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8

Solution of a Linearized Model of Heisenberg's Fundamental Equation

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1. Introduction

The theory of quantized fields has its roots in the quantum theory of M. Planck [Planck (1900)]. For the solution of the problem of black body radiation, he introduced the universal quantum of action to the theory of electromagnetic fields. Nonrelativistic quantum mechanics is established by W. Heisenberg [Heisenberg (1925)] and E. Schrödinger [Schrödinger (1926)]. M. Born, W. Heisenberg and P. Jordan [Born et al. (1926)] realized the quantization of electromagnetic fields, and P.A.M. Dirac [Dirac (1927)] quantized electromagnetic fields in interaction with a material system. But P. Ehrenfest noticed soon that the theory had to lead to infinities. The existence of the positron is suggested by the theory of P.A.M. Dirac [Dirac (1928; 1931)]. The discovery of the positron by C.D. Andersen (1932) established the theory of quantum electrodynamics which treats the behavior of electron, positron and electromagnetic fields. But this theory still has to lead to infinities, and these difficulties are (partially) removed by S. Tomonaga [Tomonaga (1946)], H.A. Bethe [Bethe (1947)] and J. Schwinger [Schwinger (1948)] using the subtraction formalism in perturbation theory. R.P. Feynman [Feynman (1948)] developed the method of path integral which simplifies the calculation and F.J. Dyson [Dyson (1949)] derives Feynman's prescription from Tomonaga-Schwinger theory. Thus the prescription of the subtraction formalism in electrodynamics was completely worked out. Those theories in which the infinity of each term in the perturbation series can be subtracted consistently are called renormalizable theories. But for these renormalizable theories, there are still doubts about summability of the series in perturbation theory. In electrodynamics, the expansion parameter (the coupling constant) is small and the sum of the first few terms gives an amazing agreement with experiments, but there are no proofs about the convergence of the series. In some theory of strong interaction the parameter is greater than 1, and this subtraction formalism does not work. The main question is: What is hidden behind these formal infinite series? To this question, one answer is given by the formalism of A.S. Wightman and L. Gårding [Wightman & Gårding (1964)], which is a mathematically rigorous study of quantum fields. Since the structure of the theory is axiomatic, the theory is called axiomatic quantum field theory. The axioms formulate the basic physical postulates (e.g., relativistic invariance of the state space, spectral property, unique existence of a vacuum state, Poincaré-covariance of the fields, locality or micro causality, etc., and some technical postulate: temperedness of the fields, etc.) in a mathematical language. Nowadays the strong interaction is described by the quantum chromodynamics. Its renormalizability was proven in ['t Hooft (1971)], and its asymptotic freedom discovered by [Gross & Wilczek (1973); Politzer (1973)] gave it excellent predictability as well as quantum electrodynamics. But we know nothing about its summability of perturbation series and the perturbation theory does not work in the low-energy region where the coupling constant becomes large. The situation is not so different from 1940's. The axioms show the way to reconcile the principles of quantum mechanics and those of special relativity. Quantum field theory is also important as effective theory in low-energy approximation to a deeper theory like a string theory where it is said that there is a length $\ell > 0$ (fundamental length) such that one cannot distinguish events which occur in a smaller distance than ℓ (see [Polchinski (1998)]). From this perspective, quantum field theory with a fundamental length becomes interesting.

In 1958, Heisenberg and Pauli introduced the equation

$$\gamma_\mu \frac{\partial}{\partial x_\mu}\psi(x) \pm l^2 \gamma_\mu \gamma_5 : \psi(x)\bar{\psi}(x)\gamma^\mu \gamma_5 \psi(x) : = 0 \tag{1.1}$$

which was later called Heisenberg's fundamental field equation or the equation of universe and studied in [Dürr et al. (1959); Heisenberg (1966)]. The equation contains a parameter l of the dimension of length and accordingly one might speculate that this parameter can play the rôle of the fundamental length of a quantum field theory with a fundamental length. In order to verify this speculation one must solve two eminent problems:

(A) Formulate a relativistic quantum field theory with a fundament length in an axiomatic way and establish its main properties;

(B) Solve this equation with a field theory according to (A).

As a mathematical theory a solution to problem (A) has been suggested in [Brüning & Nagamachi (2004)]. According to this suggestion a *relativistic quantum field theory with a fundamental length* is a relativistic quantum field theory similar to the Gårding - Wightman theory (see [Wightman & Gårding (1964)]) where the fields are operator valued tempered ultra-hyperfunctions instead of operator valued tempered (Schwartz) distributions. The physically important aspect of this theory is that for the first time in a mathematical rigorous way, the fundamental length is realized on the level of the fields, not on the level of the geometry of the space-time on which these fields are defined. This allows to rely on the established concepts and theories of Physics based on a standard space-time. The fundamental length is introduced through the use of a class of generalized functions (tempered ultrahyperfunctions) which distinguish events in space-time only when they are separated by more than a certain length, the fundamental length ℓ. Certainly, up to now there is now no? physical evidence that this is the appropriate way of defining the fundamental length.

Concerning problem (B) we recall that nobody knows to solve Heisenberg's fundamental field equation. However there is a simplification of this equation which is solvable in the sense of classical field theory, namely the system of equations ($: \cdot :$ indicates the Wick product)

$$\begin{cases} (\Box + m^2)\phi(x) = 0 \\ \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - M\right)\psi(x) = -2l^2\gamma^\mu : \frac{\partial \phi(x)}{\partial x^\mu}\phi(x)\psi(x) : \end{cases} \tag{1.2}$$

for a Klein-Gordon field ϕ and a spinor field ψ. It is this system of coupled equations which we discuss in the framework of [Brüning & Nagamachi (2004)], i.e., for a given Klein-Gordon field ϕ we define an operator valued tempered ultrahyperfunction ψ such that the pair (ϕ, ψ) satisfies equation (1.2). This system has been studied first by [Okubo (1961)] as a quantum field theory although this interaction is un-renormalizable in the usual sense of perturbation theory, and Green's functions are calculated. There are some interactions which look un-renormalizable but actually are renormalizable if the use of perturbations is avoided (see [Okubo (1954)]).

2. Overview

The basic idea to solve the system (1.2) is quite natural:

Take a Klein-Gordon field of mass m and suppose that we can show the following statements:

A) the Wick power series

$$\rho(x) =: e^{il^2\phi(x)^2} := \sum_{n=0}^{\infty} i^n l^{2n} : \phi(x)^{2n} : /n! \tag{2.1}$$

and

$$\rho^*(x) =: e^{-il^2\phi(x)^2} := \sum_{n=0}^{\infty} (-i)^n l^{2n} : \phi(x)^{2n} : /n!$$

are well-defined as operator-valued ultra-hyperfunctions.

B) $\rho(x)$ satisfies

$$\frac{\partial}{\partial x^\mu}\rho(x) = 2il^2 : \frac{\partial\phi(x)}{\partial x^\mu}\phi(x)e^{il^2\phi(x)^2} := 2il^2 : \frac{\partial\phi(x)}{\partial x^\mu}\phi(x)\rho(x) : . \tag{2.2}$$

C) the free Dirac field $\psi_0(x)$ is a multiplier for the field ρ and thus we can define the field

$$\psi(x) = \rho(x)\psi_0(x). \tag{2.3}$$

D) Show that the field ψ defined in C) is indeed al relativistic quantum field with a fundamental length.

E) Calculate

$$\begin{aligned}\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - M\right)\psi(x) &= \rho(x)\left[\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - M\right)\psi_0(x)\right] + \gamma^\mu \frac{\partial\rho(x)}{\partial x^\mu}\psi_0(x) \\ &= -2l^2\gamma^\mu : \frac{\partial\phi(x)}{\partial x^\mu}\phi(x)\rho(x)\psi_0(x) := -2l^2\gamma^\mu : \frac{\partial\phi(x)}{\partial x^\mu}\phi(x)\psi(x) : .\end{aligned}$$

Thus, if A) – E) hold, the operator-valued ultra-hyperfunctions $\phi(x), \psi(x)$ satisfy the system of equations (1.2).

Note that the above Wick power series do not converge in the sense tempered distributions. They even do not converge in the sense Fourier hyperfunctions (see [Ito (1988); Nagamachi (1981a;b)]). But as we show they converge in the sense of tempered ultra-hyperfunctions.

Remark 2.1.

The definition of a relativistic quantum field with a fundamental length has been proposed in [Brüning & Nagamachi (2004)] and there the functional characterization has been derived for the case of a scalar field. The functional characterization for a spinor field is given in [Brüning & Nagamachi (2008)].

Naturally the localization properties of a relativistic quantum field with a fundament length $\ell > 0$ are very different from those of a standard quantum field. According to their definition these fields do not distinguish events in space-time which are separated by $\ell' < \ell$ (see Remark 3.11).

According to these new localization properties the counter part of the "locality or causality condition for standard fields" looks quite different. It is called "extended causality" and the verification of this condition is the major difficulty in verifying that the fields ρ and ψ as introduced above are indeed relativistic quantum fields with a fundamental length.

Remark 2.2. The key to our approach is the use of tempered ultra-hyperfunctions (see [Morimoto (1970; 1975a;b)]) and their localization properties (see [Nagamachi & Brüning (2003)]). This has first been suggested in [Brüning & Nagamachi (2004)]. This class of generalized functions is not too well known and therefore we will briefly explain what tempered ultra-hyperfunctions are and that and how these localization properties arise.

3. Tempered ultra-hyperfunctions

For any subset A of $\mathbb{R}^n$, denote by $T(A) = \mathbb{R}^n + iA \subset \mathbb{C}^n$ the tubular set with base A. For a convex compact set K of $\mathbb{R}^n$, $\mathcal{T}_b(T(K))$ is, by definition, the space of all continuous functions f on $T(K)$ which are holomorphic in the interior of $T(K)$ and which satisfy

$$\|f\|^{T(K),j} = \sup\{|z^p f(z)|; z \in T(K), |p| \le j\} < \infty, \; j = 0, 1, \dots \tag{3.1}$$

where $p = (p_1, \dots, p_n)$ and $z^p = z_1^{p_1} \cdots z_n^{p_n}$. $\mathcal{T}_b(T(K))$ is a Fréchet space with the semi-norms $\|f\|^{T(K),j}$. If $K_1 \subset K_2$ are two compact convex sets, we have the canonical mapping:

$$\mathcal{T}_b(T(K_2)) \to \mathcal{T}_b(T(K_1)). \tag{3.2}$$

For a convex open set O in $\mathbb{R}^n$ we define

$$\mathcal{T}(T(O)) = \lim_{\leftarrow} \mathcal{T}_b(T(K)), \tag{3.3}$$

where K runs through the convex compact sets contained in O and the projective limit is taken following the restriction mappings (3.2).

Definition 3.1. *A* **tempered ultra-hyperfunction** *is by definition a continuous linear functional on* $\mathcal{T}(T(\mathbb{R}^n))$.

The Fourier transformation $\mathcal{F}$ is well defined on $\mathcal{T}(T(\mathbb{R}^n))$ by the standard formula (3.8). In order to determine the range of $\mathcal{F}$ on $\mathcal{T}(T(\mathbb{R}^n))$ we introduce another function space.

The gauge functional h_K of a compact convex set $K \subset \mathbb{R}^n$ is defined by

$$h_K(x) = \sup\{\langle x, \xi \rangle; \xi \in K\}. \tag{3.4}$$

For a convex compact set K of $\mathbb{R}^n$, denote by $H_b(\mathbb{R}^n; K)$ the space of all C^∞ functions f on $\mathbb{R}^n$ which satisfy, for $j = 0, 1, \ldots$,

$$\|f\|_{K,j} = \sup\{\exp(h_K(x))|D^p f(x)|; x \in \mathbb{R}^n,\ |p| \le j\} < \infty. \tag{3.5}$$

Equipped with the system of semi-norms $\|f\|_{K,j}$, $H_b(\mathbb{R}^n; K)$ is a Fréchet space. If $K_1 \subset K_2$ are two compact convex sets, then $h_{K_1} \le h_{K_2}$ and thus one has the canonical mappings:

$$H_b(\mathbb{R}^n; K_2) \to H_b(\mathbb{R}^n; K_1). \tag{3.6}$$

For a convex open set $O \subset \mathbb{R}^n$ the space $H(\mathbb{R}^n; O)$ is the projective limit of the spaces $H_b(\mathbb{R}^n; K)$ along the restriction mappings (3.6), i.e.,

$$H(\mathbb{R}^n; O) = \lim_{\leftarrow} H_b(\mathbb{R}^n; K), \tag{3.7}$$

where K runs through the convex compact sets contained in O.

