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Quantum Field Theory and Knot Invariants

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1. Introduction

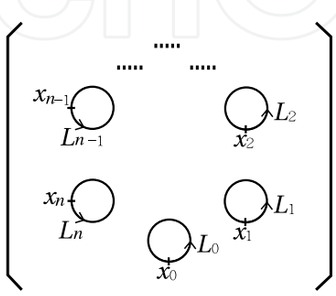
Massey-Milnor linking theory was developed by J. Milnor [Mi1, Mi2] and W. Massey [Ma, Po] algebraically and homological-theoretically in 1960's, but still remains quite mysterious as the explicit formulae thereof is missing.

Chern-Simons-Witten configuration space integrals are the Feynman graphs in the aspect of perturbative quantum field theory, and are developed by E. Witten [ADW, AF1, AF2, At, Aw, Ba1, Ba2, HM, MV, RT, Tu, Wi] in 1990's. But, as in almost all quantum field theories to compute Feynman graphs explicitly is always beyond any rigorous mathematical attack for the time being [PS]. Nevertheless in this paper, in the aspect of the first nonvanishing Massey-Milnor linking [Ma, Mi1, Mi2, Po] we compute explicitly the related Chern-Simons-Witten configuration space integrals [HKT, Hs1, Hs2, Hs3, Hs4, Hs5, Hs6, Hs7, HY], from which we derive the combinatorial formulae of the Massey-Milnor linking when the link under study is represented as a link diagram on the plane $\mathbb{R}^2$.

2. Set-up

In this section for the forthcoming presentation of Massey-Milnor linking theory and Chern-Simons-Witten graphs in perturbative quantum field theory [AF1, AF2, Ba2, Wi], we define the related concept as follows.

For the set-up suppose that a given link $L = \{L_0, L_1, \ldots, L_n\}$ oriented with base points $\{x_j \in L_j | j = 0, 1, \ldots, n\}$, is in a general position with pairwise crossings specified in $\mathbb{R}^2$ and is represented schematically as



To be more precise, each component L_j is represented schematically by a trivial circle with the base point x_j placed outer the most on L_j of $L = \{L_0, L_1, \ldots, L_n\}$ which is arranged

counter-clockwise with L_0 as the root. Moreover each L_j is oriented counter-clockwise as shown schematically.

Also for the link $L = \{L_0, L_1, \ldots, L_n\}$ of $(n+1)$ components, to define the invariant $L_{n,n-1,\ldots,1,0}$ below we assume that all invariants of strictly lower degrees vanish namely that $L_{m,m-1,\ldots,1,0} = 0$, for any permutation of any subset $\{L_0^*, L_1^*, \ldots, L_m^*\} \subseteq L$ of $(m+1)$ components and for any $m \leq n-1$.

3. Chern-Simons-Witten graphs

In this section we present the key concept of Chern-Simons-Witten configuration space integrals in the framework of perturbative quantum field theory. Beyond that we define our first knot invariant $L_{n,n-1,\ldots,1}$ for a link $\{L_0, L_1, \ldots, L_n\}$, for which all invariants of strictly lower degrees vanish as in the setup.

Definition 1. *(1) Given an oriented link $L = \{L_0, L_1, \ldots, L_{n-1}, L_n\}$ as above, a Chern-Simons-Witten graph Γ supported on L is a uni-trivalent rooted tree with all univalent vertices supported on L.—Notice that our trees are "honest" trees in strict sense that all edges rooted at a vertex are all going upward therefrom.*

(2) Given a Chern-Simons-Witten graph on L we define its degree to be

$$\text{degree}\,\Gamma = \#\{\text{ edges of }\Gamma\} - \#\{\text{ trivalent vertices of }\Gamma\}.$$

(3) Given a Chern-Simons-Witten graph Γ supported on L we define the associated Chern-Simons-Witten configuration space to be the space as follows.

(3-1) For each trivalent vertex we assign a copy of $\mathbb{R}^3$.

(3-2) For a univalent vertex supported on the component L_j of L we assign L_j to it.

And if some univalent vertices $\{U_1, U_2, \ldots, U_k\}$ ordered linearly with respect to the orientation and the base point $x_i \in L_j$ are supported on L_j, then we assign to $\{U_1, U_2, \ldots, U_k\}$ the subset of $(L_j)^k$ which respects the linear order of L_j, namely the subset $\{(y_1, y_2, \ldots, y_k) | x_j \leq y_1 \leq y_2 \leq \cdots \leq y_k \leq x_j\} \subseteq (L_j)^k$ with the induced orientation.

As the configuration space of Γ we take the abstract product of the spaces in (3-1) and (3-2), but with the orientation specified (or the ordering of the factors of the product) as follows:

(3-2-1) Always start with the root L_0.

(3-2-2) Going up for each edge.

(3-2-3) From the right edge to the left edge at each trivalent vertex.

(3-2-4) Endow the connected subgraphs with the above orientations and then take the product of the orientation of all components. It does not matter how to get the product of the orientation of the components as all components are even dimensional spaces.

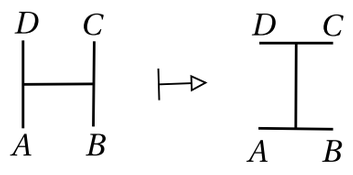
(4) For an edge joining vertices A and B in Γ, we assign a differential 2-form to it by pulling-back the standard area form on the unit sphere in $\mathbb{R}^3$ by the map $\dfrac{A-B}{|A-B|}$ where vertex A sits below vertex B in Γ. And we define the differential form associated to Γ to be the product of the 2-forms indexed by all edges of Γ. Notice that it does not matter how we "arrange" the order of the product, as all these

differential forms are 2*-forms. Also notice that if univalent vertices* $U_1, \ldots, U_k$ *are supported on a component* L_j*, then they are "positioned" on* L_j *with the same "height".*

(5) Finally for a Chern-Simons-Witten graph Γ *supported on a given link* $L = \{L_0, L_1, \ldots, L_{n-1}, L_n\}$*, we define the associted Chern-Simons-Witten configuration space integral to be the integral of the differential form constructed in (4) over the configuration space constructed in (3).*

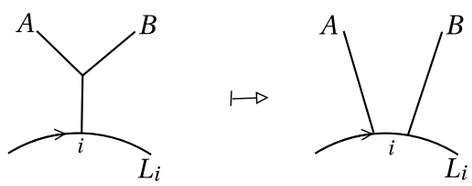
Next in Definition 2 we define the first non-vanishing invariant $L_{n,n-1,\ldots,1,0}$ for the link $\{L_0, L_1, \ldots, L_n\}$ represented diagrammatically as in the setup. We coin the construction as HIST-transform where HI comes from the IHX-relation and ST comes from the STU-relation in the perturbative Chern-Simons-Witten quantum field theory [Oh, Wi, Ye].

