appendix)

$$\begin{aligned} A^{[v]}(x) &= \sum_n \phi_n^+(x) \Gamma^{[v]} \phi_n(x) \\ &= \langle x | \Gamma^{[v]} | x \rangle \end{aligned} \quad (24)$$

In virtue of path-integral technique, the anomaly emerging from the functional measure for the fermions arises from regularization of the covariant derivative acting on the fermions. Thus the anomaly function becomes

$$A^{[v]}(x) = \lim_{M \rightarrow \infty} \langle x | \Gamma^{[v]} f \left( \frac{-D^2}{M^2} \right) | x \rangle \quad (25)$$

with regulator  $f \left( \frac{-D^2}{M^2} \right) = e^{-\frac{D^2}{M^2}}$ .

We noticed that the quantity  $\langle x | \Gamma^{[\nu]} f \left( \frac{-D^2}{M^2} \right) | x \rangle$  is a kernel function of Dirac operator, which is also a section of the endomorphisms  $End(E)$  on Euclidean manifold  $E^4$ .

In analogy to the axial-vector current case, we have a generalization of the local index theorem for  $D$  which enable us to calculate the  $\langle x | \Gamma^{[\nu]} f \left( \frac{-D^2}{M^2} \right) | x \rangle$  for arbitrary  $\Gamma^{[\nu]}$ . Here the element  $\Gamma^{[\nu]}$  (Dirac matrix) lies in a compact group, which is a representation of Clifford algebra.

According to the local Atiyah-Segal-Singer index theorem, the existence of an asymptotic expansion for  $\langle x | \Gamma^{[\nu]} f \left( \frac{-D^2}{M^2} \right) | x \rangle$  at  $M \rightarrow \infty$  is presented by the index  $Ind(\Gamma^{[\nu]}, D)$  of Dirac operator  $D$  on the manifold as follows

$$\begin{aligned} Ind(\Gamma^{[\nu]}, D) &= \lim_{M \rightarrow \infty} \int d^4x \langle x | \Gamma^{[\nu]} f \left( \frac{-D^2}{M^2} \right) | x \rangle \\ &= \int d^4x A^{[\nu]}(x) \end{aligned} \quad (26)$$

Here the kernel function in Eq.(26) is no other than the anomaly function for Jacobian of functional measure.

This is what we do. It is shown that the index theorem for the operator  $\Gamma^{[\nu]} e^{-D^2/M^2}$  associated to the square of Dirac operator  $D$  is a statement about the relationship between the kernel and the topological character of the associated connection in gauge theory.

Fortunately, the asymptotic expansion of the kernel of the operator  $\Gamma^{[\nu]} e^{-D^2/M^2}$  on the right of the Eq.(26) can be evaluated by Fujikawa's method (see appendix) for these fermion currents.

The computation shows that anomaly functions for many fermion currents vanish. In other words, there is no topological singularity for these currents.

#### 4. An Example of Abelian gauge theory

As one see from the following quantum electrodynamics case, the above argument provides a approach to examine the topological singularity in the operator products of the fermion current. Let us consider an Abelian gauge field to show our argument. The Lagrangian density based on the minimal coupling ansatz for quantum electrodynamics is the following [4]

$$L_{eff} = \bar{\psi}(x) \gamma^\mu (\partial_\mu - igB_\mu(x)) \psi(x) - \bar{\psi}(x) m \psi(x) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\xi} (\partial^\mu B_\mu)^2 \quad (27)$$

where  $g$  and  $m$  denote, respectively, the charge and mass of the electron. In this case, the gauge field is just the photon field  $B_\mu(x)$ ,  $\xi$  is the gauge constant.

Then the generating functional of QED is given by

$$Z(J_\mu, \bar{\eta}, \eta) = \int D(B_\mu, \bar{\psi}, \psi) e^{i \int d^4x [L + B_\mu J^\mu + \bar{\eta} \psi + \bar{\psi} \eta]} \tag{28}$$

Noted that the path integral method allows us to generalize WT identities relative to fermion currents, because the functional  $Z(B_\mu, \bar{\eta}, \eta)$  is gauge invariant under gauge transformations. This means that the fact holds irrespective whether these re-naming field variables take the form of symmetry of action, or not. Therefore, this implicitly expect that a anomaly might take place at the quantum measure, which is the failure of a class symmetry to survive the process of quantization and regularization. Normally, under gauge transformation, the determinant in the measure transformation is discarded because it appears to be a constant. However, closer analysis of this term shows that it is actually divergent and here requires regularization.