In order to relate the space $H(\mathbb{R}^n; \mathbb{R}^n)$ to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ we derive a more direct characterization of $H(\mathbb{R}^n; \mathbb{R}^n)$. Observe that for any convex compact set $K \subset \mathbb{R}^n$ there is a number $k > 0$ such that $K \subseteq [-k, k]^n$. For the sets $K = [-k, k]^n$ the gauge function h_K is easily determined:

$$h_K(x) = \sup\{\langle x, \xi\rangle;\ \xi \in K\} = k \sum_{i=1}^{n} |x_i|,$$

and the system of continuous norms takes the form, using the notation $|x| = \sum_{i=1}^n |x_i|$,

$$\begin{aligned}\|f\|_{K,j} &= \sup\{\exp(h_K(x))|D^p f(x)|;\ |p| \le j,\ x \in \mathbb{R}^n\} \\ &= \sup\{e^{k|x|}|D^p f(x)|;\ |p| \le j,\ x \in \mathbb{R}^n\}.\end{aligned}$$

Thus, the space $H(\mathbb{R}^n; \mathbb{R}^n)$ can be defined as the projective limit of the spaces $H_b(\mathbb{R}^n; K)$ along the restriction mappings (3.6), where $K = [-k, k]^n$, $0 < k < \infty$. Accordingly, the space $H(\mathbb{R}^n; \mathbb{R}^n)$ is the space of all $\mathcal{C}^\infty$-functions on $\mathbb{R}^n$ which, together with all derivatives, decrease faster than any (linear) exponential. An easy consequence is

Corollary 3.2. *1. The space $H(\mathbb{R}^n; \mathbb{R}^n)$ is continuously embedded into the Schwartz space $\mathcal{S}(\mathbb{R}^n)$;*

2. *The elements of $\mathcal{S}(\mathbb{R}^n)$ are multipliers for the space $H(\mathbb{R}^n; \mathbb{R}^n)$, and for each $g \in \mathcal{S}(\mathbb{R}^n)$ the map $f \mapsto gf$ is a continuous linear map of $H(\mathbb{R}^n; \mathbb{R}^n)$ into itself.*

Proof: See [Hasumi (1961); Morimoto (1975b)]. □

The following theorem collects the basic facts about the spaces introduced above.

Theorem 3.3. *For the spaces introduced above the following statements hold, for any convex compact set K respectively convex open set O.*

1. *The space of $\mathcal{D}(\mathbb{R}^n)$ all C^∞ functions with compact support is dense in $H(\mathbb{R}^n; O)$.*
2. *The space $H(\mathbb{R}^n; \mathbb{R}^n)$ is dense in $H(\mathbb{R}^n; O)$ and in $H(\mathbb{R}^n; K)$.*
3. *$H(\mathbb{R}^m; \mathbb{R}^m) \otimes H(\mathbb{R}^n; \mathbb{R}^n)$ is dense in $H(\mathbb{R}^{m+n}; \mathbb{R}^{m+n})$.*

Proof: For the proof of the first two items we refer to [Hasumi (1961); Morimoto (1975b)]. The proof the last item can be found in [Brüning & Nagamachi (2004)]. □

Proposition 3.4. *The Fourier transformation $f \mapsto \tilde{f} \equiv \mathcal{F}f$,*

$$\tilde{f}(p) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(z) e^{i\langle p,z\rangle} dz \tag{3.8}$$

is a topological isomorphism between the spaces $\mathcal{T}(T(O))$ and $H(\mathbb{R}^n; O)$, for any open convex nonempty set $O \subset \mathbb{R}^n$. The inverse transformation is

$$f(z) = \bar{\mathcal{F}}\tilde{f} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \tilde{f}(p) e^{-i\langle p,z\rangle} dp. \tag{3.9}$$

Proof: See [Hasumi (1961); Morimoto (1975b)]. □

Proposition 3.5. *Let $O \subset \mathbb{R}^n$ be a nonempty convex open subset. Then the spaces $H(\mathbb{R}^n; O)$ and $\mathcal{T}(T(O))$ are nuclear Fréchet spaces and thus, in particular, reflexive.*

Proof: In the case of $O = \mathbb{R}^n$ Hasumi [Hasumi (1961)] proved this result, and his proof is valid in the general case. A sketch of the proof for $H(\mathbb{R}^n; O)$ is provided in [Brüning & Nagamachi (2004)]. □

Theorem 3.6 (Corollary of Theorem 34.1 of [Treves (1967)]). *Let E be a Fréchet space, E_1 a metrizable space, G a locally convex space. Then a separately continuous bilinear map of $E \times E_1$ into G is continuous.*

Theorem 3.7 (Kernel theorem for ultra-hyperfunctions). *Let M be a separately continuous multi-linear map of $[\mathcal{T}(T(\mathbb{R}^4))]^n$ into a Banach space G. Then there is a unique continuous linear map F of $\mathcal{T}(T(\mathbb{R}^{4n}))$ into G such that, for all $f_i \in \mathcal{T}(T(\mathbb{R}^4))$, $i = 1, \ldots, n$,*

$$M(f_1, \ldots, f_n) = F(f_1 \otimes \cdots \otimes f_n).$$

Proof: The proof is quite involved and lengthy. Details are again given in [Brüning & Nagamachi (2004)]. □

For an open set V in $\mathbb{R}^n$ and a positive number ϵ introduce the set V^ϵ defined by

$$V^\epsilon = \{z \in \mathbb{C}^n; \exists\, x \in V, |\mathrm{Re}\, z - x| < \epsilon, |\mathrm{Im}\, z|_\beta < \epsilon\},$$

where $|y|_\beta$ is a norm of $\mathbb{R}^n$ satisfying $|y|_\beta \geq |y|$ for the Euclidean norm $|y|$. Let K_p be the closure of $V^{\epsilon/(1+1/p)}$ in $\mathbb{C}^n$ and $L_p = \{w \in \mathbb{C}^m; |\mathrm{Im}\, w| \leq p\}$. Denote $U = V^\epsilon \times \mathbb{C}^m$ and $M_p = K_p \times L_p$. $\mathcal{T}_b(M_p)$ is, by definition, the space of all continuous functions f on M_p which are holomorphic in the interior of M_p and satisfy, for $k = 1, 2, \ldots$,

$$\|f\|^{M_p,k} = \sup\{|z^s w^t f(z,w)|; (z,w) \in M_p, |s| + |t| \leq k\} < \infty;$$

$\mathcal{T}_b(M_p)$ is a Fréchet space with the seminorms $\|f\|^{M_p,k}$.

If $k < m$, then we have the canonical mappings:

$$\mathcal{T}_b(M_m) \to \mathcal{T}_b(M_k). \tag{3.10}$$

We define

$$\mathcal{T}(U) = \lim_{\leftarrow} \mathcal{T}_b(M_m), \tag{3.11}$$

where the projective limit is taken following the restriction mappings (3.10).

Theorem 3.8. *$\mathcal{T}(T(\mathbb{R}^{n+m}))$ is dense in $\mathcal{T}(U)$.*

Proof: The proof is similar to the proofs of Proposition 2.4 of [Nishimura & Nagamachi (1990)] and Proposition 9.1.2 of [Hörmander (1983)]. For more details we refer to the appendix of [Brüning & Nagamachi (2004)]. □

Theorem 3.9. *Let V be a closed convex cone and K a convex compact set in $\mathbb{R}^n$. Define a function $h_{K,V}(\xi)$, $\xi \in \mathbb{R}^n$, and a set V_K^0 as follows (see Equation (3.4) for the definition of h_K):*

$$h_{K,V}(\xi) = \sup_{x \in V} h_K(x) - \langle x, \xi \rangle, \quad \textit{and} \quad V_K^0 = \{\xi \in \mathbb{R}^n; h_{K,V}(\xi) < \infty\}.$$

Then for every $\mu \in H(\mathbb{R}^n; O)'$ with support in the cone V there is a function

$$\hat{\mu}(\zeta) = \langle \mu, e^{i\langle \cdot, \zeta \rangle} \rangle \tag{3.12}$$

with the following properties: $\hat{\mu}$ is well defined and holomorphic in the interior of $\mathbb{R}^n \times iV_K^0$ and satisfies there the following estimate, for a suitable $K \subset O$.

$$|\hat{\mu}(\zeta)| \leq C(1 + |\zeta|)^j \exp(h_{K,V}(\operatorname{Im} \zeta)). \tag{3.13}$$

$\hat{\mu}$ is called the Laplace transform of the tempered ultra-hyperfunction μ.

Proof: See [Brüning & Nagamachi (2004)]. □

Remark 3.10. Let $|x|_\infty = \max\{|x^0|, |\boldsymbol{x}|\}$ be a norm in $\mathbb{R}^4$ and $\bar{V}_+$ the closed forward light-cone in $\mathbb{R}^4$. Abbreviate $V = \bar{V}_+^n$ and for $\ell_i > 0$ introduce $h_K(x) = \sum_{i=1}^n \ell_i |x_i|_\infty$. Then we estimate

$$h_{K,V}(\xi) = \sup_{x_i \in \bar{V}_+} \sum_{i=1}^n (\ell_i |x_i|_\infty - \langle x_i, \xi_i \rangle) \leq \sum_{i=1}^n \sup_{x_i \in \bar{V}_+} (\ell_i |x_i|_\infty - \langle x_i, \xi_i \rangle).$$

Let V_+ be the open forward light-cone. It follows

$$\sup_{x \in \bar{V}_+} -\langle x, \eta \rangle < \infty$$

for $\eta \in V_+$. Let $\xi_i = \eta_i + (\ell_i, \mathbf{o}) \in V_+ + (\ell_i, \mathbf{o})$. Since $|x|_\infty = x^0$ in $\bar{V}_+$, we find

$$\sup_{x_i \in \bar{V}_+} (\ell_i |x_i|_\infty - \langle x_i, \xi_i \rangle) = \sup_{x_i \in \bar{V}_+} (\ell_i x_i^0 - \langle x_i, \eta_i \rangle - x_i^0 \ell_i) = \sup_{x_i \in \bar{V}_+} -\langle x_i, \eta_i \rangle < \infty.$$

Thus the set

$$V_+(\ell_1, \ldots, \ell_n) = \{(\xi_1, \ldots, \xi_n) \in \mathbb{R}^{4n}; \xi_i \in V_+ + (\ell_i, \mathbf{o})\} \tag{3.14}$$

is contained in V_K^0.

The reason why we use tempered ultra-hyperfunctions for the formulation of relativistic quantum field theory with a fundamental length is illustrated in the following remark.

Remark 3.11. If $f(z)$ is a holomorphic function in the strip $|\mathrm{Im}\, z| < \ell$ around the real axis, then, for $|a| < \ell$, we have

$$\langle \sum_{n=0}^{\infty} \frac{a^n}{n!} \delta^{(n)}(x), f(x) \rangle = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} f^{(n)}(0) = f(0-a) = \langle \delta(x+a), f(x) \rangle, \tag{3.15}$$

that is, as an equation for functionals defined on the function space $\mathcal{T}(T(-\ell,\ell))$ whose elements are holomorphic functions in $T(-\ell,\ell) = \mathbb{R} + i(-\ell,\ell) \subset \mathbb{C}$, the identity

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} \delta^{(n)}(x) = \delta(x+a)$$

holds, i.e., the sequence of generalized functions $S_N = \sum_{n=0}^{N} \frac{a^n}{n!} \delta^{(n)}(x)$ with support $\{0\}$ converges (weakly, in the dual space of $\mathcal{T}(T(-\ell,\ell))$) to the generalized function $\delta(x+a)$ with support $\{-a\}$, as $N \to \infty$. However, if $|a| > \ell$, then this sequence does not converge in $\mathcal{T}(T(-\ell,\ell))'$. This phenomenon can be understood as follows. If $|a| < \ell$, then elements in $\mathcal{T}(T(-\ell,\ell))'$ do not distinguish between the points $\{0\}$ and $\{-a\}$, but if $|a| > \ell$ then elements in $\mathcal{T}(T(-\ell,\ell))'$ can distinguish between the points $\{0\}$ and $\{-a\}$. Since $|a| < \ell$ is arbitrary, one can say that elements in $\mathcal{T}(T(-\ell,\ell))'$ do not distinguish between points which are separated by less than ℓ.

Remark 3.12. Let $U = (-\ell,\ell) + i(-\ell,\ell)$. Then the functional (3.15) is considered to be a functional on $\mathcal{T}(U)$ (the space of holomorphic functions on U), i.e., it is continuously extendable to $\mathcal{T}(U)$. If a functional μ on $\mathcal{T}(T(\mathbb{R}))$ is continuously extendable to $\mathcal{T}(U)$, one says that a carrier of μ is contained in U. The notion of carrier for tempered ultra-hyperfunctions is the counterpart of the notion of support for distributions. We can recognize the similarity of these notions: If a distribution $\mu \in \mathcal{S}'(\mathbb{R})$ is continuously extendable to $\mathcal{E}(U)$ for some open set $U \subset \mathbb{R}$, then we know that the support of μ is contained in U.

4. Relativistic quantum fields with a fundamental length and their functional characterization

4.1 Wightman's Axioms for relativistic quantum fields with a fundamental length

In Wightman's scheme, the concept of a relativistic quantum field $\phi^{(\kappa)}$ of type κ plays a fundamental role. Such a field, for example a scalar, tensor or spinor field, has a finite number of Lorentz components $\phi_j^{(\kappa)}$ $(j = 1, \dots, r_\kappa)$.

The field components $\phi_j^{(\kappa)}(x)$ are operator-valued tempered ultra-hyperfunctions, i.e., for $f \in \mathcal{T}(T(\mathbb{R}^4))$,

$$\phi_j^{(\kappa)}(f) = \int \phi_j^{(\kappa)}(x) f(x) d^4x$$

are densely defined linear operators in a complex Hilbert space $\mathcal{H}$. They are not assumed to be bounded.

Here we state Wightman's axioms for the ultra-hyperfunction quantum field theory [Brüning & Nagamachi (2008)]. For the neutral scalar fields, these axioms are the axioms listed in [Brüning & Nagamachi (2004)].

W.I. **Relativistic invariance and state space**: There is a complex Hilbert space $\mathcal{H}$ with positive metric in which a unitary representation $U(a, A)$ of the Poinaré spinor group $\mathcal{P}_0$ acts. $(a, A) \mapsto U(a, A)$ is weakly continuous.

W.II. **Spectral property**: The spectrum Σ of the energy-momentum operator P which generates the translations in this representation, i.e., $e^{iaP} = U(a, 1)$, is contained in the closed forward light cone
$\bar{V}_+ = \{p = (p^0, \dots, p^3) \in \mathbb{R}^4; p^0 \geq |\boldsymbol{p}|\}$.