Definition 2. *(1) We define the HI-transform as:*



where vertices A, B, C and D are generic vertices of a Chern-Simons-Witten graph which are not necessarily uni-valent ones.

(2) We define the ST-transform as:



where vertices A and B are generic vertices and vertex i is a univalent vertex sitting on some knot component.

(3) For the construction of connected Chern-Simons-Witten graphs of degree n supported on $\{L_0, L_1, \ldots, L_n\}$*, and to ease the notation, we define "double round brackets"* $((n, n-1, \ldots, 2, 1))$ *as follows.*

(3-1) $((2,1)) = 2 \wedge 1$

(3-2) $((3,2,1)) = ((3,2)) \wedge 1 + 3 \wedge ((2,1))$

(3-3) $((4,3,2,1)) = ((4,3,2)) \wedge 1 + ((4,3)) \wedge ((2,1)) + 4 \wedge ((3,2,1))$

$\ldots \qquad \ldots$

(3-4) $((n, n-1, \ldots, 2, 1)) = ((n, n-1, \ldots, 2)) \wedge 1 + ((n, n-1, \ldots, 3)) \wedge ((2,1)) + \cdots + ((n, n-1)) \wedge ((n-2, n-3, \ldots, 2, 1)) + n \wedge ((n-1, n-2, \ldots, 2, 1))$

(4) The construction of connected Chern-Simons-Witten graphs of degree n on $\{L_0, L_1, \ldots, L_n\}$ *is as follows.*

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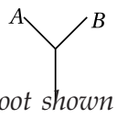
(4-1) For generic vertices A and B, we assign 

to $(A \wedge B)$, this is we connect vertices A and B arranged as above, to a new vertex—the root shown here—by a "Y". And by definition the new vertex—the root—is denoted as $(A \wedge B)$, if we need to continue doing this construction therefrom.

(4-2) For two connected Chern-Simons-Witten graphs A and B, to $(A + B)$ we assign the disjoint copies of A and B.

(5) On $\{L_0, L_1, \ldots, L_n\}$ for the construction of the first non-vanishing Chern-Simons-Witten graph $L_{n,n-1,\ldots,1,0}$: Start with the connected ones in (4) and apply ST-transform exactly once to get the set of all Chern-Simons-Witten graphs of two components; repeat ST-transforms till we get finally the Chern-Simons-Witten graphs of exactly n components. And $L_{n,n-1,\ldots,1,0}$ is the "sum" of the Chern-Simons-Witten graphs constructed above.

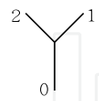
Before giving some examples we make the following remarks to make more sense of Definition 2.

Note 1. *(1) It is easy to see that the connected Chern-Simons-Witten graphs constructed in (4) of Definition 2 is closed under the HI-transforms.*

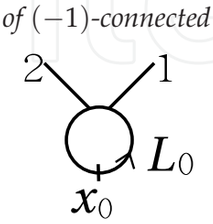
(2) If we coin the connected Chern-Simons-Witten graphs as 0-connected, those ones of two components as (-1)-connected, the ones of three components as (-2)-connected and so forth and so on, then it is easy to see that for any l the set of l-connected Chern-Simons-Witten graphs of degree n supported on $L = \{L_1, \ldots, L_n\}$ are closed under HI-transform; moreover, the set of l-connected Chern-Simons-Witten graphs constructed as above will produce exactly the set of $(l-1)$-connected ones after doing ST-transform exactly once.

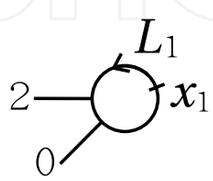
(3) Our Chern-Simons-Witten graphs are always not edge-overlapping, that is when edges of the graphs are represented as line segment in $\mathbb{R}^2$ they never intersect with one another except obviously at the vertices proper.

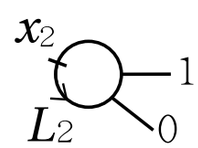
Here are some examples to show the idea.

Example 1. *For $n = 2$, $L = \{L_0, L_1, L_2\}$, the connected Chern-Simons-Witten graph of degree 2 is:* 

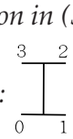
. As usual to ease the notation we use numerals $i, j, k, \ldots$ for the knot component L_i, L_j, L_k etc. The set of (-1)-connected ones are

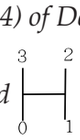






It is easy to see that starting with the connected Chern-Simons-Witten graphs we get the set of (-1)-connected ones after doing ST-transform exactly once.