Following Fujikawa's prescription[6,20], the regularization procedure for the variation of the integral measure can provide access to a wider class of such anomaly objects. In order to analyze the topology property of various fermion current, the anomaly functions  $A^{[v]}(x)$  corresponding to Dirac matrix  $\Gamma^{[v]}$  in the transformation Eq.(3) can be written as the limit of a manifestly convergent integral[14]

$$A^{[v]}(x) = \lim_{M \rightarrow \infty} \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \Gamma^{[v]} f \left( \frac{(iD)^2}{M^2} \right) e^{ikx} \tag{29}$$

$$\bar{A}^{[v]}(x) = \lim_{M \rightarrow \infty} \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \gamma^0 \Gamma^{[v]+} \gamma^0 f \left( \frac{(iD)^2}{M^2} \right) e^{ikx} \tag{30}$$

Thus the Jacobian  $J_{[v]}$  of measure can be put in the form

$$J_{[v]} = e^{i \int d^4x A^{[v]}(x) \theta_{[v]}(x) - i \int d^4x \bar{A}^{[v]}(x) \theta_{[v]}(x)} \tag{31}$$

The corresponding Jacobian  $J_{[v]}$  is evaluated below for various fermion currents

i.  $\Gamma^{[1]} = I$  (unit - matrix);

$$A^{[1]}(x) = \frac{-i}{2^9 \pi^2} \text{tr} (F_{\mu\nu} F_{\lambda\rho}) (g_{\mu\rho} g_{\nu\lambda} - g_{\mu\lambda} g_{\nu\rho}) \text{ and } J_{[1]} = 1. \tag{32}$$

ii.  $\Gamma^{[5]} = \gamma^{[5]}$ ;  $A^{[5]}(x) = \frac{i}{16\pi^2} \text{tr} (F_{\mu\nu} \tilde{F}_{\rho\sigma})$  with  $\tilde{F}_{\rho\sigma}(x) = -\frac{i}{2} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}$  ;

$$J_{[5]} = e^{-\int d^4x \theta_5(x) \left( \frac{i}{16\pi^2} \text{tr} (F_{\mu\nu} \tilde{F}_{\rho\sigma}) \right)} \tag{33}$$

$$\text{iii. } \Gamma^{[v]} = \gamma^\mu ;$$

$$A^{[\mu]}(x) = 0 \quad \text{and} \quad J_{[\mu]} = 1. \quad (34)$$

$$\text{iv. } \Gamma^{[\mu 5]} = \gamma^\mu \gamma_5$$

$$A^{[\mu 5]}(x) = 0 \quad \text{and} \quad J_{[\mu 5]} = 1. \quad (35)$$

$$\text{v. } \Gamma^{[\lambda \mu \nu 5]} = \varepsilon^{\lambda \mu \nu \rho} \gamma_\rho \gamma_5 ;$$

$$A^{[\lambda \mu \nu 5]}(x) = 0 \quad \text{and} \quad J_{[\lambda \mu \nu]} = 1. \quad (36)$$

$$\text{vi. } \Gamma^{[\mu \nu]} = \sigma^{\mu \nu} ; \quad A^{[\mu \nu]}(x) = 0$$

$$J_{[\mu \nu]} = 1 \quad (37)$$

$$\text{vii. } \Gamma^{[\mu \nu 5]} = \sigma^{\mu \nu} \gamma_5 \quad A^{[\mu \nu 5]}(x) = -\bar{A}^{[\mu \nu 5]}(x)$$

$$J_{[\mu \nu 5]} = 1 \quad (38)$$

The above computation shows that the anomaly function in Jacobian vanishes for many fermion currents, (in other words, they cancel each other in regularization).