W.III. **Existence and uniqueness of the vacuum**: In $\mathcal{H}$ there exists unit vector Φ_0 (called the vacuum vector) which is unique up to a phase factor and which is invariant under all space-time translations $U(a, 1)$, $a \in \mathbb{R}^4$.

W.IV. **Fields as operator-valued tempered ultra-hyperfunctions**: The components $\phi_j^{(\kappa)}$ of the quantum field $\phi^{(\kappa)}$ are operator-valued generalized functions $\phi_j^{(\kappa)}(x)$ over the space $\mathcal{T}(T(\mathbb{R}^4))$ with common dense domain $\mathcal{D}$; i.e., for all $\Psi \in \mathcal{D}$ and all $\Phi \in \mathcal{H}$,

$$\mathcal{T}(T(\mathbb{R}^4)) \ni f \to (\Phi, \phi_j^{(\kappa)}(f)\Psi) \in \mathbb{C}$$

is a tempered ultra-hyperfunction. It is supposed that the vacuum vector Φ_0 is contained in $\mathcal{D}$ and that $\mathcal{D}$ is invariant under the action of the operators $\phi_j^{(\kappa)}(f)$ and $U(a, A)$, i.e.,

$$\phi_j^{(\kappa)}(f)\mathcal{D} \subset \mathcal{D},\ U(a, A)\mathcal{D} \subset \mathcal{D}.$$

Moreover it is assumed that there exist indices $\bar{\kappa}$, $\bar{j}$ such that $\phi_{\bar{j}}^{(\bar{\kappa})}(\bar{f}) \subset \phi_j^{(\kappa)}(f)^*$ where * indicates the Hilbert space adjoint of the operator in question.

W.V. **Poincaré-covariance of the fields**: According to the type of the field, there is a finite dimensional real or complex matrix representation $V^{(\kappa)}(A)$ of $SL(2, \mathbb{C})$ such that

$$U(a, A)\phi_j^{(\kappa)}(x)U(a, A)^{-1} = \sum_{\ell} V_{j,\ell}^{(\kappa)}(A^{-1})\phi_\ell^{(\kappa)}(\Lambda(A)x + a),$$

i.e., for any $f \in \mathcal{T}(T(\mathbb{R}^4))$ and $\Psi \in \mathcal{D}$,

$$U(a, A)\phi_j^{(\kappa)}(f)U(a, A)^{-1}\Psi = \sum_{\ell} V_{j,\ell}^{(\kappa)}(A^{-1})\phi_\ell^{(\kappa)}(f_{(a,A)})\Psi,$$

where $f_{(a,A)}(x) = f(\Lambda(A)^{-1}(x - a))$. We have $V^{(\kappa)}(-1) = \pm 1$. If $V^{(\kappa)}(-1) = 1$, then the field is called a tensor field. If $V^{(\kappa)}(-1) = -1$, then the field is called a spinor field.

W.VI. **Extended causality or extended local commutativity**: Any two field components $\phi_j^{(\kappa)}(x)$ and $\phi_l^{(\kappa')}(y)$ either commute or anti-commute if the space-like distance between x

and y is greater than ℓ: In some Lorentz frame[1] , for any $\ell' > \ell$ and arbitrary elements Φ, Ψ in $\mathcal{D}$,

a) the functionals

$$\mathcal{T}(T(\mathbb{R}^4)) \otimes \mathcal{T}(T(\mathbb{R}^4)) \ni f \otimes g \to (\Phi, \phi_j^{(\kappa)}(f)\phi_l^{(\kappa')}(g)\Psi)$$

and

$$\mathcal{T}(T(\mathbb{R}^4)) \otimes \mathcal{T}(T(\mathbb{R}^4)) \ni f \otimes g \to (\Phi, \phi_l^{(\kappa')}(g)\phi_j^{(\kappa)}(f)\Psi)$$

can be extended continuously to $\mathcal{T}(T(L^{\ell'}))$, where

$$T(L^\ell) = \{(z_1, z_2) \in \mathbb{C}^{4\cdot 2}; |\mathrm{Im}\, z_1 - \mathrm{Im}\, z_2|_1 < \ell\},$$

with $|y|_1 = |y^0| + \sqrt{\sum_{i=1}^3 (y^i)^2}$, and moreover,

b) the carrier of the functional

$$f \otimes g \to (\Phi, [\phi_j^{(\kappa)}(f), \phi_l^{(\kappa')}(g)]_{\mp}\Psi)$$

on $\mathcal{T}(T(\mathbb{R}^4)) \otimes \mathcal{T}(T(\mathbb{R}^4))$ is contained in the set

$$W^{\ell'} = \{(z_1, z_2) \in \mathbb{C}^{4\cdot 2}; z_1 - z_2 \in V^{\ell'}\},$$

where

$$V^\ell = \{z \in \mathbb{C}^4; \exists\, x \in V, |\mathrm{Re}\, z - x| < \ell, |\mathrm{Im}\, z|_1 < \ell\}$$

with $|y| = \sqrt{\sum_{i=0}^3 (y^i)^2}$ is a complex neighborhood of light cone V, i.e., this functional can be extended continuously to $\mathcal{T}(W^{\ell'})$.

W.VII. **Cyclicity of the vacuum**: The set $\mathcal{D}_0$ of finite linear combinations of vectors of the form

$$\phi_{j_1}^{(\kappa_1)}(f_1) \cdots \phi_{j_n}^{(\kappa_n)}(f_n)\Phi_0,\ f_j \in \mathcal{T}(T(\mathbb{R}^4))\ (n = 0, 1, \dots)$$

is dense in $\mathcal{H}$.

Remark 4.1. Condition a) of axiom W.VI expresses the fact that if the distance between x and y is greater than ℓ then x and y are distinguishable (see Remark 3.11). Condition b) of axiom W.VI corresponds to the locality condition of ordinary quantum field theory (see Remark 3.12).

4.2 Main properties of the system of vacuum expectation values

A vector-valued generalized function $\Phi_{\underline{\mu}_n}^{(\underline{\kappa}_n)}(f)$ is defined as follows: First, for $g(x_1, \dots, x_n) = f_1(x_1) \cdots f_n(x_n)$, $f_j \in \mathcal{T}(T(\mathbb{R}^4))$, define $\Phi_{\mu_1 \dots \mu_n}^{(\kappa_1 \dots \kappa_n)}(g)$ by:

$$\Phi_{\mu_1 \dots \mu_n}^{(\kappa_1 \dots \kappa_n)}(g) = \phi_{\mu_1}^{(\kappa_1)}(f_1) \cdots \phi_{\mu_j}^{(\kappa_j)}(f_j) \cdots \phi_{\mu_n}^{(\kappa_n)}(f_n)\Phi_0.$$

[1] In [Nagamachi & Brüning (2010)] it is shown that the fundamental length does actually not dependent on the Lorentz frame.

This mapping is naturally extended to $\mathcal{T}(T(\mathbb{R}^4))^{\otimes n}$ by linearity. Then, by the same argument as Proposition 4.1 of [Brüning & Nagamachi (2004)]and using Theorem 3.7, $\Phi^{(\kappa_1\ldots\kappa_n)}_{\mu_1\ldots\mu_n}(g)$ is extended to a continuous mapping

$$\mathcal{T}(T(\mathbb{R}^{4n})) \ni f \to \Phi^{(\kappa_1\ldots\kappa_n)}_{\mu_1\ldots\mu_n}(f) \in \mathcal{H}.$$

The Wightman (generalized) function $\mathcal{W}^{(\kappa_1\ldots\kappa_n)}_{\mu_1\ldots\mu_n}(f)$ is defined by

$$\mathcal{T}(T(\mathbb{R}^{4n})) \ni f \to \mathcal{W}^{(\kappa_1\ldots\kappa_n)}_{\mu_1\ldots\mu_n}(f) = (\Phi_0, \Phi^{(\kappa_1\ldots\kappa_n)}_{\mu_1\ldots\mu_n}(f)) \in \mathbb{C}.$$

With the definition of the Fourier transform $\tilde{\Phi}^{(\underline{\kappa}_n)}_{\underline{\mu}_n}$ of $\Phi^{(\underline{\kappa}_n)}_{\underline{\mu}_n}$ by

$$\Phi^{(\underline{\kappa}_n)}_{\underline{\mu}_n}(f) = \tilde{\Phi}^{(\underline{\kappa}_n)}_{(\underline{\mu}_n)}(\tilde{f}).$$

we find

$$U(a,1)\tilde{\Phi}^{(\underline{\kappa}_n)}_{\underline{\mu}_n}(\tilde{f}) = U(a,1)\Phi^{(\underline{\kappa}_n)}_{\underline{\mu}_n}(f) = \Phi^{(\underline{\kappa}_n)}_{\underline{\mu}_n}(f_{(a,1)}) = \tilde{\Phi}^{(\underline{\kappa}_n)}_{\underline{\mu}_n}\left(\tilde{f}e^{[i(\sum_{k=1}^n p_k a)]}\right).$$

According to the standard strategy we use this identity to determine support properties of the Fourier transforms of the field operators. For $h \in \mathcal{T}(T(\mathbb{R}^4))$ calculate

$$(2\pi)^2\tilde{h}(P)\tilde{\Phi}^{(\underline{\kappa}_n)}_{\underline{\mu}_n}(\tilde{f}) = \int_{\mathbb{R}^4} h(a)U(a,1)da\tilde{\Phi}^{(\underline{\kappa}_n)}_{\underline{\mu}_n}(\tilde{f})$$

$$= (2\pi)^2\langle\tilde{\Phi}^{(\underline{\kappa}_n)}_{\underline{\mu}_n}(p_1,\ldots,p_n), \tilde{h}(p_1+\cdots+p_n)\cdot\tilde{f}(p_1,\ldots,p_n)\rangle.$$

Let χ_n be the linear mapping defined by

$$(p_1,\ldots,p_n) = \chi_n(q_0,\ldots,q_{n-1}),\ p_k = q_{k-1} - q_k (k=1,\ldots,n-1),\ p_n = q_{n-1}.$$

The inverse mapping χ_n^{-1} is:

$$q_k = \sum_{j=k+1}^{n} p_j \quad (k=0,\ldots,n-1).$$

Define $\tilde{Z}^{(\underline{\kappa}_n)}_{\underline{\mu}_n}$ by

$$\tilde{Z}^{(\underline{\kappa}_n)}_{\underline{\mu}_n}(\tilde{f}\circ\chi_n) = \tilde{\Phi}^{(\underline{\kappa}_n)}_{\underline{\mu}_n}(\tilde{f}).$$

Then

$$\tilde{Z}^{(\underline{\kappa}_n)}_{\underline{\mu}_n}(\tilde{g}) = \tilde{\Phi}^{(\underline{\kappa}_n)}_{\underline{\mu}_n}(\tilde{g}\circ\chi_n^{-1}).$$

In particular, for $\tilde{g}_2 \in H(\mathbb{R}^{4(n-1)};\mathbb{R}^{4(n-1)})$ and $\tilde{g}_1 \in H(\mathbb{R}^4;\mathbb{R}^4)$ we find

$$\tilde{h}(P)\tilde{Z}^{(\underline{\kappa}_n)}_{\underline{\mu}_n}(\tilde{g}_1\otimes\tilde{g}_2)) = \tilde{Z}^{(\underline{\kappa}_n)}_{\underline{\mu}_n}(\tilde{h}\cdot\tilde{g}_1\otimes\tilde{g}_2)) = \tilde{g}_1(P)\tilde{Z}^{(\underline{\kappa}_n)}_{\underline{\mu}_n}(\tilde{h}\otimes\tilde{g}_2)).$$

These identities show that the vector-valued generalized function

$$H(\mathbb{R}^4;\mathbb{R}^4) \ni \tilde{g}_1 \to \tilde{Z}^{(\kappa_n)}_{\underline{\mu}_n}(\tilde{g}_1 \otimes \tilde{g}_2)) \in \mathcal{H}$$

has its support contained in the spectrum Σ of energy-momentum operator P (see Proposition 4.5 of [Brüning & Nagamachi (2004)]), moreover we can define a functional $\tilde{W}^{(\kappa_n)}_{\underline{\mu}_n}$ by

$$(2\pi)^2\tilde{g}_1(0)\tilde{W}^{(\kappa_n)}_{\underline{\mu}_n}(\tilde{g}_2) = (\Phi_0, \tilde{Z}^{(\kappa_n)}_{\underline{\mu}_n}(\tilde{g}_1 \otimes \tilde{g}_2)),$$

and we have

$$(\tilde{Z}^{(\kappa_m)}_{\underline{\mu}_m}(\tilde{g}_1), \tilde{Z}^{(\kappa_n)}_{\underline{\mu}_n}(\tilde{g}_2)) = (2\pi)^2\langle \tilde{W}^{(\kappa_{m+n})}_{\underline{\mu}_{m+n}}(q_1,\dots,q_{m+n-1}), \overline{\tilde{g}}_1(q_m,\dots,q_1)\tilde{g}_2(q_m,\dots,q_{m+n-1})\rangle.$$

This identity implies that the support of $\tilde{W}^{(\kappa_n)}_{\underline{\mu}_n}(q_1,\dots,q_{n-1})$ is contained in Σ^{n-1} (see Proposition 4.6 of [Brüning & Nagamachi (2004)]). Moreover, the equality

$$(\tilde{Z}^{(\kappa_n)}_{\underline{\mu}_n}(\tilde{g}), \tilde{Z}^{(\kappa_n)}_{\underline{\mu}_n}(\tilde{g})) = (2\pi)^2\langle \tilde{W}^{(\kappa_{2n})}_{\underline{\mu}_{2n}}(q_1,\dots,q_{2n-1}), \overline{\tilde{g}}(q_n,\dots,q_1)\tilde{g}(q_n,\dots,q_{2n-1})\rangle$$

shows that the support of $\tilde{Z}^{(\kappa_n)}_{\underline{\mu}_n}(q_0,\dots,q_{n-1})$ is contained in Σ^n.