Example 2. *For $n = 3$, $L = \{L_0, L_1, L_2, L_3\}$, from the construction in (3), (4) of Definition 2 it is to see that the connected Chern-Simons-Witten graphs of degree 3 are:* 

and 

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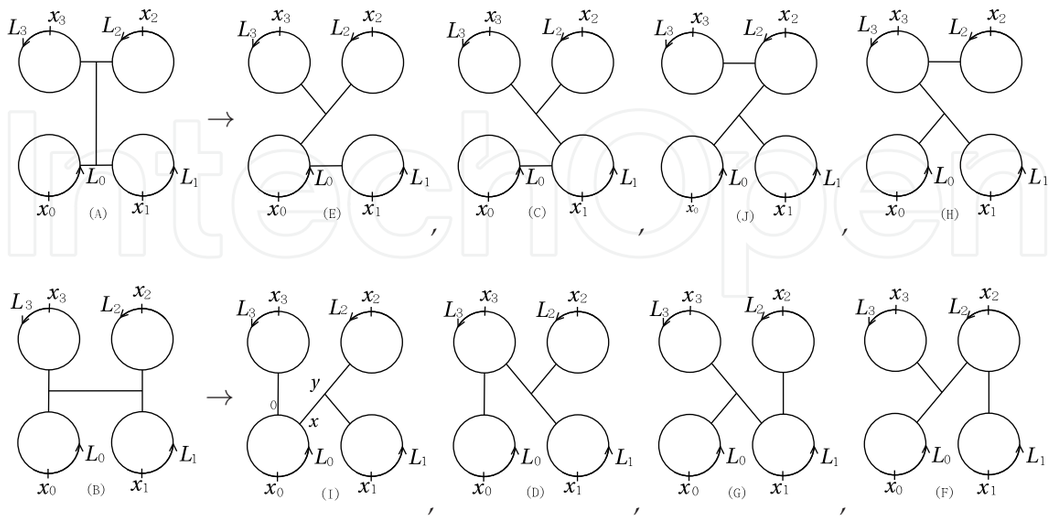
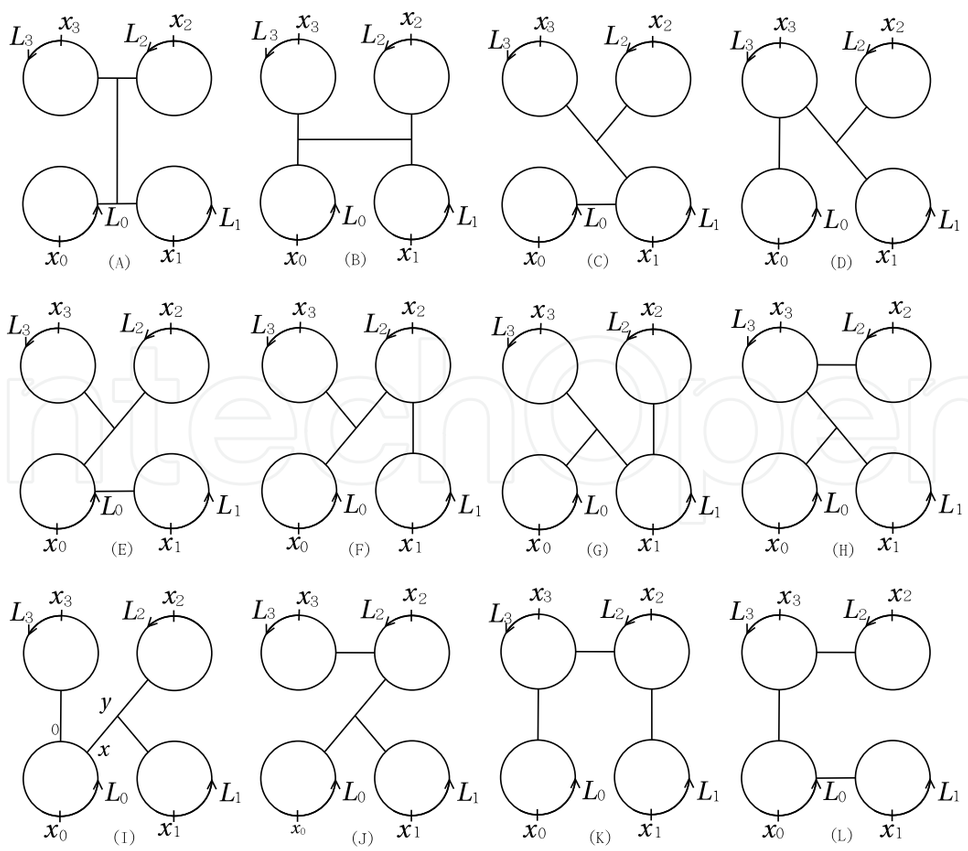
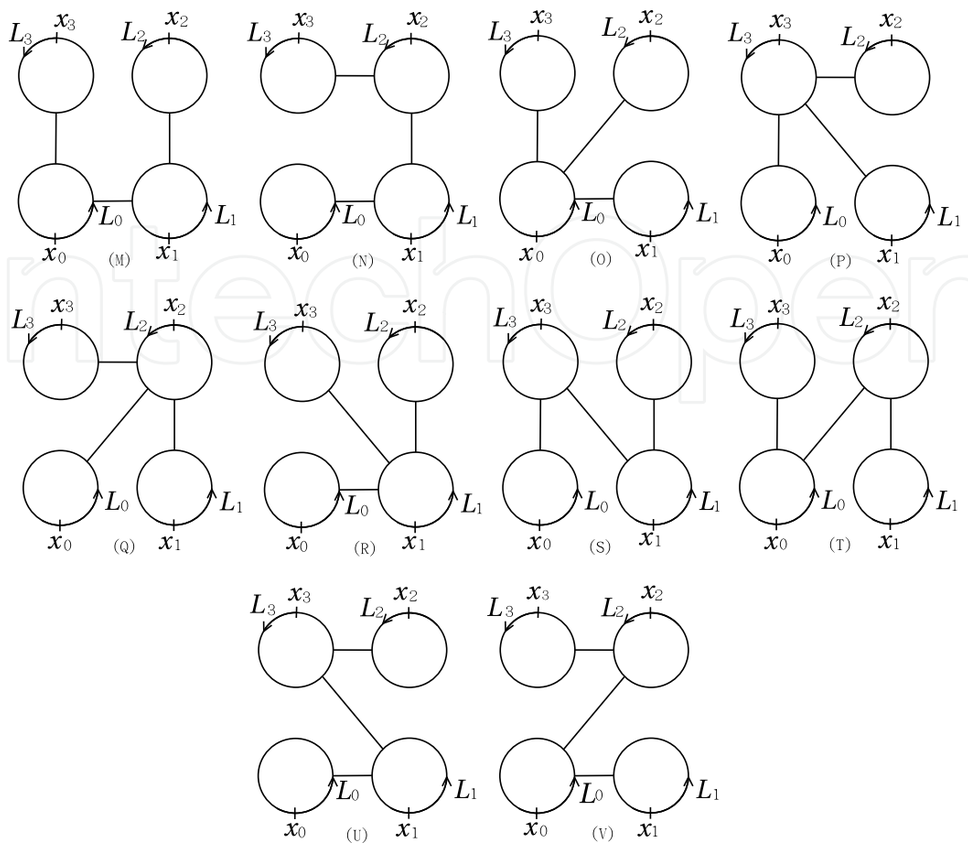
For the (-1)-connected Chern-Simons-Witten graphs,we apply ST-transform exactly once to the set of connected ones to get:



For the set of (-2)-connected Chern-Simons-Witten graphs, we apply ST-transform exactly once to the set of (-1)-connected ones to get the graphs of three components in $L_{3,2,1,0}$ listed below.