For simplicity, we take a two-dimensional eigenspace ( $d=2$ ) of the regulation operator

$e^{-\frac{D^2}{M^2}}$  as an following example to illustrate in detail the link between the Jacobian of the integral measure and the topological property of the fermion current. In the path-integral formulation of gauge theory, the functional measure over  $\psi'(x)$  and  $\bar{\psi}'(x)$ . can be read off

$$d\mu'_{[v]} = d\bar{c}'_1 d\bar{c}'_2 dc'_1 dc'_2 \quad (39)$$

and operator product  $\mu'$  in Eq.(9) is given by

$$\begin{aligned} \mu'_{[v]} &= \bar{c}'_1 \bar{c}'_2 c'_1 c'_2 \\ &= \tilde{J}'_{[v]} \mu \end{aligned} \quad (40)$$

with the Grassmann expansion coefficients of  $\psi'(x)$  and  $\bar{\psi}'(x)$

$$\begin{aligned}
 c'_1 &= c_1 \int d^4x \phi_1^+ e^{-i\theta_{[v]}(x)\Gamma^{[v]}} \phi_1 + c_2 \int d^4x \phi_1^+ e^{-i\theta_{[v]}(x)\Gamma^{[v]}} \phi_2 \\
 c'_2 &= c_1 \int d^4x \phi_2^+ e^{-i\theta_{[v]}(x)\Gamma^{[v]}} \phi_1 + c_2 \int d^4x \phi_2^+ e^{-i\theta_{[v]}(x)\Gamma^{[v]}} \phi_2 \\
 \bar{c}'_1 &= \bar{c}_1 \int d^4x \phi_1^+ \gamma^0 e^{i\theta_{[v]}(x)\Gamma^{[v]\dagger}} \gamma^0 \phi_1 + \bar{c}_2 \int d^4x \phi_1^+ \gamma^0 e^{-i\theta_{[v]}(x)\Gamma^{[v]\dagger}} \gamma^0 \phi_2 \\
 \bar{c}'_2 &= \bar{c}_1 \int d^4x \phi_1^+ \gamma^0 e^{i\theta_{[v]}(x)\Gamma^{[v]\dagger}} \gamma^0 \phi_2 + \bar{c}_2 \int d^4x \phi_2^+ \gamma^0 e^{-i\theta_{[v]}(x)\Gamma^{[v]\dagger}} \gamma^0 \phi_2
 \end{aligned}
 \tag{41}$$

The corresponding expansion of the fermion current can be performed in the eigenspace

$$\begin{aligned}
 I^{[\mu\nu]'}(x) &= (\bar{c}'_1\phi_1^+ + \bar{c}'_2\phi_2^+) \gamma^\mu \Gamma^{[v]} (c'_1\phi_1 + c'_2\phi_2) \\
 &= \bar{c}'_1 c'_1 \phi_1^+ \gamma^\mu \Gamma^{[v]} \phi_1 + \bar{c}'_1 c'_2 \phi_1^+ \gamma^\mu \Gamma^{[v]} \phi_2 + \bar{c}'_2 c'_1 \phi_2^+ \gamma^\mu \Gamma^{[v]} \phi_1 + \bar{c}'_2 c'_2 \phi_2^+ \gamma^\mu \Gamma^{[v]} \phi_2
 \end{aligned}
 \tag{42}$$

In accordance with Eq.(11), the square of the fermion current  $I^{[\mu\nu]'}(x)$  is written out

$$\begin{aligned}
 I^{[\mu\nu]'^2}(x) &= (\bar{c}'_1\phi_1^+ + \bar{c}'_2\phi_2^+) \gamma^\mu \Gamma^{[v]} (c'_1\phi_1 + c'_2\phi_2) (\bar{c}'_1\phi_1^+ + \bar{c}'_2\phi_2^+) \gamma^\mu \Gamma^{[v]} (c'_1\phi_1 + c'_2\phi_2) \\
 &= \bar{c}'_1 \bar{c}'_1 c'_1 c'_1 \phi_1^+ \gamma^\mu \Gamma^{[v]} \phi_1 \phi_1^+ \gamma^\mu \Gamma^{[v]} \phi_1 + \bar{c}'_2 \bar{c}'_1 c'_1 c'_1 \phi_2^+ \gamma^\mu \Gamma^{[v]} \phi_1 \phi_1^+ \gamma^\mu \Gamma^{[v]} \phi_1 + \dots \\
 &\quad + \bar{c}'_2 \bar{c}'_2 c'_2 c'_2 \phi_2^+ \gamma^\mu \Gamma^{[v]} \phi_2 \phi_2^+ \gamma^\mu \Gamma^{[v]} \phi_2
 \end{aligned}
 \tag{43}$$