From this support property it follows that $\tilde{Z}^{(\kappa_n)}_{\underline{\mu}_n}(\tilde{g})$ exists for a much wider class of test functions $\tilde{g}$ than was originally considered. For example, the function

$$\tilde{g}_\zeta(q) = (2\pi)^{-2n}e^{i[\sum_{j=0}^{n-1} q_j\zeta_j]}, \quad \mathrm{Im}\,\zeta_j \in V_+ + \ell_j(1,\mathbf{o})$$

belongs to the class of test functions for sufficiently large ℓ_j. We investigate the region of holomorphy of the following function

$$\begin{aligned}
&\langle \tilde{W}^{(\kappa_{2n})}_{\underline{\mu}_{2n}}(q_1,\dots,q_{2n-1}), \tilde{g}^*_{\zeta'}(q_1,\dots,q_n)\tilde{g}_\zeta(q_n,\dots,q_{2n-1})\rangle \\
&= \frac{1}{(2\pi)^{4n}}\langle \tilde{W}^{(\kappa_{2n})}_{\underline{\mu}_{2n}}(q_1,\dots,q_{2n-1}), e^{-i[\sum_{j=1}^{n} q_{n+1-j}\bar{\zeta}'_{j-1}]}e^{i[\sum_{k=1}^{n} q_{n+k-1}\zeta_{k-1}]}\rangle \\
&= W^{(\kappa_{2n})}_{\underline{\mu}_{2n}}(-\bar{\zeta}'_{n-1},\dots,-\bar{\zeta}'_0+\zeta_0,\dots,\zeta_{n-1}).
\end{aligned}$$

Now recall the following proposition.

Proposition 4.2 (Proposition 4.7 of [Brüning & Nagamachi (2004)])**.** *There exist decreasing functions $R_{ij}(r)$ defined for $\ell < r$ such that $W^{(\kappa_{2n})}_{\underline{\mu}_{2n}}(\zeta_1,\dots,\zeta_{2n-1})$ is holomorphic in*

$$\bigcup_{i=1}^{2n-1}\{\zeta \in \mathbb{C}^{4(2n-1)}; \mathrm{Im}\,\zeta_i \in V_+ + (\ell',\mathbf{o}), \mathrm{Im}\,\zeta_j \in V_+ + (R_{ij}(\ell'),\mathbf{o}), \ell < \ell', j \neq i\}.$$

This proposition shows that $Z^{(\kappa_n)}_{\underline{\mu}_n}(\zeta_0,\dots,\zeta_{n-1})$ is holomorphic in the domain $\mathrm{Im}\,\zeta_0 \in V_+ + (\ell,\mathbf{o})/2$ and $\mathrm{Im}\,\zeta_k \in V_+ + (\ell_k,\mathbf{o})$ for sufficiently large ℓ_k for $k = 1,\dots,n$.

Note that

$$(\tilde{g}_\zeta \circ \chi_n^{-1})(p_1,\dots,p_n) = (2\pi)^{-2n} e^{i\langle \zeta, \chi_n^{-1} p\rangle} = (2\pi)^{-2n} e^{i\langle \chi_n^{-1T}\zeta, p\rangle} = (2\pi)^{-2n} e^{i\langle z,p\rangle},$$

where $z = \chi_n^{-1T}\zeta$ and $\zeta = \chi_n^T z$, that is,

$$\zeta_0 = z_1,\ \zeta_j = z_{j+1} - z_j\ (j = 1,\dots,n-1),\ z_1 = \zeta_0,\ z_j = \sum_{k=0}^{j-1} \zeta_k\ (j = 2,\dots,n).$$

Therefore we get

$$Z^{(\kappa_n)}_{\underline{\mu}_n}(\zeta_0,\dots,\zeta_{n-1}) = \tilde{Z}^{(\kappa_n)}_{\underline{\mu}_n}(\tilde{g}_\zeta) = \tilde{\Phi}^{(\kappa_n)}_{\underline{\mu}_n}(\tilde{g}_\zeta \circ \chi_n^{-1}) = \Phi^{(\kappa_n)}_{\underline{\mu}_n}(z_1,\dots,z_n),$$

and

$$\Phi^{(\kappa_n)}_{\underline{\mu}_n}(f) = \int \Phi^{(\kappa_n)}_{\underline{\mu}_n}(x_1 + i\ell_0,\dots,x_n + i\sum_{k=1}^{n} \ell_{k-1}) f(x_1 + i\ell_0,\dots,x_n + i\sum_{k=1}^{n} \ell_{k-1}) dx_1 \cdots dx_n,$$

where $\ell_0 = \ell/2 + \epsilon$ for any $\epsilon > 0$.

Note that the Poincaré group acts on $\tilde{g}_\zeta(q)$ as

$$(a,A) : \tilde{g}_\zeta(q) \to \tilde{g}_\zeta(\Lambda(A)^{-1}q)e^{iaq_0} = (2\pi)^{-2n} e^{i[\sum_{j=0}^{n-1} \Lambda(A)^{-1} q_j \zeta_j]} e^{iaq_0}$$

$$= (2\pi)^{-2n} \exp i[\sum_{j=0}^{n-1} q_j \Lambda(A)\zeta_j] e^{iaq_0} = \tilde{g}_{\Lambda(A)\zeta}(q) \exp iaq_0.$$

Then the formula of covariance

$$U(a,A)\Phi^{(\kappa_n)}_{\underline{\mu}_n}(f) = \sum_{\nu_1,\dots,\nu_n} \prod_{j=1}^{n} V^{(\kappa_j)}_{\mu_j,\nu_j}(A^{-1}) \Phi^{(\kappa_1\dots\kappa_n)}_{\nu_1\dots\nu_n}(f_{(a,A)})$$

implies the following simple formula of covariance in the domain of holomorphy of $\Phi^{(\kappa_n)}_{\underline{\mu}_n}(z_1,\dots,z_n)$ in complex space:

$$U(a,A)\Phi^{(\kappa_n)}_{\underline{\mu}_n}(z_1,\dots,z_n)$$

$$= \sum_{\nu_1,\dots,\nu_n} \prod_{j=1}^{n} V^{(\kappa_j)}_{\mu_j,\nu_j}(A^{-1}) \Phi^{(\kappa_1\dots\kappa_n)}_{\nu_1\dots\nu_n}(\Lambda(A)z_1 + a,\dots,\Lambda(A)z_n + a). \tag{4.1}$$

4.3 Functional characterization of fundamental length quantum fields

The analysis of the previous sections has shown that the sequence of vacuum expectation values of an ultra-hyperfunction quantum field theory has a number of specific properties. In analogy to standard quantum field theory we single out a set of properties of these vacuum expectation values which actually characterizes an ultra-hyperfunction quantum field theory up to isomorphisms. For the use in Section 5 (and because of space restrictions), we state them only in case of a scalar field.

Properties of UHQFT functionals:

(R1) $\mathcal{W}_0 = 1$, $\mathcal{W}_n \in \mathcal{T}(T(\mathbb{R}^{4n}))'$ for $n \geq 1$, and $\mathcal{W}_n(f^*) = \overline{\mathcal{W}_n(f)}$, for all $f \in \mathcal{T}(T(\mathbb{R}^{4n})) \equiv E(n)$, where $f^*(z_1, \ldots, z_n) = \overline{f(\bar{z}_n, \ldots, \bar{z}_1)}$.

(R2) For the Fourier transform $\tilde{\mathcal{W}}_n \in H(\mathbb{R}^{4n}; \mathbb{R}^{4n})'$ of $\mathcal{W}_n$, there exists $\tilde{W}_{n-1} \in H(\mathbb{R}^{4(n-1)}; \mathbb{R}^{4(n-1)})'$ such that

$$\tilde{\mathcal{W}}_n \circ \chi_n(q_0, \ldots, q_{n-1}) = (2\pi)^2 \delta(q_0) \tilde{W}_{n-1}(q_1, \ldots, q_{n-1})$$

and supp $\tilde{W}_{n-1} \subset \Sigma^{n-1}$.

(R3) For a space-like vector $\boldsymbol{a} \in \mathbb{R}^4$ and $g_n \in E(n)$ introduce, for all $\lambda > 0$,

$$g_{n,\lambda}(x_1, \ldots, x_n) = g_n(x_1 - \lambda \boldsymbol{a}, \ldots, x_n - \lambda \boldsymbol{a}).$$

Then, for every $f_m \in E(m)$ and $g_n \in E(n)$ as $\lambda \to \infty$,

$$\mathcal{W}_{m+n}(f_m \otimes g_{n,\lambda}) \to \mathcal{W}_m(f_m)\mathcal{W}_n(g_n).$$

(R4) For any finite set $f_0, f_1, \ldots, f_N$ of test functions such that $f_0 \in \mathbb{C}$, $f_n \in \mathcal{T}(T(\mathbb{R}^{4n}))$ for $1 \leq n \leq N$, one has

$$\sum_{m,n=0}^{N} \mathcal{W}_{m+n}(f_m^* \otimes f_n) \geq 0.$$

(R5) $\mathcal{W}_n(f) = \mathcal{W}_n(f_{(a,\Lambda)})$ for all $(a, \Lambda) \in \mathcal{P}_+^\uparrow$, all $f \in \mathcal{T}(T(\mathbb{R}^{4n}))$, and all $n = 1, 2, \ldots$.

(R6) For all $n = 2, 3, \ldots$ and all $i = 1, \ldots, n-1$ denote

$$L_i^\ell = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^{4n}; |x_i - x_{i+1}|_1 < \ell\},$$
$$W_i^\ell = \{(z_1, \ldots, z_n) \in \mathbb{C}^{4n}; z_i - z_{i+1} \in V^\ell\}.$$

Then, for any $\ell' > \ell$,

(i) $\mathcal{W}_n \in \mathcal{T}(T(\mathbb{R}^{4n}))'$ belongs to $\mathcal{T}(T(L_i^{\ell'}))'$ and

(ii) $\mathcal{W}_n \circ c_i^n$ belongs to $\mathcal{T}(W_i^{\ell'})'$,

where

$$(\mathcal{W}_n \circ c_i^n)(f) = \mathcal{W}_n(c_i^n(f)),$$
$$c_i^n(f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n).$$

The derivation of the properties (R1) - (R6) is found in [Brüning & Nagamachi (2004)].

Theorem 4.3 (reconstruction theorem)**.** *To a given sequence $(\mathcal{W}_n)_{n\in\mathbb{N}}$ of tempered ultra-hyperfunctions satisfying the conditions (R1) - (R6), there corresponds a neutral scalar field $A(f)$ which obeys all the axioms W.I - W.VII and has the given tempered ultra-hyperfunctions as vacuum expectation values. The field A is unique up to isomorphisms.*

Proof: The proof of the theorem is found in [Brüning & Nagamachi (2004)].
□

5. $:\exp\left(\mathrm{i}l^2\phi(x)^2\right):$ as a fundamental length quantum field

We are going to construct models of relativistic quantum fields with a fundamental length by constructing a sequence of n-point functionals which satisfies conditions (R1) – (R6) and then applying the reconstruction theorem (Theorem 4.3). Our starting point are the well-known results of Jaffe [Jaffe (1965)] on formal Wick power series of free fields. If we consider the power series of a free field ϕ

$$\rho^{(i)}(x) = \sum_{n=0}^{\infty} a_n^{(i)} \frac{:\phi(x)^n:}{n!}, \tag{5.1}$$

then we have the following theorem.

Theorem 5.1 (Theorem A.1 of [Jaffe (1965)]). *As a formal power series*

$$(\Phi_0, \rho^{(1)}(x_1)\cdots\rho^{(n)}(x_n)\Phi_0) = \sum_{r_{ij}=0;\,1\le i<j\le n}^{\infty} \frac{A(R)T^R}{R!} \tag{5.2}$$

$$r_{ij} = r_{ji},\ r_{ii} = 0,\ R_i = \sum_{j=1}^{n} r_{ij},\ A(R) = \prod_{j=1}^{n} a_{R_j}^{(j)}$$

$$R! = \prod_{1\le i<j\le n} (r_{ij})!,\quad T^R = \prod_{1\le i<j\le n} (t_{ij})^{r_{ij}} \tag{5.3}$$

$$t_{ij} = (\Phi_0, \phi(x_i)\phi(x_j)\Phi_0) = D_m^{(-)}(x_i - x_j).$$

Therefore

$$(\Phi_0, \rho^{(i)}(x)\rho^{(i)}(y)\Phi_0) = \sum_{n=0}^{\infty} \frac{a_n^{(i)2}}{n!} D_m^{(-)}(x-y)^n,$$

$$D_m^{(-)}(x) = (2\pi)^{-3}\int_{\boldsymbol{R}^3} [2\omega(\boldsymbol{k})]^{-1} e^{-i\omega(\boldsymbol{k})x^0} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{k}$$

$$(k\cdot x = k^0x^0 - \boldsymbol{k}\cdot\boldsymbol{x},\ \omega(\boldsymbol{k}) = \sqrt{\boldsymbol{k}^2 + m^2}).$$

If the coefficients $\{a_n^{(i)}\}$ satisfy $\lim_{n\to\infty}[|a_n^{(i)}|^2/n!]^{1/n} = 0$ then the series (5.1) defines a hyperfunction quantum field (see [Nagamachi & Mugibayashi (1986)]).