Also by Definition 1, the relevant Chern-Simons-Witten configuration space integrals of the above example are shown in example 5.

Example 3. *For $n = 1$, $L = \{L_0, L_1\}$, the invariant $L_{1,0}$ is nothing but the integration of the 2-form corresponding to the edge, over the configuration space $L_0 \times L_1$:*

$$L_{1,0} = \frac{1}{4\pi} \int_{L_0} \int_{L_1} \det \begin{pmatrix} y_0 - y_1 \\ dy_0 \\ dy_1 \end{pmatrix} \frac{1}{|y_0 - y_1|^3},$$

which is exactly the classic Gauss linking.

Example 4. *$n = 2$, $L = \{L_0, L_1, L_2\}$ for which the invariants of strictly lower degrees $L_{i,j} = 0$, $\forall i \neq j$, then $L_{2,1,0}$ is the sum of the Chern-Simons-Witten configuration space integrals corresponding to the Chern-Simons-Witten graphs listed in Example 1.*

$$\int_{L_0} \int_{\mathbb{R}^3} \int_{L_1} \int_{L_2} (0 - x) \wedge (x - 1) \wedge (x - 2),$$

$$\int_{L_0} \int_{x_0}^{0} \int_{L_1} \int_{L_2} (x - 1) \wedge (x - 2),$$

$$\int_{L_0} \int_{L_1} \int_{x_1}^{1} \int_{L_2} (0 - 1) \wedge (x - 2),$$

$$\int_{L_0}\int_{L_1}\int_{L_2}\int_{x_2}^{2}(0-2)\wedge(x-1).$$

Example 5. *$n = 3$, $L = \{L_0, L_1, L_2, L_3\}$ for which the invariants of strictly lower degrees vanish: $L_{i,j} = 0$, $L_{i,j,k} = 0$ for all distinct i, j, k, then the invariant $L_{3,2,1,0}$ is the sum of the following 22 Chern-Simons-Witten configuration space integrals which are the relevant integrals of the Chern-Simons-Witten graphs listed in Example 2.*

$(A) = \int_{L_0}\int_{\mathbb{R}^3}\int_{L_1}\int_{\mathbb{R}^3}\int_{L_2}\int_{L_3}(0-x)\wedge(x-1)\wedge(x-y)\wedge(y-2)\wedge(y-3),$

$(B) = -\int_{L_0}\int_{\mathbb{R}^3}\int_{L_3}\int_{\mathbb{R}^3}\int_{L_1}\int_{L_2}(0-x)\wedge(x-3)\wedge(x-y)\wedge(y-1)\wedge(y-2),$

$(C) = \int_{L_0}\int_{L_1}\int_{x_1}^{1}\int_{\mathbb{R}^3}\int_{L_2}\int_{L_3}(0-1)\wedge(x-y)\wedge(y-2)\wedge(y-3),$

$(D) = \int_{L_0}\int_{L_3}\int_{x_1}^{3}\int_{\mathbb{R}^3}\int_{L_1}\int_{L_2}(0-x)\wedge(3-y)\wedge(y-1)\wedge(y-2),$

$(E) = \int_{L_0}\int_{x_0}^{0}\int_{L_1}\int_{\mathbb{R}^3}\int_{L_2}\int_{L_3}(x-1)\wedge(0-y)\wedge(y-2)\wedge(y-3),$

$(F) = (+)\int_{L_0}\int_{\mathbb{R}^3}\int_{L_2}\int_{x_2}^{2}\int_{L_1}\int_{L_3}(0-x)\wedge(x-y)\wedge(y-1)\wedge(x-2),$

$(G) = (+)\int_{L_0}\int_{\mathbb{R}^3}\int_{L_1}\int_{x_1}^{1}\int_{L_2}\int_{L_3}(0-x)\wedge(x-1)\wedge(y-2)\wedge(x-3),$

$(H) = \int_{L_0}\int_{\mathbb{R}^3}\int_{L_1}\int_{L_3}\int_{x_3}^{3}\int_{L_2}(0-x)\wedge(x-1)\wedge(x-y)\wedge(y-2),$

$(I) = \int_{L_0}\int_{x_0}^{0}\int_{\mathbb{R}^3}\int_{L_1}\int_{L_2}\int_{L_3}(x-y)\wedge(y-1)\wedge(y-2)\wedge(0-3),$

$(J) = \int_{L_0}\int_{\mathbb{R}^3}\int_{L_1}\int_{L_2}\int_{x_2}^{2}\int_{L_3}(0-x)\wedge(x-1)\wedge(x-2)\wedge(y-3),$

$(K) = \int_{L_0}\int_{L_3}\int_{x_3}^{3}\int_{L_2}\int_{x_2}^{2}\int_{L_1}(0-x)\wedge(3-y)\wedge(2-1),$

$(L) = \int_{L_0}\int_{x_0}^{0}\int_{L_1}\int_{Lx_3}\int_{x_3}^{3}\int_{L_2}(x-1)\wedge(0-y)\wedge(3-2),$

$(M) = \int_{L_0} int_{x_0}^{0}\int_{L_1}\int_{x_1}^{1}\int_{L_2}\int L_3(x-1)\wedge(y-2)\wedge(1-3),$

$(N) = \int_{L_0}\int_{L_1}\int_{x_1}^{1}\int_{L_2}\int_{x_2}^{2}\int_{L_3}(0-1)\wedge(x-2)\wedge(y-3),$