Further by taking the property of Grassmann algebra into account, the calculus leads straightforwardly to the following result

$$\begin{aligned}
 I^{[\mu\nu]'^2}(x) &= 2\bar{c}'_1 \bar{c}'_2 c'_1 c'_1 \phi_1^+ \gamma^\mu \Gamma^{[v]} \phi_1 \phi_2^+ \gamma^\mu \Gamma^{[v]} \phi_2 + 2\bar{c}'_1 \bar{c}'_2 c'_1 c'_2 \phi_1^+ \gamma^\mu \Gamma^{[v]} \phi_2 \phi_2^+ \gamma^\mu \Gamma^{[v]} \phi_1 \\
 &= -2\mu'_\nu \phi_1^+ \gamma^\mu \Gamma^{[v]} \phi_1 \phi_2^+ \gamma^\mu \Gamma^{[v]} \phi_2 + 2\mu'_\nu \phi_1^+ \gamma^\mu \Gamma^{[v]} \phi_2 \phi_2^+ \gamma^\mu \Gamma^{[v]} \phi_1 \\
 &= -2\tilde{J}_{[v]} \mu \phi_1^+ \gamma^\mu \Gamma^{[v]} \phi_1 \phi_2^+ \gamma^\mu \Gamma^{[v]} \phi_2 + 2\tilde{J}_{[v]} \mu \phi_1^+ \gamma^\mu \Gamma^{[v]} \phi_2 \phi_2^+ \gamma^\mu \Gamma^{[v]} \phi_1 \\
 &= -2J_{[v]}^{-1} \mu \phi_1^+ \gamma^\mu \Gamma^{[v]} \phi_1 \phi_2^+ \gamma^\mu \Gamma^{[v]} \phi_2 + 2J_{[v]}^{-1} \mu \phi_1^+ \gamma^\mu \Gamma^{[v]} \phi_2 \phi_2^+ \gamma^\mu \Gamma^{[v]} \phi_1 \\
 &= J_{[v]}^{-1} \left( -2\mu \phi_1^+ \gamma^\mu \Gamma^{[v]} \phi_1 \phi_2^+ \gamma^\mu \Gamma^{[v]} \phi_2 + 2\mu \phi_1^+ \gamma^\mu \Gamma^{[v]} \phi_2 \phi_2^+ \gamma^\mu \Gamma^{[v]} \phi_1 \right) \\
 &= J_{[v]}^{-1} I^{[\mu\nu]'^2}(x)
 \end{aligned}
 \tag{44}$$

From this example, we see clearly that the topological property of products of the fermion currents  $I^{[\mu\nu]'^2}(x)$  (i.e.  $I^{[\mu\nu]'}(x)$ ) relates with the corresponding Jacobian factor. Combining the evaluated Jacobians of various fermion currents Eq. (32~38) and the Atiyah-Singer local index theorem Eq.(26), we find that only axial-vector current  $I^{[\mu 5]}(x) = \bar{\psi}(x) i\gamma^\mu \gamma^5 \psi(x)$  has topological singularity, which is coincident with previous analysis made by perturbational methods.

### 5. Conclusion

We have presented that the topological singularity in operator product of various fermion currents coupling to a gauge field is characterized by the topological properties of anomaly function in a quantum gauge background in terms of Atiyah-Singer index theorem for Dirac

operator. The anomaly functions corresponding to various fermion currents have been evaluated through the calculus of the kernel of Dirac operator.

As the above illustration, the topological singularity of various fermion currents coupling gauge field is indeed understood on Atiyah-Singer index theorem in quantum field theory as a consequence of the fact that the Jacobian of integration measure possesses anomaly terms. That is, the kernel of the Dirac operator may have short distance singularities.