Now we assume that for some $\sigma > 0$

$$\limsup_{n\to\infty} [|a_n^{(i)}|^2/n!]^{1/n} = \sigma. \tag{5.4}$$

For example, consider

$$\rho(x) =: e^{g\phi(x)^2} := \sum_{n=0}^{\infty} g^n \frac{:\phi(x)^{2n}:}{n!} = \sum_{n=0}^{\infty} g^n \frac{(2n)!}{n!} \frac{:\phi(x)^{2n}:}{(2n)!}. \tag{5.5}$$

Then

$$\sigma = \lim_{n\to\infty} \left[|g^n|^2 \frac{(2n)!}{(n!)^2} \right]^{1/2n} = 2|g| \tag{5.6}$$

and

$$(\Phi_0, \rho(x)\rho(y)\Phi_0) = \sum_{n=0}^{\infty} \left(g^n \frac{(2n)!}{n!} \right)^2 \frac{1}{(2n)!} D_m^{(-)}(x-y)^{2n}.$$

Since

$$(1-x)^{-\alpha} = 1 + \alpha x + \frac{\alpha(\alpha+1)}{2!} x^2 + \ldots + \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} x^n + \ldots,$$

and for $\alpha = 1/2$

$$\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} = \frac{(2n)!}{4^n n!} \frac{1}{n!},$$

we get, in the sense of formal power series,

$$(\Phi_0, \rho(x)\rho(y)\Phi_0) = [1 - 4g^2 D_m^{(-)}(x-y)^2]^{-1/2}. \tag{5.7}$$

Now we investigate the convergence of this power series, in the sense of tempered ultra-hyperfunctions. To this end consider the power series

$$\sum_{r_{ij}=0;\, 1 \le i < j \le n}^{\infty} \frac{A(R) Z^R}{R!} \tag{5.8}$$

in the variables z_{ij} $(1 \le i < j \le n)$, where $Z^R = \prod_{1 \le i < j \le n} (z_{ij})^{r_{ij}}$. Let

$$\|R\| = \sum_{1 \le i < j \le n} r_{ij} \tag{5.9}$$

and $t_{ij}(1 \le i < j \le n)$ be positive constants. Suppose

$$\limsup_{\|R\| \to \infty} \left[\frac{|A(R)| T^R}{R!} \right]^{1/\|R\|} \le 1.$$

Then the fact that the series (5.8) converges if $|z_{ij}| < t_{ij}$ $(1 \le i < j \le n)$ follows from the following theorem of Lemire.

Theorem 5.2. *The associate convergence radii* $(r_1, \ldots, r_n)$ *of a series* $\sum a_{\nu_1,\ldots,\nu_n} z_1^{\nu_1} \cdots z_n^{\nu_n}$ *satisfy*

$$\limsup_{\nu_1+\cdots+\nu_n \to \infty} [|a_{\nu_1,\ldots,\nu_n}| r_1^{\nu_1} \cdots r_n^{\nu_n}]^{1/(\nu_1+\cdots+\nu_n)} = 1.$$

The multinomial theorem implies

$$R_i! \prod_{j=1}^{n} \frac{t_{ij}^{r_{ij}}}{(r_{ij})!} \le \left(\sum_{j=1}^{n} t_{ij} \right)^{R_i}.$$

and according to equations (5.3) and (5.9) we know

$$\sum_{i=1}^{n} R_i = 2\|R\|, \quad \prod_{i=1}^{n}\prod_{j=1}^{n} (r_{ij})! = (R!)^2, \quad \prod_{i=1}^{n}\prod_{j=1}^{n} t_{ij}^{r_{ij}} = (T^R)^2;$$

hence

$$\left[\frac{|A(R)|T^R}{R!}\right]^2 = \frac{\prod_{i=1}^n |a_{R_i}^{(i)}|^2 (T^R)^2}{(R!)^2} = \prod_{i=1}^n \left(|a_{R_i}^{(i)}|^2 \prod_{j=1}^n \frac{t_{ij}^{r_{ij}}}{(r_{ij})!} \right)$$

and

$$\left[\frac{|A(R)|T^R}{R!}\right]^{1/\|R\|} = \prod_{i=1}^n \left[|a_{R_i}^{(i)}|^2 \prod_{j=1}^n \frac{t_{ij}^{r_{ij}}}{(r_{ij})!} \right]^{1/2\|R\|} = \prod_{i=1}^n \left[\frac{|a_{R_i}^{(i)}|^2}{R_i!} R_i! \prod_{j=1}^n \frac{t_{ij}^{r_{ij}}}{(r_{ij})!} \right]^{1/2\|R\|}$$

$$\leq \prod_{i=1}^n \left[\frac{|a_{R_i}^{(i)}|^2}{R_i!} \left(\sum_{j=1}^n t_{ij} \right)^{R_i} \right]^{1/2\|R\|} = \prod_{i=1}^n \left[\left(\frac{|a_{R_i}^{(i)}|^2}{R_i!} \right)^{1/R_i} \left(\sum_{j=1}^n t_{ij} \right) \right]^{R_i/2\|R\|}.$$

Suppose that $t_{kk+1} < 1/\sigma$, and the other t_{ij}'s are so small that

$$\sum_{1\leq i<j\leq n} t_{ij} < \frac{1}{\sigma}.$$

This then implies

$$\limsup_{\|R\|\to\infty} \prod_{i=1}^n \left[\left(\frac{|a_{R_i}^{(i)}|^2}{R_i!} \right)^{1/R_i} \left(\sum_{j=1}^n t_{ij} \right) \right]^{R_i/2\|R\|} \leq 1$$

and the power series (5.8) is convergent for $|z_{ij}| < t_{ij} (1 \leq i < j \leq n)$. This shows the convergence of the vacuum expectation value

$$(\Phi_0, \rho^{(1)}(x_1) \cdots \rho^{(n)}(x_n)\Phi_0)$$

in the sense of tempered ultra-hyperfunctions.
Now we consider the case of $m = 0$ for simplicity. In this case the growth of the two-point function of the free field is easier to estimate. In the case of $m > 0$, see Proposition 8.3. Recall

$$D_0^{(-)}(x) = \lim_{\epsilon\to+0} (2\pi)^{-2}[(x^0 - i\epsilon)^2 - \boldsymbol{x}^2]^{-1},$$

$$|(x^0 - i\epsilon)^2 - \boldsymbol{x}^2| = |x^2 - \epsilon^2 - 2i\epsilon x^0|.$$

We claim that we can find $\epsilon \geq 0$ such that

$$|(2\pi)^{-2}[(x^0 - i\epsilon)^2 - \boldsymbol{x}^2]^{-1}| < 1/\sigma = 1/(2|g|),$$

where we used the relation (5.6). For $x^2 \leq 0$, $|x^2 - \epsilon^2 - 2i\epsilon x^0| \geq |x^2 - \epsilon^2| \geq |x^2| + \epsilon^2$, and for $x^2 \geq 0$, $|x^2 - \epsilon^2 - 2ix^0| \geq |x^2 - \epsilon^2 - 2i\epsilon\sqrt{x^2}| = x^2 + \epsilon^2$, and $(|x^2| + \epsilon^2)^{-1} < (2\pi)^2/\sigma$ is equivalent to $\epsilon^2 > \sigma/(2\pi)^2 - |x^2|$. Choose a number $r' > \sqrt{\sigma}/(2\pi)$ and define

$$\epsilon(x) = \sqrt{\max\{r'^2 - |x^2|, 0\}}. \qquad (5.10)$$

For such a choice one has

$$|(2\pi)^{-2}[(x^0 - i\epsilon(x))^2 - \boldsymbol{x}^2]^{-1}| < 1/\sigma.$$

Finally we fix the fundamental length for these models:

$$\ell = \sqrt{\sigma}/(2\pi) = \sqrt{2|g|}/(2\pi) = l/(\sqrt{2}\pi), \tag{5.11}$$

where we used the relation $g = il^2$ for $l > 0$. It is easily seen that for any $\ell' > \ell$ there exist $\epsilon(x)$ such that

$$\{(x^0 + i\epsilon(x), x^1, x^2, x^3); x \in \mathbb{R}^4\} \subset V^{\ell'}.$$

Therefore, for any $\ell' > \ell$ there exists $R > 0$ such that (in formal but suggestive notation)

$$W_{n-1}(\zeta) = \mathcal{W}_n(z) = (\Phi_0, \rho(z_1)\cdots\rho(z_k)\rho(z_{k+1})\cdots\rho(z_n)\Phi_0)$$

is a well-defined holomorphic function for

$$\mathrm{Im}\,\zeta_k = \mathrm{Im}\,(z_{k+1} - z_j) \in V_+ + (\ell', 0, 0, 0)$$

and

$$\mathrm{Im}\,\zeta_j = \mathrm{Im}\,(z_{j+1} - z_j) \in V_+ + (R, 0, 0, 0),\ (j \neq k). \tag{5.12}$$

This implies that $\mathcal{W}_n$ satisfies the condition (i) of the axiom (R6). That is, the mapping

$$\mathcal{T}(T(\mathbb{R}^{4n})) \ni f \to \mathcal{W}(f) = \int_{\prod_{i=0}^{n-1}\Gamma_i} W_{n-1}(\zeta)g(\zeta)d\zeta_0\cdots d\zeta_{n-1}$$

is continuous and can be extended continuously to

$$\mathcal{T}(T(L_k^{\ell'})) \ni f \to \mathcal{W}(f) = \int_{\prod_{i=0}^{n-1}\Gamma_i} W_{n-1}(\zeta)g(\zeta)d\zeta_0\cdots d\zeta_{n-1},$$

where, with $\epsilon(x)$ according to (5.10) and R sufficiently large,

$$\Gamma_k = \{(x^0 + i\epsilon(x), x^1, x^2, x^3); x \in \mathbb{R}^4\},\ \Gamma_j = \{(x^0 + iR, x^1, x^2, x^3); x \in \mathbb{R}^4\}$$

and $g(\zeta) = f(\zeta_0, \zeta_0 + \zeta_1, \ldots, \zeta_0 + \cdots + \zeta_{n-1})$. Now consider the formula

$$\begin{aligned}\prod_{1\le i<j\le n}(t_{i,j})^{r_{i,j}} = {} & (t_{k,k+1})^{r_{k,k+1}} \prod_{1\le i<j\le n,\, i\neq k,\, j\neq k+1}(t_{i,j})^{r_{i,j}} \\ & \times \prod_{1\le i<k}(t_{i,k})^{r_{i,k}} \prod_{k+1<j\le n}(t_{k,j})^{r_{k,j}} \\ & \times \prod_{1\le i<k}(t_{i,k+1})^{r_{i,k+1}} \prod_{k+1<j\le n}(t_{k+1,j})^{r_{k+1,j}}.\end{aligned}$$

The transposition of x_k and x_{k+1} causes the transposition of $(t_{k,k+1})^{r_{k,k+1}}$ and $(t_{k+1,k})^{r_{k+1,k}}$ in the first line, and the transposition of the second line and the third line. If x_k and x_{k+1} are space-like separated, then $t_{k,k+1} = t_{k+1,k}$. The function

$$W_{n-1}^k(\zeta) = (\Phi_0, \rho(z_1)\cdots\rho(z_{k+1})\rho(z_k)\cdots\rho(z_n)\Phi_0)$$

is also holomorphic in a domain defined by (5.12) and

$$-\mathrm{Im}\,\zeta_k \in V_+ + (\ell', 0, 0, 0).$$

Moreover, if ζ_k lies in $\mathbb{R}^4 \backslash V^{\ell'}$, the functions $W_{n-1}(\zeta)$ and $W^k_{n-1}(\zeta)$ are well-defined and coincide. Thus we have

$$\begin{aligned}(\mathcal{W}_n \circ c^n_k)(f) &= \int_{\prod_{i=0}^{n-1}\Gamma_i} W_{n-1}(\zeta)g(\zeta)d\zeta_0 \cdots d\zeta_{n-1} - \int_{-\Gamma_k \prod_{i\neq k}\Gamma_i} W^k_{n-1}(\zeta)g(\zeta)d\zeta_0 \cdots d\zeta_{n-1} \\ &= \int_{\Gamma^{\ell'}_k \prod_{i\neq k}\Gamma_i} W_{n-1}(\zeta)g(\zeta)d\zeta_0 \cdots d\zeta_{n-1} - \int_{-\Gamma^{\ell'}_k \prod_{i\neq k}\Gamma_i} W^k_{n-1}(\zeta)g(\zeta)d\zeta_0 \cdots d\zeta_{n-1},\end{aligned}$$

where

$$\Gamma^{\ell}_k = \{(x^0 + i\epsilon(x), x^1, x^2, x^3); x \in \mathbb{R}^4 \cap V^{\ell}\}$$

and we used the fact that $W_{n-1}(\zeta)$ and $W^k_{n-1}(\zeta)$ coincides for $\zeta_k \in \mathbb{R}^4 \backslash V^{\ell'}$. The above formula shows that the functional $\mathcal{W}_n \circ c^n_k$ belongs to $\mathcal{T}(T(W^{\ell'}_k))'$ for any $\ell' > \ell$ which shows condition (ii) of axiom (R6). We can show that $\mathcal{W}_n$'s satisfy the axioms (R1), (R3), (R4) and (R5) in a similar way as [Brüning & Nagamachi (2001)] where it is shown that if the coefficients $\{a^{(i)}_n\}$ satisfy $\lim_{n\to\infty}[|a^{(i)}_n|^2/n!]^{1/n} = 0$ then the series (5.1) define hyperfunction quantum fields. There, a Wick polynomial $\rho_N(x)$ is introduced as a truncation of $\rho(x)$,

$$\rho_N(x) = \sum_{n=0}^{N} g^n \frac{:\phi(x)^{2n}:}{n!}.$$

Then the Wightman functions $\mathcal{W}^N_n(x) = (\Phi_0, \rho_N(x_1)\cdots\rho_N(x_n)\Phi_0)$ for $\rho_N(x)$ satisfy all the standard Wightman axioms, and they converge weakly to $\mathcal{W}_n(x) = (\Phi_0, \rho(x_1)\cdots\rho(x_n)\Phi_0)$ as $N \to \infty$ in the sense of tempered ultra-hyperfunctions. Thus they satisfy the above axioms. The proof of the spectral condition (R2) is easier than in the case of hyperfunction quantum field theory because $\tilde{W}^N_{n-1}(q)$ and $\tilde{W}_{n-1}(q)$ are distributions, and $\tilde{W}^N_{n-1}(q)$ converge weakly to $\tilde{W}_{n-1}(q)$ as $N \to \infty$ in the sense of distributions. Since the limit in the sense of distributions preserves the support, (R2) is valid for $\rho(x)$. Accordingly we formulate the main result of the section.