$(O) = \int_{L_0}\int_{x_0}^{0}\int_{x_0}^{y}\int_{L_1}\int_{L_2}\int_{L_3}(x-1)\wedge(y-2)\wedge(0-3),$

$(P) = \int_{L_0}\int_{L_3}\int_{x_3}^{3}\int_{x_3}^{y}\int_{L_1}\int_{L_2}(0-x)\wedge(y-1)\wedge(3-2),$

$(Q) = \int_{L_0}\int_{L_2}\int_{x_2}^{2}\int_{x_2}^{y}\int_{L_1}\int_{L_3}(0-y)\wedge(x-3)\wedge(2-1),$

$(R) = \int_{L_0}\int_{L_1}\int_{x_1}^{1}\int_{x_1}^{y}\int_{L_2}\int_{L_3}(0-1)\wedge(x-2)\wedge(y-3),$

$(S) = \int_{L_0}\int_{L_3}\int_{x_3}^{3}\int_{L_1}\int_{x_1}^{1}\int_{L_2}(0-x)\wedge(3-1)\wedge(y-2),$

$(T) = \int_{L_0}\int_{x_0}^{0}\int_{L_3}\int_{L_2}^{y}\int_{x_2}^{2}\int_{L_1}(0-3)\wedge(x-y)\wedge(2-1),$

$(U) = \int_{L_0}\int_{L_1}\int_{x_1}^{1}\int_{L_3}\int_{x_3}^{3}\int_{L_2}(0-1)\wedge(x-y)\wedge(3-2),$

$(V) = \int_{L_0}\int_{x_0}^{0}\int_{L_1}\int_{L_2}\int_{x_2}^{2}\int_{L_3}(x-1)\wedge(0-2)\wedge(y-3),$

4. Massey-Milnor linking theory

Although Massey-Milnor linking theory was developed in 1960's [Fe, Ma, Mi1, Mi2, Po], the explicit and combinatorial formulae thereof is still missing. In this section we develop the "absolute version" of homological theory of Massey-Milnor linking [HKT, Hs1, Hs2, Hs3, Hs4, Hs5, Hs6, Hs7, HY] in contrast to the relative homological theory of Massey's [Fe, Ma, Po] for the purpose of the first non-vanishing linking $L^*_{n,n-1,\ldots,1,0}$ and its combinatorial formulae for link $L = \{L_0, L_1, \ldots, L_n\}$ as in the set-up. First we need the following definition.

Definition 3. *(1) For each component $L_j \in L = \{L_0, L_1, \ldots, L_n\}$, we define a closed 1-form denoted as $j(x)$ associated to it as:*

$$j(x) = \frac{1}{4\pi} \int_{L_j} \det \begin{pmatrix} x-y \\ dx \\ dy \end{pmatrix} \frac{1}{|x-y|^3} \stackrel{\triangle}{=} \int_{L_j} (x-y),$$

which is a smooth 1-form as long as $x \notin L_j$. Notice also that by convention and by notation we set

$$\begin{aligned}(x-y) \stackrel{def}{=} & \frac{1}{4\pi} \det \begin{pmatrix} x-y \\ dx \\ dy \end{pmatrix} \frac{1}{|x-y|^3} \stackrel{def}{=} \frac{1}{4\pi} \det \begin{pmatrix} x_1-y_1 & x_2-y_2 & x_3-y_3 \\ dx_1 & dx_2 & dx_3 \\ dy_1 & dy_2 & dy_3 \end{pmatrix} \frac{1}{|x-y|^3} \\ = & \frac{1}{4\pi} \frac{1}{|x-y|^3} ((x_1-y_1)(dx_2 \wedge dy_3 - dx_3 \wedge dy_2) \\ & + (x_2-y_2)(dx_3 \wedge dy_1 - dx_1 \wedge dy_3) \\ & + (x_3-y_3)(dx_1 \wedge dy_x - dx_2 \wedge dy_1)).\end{aligned}$$

This is nothing but the pull-back of the standard area form of the unit sphere $S^2 \subseteq \mathbb{R}^3$ by the map $\frac{x-y}{|x-y|}$, x, y in $\mathbb{R}^3$.

(2) Whenever the Gauss linking (here for convenience, we coin it as the Massey-Milnor linking of degree one)

$$L^*_{i,j} = \int_{L_i} \int_{L_j} (x-y) = \frac{1}{4\pi} \int_{L_i} \int_{L_j} \det \begin{pmatrix} x-y \\ dx \\ dy \end{pmatrix} \frac{1}{|x-y|^3}$$

vanishes, we define a well-defined function as: $d^{-1}i(x) = \int_{x_j}^{x} \int_i (t-y)$, where $t \in L_i$, $x \in L_j$ and $x_j \in L_j$ is the base point of L_j.

Similarly, whenever a given 1-form ψ defined on L_j with $\int_{L_j} \psi = 0$, we define the relevant function $\overline{\psi}(x)$, $x \in L_j$ as: $\overline{\psi}(x) = d^{-1}(\psi) = \int_{x_j}^{x} \psi$, where $x_j \in L_j$ is the base point. And we also coin a function of this sort as linking function.

(3) Given a closed 2-form ϕ in $\mathbb{R}^3$, for simplicity we set 1-form $\overline{\phi} \stackrel{def}{=} d^{-1}(\phi)(x,y,z) = \int_{\infty}^{x} \phi(t,y,z)$, where the path of integration is taken along the $\frac{\partial}{\partial x}$-direction starting from $(-\infty, y, z)$ up to (x, y, z), the point of interests.