No doubt, the singularity of the fermion current has to be considered when dealing with reduction of the interaction vertex by using the Dirac differential equation of motion in the Dyson-Schwinger equation. Also in the non-abelian gauge case, the relation of anomalies in conservation of general axial currents to the index of the Dirac operator in a gauge background needs to be discussed further. In addition, the property of quantum anomaly associated with Ward-Takahashi relation plays an important role in the nonperturbative study of gauge theories, such as the dynamical chiral symmetry breaking.

## 6. Appendix: Regularization of measure for anomaly

### a. regularization prescription for low-rank tensor current $(\bar{\psi}(x)\gamma^\mu\Gamma^{[\nu]}\psi(x))$

Following Fujikawa's method, we work out a calculation of Jacobian of anomaly function in the transformation of measure due to the Eq.(3). The change in functional measure is in the form

$$d\mu \rightarrow d\mu' = \prod_n d\bar{c}_n' \prod_n dc_n' = (\det f_{nm})^{-1} (\det f'_{nm})^{-1} \prod_m d\bar{c}_m \prod_m dc_m \quad (A1)$$

with

$$f_{nm} = \int d^4x \phi_n^+(x) e^{-i\theta_{[\nu]}(x)\Gamma^{[\nu]}} \phi_m(x). \quad (A2)$$

The corresponding Jacobian factor is given explicitly by

$$(\det f'_{nm})^{-1} = e^{-i \int d^4x \gamma^0 A^{[\nu]+}(x) \gamma^0 \theta_{[\nu]}(x)}, \quad (A3)$$

where anomaly function  $A^{[\nu]}(x)$  denotes the trace of Dirac matrix  $\Gamma^{[\nu]}$  in the function space above

$$A^{[\nu]}(x) = \sum_n \phi_n^+(x) \Gamma^{[\nu]} \phi_n(x). \quad (A4)$$

Based on the use of path integrals in Euclidean space, regularization of the anomaly function is achieved by inserting the convergent factor  $f\left(-\frac{D^2}{M^2}\right) = e^{-D^2/M^2}$  and taking the limit as

$M \rightarrow \infty$ . To do this the above anomaly function can be written as the limit of a manifestly convergent integral

$$\begin{aligned}
 A^{[\nu]}(x) &= \lim_{M \rightarrow \infty} \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \Gamma^{[\nu]} f\left(\frac{-D^2}{M^2}\right) e^{ikx} \\
 &= \lim_{M \rightarrow \infty} \frac{1}{32M^2} \text{Tr} \left[ -\Gamma^{[\nu]} \left( [\gamma^\mu, \gamma^\nu] F_{\mu\nu} \right)^2 \right] \int \frac{d^4k}{(2\pi)^4} f''\left(\frac{k^2}{M^2}\right)
 \end{aligned}
 \tag{A5}$$

In terms of the trace over Dirac indices, when we expand the regularization operator  $f(-D^2 / M^2)$  in  $D$ , the terms with less than four gamma matrices evaluate to zero. In the light of the limit over  $M$  (mass), terms with more than four  $D$  s will also drop out.

In the last step, we have used the operator identities

$$(\gamma^\mu D_\mu)^2 = D_\mu D_\mu - i \frac{1}{4} [\gamma^\mu, \gamma^\nu] F_{\mu\nu}
 \tag{A6}$$

$$f\left(\frac{(i\gamma^\mu D_\mu)^2}{M^2}\right) = \sum_n \frac{1}{n!} f^{(n)}\left(\frac{D_\mu^2}{M^2}\right) \left(-i \frac{1}{4M^2} [\gamma^\mu, \gamma^\nu] F_{\mu\nu}\right)^n
 \tag{A7}$$

In the case of  $\Gamma^{[\nu]} = \gamma^5$  for the chiral anomaly, the  $A^{[5]}(x)$  becomes into

$$\begin{aligned}
 A^{[5]}(x) &= \lim_{M \rightarrow \infty} \frac{1}{32M^2} \text{Tr} \left[ -\gamma^5 \left( [\gamma^\mu, \gamma^\nu] F_{\mu\nu} \right)^2 \right] \int \frac{d^4k}{(2\pi)^4} f''\left(\frac{k^2}{M^2}\right) \\
 &= -\frac{1}{32M^2} \varepsilon^{\mu\nu\rho\sigma} \text{Tr} [F_{\mu\nu} F_{\rho\sigma}]
 \end{aligned}
 \tag{A8}$$

It is just the well-known result.