Theorem 5.3 (Existence of fields with fundamental length)**.** *For a free field ϕ of mass $m \geq 0$ the Wick power series (5.1) (or more specifically (5.5)) define ultra-hyperfunction quantum fields with a fundamental length ℓ given by equations (5.4) and (5.11).*

Remark 5.4. We explained this only in the case of $m = 0$, but the above theorem is valid for $m \geq 0$. For details we have to refer to [Brüning & Nagamachi (2008)].

For the explicit form of $(\Phi_0, \rho^{(1)}(x_1)\cdots\rho^{(n)}(x_n)\Phi_0)$, we have the following proposition.

Proposition 5.5. *Abbreviate*

$$\rho^{(j)}(x_j) =: e^{-r_j i l^2 \phi(x_j)^2}:$$

with $r_j = \pm 1$. Then the vacuum expectation values of these fields are given by

$$(\Phi_0, \rho^{(1)}(x_1)\cdots\rho^{(n)}(x_n)\Phi_0) = (\det A)^{-1/2}, \tag{5.13}$$

where A is the $n \times n$ symmetric matrix whose entries $a_{j,k}$ are given by

$$a_{j,k} = a_{k,j} = 2h_{r_j}h_{r_k}l^2 D_m^{(-)}(x_j - x_k)$$

for $h_{\pm 1} = e^{\pm i\pi/4}$, $j < k$ and $a_{j,j} = 1$.

Proof. The proof is given in [Brüning & Nagamachi (2008)].

$\square$

6. Proof of equation 2.2

In order to prove statement B) we need some further properties of Wick products. Thus we begin by recalling some basic facts about Wick products of free fields which are then used to derive this statement.

Let $\mathcal{H}$ be the Hilbert space defined by

$$\mathcal{H} = \oplus_{n=0}^{\infty} \mathcal{H}_n.$$

Here, $\mathcal{H}_n$ is the set of symmetric square-integrable functions on the direct product of the momentum space hyperboloids

$$\xi_k^2 = m^2,\ \xi_k^0 > 0,\ k = 1, \ldots, n \tag{6.1}$$

with respect to the Lorentz invariant measure $\prod_{k=1}^{n} d\Omega_m(\xi_k)$, given by

$$d\Omega_m(\xi) = \frac{d\xi^1 d\xi^2 d\xi^3}{\sqrt{\sum_{k=1}^3 (\xi^k)^2 + m^2}}.$$

In the fundamental paper [Wightman & Gårding (1964)], we find the following quite general formula (3.44) for the definition of Wick products of a free field ϕ of mass m as operators in $\mathcal{H}$: For $f \in \mathcal{S}(\mathbb{R}^4)$ and $\Phi \in \mathcal{H}$ one has:

$$\begin{aligned}
&(: D^{\alpha^{(1)}}\phi D^{\alpha^{(2)}}\phi \cdots D^{\alpha^{(l)}}\phi : (f)\Phi)^{(n)}(\xi_1, \ldots, \xi_n) \qquad (3.44)\\
&= \frac{\pi^{l/2}}{(2\pi)^{2(l-1)}} \sum_{j=0}^{l} \left[\frac{(n-l+2j)!}{n!}\right]^{1/2} \int \cdots \int \left(\prod_{k=1}^{j} d\Omega_m(\eta_k)\right) \times \\
&\sum_{1 \le k_1 < k_2 < \ldots < k_{l-j} \le n} (j!)^{-1} \sum_P P\left((-i\eta_1)^{\alpha^{(1)}} \cdots (-i\eta_j)^{\alpha^{(j)}} (i\xi_{k_1})^{\alpha^{(j+1)}} \cdots \right. \\
&\left. \cdots (i\xi_{k_{l-j}})^{\alpha^{(l)}}\ \tilde{f}\left(\sum_{r=1}^{j} \eta_r - \sum_{r=1}^{l-j} \xi_{k_r}\right)\right) \Phi^{(n-l+2j)}(\eta_1, \ldots, \eta_j, \xi_1, \ldots, \hat{\xi}_{k_1}, \ldots, \hat{\xi}_{k_{l-j}}, \ldots, \xi_n),
\end{aligned}$$

where in the summation $\sum_{j=0}^{l}$, only those terms are to be retained for which $n-l+2j \geq 0$, and the sum $\sum_{P}$ is over all permutation P of the variables $\eta_1, \ldots, \eta_j, (-\xi_{k_1}), \ldots, (-\xi_{k_{l-j}})$. We reconsider this formula in the sense of operator-valued ultra-hyperfunctions. Let $|\beta| = 1$ and $|\alpha^{(1)}| = |\alpha^{(2)}| = \ldots = |\alpha^{(l)}| = 0$. Then we have from (3.44)

$$\begin{aligned}
(:\phi^l : (-D^\beta f)\Phi)^{(n)}(\xi_1, \ldots, \xi_n) &= \frac{\pi^{l/2}}{(2\pi)^{2(l-1)}} \sum_{j=0}^{l} \left[\frac{(n-l+2j)!}{n!}\right]^{1/2} \int \cdots \int \left(\prod_{k=1}^{j} d\Omega_m(\eta_k)\right) \\
&\times \sum_{1\leq k_1<k_2<\ldots<k_{l-j}\leq n} (j!)^{-1} \sum_P P\left(i\left(\sum_{r=1}^{j}\eta_r - \sum_{r=1}^{l-j}\xi_{k_r}\right)^\beta \tilde{f}\left(\sum_{r=1}^{j}\eta_r - \sum_{r=1}^{l-j}\xi_{k_r}\right)\right) \times \\
&\times \Phi^{(n-l+2j)}(\eta_1, \ldots, \eta_j, \xi_1, \ldots, \hat{\xi}_{k_1}, \ldots, \hat{\xi}_{k_{l-j}}, \ldots, \xi_n). \\
&= \frac{\pi^{l/2}}{(2\pi)^{2(l-1)}} \sum_{j=0}^{l} \left[\frac{(n-l+2j)!}{n!}\right]^{1/2} \int \cdots \int \left(\prod_{k=1}^{j} d\Omega_m(\eta_k)\right) \\
&\times \sum_{1\leq k_1<k_2<\cdots<k_{l-j}\leq n} (j!)^{-1} \sum_P P\left(l(i\eta_1)^\beta \tilde{f}\left(\sum_{r=1}^{j}\eta_r - \sum_{r=1}^{l-j}\xi_{k_r}\right)\right) \\
&\times \Phi^{(n-l+2j)}(\eta_1, \ldots, \eta_j, \xi_1, \ldots, \hat{\xi}_{k_1}, \ldots, \hat{\xi}_{k_{l-j}}, \ldots, \xi_n).
\end{aligned}$$

Observe that

$$\sum_P P(\eta_i) = \sum_P P(-\xi_{k_r})$$

for any i and r. This implies for $|\beta| = 1$,

$$\sum_P P\left((\eta_i)^\beta\right) = \sum_P P\left((-\xi_{k_r})^\beta\right)$$

and therefore

$$\sum_P P\left(i\left(\sum_{r=1}^{j}\eta_r - \sum_{r=1}^{l-j}\xi_{k_r}\right)^\beta \tilde{f}\left(\sum_{r=1}^{j}\eta_r - \sum_{r=1}^{l-j}\xi_{k_r}\right)\right) = \sum_P P\left(l(i\eta_1)^\beta \tilde{f}\left(\sum_{r=1}^{j}\eta_r - \sum_{r=1}^{l-j}\xi_{k_r}\right)\right).$$

On the other hand, we also have from (3.44), for $|\alpha^{(1)}| = 1$, and $|\alpha^{(2)}| = \ldots = |\alpha^{(l)}| = 0$

$$\begin{aligned}
&(:(D^{\alpha^{(1)}}\phi)\phi^{l-1} : (f)\Phi)^{(n)}(\xi_1, \ldots, \xi_n) \\
&= \frac{\pi^{l/2}}{(2\pi)^{2(l-1)}} \sum_{j=0}^{l} \left[\frac{(n-l+2j)!}{n!}\right]^{1/2} \int \cdots \int \left(\prod_{k=1}^{j} d\Omega_m(\eta_k)\right) \\
&\times \sum_{1\leq k_1<k_2<\ldots<k_{l-j}\leq n} (j!)^{-1} \sum_P P\left((i\eta_1)^{\alpha^{(1)}} \tilde{f}\left(\sum_{r=1}^{j}\eta_r - \sum_{r=1}^{l-j}\xi_{k_r}\right)\right)
\end{aligned}$$

$$\times\Phi^{(n-l+2j)}(\eta_1,\ldots,\eta_j,\xi_1,\ldots,\hat{\xi}_{k_1},\ldots,\hat{\xi}_{k_{l-j}},\ldots,\xi_n).$$

This shows that

$$(:\phi^l:(-D^{\alpha^{(1)}}f)\Phi)^{(n)} = l(:(D^{\alpha^{(1)}}\phi)\phi^{l-1}:(f)\Phi)^{(n)}, \tag{6.2}$$

that is,

$$D^{\alpha^{(1)}}:\phi(x)^l := l:(D^{\alpha^{(1)}}\phi(x))\phi^{l-1}(x):. \tag{6.3}$$

Let $\mathcal{D}_0$ be the set generated by the vectors of the form

$$\rho^{(1)}(f_1)\cdots\rho^{(n)}(f_n)\Phi_0, f_k \in \mathcal{T}(T(\mathbb{R}^4)),$$

where $\rho^{(k)}(x)$ is one of $\phi(x)$, $\rho(x)$ and $\rho^*(x)$, and $\Phi \in \mathcal{D}_0$. Then it follows from the weak conveargence of

$$\rho(-D^{\alpha^{(1)}}f)\Phi =: e^{ig\phi^2}:(-D^{\alpha^{(1)}}f)\Phi = \sum_{l=0}^{\infty}\frac{(ig)^l}{l!}:\phi^{2l}:(-D^{\alpha^{(1)}}f)\Phi$$

which we have seen in the previous section that the above series is also strongly convergent, and by (6.2)

$$:\phi^{2l}:(-D^{\alpha^{(1)}}f)\Phi = l:(D^{\alpha^{(1)}}\phi)\phi^{l-1}:(f)\Phi.$$

This shows that

$$\sum_{l=0}^{\infty}\frac{(ig)^l}{l!}:\phi^{2l}:(-D^{\alpha^{(1)}}f)\Phi = \sum_{l=1}^{\infty}\frac{(ig)^l}{(l-1)!}2:(D^{\alpha^{(1)}}\phi)\phi\phi^{2(l-1)}:(f)\Phi$$

$$= \sum_{l=0}^{\infty}2(ig)\frac{(ig)^l}{l!}:(D^{\alpha^{(1)}}\phi)\phi\phi^{2l}:(f)\Phi.$$

We write the last expression as

$$= 2(ig):(D^{\alpha^{(1)}}\phi)\phi\sum_{l=0}^{\infty}\frac{(ig)^l}{l!}\phi^{2l}:(f)\Phi = 2ig:(D^{\alpha^{(1)}}\phi)\phi\rho:(f)\Phi.$$

That is, the formal expression (which is difficult to give a direct meaning)

$$2ig:(D^{\alpha^{(1)}}\phi(x))\phi(x)(:e^{ig\phi(x)^2}:):\Phi = 2ig:(D^{\alpha^{(1)}}\phi(x))\phi(x)\sum_{l=0}^{\infty}\frac{(ig)^l}{l!}:\phi^{2l}(x)::\Phi$$

should be understood to be

$$\sum_{l=0}^{\infty}2ig:(D^{\alpha^{(1)}}\phi(x))\phi(x)\frac{(ig)^l}{l!}\phi^{2l}(x):\Phi = \sum_{l=1}^{\infty}2:(D^{\alpha^{(1)}}\phi(x))\frac{(ig)^l}{(l-1)!}\phi^{2l-1}(x):\Phi.$$

Then by (6.3), the above expression equals

$$\sum_{l=1}^{\infty}\frac{(ig)^l}{l!}D^{\alpha^{(1)}}:\phi^{2l}(x):\Phi,$$

and this is equal to

$$D^{\alpha^{(1)}} \sum_{l=1}^{\infty} \frac{(ig)^l}{l!} : \phi^{2l}(x) : \Phi = D^{\alpha^{(1)}} \rho(x)\Phi$$

in the sense of generalized functions. In the above understanding, we have

$$D^{\alpha^{(1)}} \rho(x)\Phi = 2ig : (D^{\alpha^{(1)}} \phi(x))\phi(x)\rho(x) : \Phi, \tag{6.4}$$

that is, if the Wick product

$$: (D^{\alpha^{(1)}} \phi(x))\phi(x)\rho(x) :$$

is defined by the Wick power series

$$\sum_{l=0}^{\infty} 2ig : (D^{\alpha^{(1)}} \phi(x))\phi(x) \frac{(ig)^l}{l!} \phi^{2l}(x) :,$$

then (6.4) holds and (2.2) follows.