(4) Using $\overline{i}(x) \stackrel{def}{=} d^{-1}(i)(x)$, $\overline{\phi}(x) = d^{-1}(\phi)(x)$, *whenever they make sense we define inductively closed* 2*-form in* $\mathbb{R}^3$*: the "round brackets"* $(m, m-1, \ldots, 1)$*, and closed* 1*-forms of interests the "sharp brackets"* $< m, m-1, \ldots, 1 >$ *as follows.*

(4-1) We define the round bracket $(j,i) = (j) \wedge (i) + \overline{j}(di) - \overline{i}(dj)$ *is a closed* 2*-form in* $\mathbb{R}^3$*. Notice that* di *is a Dirac-like singular* 2*-form supported on the knot component* L_i*, and that* $\overline{j}(x)$ *in* $\overline{j}(x)(di)$ *refers to* $x \in L_i$ *and is a well-defined function thereon (a so-called linking function) as we assume that* $L^*_{i,j} = 0$*. Similar remarks hold for* $\overline{i}(x)(dj)$*, with* $x \in L_j$*.*

Next, we define the sharp bracket $< j,i >= \overline{(j,i)} + j(x)\overline{i}(x)$ *which is a closed* 1*-form in a tubular neighborhood of* L_k *with* $k \neq i$, $k \neq j$*.*

(4-2) We define the round bracket

$$(k,j,i) = \overline{(k,j)} \wedge i + k \wedge \overline{(j,i)} + \overline{< k,j >}(di) - \overline{ki}(dj) + \overline{< i,j >}(dk)$$

which is a closed 2*-form in* $\mathbb{R}^3$*. We notice that* $(di)(x)$ *is a Dirac-like singular* 2*-form supported on the knot component* L_i*, and that* $\overline{< k,j >}(x)$ *refers to* $x \in L_i$ *and is a well-defined function on* L_i*(a linking function), as we assume that the Massey-Milnor linkings of degree* 2 *vanish there. Similar remarks hold for* $\overline{ki}(dj)$ *and* $\overline{< i,j >}(dk)$*.*

Next, we define the sharp bracket which is a closed 1*-form* $< k,j,i > (x)$ *for* $x \in L_l$, $l \neq i,j,k$ *as:*

$$< k,j,i > (x) = \overline{(k,j,i)}(x) + \overline{(k,j)}(x)i(x) + k(x)\overline{< j,i >}(x)$$

which is well-defined as $L^*_{i.l} = 0$ *and* $L^*_{j,i,l} = 0$ *by the assumption of vanishing of Massey-Milnor linkings of lower degrees.*

(4-3) Inductively, we define the closed 2*-form* $(j, j-1, \ldots, 2, 1)$ *in* $\mathbb{R}^3$ *as*

$$\begin{aligned}
&(j, j-1, \ldots, 2, 1)\\
=&\overline{(j, j-1, \ldots, 2)} \wedge 1 + \overline{(j, j-1, \ldots, 3)} \wedge \overline{(2,1)}\\
&+ \cdots + \overline{(j, j-1)} \wedge \overline{(j-2, \ldots, 1)} + j \wedge \overline{(j-1, j-2, \ldots, \ldots, 1)}\\
&+ \overline{< j, j-1, \ldots, 2 >}(d1) - \overline{< j, j-1, \ldots, 3 >}i(d2)\\
&+ \overline{< j, j-1, \ldots, 4 \cdot < 1,2 >}(d3) + \cdots + (01)^j \overline{< 1,2, \ldots, j-1 >}(dj),
\end{aligned}$$

which is well-defined by the assumption of vanishing of Massey-Milnor linkings of strictly lower degrees.

Next, we define the closed 1*-form* $< j, j-1, \ldots, 1 > (x)$, $x \in L_l$ *with* $l \neq 1, 2, \ldots, j$ *as:*

$$\begin{aligned}
&< j, j-1, \ldots, 1 > (x)\\
=&\overline{(j, j-1, \ldots, 1)}(x) + \overline{(j, j-1, \ldots, 2}(x)\overline{1}(x)\\
&+ \overline{(j, j-1, \ldots, 3)}(x)\overline{< 2,1 >}(x) + \ldots\\
&+ \overline{(j, j-1)}(x)\overline{(j-2, j-3, \ldots, 1)}(x)\\
&+ j(x)\overline{< j-1, j-2, \ldots, 1 >}(x),
\end{aligned}$$

which is a well-defined function on L_l*, as we assume that Massey-Milnor linkings of strictly lower degrees vanish.*

(5) Finally for a link $L = \{L_0, L_1, \ldots, L_n\}$ of $(n+1)$ components for which all Massey-Milnor linkings L^*'s of strictly lower degrees vanish, we define the first non-vanishing Massey-Milnor linking

$$L^*_{n,n-1,\ldots,1,0} = \int_{L_0} < n, n-1, \ldots, 2, 1 > .$$

To conclude the presentation of Massey-Milnor linkings $L^*_{n,n-1,\ldots,1}$, we notice that $L^*_{n,n-1,\ldots,1,0}$ is an ambient isotopy invariant with respect to L_0 as $< n, n-1, \ldots, 1 >$ is a closed 1-form in a tubular neighborhood of L_0 which is disjoint from $\{L_1, L_2, \ldots, L_n\}$. But a beautiful theorem of J. Milnor [Mi1, Mi2] asserting that $L^*_{n,n-1,\ldots,0} = L^*_{n-1,n-2,\ldots,0,n} = L^*_{n-2,n-3,\ldots,0,n,n-1} = L^*_{0,n,n-1,\ldots,1}$ implies that the first non-vanishing Massey-Milnor linking $L^*_{n,n-1,\ldots,1,0}$ is an ambient isotopy invariant with respect to L_i for $i = 0, 1, \ldots, n$. And notice that the "double round brackets" $((n, n-1, \ldots, 1))$ defined in (3) of Definition 2 are nothing but the "connected" part of $(n, n-1, \ldots, 1)$, and also notice that the difference between $((n, n-1, \ldots, 1))$ and $(n, n-1, \ldots, 1)$ is a Dirac-like singular 2-form supported on $\{L_1, L_2, \ldots, L_n\}$.

5. Some calculus lemma

Armed with the link $L = \{L_0, L_1, \ldots, L_n\}$ as in the setup—for example the cyclic arrangement of L as shown there and the assumption of the vanishing of both invariants L's (Chern-Simons-Witten invariants defined in Section 3) and L^*'s (Massey-Milnor linkings defined in Section 4) of strictly lower degrees—and for the purpose of the equality $L = L^*$ and the combinatorial formulae thereof, we prepare some calculus lemmas which mostly are localization computation as our link L is represented as a link diagram lying entirely on the plane $\mathbb{R}^2$, except possibly the infinitesimally small neighborhood of crossings of L.