**b. regularization description for high rank tensor current**

In a similar way, we discuss the regularization of the anomaly fraction  $A^{[\nu]}(x)$  for high rank Dirac matrix transformation. The expression of the anomaly function  $A^{[\alpha\beta]}(x)$  for the case of Eq.(37) can be put in the regulating form

$$\begin{aligned}
 A^{[\alpha\beta]}(x) &= \lim_{M \rightarrow \infty} \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \sigma^{\alpha\beta} f\left(\frac{-D^2}{M^2}\right) e^{ikx} \\
 &= \lim_{M \rightarrow \infty} \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \sigma^{\alpha\beta} \sum_n \frac{1}{n!} f^{(n)}\left(\frac{D_\mu^2}{M^2}\right) \left(-i \frac{1}{4M^2} [\gamma^\mu, \gamma^\nu] F_{\mu\nu}\right)^n e^{ikx} \\
 &= \lim_{M \rightarrow \infty} \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \text{Tr} \left[ \sigma^{\alpha\beta} f^{(2)}\left(\frac{-D^2}{M^2}\right) \left(\frac{i}{4M^2} [\gamma^\mu, \gamma^\nu] F_{\mu\nu}\right)^2 \right] \\
 &= \frac{-1}{32\pi^2} \left( g^{\alpha\mu} g^{\beta\rho} g^{\nu\sigma} - g^{\alpha\mu} g^{\beta\sigma} g^{\nu\rho} + g^{\alpha\nu} g^{\beta\sigma} g^{\mu\rho} - g^{\alpha\nu} g^{\beta\rho} g^{\mu\sigma} \right. \\
 &\quad \left. + g^{\beta\mu} g^{\alpha\sigma} g^{\nu\rho} - g^{\beta\mu} g^{\alpha\rho} g^{\nu\sigma} + g^{\beta\nu} g^{\alpha\rho} g^{\mu\sigma} - g^{\beta\nu} g^{\alpha\sigma} g^{\mu\rho} \right) \text{Tr} [F_{\mu\nu} F_{\rho\sigma}]
 \end{aligned}
 \tag{A9}$$

In terms of the symmetry of metric and antisymmetry of 4-dimensional field strength tensor, we expand the anomaly function and find that it equals zero. So that, the Jacobian becomes

$$J^{[\alpha\beta]} = 1 \quad (\text{A10})$$

By the parallel procedure, for the case of the transformation Eq.(38), the axial vector anomaly function is given by

$$\begin{aligned} A^{[\alpha\beta 5]}(x) &= \lim_{M \rightarrow \infty} \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \sigma^{\alpha\beta} \gamma^5 f \left( \frac{(i\gamma^\mu D_\mu)^2}{M^2} \right) e^{ikx} \\ &= \lim_{M \rightarrow \infty} \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \sigma^{\alpha\beta} \gamma^5 \sum_n \frac{1}{n!} f^{(n)} \left( \frac{D^2}{M^2} \right) \left( -i \frac{1}{4M^2} [\gamma^\mu, \gamma^\nu] F_{\mu\nu} \right)^n e^{ikx} \\ &= \lim_{M \rightarrow \infty} \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \text{Tr} \left[ \sigma^{\alpha\beta} \gamma^5 f^{(2)} \left( \frac{-D^2}{M^2} \right) \left( \frac{i}{4M^2} [\gamma^\mu, \gamma^\nu] F_{\mu\nu} \right)^2 \right] \\ &= \frac{-i}{32\pi^2} \left( -g^{\alpha\mu} \varepsilon^{\beta\nu\rho\sigma} + g^{\alpha\nu} \varepsilon^{\beta\mu\rho\sigma} + g^{\beta\mu} \varepsilon^{\alpha\nu\rho\sigma} - g^{\beta\nu} \varepsilon^{\alpha\mu\rho\sigma} \right) \\ &\quad \left( +g^{\nu\rho} \varepsilon^{\alpha\beta\mu\sigma} - g^{\mu\rho} \varepsilon^{\alpha\beta\nu\sigma} + g^{\mu\sigma} \varepsilon^{\alpha\beta\nu\rho} - g^{\nu\sigma} \varepsilon^{\alpha\beta\mu\rho} \right) \text{Tr} [F_{\mu\nu} F_{\rho\sigma}] \\ &= -\bar{A}^{[\alpha\beta 5]}(x) \end{aligned} \quad (\text{A11})$$