7. Multiplier

As stated at the end of Section 5, $\{\mathcal{H}, \Phi_0, U(a,\Lambda), \phi(x), \rho(x), \rho^*(x)\}$ satisfies the axioms of UHFQFT (= ultrahyperfunction quantum field theory). Let $\rho^{(\kappa)}(x) = \rho(x)$ and $\rho^{(\bar{\kappa})}(x) = \rho^*(x)$. Then, as we learned in Section 4.2, the vector-valued function $\rho^{(\lambda_1)}(z_1) \cdots \rho^{(\lambda_n)}(z_n)\Phi_0$ is holomorphic in

$$\{(z_1, \ldots, z_n) \in \mathbb{C}^{4n}; \operatorname{Im} z_1 \in V_+ + (\ell/2, \mathbf{o}), \operatorname{Im}(z_{j+1} - z_j) \in V_+ + (\ell_j, \mathbf{o})\}$$

for some $\ell_j > \ell > 0$ $(j = 1, \ldots, n-1)$, where $\rho^{(\lambda)}(x)$ is one of $\rho^{(\kappa)}(x)$, $\rho^{(\bar{\kappa})}(x)$ and $\phi(x)$. Let $\psi_{0,\alpha}^{(\kappa)}(x) = \psi_{0,\alpha}(x)$ and $\psi_{0,\bar{\alpha}}^{(\bar{\kappa})}(x) = \bar{\psi}_{0,\bar{\alpha}}(x)$ be free Dirac fields of mass M. Then the system

$$\{\mathcal{K}, \Psi_0, V(a,\Lambda), \psi_{0,\alpha}^{(\kappa)}(x), \psi_{0,\bar{\alpha}}^{(\bar{\kappa})}(x)\}$$

satisfies the axioms of standard quantum field theory in terms of tempered distributions (and consequently, that of UHFQFT), and therefore $\psi_{0,\beta_1}^{(\lambda_1)}(z_1) \cdots \psi_{0,\beta_n}^{(\lambda_n)}(z_n)\Psi_0$ is holomorphic in

$$\{(z_1, \ldots, z_n) \in \mathbb{C}^{4n}; \operatorname{Im} z_1 \in V_+, \operatorname{Im}(z_j - z_{j-1}) \in V_+\},$$

where $\lambda = \kappa$, $\beta = \alpha$ or $\lambda = \bar{\kappa}$, $\beta = \bar{\alpha}$. Therefore, $\rho(z)\Phi$ for $\Phi = \rho^{(\lambda_2)}(f_2) \cdots \rho^{(\lambda_n)}(f_n)\Phi_0$, $f_j \in \mathcal{T}(T(\mathbb{R}^4))$ is holomorphic in

$$\{z \in \mathbb{C}^4; \operatorname{Im} z \in V_+ + (\ell/2, \mathbf{o})\}$$

and $\psi_{0,\alpha_1}(z)\Psi$ for $\Psi = \psi_{0,\beta_2}^{(\lambda_2)}(g_2) \cdots \psi_{0,\beta_n}^{(\lambda_n)}(g_n)\Psi_0$, $g_j \in \mathcal{S}(\mathbb{R}^4)$ is holomorphic there too.
The composite system

$$\{\mathcal{H} \otimes \mathcal{K}, \Phi_0 \otimes \Psi_0, U(a,\Lambda) \otimes V(a,\Lambda), \phi(x) \otimes I_{\mathcal{K}}, \rho(x) \otimes I_{\mathcal{K}}, \\ \rho^*(x) \otimes I_{\mathcal{K}}, I_{\mathcal{H}} \otimes \psi_{0,\alpha}(y), I_{\mathcal{H}} \otimes \bar{\psi}_{0,\bar{\alpha}}(y)\}$$

is the tensor product of two systems and thus satisfies all the axioms of UHFQFT. Although the tensor product is well-defined, the pointwise product is not necessarily well-defined for generalized (vector-valued) functions. In the category of distributions, the following theorem is well-known:

Theorem 7.1 (Theorem 8.2.10 of [Hörmander (1983)]). *If $u, v \in \mathcal{D}'(X)$ then the product uv can be defined as the pullback of the tensor product $u \otimes v$ by the diagonal map $\delta : X \to X \times X$ unless $(x, \xi) \in WF(u)$ and $(x, -\xi) \in WF(v)$.*

In our case, the condition that $\rho(z)\Phi$ and $\psi_{0,\alpha_1}(z)\Psi$ have the common domain of holomorphy,

$$\{z \in \mathbb{C}^4; \operatorname{Im} z \in V_+ + (\ell/2, \mathbf{o})\},$$

which corresponds to the condition of the wave front sets $WF(u)$ and $WF(v)$ of distributions, implies that the product $(\rho\psi_{0,\alpha})(f)$ is well-defined by the formula

$$(\rho\psi_{0,\alpha})(f)(\Phi \otimes \Psi) = \int_{\Gamma_N} f(z)\rho(z)\Phi \otimes \psi_{0,\alpha}(z)\Psi dz, \quad \Gamma_N = \{z \in \mathbb{C}^4; z = x + i(N, \mathbf{o})\}$$

for suitable $N > 0$. Thus the field $\psi_0(x)$ is a multiplier of the field $\rho(x)$. Similarly one can show that $\frac{\partial}{\partial x^\mu}\psi_{0,\alpha}$ is a multiplier for $\rho(x)$ and then we calculate

$$\begin{aligned}
(\frac{\partial}{\partial x^\mu}(\rho\psi_{0,\alpha}))(f)\Phi \otimes \Psi &= (\rho\psi_{0,\alpha})(-\frac{\partial}{\partial x^\mu}f)\Phi \otimes \Psi = \int_{\Gamma_N}(-\frac{\partial}{\partial x^\mu}f(z))\rho(z)\Phi \otimes \psi_{0,\alpha}(z)\Psi dz \\
&= \int_{\Gamma_N} f(z)\{\rho(z)\Phi \otimes \frac{\partial}{\partial x^\mu}\psi_{0,\alpha}(z)\Psi + \frac{\partial}{\partial x^\mu}\rho(z)\Phi \otimes \psi_{0,\alpha}(z)\Psi\}dz \\
&= (\rho\frac{\partial}{\partial x^\mu}\psi_{0,\alpha})(f)\Phi \otimes \Psi + ((\frac{\partial}{\partial x^\mu}\rho)\psi_{0,\alpha})(f)\Phi \otimes \Psi.
\end{aligned}$$

This gives

$$\frac{\partial}{\partial x^\mu}(\rho(x)\psi_{0,\alpha}(x))(\Phi \otimes \Psi) = \rho(x)\frac{\partial}{\partial x^\mu}\psi_{0,\alpha}(x)\Phi \otimes \Psi + (\frac{\partial}{\partial x^\mu}\rho(x))\psi_{0,\alpha}(x)\Phi \otimes \Psi.$$

Let $\psi(x) = \rho(x)\psi_0(x)$ and $\bar{\psi}(x) = \rho^*(x)\bar{\psi}_0(x)$. We can easily see that the fields $\psi(x), \bar{\psi}(x), \phi(x)$ satisfy the axioms of UHFQFT except for the condition of extended causality, which is proven in the next section. In fact, W.I - W.V follow from those of the systems $\{\mathcal{H}, \Phi_0, U(a, \Lambda), \phi(x), \rho(x), \rho^*(x)\}$ and $\{\mathcal{K}, \Psi_0, V(a, \Lambda), \psi^{(\kappa)}_{0,\alpha}(x), \psi^{(\bar{\kappa})}_{0,\bar{\alpha}}(x)\}$ (for W.V the relation (4.1) is used). For W.VI, we have only to restrict the Hilbert space $\mathcal{H} \otimes \mathcal{K}$ to the subspace generated by

$$\phi^{(\kappa_1)}_{j_1}(f_1) \cdots \phi^{(\kappa_n)}_{j_n}(f_n)\Phi_0 \otimes \Psi_0,\ f_j \in \mathcal{T}(T(\mathbb{R}^4))\ (n = 0, 1, \ldots),$$

where $\phi^{(\kappa)}_j(x)$ is $\psi_\alpha(x) = (\rho(x) \otimes I_{\mathcal{K}}) \cdot (I_{\mathcal{H}} \otimes \psi_{0,\alpha}(x)) = \rho(x) \otimes \psi_{0,\alpha}(x)$ or $\bar{\psi}_{\bar{\alpha}}(x) = (\rho^*(x) \otimes I_{\mathcal{K}}) \cdot (I_{\mathcal{H}} \otimes \bar{\psi}_{0,\bar{\alpha}}(x)) = \rho^*(x) \otimes \bar{\psi}_{0,\bar{\alpha}}(x)$ or $\phi(x) \otimes I_{\mathcal{K}}$.

8. Fundamental length quantum fields

In this section we are going to prove the condition of extended causality (the axiom W.VI). In a first step we prove that Axiom W.VI is equivalent to a condition **R6** for the Wightman functionals. Then we proceed to verify condition **R6**.

Proposition 8.1. *Assuming the validity of the other axioms, the axiom of extended causality WVI is equivalent to the following condition*

R6 *For all $n = 2, 3, \dots$ and all $i = 1, \dots, n-1$ denote*

$$L_i^\ell = \{x = (x_1, \dots, x_n) \in \mathbb{R}^{4n}; |x_i - x_{i+1}|_1 < \ell\},$$

$$W_i^\ell = \{x = (z_1, \dots, z_n) \in \mathbb{C}^{4n}; z_i - z_{i+1} \in V^\ell\},$$

$$V^\ell = \{z \in \mathbb{C}^4; \exists\, x \in V, |\mathrm{Re}\, z - x| < \ell, |\mathrm{Im}\, z|_1 < \ell\}. \tag{8.1}$$

Then, for any $\ell' > \ell$,

a) the functional

$$\mathcal{T}(T(\mathbb{R}^{4n})) \ni f \to \mathcal{W}_{\mu_1 \dots \mu_n}^{(\kappa_1 \dots \kappa_n)}(f) \in \mathbb{C}$$

is extended continuously to $\mathcal{T}(T(L_i^{\ell'}))$, and

b) the functional on $\mathcal{T}(T(\mathbb{R}^{4n}))$

$$f \to \mathcal{W}_{\mu_1 \dots \mu_j \mu_{j+1} \dots \mu_n}^{(\kappa_1 \dots \kappa_j \kappa_{j+1} \dots \kappa_n)}(f) + \mathcal{W}_{\mu_1 \dots \mu_{j+1} \mu_j \dots \mu_n}^{(\kappa_1 \dots \kappa_{j+1} \kappa_{j+} \dots \kappa_n)}(f) \in \mathbb{C}$$

is extended continuously to $\mathcal{T}(W_i^{\ell'})$.

Proof. Since the spinor/tensor indices do not play a role in this statement the proof given in [Brüning & Nagamachi (2004)] for the scalar case applies (see Propositions 4.3, 4.4 and Theorem 5.1 of [Brüning & Nagamachi (2004)]). □

Proposition 8.2. *The Wightman functions $\mathcal{W}_\alpha^r$ as given in [Nagamachi & Brüning (2008)] or in formula (8.6) below satisfy condition* **R6***.*

Proof. The determinant $\det A$ of (5.13) can be expressed as

$$\det A = 1 + P_n(a_{j,k}) \tag{8.2}$$

where $P_n(a_{j,k})$ is the sum of homogeneous polynomials of degrees $m = 2, \cdots, n$ in the entries $a_{j,k}$, $1 \le j < k \le n$ with integer coefficients.