Lemma 1.

$$\begin{aligned} & d_A(A-B) - d_B(\underline{A}-B) \\ =& (-)\delta(A-B)(dA_1 \wedge dA_2 \wedge dB_3 + dA_2 \wedge dA_3 \wedge dB_1 + dA_3 \wedge dA_1 \wedge dB_2), \end{aligned}$$

where $A, B \in \mathbb{R}^3$ are dummy variables,

$$(A-B) \stackrel{def}{=} \left(\frac{1}{4\pi}\right) \frac{1}{|A-B|^3} \det \begin{pmatrix} A-B \\ dA \\ dB \end{pmatrix}$$

and

$$(\underline{A}-B) \stackrel{def}{=} \left(\frac{1}{8\pi}\right) \frac{1}{|A-B|^3} \det \begin{pmatrix} A-B \\ dA \\ dA \end{pmatrix};$$

and $\delta(A-B)$ is the Dirac function in $\mathbb{R}^3$.

Proof.

$$\begin{aligned} & d_A(A-B) \\ =& d_A(\frac{1}{4\pi}) \frac{1}{|A-B|^3} \begin{pmatrix} A_1-B_1 & A_2-B_2 & A_3-B_3 \\ dA_1 & dA_2 & dA_3 \\ dB_1 & dB_2 & dB_3 \end{pmatrix} = d_A \begin{pmatrix} \Gamma_1(A-B) & \Gamma_2(A-B) & \Gamma_3(A-B) \\ dA_1 & dA_2 & dA_3 \\ dB_1 & dB_2 & dB_3 \end{pmatrix}, \end{aligned}$$

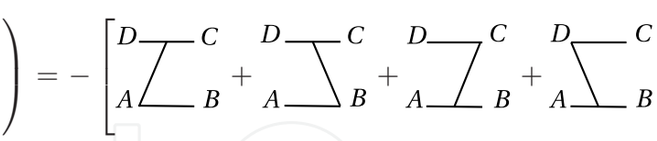
if we set $\Gamma(x) = \frac{1}{4\pi|x|}$

$$
\begin{aligned}
=& d_A[\Gamma_1(dA_2 \wedge dB_3 - dA_3 \wedge dB_2) + \Gamma_2(dA_3 \wedge dB_1 - dA_1 \wedge dB_3) \\
&+ \Gamma_3(dA_1 \wedge dB_2 - dA_2 \wedge dB_1)] \\
=& (\Gamma_{11} dA_1 + \Gamma_{12} dA_2 + \Gamma_{13} dA_3) \wedge (dA_2 \wedge dB_3 - dA_3 \wedge dB_2) \\
&+ (\Gamma_{21} dA_1 + \Gamma_{22} dA_2 + \Gamma_{23} dA_3) \wedge (dA_3 \wedge dB_2 - dA_1 \wedge dB_3) \\
&+ (\Gamma_{31} dA_1 + \Gamma_{32} dA_2 + \Gamma_{33} dA_3) \wedge (dA_1 \wedge dB_2 - dA_2 \wedge dB_1) \\
=& (\Gamma_{11} + \Gamma_{22} + \Gamma_{33}(dA_1 \wedge dA_2 \wedge dB_3 + dA_2 \wedge dA_3 \wedge dB_1) \\
&+ dA_3 \wedge dA_1 \wedge dB_2) \\
&- (\Gamma_{11} dA_2 \wedge dA_3 \wedge dB_1 + \Gamma_{12} dA_2 \wedge dA_3 \wedge dB_2 + \Gamma_{13} dA_2 \wedge dA_3 \wedge dB_3) \\
&- \Gamma_{21} dA_3 \wedge dA_1 \wedge dB_1 + \Gamma_{22} dA_3 \wedge dA_1 \wedge dB_2 + \Gamma_{23} dA_3 \wedge dA_1 \wedge dB_3) \\
&- (\Gamma 31 dA_1 \wedge dA_2 \wedge dB_1 + \Gamma_{32} dA_1 \wedge dA_2 \wedge dB_2 + \Gamma_{33} dA_1 \wedge dA_2 \wedge dB_3) \\
=& -\delta(A - B) + d_B(\Gamma_1 dA_2 \wedge dA_3 + \Gamma_2 dA_3 \wedge dA_1 + \Gamma_3 dA_1 \wedge dA_2) \\
=& -\delta(A - B) = \frac{1}{8\pi} d_B \frac{1}{|A - B|^3} \det \begin{pmatrix} A - B \\ dA \\ dA \end{pmatrix}
\end{aligned}
$$

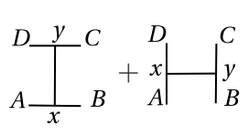
□

Lemma 2. *For the variation with respect to A, we have*

$$d_A \left(\begin{array}{c} \\ \end{array} \right) = -$$



where A, B, C, and D are distinct indices (or dummy variables) in $\{0, 1, 2, \ldots, n\}$.

Proof. By definition, the configuration space integrals corresponding to 

are:

$$
\begin{aligned}
&\int_{A_x B_y CD} (A - x) \wedge (x - B) \wedge (x - y) \wedge (y - C) \wedge (y - D) \\
&\qquad + \int_{D_x A_y BC} (D - x_\wedge (x - A) \wedge (x - y) \wedge (y - B) \wedge (y - C).
\end{aligned}
$$

Taking d_A, the exterior derivative of the LHS with respect to A, by our convention, is to take d_A of the associative configuration space integrals omitting the integration over A-chain, thereby we get a 2-form in dA. To be more explicit,