The corresponding Jacobian is

$$J^{[\alpha\beta 5]} = 1 \quad (\text{A12})$$

In the above calculation, we have employed the following operator identities

$$\begin{aligned} \text{Tr} \left( \prod_{j=1}^6 \gamma_{\mu_j} \right) &= 4g_{\mu_1\mu_2} \left\{ g_{\mu_3\mu_4} g_{\mu_5\mu_6} + g_{\mu_3\mu_6} g_{\mu_4\mu_5} - g_{\mu_3\mu_5} g_{\mu_4\mu_6} \right\} \\ &\quad - 4g_{\mu_1\mu_3} \left\{ g_{\mu_2\mu_4} g_{\mu_5\mu_6} + g_{\mu_2\mu_6} g_{\mu_4\mu_5} - g_{\mu_2\mu_5} g_{\mu_4\mu_6} \right\} \\ &\quad + 4g_{\mu_1\mu_4} \left\{ g_{\mu_2\mu_3} g_{\mu_5\mu_6} + g_{\mu_2\mu_6} g_{\mu_3\mu_5} - g_{\mu_2\mu_5} g_{\mu_3\mu_6} \right\} \\ &\quad - 4g_{\mu_1\mu_5} \left\{ g_{\mu_2\mu_3} g_{\mu_4\mu_6} + g_{\mu_2\mu_6} g_{\mu_3\mu_4} - g_{\mu_2\mu_4} g_{\mu_3\mu_6} \right\} \\ &\quad - 4g_{\mu_1\mu_6} \left\{ g_{\mu_2\mu_3} g_{\mu_4\mu_5} + g_{\mu_2\mu_5} g_{\mu_3\mu_4} - g_{\mu_2\mu_4} g_{\mu_3\mu_5} \right\} \\ \text{Tr} \left( \gamma^5 \prod_{j=1}^6 \gamma_{\mu_j} \right) &= 4i \left\{ g_{\mu_1\mu_2} \varepsilon_{\mu_3\mu_4\mu_5\mu_6} - g_{\mu_1\mu_3} \varepsilon_{\mu_2\mu_4\mu_5\mu_6} + g_{\mu_2\mu_3} \varepsilon_{\mu_1\mu_4\mu_5\mu_6} \right. \\ &\quad \left. + g_{\mu_4\mu_5} \varepsilon_{\mu_1\mu_2\mu_3\mu_6} - g_{\mu_4\mu_6} \varepsilon_{\mu_1\mu_2\mu_3\mu_5} + g_{\mu_5\mu_6} \varepsilon_{\mu_2\mu_2\mu_3\mu_4} \right\} \end{aligned} \quad (\text{A14})$$

Obviously the results in Eq.(A10) and Eq.(A12) is perfectly consistent with result of derivation of transverse vector and axial vector anomalies in four-dimensional  $U(1)$  gauge theory using perturbative methods[21,22].

Now we have completed our computation in Euclidean space, it is necessary to transform the conclusions back into Minkowski space. According to the Wick substitutions, then, there is no change in the form of the final formula.

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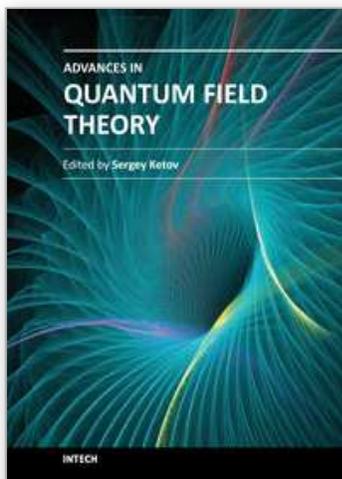
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