Introduce

$$Q_{n,j}(a_{i,k}) = \sum_{\substack{(i,k,\dots,l) \neq (1,2,\dots,n) \\ (i,k,\dots,l) \neq (1,2,\dots,j+1,j,\dots,n)}} \mathrm{sgn}\,(i,k,\dots,l) a_{1,j} a_{2,k} \cdots a_{n,l} \tag{8.3}$$

and denote by $\sigma(j+1,j)$ the permutation $(1,\dots,j-1,j,j+1,\dots,n) \longrightarrow (1,\dots,j-1,j+1,j,\dots,n)$. Then we have

$$P_n(a_{i,k}) = \mathrm{sgn}\,(\sigma(j+1,j))a_{1,1}a_{2,2}\cdots a_{j-1,j-1}a_{j,j+1}a_{j+1,j}a_{j+1,j+1}\cdots a_{n,n} + Q_{n,j}(a_{i,k}) = -a^2_{j,j+1} + Q_{n,j}(a_{i,k}) = \pm 4l^2 D_m^{(-)}(z_j - z_{j+1})^2 + Q_{n,j}(a_{i,k}).$$

Hence we can rewrite (8.2) as

$$\det A = 1 + P_n(a_{i,k}) = 1 \pm 4l^4 D_m^{(-)}(z_j - z_{j+1})^2 + Q_{n,j}(a_{i,k}).$$

It is clear from (8.2), (8.3) and the details provided about the polynomial P_n that each term of $Q_{n,j}(a_{i,k})$ contains products of 2-points functions $D_m^{(-)}$ at arguments different from $z_j - z_{j+1}$. Assume

$$y^0_{j+1} - y^0_j > \ell = l/(\sqrt{2}\pi). \tag{8.4}$$

Then we have $|4l^4 D_m^{(-)}(z_j - z_{j+1})^2| < 1$ by the estimate

$$|D_m^{(-)}(x^0 - i\epsilon, \boldsymbol{x})| \leq (2\pi\epsilon)^{-2} \quad \text{for all } x \in \mathbb{R}^4. \tag{8.5}$$

If we choose the arguments $y^0_k - y^0_i$ ($i < k$) in these 2-points functions sufficiently large, $Q_{n,j}(a_{i,k})$ becomes very small; and for these points z_j the determinant $(\det A(z))^{-1/2}$ is holomorphic and the function $(\det A(z))^{-1/2}\mathcal{W}^r_{0,\alpha}(z_1,\ldots,z_n)$ defines a functional in $\mathcal{T}(T(L^{\ell'}_j))'$ for any $\ell' > \ell$ by the formula

$$\mathcal{W}^r_\alpha(f) = \int_{\prod_{j=1}^n \Gamma_j} (\det A(z))^{-1/2}\mathcal{W}^r_{0,\alpha}(z_1,\ldots,z_n) f(z) dz \tag{8.6}$$

for all $f \in \mathcal{T}(T(L^{\ell'}_j))$, where $\Gamma_j = \mathbb{R}^4 + i(y^0_j, 0, 0, 0)$ and

$$\mathcal{W}^r_{0,\alpha}(z_1,\ldots,z_n) = (\Psi_0, \psi^{(r_1)}_{0,\alpha_1}(z_1)\cdots\psi^{(r_n)}_{0,\alpha_n}(z_n)\Psi_0)$$

is the Wightman function of free Dirac field. In fact, for $\ell' > \ell$, we choose $\ell' > y^0_{j+1} - y^0_j > \ell$ and the other $y^0_k - y^0_i$ sufficiently large so that $(\det A(z))^{-1/2}$ is a bounded function of x. Then the corresponding integration path $\prod_{j=1}^n \Gamma_j$ of (8.6) is contained in

$$T(L^{\ell'}_j) = \{z = x + iy \in \mathbb{C}^{4n}; |y_j - y_{j+1}|_1 < \ell'\},$$

where $|y|_1 = |y^0| + |\boldsymbol{y}|$. We conclude that the functional defined by

$$(\det A(z))^{-1/2}\mathcal{W}^r_{0,\alpha}(z_1,\ldots,z_n)$$

satisfies condition a) of **R6**.

The transposition of z_j and z_{j+1} causes the change of $a_{j,j+1} = a_{j+1,j}$:

$$D_m^{(-)}(z_j - z_{j+1}) \to D_m^{(-)}(z_{j+1} - z_j)$$

and for an index k with $j < k \neq j+1$ the change

$$a_{j,k} = a_{k,j} = D_m^{(-)}(z_j - z_k) \to D_m^{(-)}(z_{j+1} - z_k) = a_{j+1,k} = a_{k,j+1},$$

$$a_{j+1,k} = a_{k,j+1} = D_m^{(-)}(z_{j+1} - z_k) \to D_m^{(-)}(z_j - z_k) = a_{j,k} = a_{k,j},$$

results while for an index k with $j > k \neq j+1$ the change is

$$a_{j,k} = a_{k,j} = D_m^{(-)}(z_k - z_j) \to D_m^{(-)}(z_k - z_{j+1}) = a_{j+1,k} = a_{k,j+1},$$

$$a_{j+1,k} = a_{k,j+1} = D_m^{(-)}(z_k - z_{j+1}) \to D_m^{(-)}(z_k - z_j) = a_{j,k} = a_{k,j}.$$

We consider the matrix $B = (b_{i,j})$ obtained from A by the change of j-th and $(j+1)$-th rows and j-th and $(j+1)$-th columns. Then we have $\det A = \det B$. Next we consider the matrix $C = (c_{j,k})$ obtained from B by changing only $b_{j,j+1} = b_{j+1,j} = a_{j,j+1} = a_{j+1,j}$, i.e., $c_{j,j+1} = c_{j+1,j} = D_m^{(-)}(z_{j+1} - z_j)$. If x_j and x_{j+1} are space-like separated, then $D_m^{(-)}(x_j - x_{j+1})$ is analytic (space-like points x are Jost points of $D_m^{(-)}(x)$) and $D_m^{(-)}(x_j - x_{j+1}) = D_m^{(-)}(x_{j+1} - x_j)$. Therefore for space-like separated x_j, x_{j+1} ($y_j^0 - y_{j+1}^0 = 0$) and the other $y_k^0 - y_i^0$ sufficiently large, we have $\det A = \det C$. Note that $\mathcal{W}_{0,\alpha}^r(z_1, \ldots, z_n)$ is also expressed by the sum of the products of the two-point functions of the free Dirac field as in the scalar case, and for space-like separated x_j, x_{j+1} ($y_j^0 - y_{j+1}^0 = 0$) and the other $y_k^0 - y_i^0$ positive, one has

$$\mathcal{W}_{0,\alpha}^r(z_1, \ldots, x_j, x_{j+1}, \ldots, z_n) = -\mathcal{W}_{0,\alpha}^r(z_1, \ldots, x_{j+1}, x_j, \ldots, z_n).$$

In order to proceed, we need some estimate for $D_m^{(-)}(x_j - x_{j+1})$.

Proposition 8.3 (Corollary 2.4 of [Brüning & Nagamachi (2008)]). *Denote by* $\mathrm{dist}\,(x, \bar{V})$ *the distance between* x *and the closed light cone* $\bar{V}$*, and for* $\ell > 0$*,*

$$V_\ell = \{x \in \mathbb{R}^4; \mathrm{dist}\,(x, V) < \ell\}.$$

Define $\epsilon_\ell(x)$ *by* $\epsilon_\ell(x) = \ell$ *if* $\mathrm{dist}\,(x, \bar{V}) < \ell/\sqrt{2}$, $\epsilon_\ell(x) = \sqrt{2\ell^2 - 2\mathrm{dist}\,(x, \bar{V})^2}$ *if* $\ell/\sqrt{2} \leq \mathrm{dist}\,(x, \bar{V}) < \ell$ *and* $\epsilon_\ell(x) = 0$ *if* $\mathrm{dist}\,(x, \bar{V}) \geq \ell$*. Then* $0 \leq \epsilon_\ell(x) \leq \ell$ *and* $\mathrm{supp}\,\epsilon_\ell(x) \subset \bar{V}_\ell$*. Let* $\ell = l/(\sqrt{2}\pi)$ *and assume* $ml < 2$*. Then, if* $\ell'' > \ell$*, the estimate*

$$2l^2 |D_m^{(-)}(x^0 - i\epsilon_{\ell''}(x), \boldsymbol{x})| < 1$$

holds.

For any $\ell' > \ell$, we choose $\ell < \ell'' < \ell'$. Let $\epsilon(x) = \epsilon_{\ell''}(x)$ and $a_{j,j+1} = D_m^{(-)}(x_j - x_{j+1} + i\epsilon(x_j - x_{j+1}))$ and for other $a_{i,k}$, $y_k^0 - y_i^0$ sufficiently large. Then $(\det A(x))^{-1/2}$ and $(\det C(x))^{-1/2}$ are well-defined continuous functions of x and $(\det A(x))^{-1/2} = (\det C(x))^{-1/2}$ if $x_j - x_{j+1} \in \mathbb{R}^4 \backslash V^{\ell'}$. Denote

$$\mathcal{W}_\alpha^r(z_1, \ldots, z_n) = (\det A(z))^{-1/2} \mathcal{W}_{0,\alpha}^r(z_1, \ldots, z_n)$$

and

$$\mathcal{W}_\alpha^{r,j}(z) = \mathcal{W}_{\alpha'}^{r'}(z'), z' = (z_1, \ldots, z_{j+1}, z_j, \ldots, z_n),$$

$$r' = (r_1, \ldots, r_{j+1}, r_j, \ldots, _n), \alpha' = (\alpha_1, \ldots, \alpha_{j+1}, \alpha_j, \ldots, \alpha_n).$$

Then, by deforming the path $\Gamma_j \times \Gamma_{j+1}$ in Eq. (8.6) into $G_{j,j+1}$, we can write

$$\mathcal{W}^r_\alpha(f) + \mathcal{W}^{r,j}_\alpha(f) = \int_{G_{j,j+1}\prod_{i\neq j,j+1}\Gamma_i} \mathcal{W}^r_\alpha(z) f(z) dz + \int_{G_{j+1,j}\prod_{i\neq j,j+1}\Gamma_i} \mathcal{W}^{r,j}_\alpha(z) f(z) dz,$$

where $y^0_j = y^0_{j+1}$ and

$$G_{j,j+1} = \{(x^0_j + iy^0_j - i\epsilon(x_j - x_{j+1}), \boldsymbol{x}_j, x^0_{j+1} + iy^0_{j+1}, \boldsymbol{x}_{j+1}); (x_j, x_{j+1}) \in \mathbb{R}^{2\cdot 4}\},$$

$$G_{j+1,j} = \{(x^0_j + iy^0_j, \boldsymbol{x}_j, x^0_{j+1} + iy^0_{j+1} - i\epsilon(x_{j+1} - x_j), \boldsymbol{x}_{j+1}); (x_j, x_{j+1}) \in \mathbb{R}^{2\cdot 4}\}.$$

Since $\mathcal{W}^r_\alpha(z) + \mathcal{W}^{r,j}_\alpha(z) = 0$ for $x_j - x_{j+1} \in \mathbb{R}^4 \backslash V^{\ell'}$,

$$\mathcal{W}^r_\alpha(f) + \mathcal{W}^{r,j}_\alpha(f) = \int_{G^{\ell'}_{j,j+1}\prod_{i\neq j,j+1}\Gamma_i} \mathcal{W}^r_\alpha(z) f(z) dz + \int_{G^{\ell'}_{j+1,j}\prod_{i\neq j,j+1}\Gamma_i} \mathcal{W}^{r,j}_\alpha(z) f(z) dz,$$

where

$$G^{\ell'}_{j,j+1} = \{(x^0_j + iy^0_j - i\epsilon(x_j - x_{j+1}), \boldsymbol{x}_j, x^0_{j+1} + iy^0_{j+1}, \boldsymbol{x}_{j+1}); (x_j, x_{j+1}) \in \mathbb{R}^{2\cdot 4} \cap V^{\ell'}\},$$

$$G^{\ell'}_{j+1,j} = \{(x^0_j + iy^0_j, \boldsymbol{x}_j, x^0_{j+1} + iy^0_{j+1} - i\epsilon(x_{j+1} - x_j), \boldsymbol{x}_{j+1}); (x_j, x_{j+1}) \in \mathbb{R}^{2\cdot 4} \cap V^{\ell'}\}.$$

Since $G^{\ell'}_{j,j+1}\prod_{i\neq j,j+1}\Gamma_i, G^{\ell'}_{j+1,j}\prod_{i\neq j,j+1}\Gamma_i \subset W^{\ell'}_j$, this shows that

$$\mathcal{T}(W^{\ell'}_j) \ni f \to \mathcal{W}^r_\alpha(f) + \mathcal{W}^{r,j}_\alpha(f) \in \mathbb{C}$$

is continuous and satisfies the condition b) of **R6**. □

Remark 8.4. In our previous paper [Brüning & Nagamachi (2004)], we defined a complex neighbourhood $V^{\ell'}$ by

$$V^{\ell'} = \{z \in \mathbb{C}^4; \exists x \in V; |\mathrm{Re}\, z - x| + |\mathrm{Im}\, z|_1 < \ell'\}. \tag{8.7}$$

But we found that to treat the present model, the neighbourhood (8.1) is convenient, and by this change of the ℓ'-neighbourhood of V, our theory [Brüning & Nagamachi (2004)] is not affected.

9. Conclusion

In the space of operator-valued tempered ultrahyperfunctions we have solved Heisenberg's linearized equation (1.2). This equation contains a parameter l which has the dimension of a length and it has been found that this parameter l is proportional to the fundamental length ℓ of our recently developed relativistic quantum field theory with a fundamental length. We found (see (5.11), (8.4) and Proposition 8.3)

$$\ell = l/(\sqrt{2}\pi).$$

The use of tempered ultrahyperfunctions was unavoidable, at least in our solution strategy. In this sense we conclude that Heisenberg's linearized equation only has a solution in the framework of relativistic quantum field theory with a fundamental length, not in the framework of ordinary relativistic quantum field theory.

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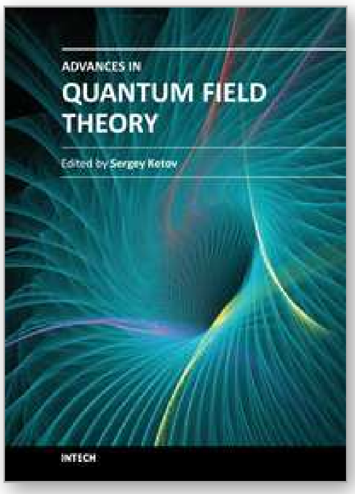

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Quantum Field Theory is now well recognized as a powerful tool not only in Particle Physics but also in Nuclear Physics, Condensed Matter Physics, Solid State Physics and even in Mathematics. In this book some current applications of Quantum Field Theory to those areas of modern physics and mathematics are collected, in order to offer a deeper understanding of known facts and unsolved problems.

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