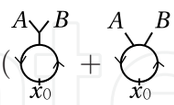
$$\begin{aligned}
& d_A \int_{xByCD} (A-x)\wedge(x-B)\wedge(x-y)\wedge(y-C)\wedge(y-C) \\
& \quad - d_A \int_{xDyBC} (A-x)\wedge(x-D)\wedge(x-y)\wedge(y-B)\wedge(y-C) \\
=& (-)\int_{ByCD} (A-B)\wedge(A-y)\wedge(y-C)\wedge(y-D) \\
& \quad + \int_{DyBC} (A-D)\wedge(A-y)\wedge(y-B)\wedge(y-C) \\
& \quad + \int_{xByCD} d_x(\underline{A}-x)\wedge(x-B)\wedge(x-y)\wedge(c-C)\wedge(y-D) \\
& \quad - \int_{xDyBC} d_x(\underline{A}-x)\wedge(x-D)\wedge(x-y)\wedge(y-B)\wedge(y-C)
\end{aligned}$$

$$\begin{aligned}
=& - \int_{ByCD} (A-B)\wedge(A-y)\wedge(y-C)\wedge(y-D) \int_{DyBC} (A-D)\wedge(A-y)\wedge(y-B)\wedge(y-C) \\
& - \int_{xByCD} (\underline{A}-x)\wedge(d_x(x-B))\wedge(x-y)\wedge(y-C)\wedge(y-D) \\
& + \int_{xDyBC} (\underline{A}-x)\wedge(d_x(x-D))\wedge(x-y)\wedge(y-D)\wedge(y-C) \\
& - \int_{xByCD} (\underline{A}-x)\wedge(x-B)\wedge(d_x(x-y))\wedge(y-C)\wedge(y-D) \\
& + \int_{xDyBC} (\underline{A}-x)\wedge(x-D)\wedge(d_x(x-y))\wedge(y-B)\wedge(y-C) \\
=& - \int_{ByCD} (A-B)\wedge(A-y)\wedge(y-C)\wedge(y-D) \\
& + \int_{DyBC} (A-D)\wedge(A-y)\wedge(y-B)\wedge(y-C) \\
& + \int_{ByCD} (\underline{A}-B)\wedge(B-y)\wedge(y-C)\wedge(y-D) \\
& - \int_{DyBC} (\underline{A}-D)\wedge(D-y)\wedge(y-B)\wedge(y-C) \\
& + \int_{ByCD} (\underline{A}-y)\wedge(y-B)\wedge(y-C)\wedge(y-D) \\
& - \int_{DyBC} (\underline{A}-y)\wedge(y-D)\wedge(y-B)\wedge(y-C) \\
& - \int_{xByCD} (\underline{A}-y)\wedge(y-D)\wedge(y-B)\wedge(y-C) \\
& - \int_{xByCD} (\underline{A}-x)\wedge(x-B)\wedge(d_y(\underline{x}-y))\wedge(y-C)\wedge(y-D) \\
& + \int_{xDyBC} (\underline{A}-x)\wedge(x-D)\wedge(D_y(\underline{x}-y))\wedge(y-B)\wedge(y-C)
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{ByCD}(A-B)\wedge(A-y)\wedge(y-C)\wedge(y-D)\\
&\quad+\int_{DyBC}(A-D)\wedge(A-y)\wedge(y-B)\wedge(y-C)\\
&\quad+\int_{ByCD}(\underline{A}-B)\wedge(B-y)\wedge(y-C)\wedge(y-D)\\
&\quad-\int_{DyBC}(\underline{A}-D)\wedge(D-y)\wedge(y-B)\wedge(y-C)\\
&\quad+\int_{xByCD}(\underline{A}-x)\wedge(x-B)\wedge(x-y)\wedge(D_y(y-C))\wedge(y-D)\\
&\quad-\int_{xDyBC}(\underline{A}-x)\wedge(x-D)\wedge(\underline{x}-y)\wedge(D_y(y-B)\wedge(y-C)\\
&\quad+\int_{xByCD}(\underline{A}-x)\wedge(x-B)\wedge(\underline{x}-y)\wedge(y-C)\wedge(D_y(y-D))\\
&\quad-\int_{xDyBC}(\underline{A}-x)\wedge(x-D)\wedge(\underline{x}-y)\wedge(y-B)\wedge(D_y(y-D))\\
&= -\int_{ByCD}(A-B)\wedge(A-y)\wedge(y-C)\wedge(y-D)\\
&\quad+\int_{DyBC}(A-D)\wedge(A-y)\wedge(y-B)\wedge(y-C)\\
&\quad+\int_{ByCD}(\underline{A}-B)\wedge(B-y)\wedge(y-C)\wedge(y-D)\\
&\quad-\int_{yBCD}(\underline{A}-D)\wedge(D-y)\wedge(y-B)\wedge(y-C)\\
&\quad-\int_{xDBC}(\underline{A}-x)\wedge(x-B)\wedge(\underline{x}-C)\wedge(C-D)\\
&\quad+\int_{xDBC}(\underline{A}-x)\wedge(x-D)\wedge(\underline{x}-B)\wedge(B-C)\\
&\quad-\int_{xBCD}(\underline{A}-x)\wedge(x-B)\wedge(\underline{x}-D)\wedge(D-C)\\
&\quad+\int_{xDBC}(\underline{A}-x)\wedge(x-D)\wedge(\underline{x}-C)\wedge(C-B).
\end{aligned}
$$

The last eight integrals are difficult to read in general. So for the purpose of book-keeping we use the graphic representation for them as in the above Lemma 2. Obviously the correspondence and the association of Chern-Simons-Witten configuration space integrals to edge-contracted Chern-Simons-Witten graphs, and vice versa are well-defined.

By definition the eight terms after the last equality sign in the above are the right hand side of Lemma 2. □

Lemma 3. $d_0($  $) = 0$, *if* $A \neq B$ *in* $\{0, 1, 2, \ldots, n\}$.

Proof. By the definition of the associated Chern-Simons-Witten configuration space integrals both sides of the above equality are 2-forms in dL_0. That is

$$\begin{aligned} LHS =& d_0 \int_{\mathbb{R}^3} \int_{BA} (0-x) \wedge (x-B) \wedge (x-A) + d_0 \int_{x_0}^{0} \int_{BA} (x-B) \wedge (0-A) \\ =& (-) \int_{BA} (0-B) \wedge (0-A) - \int_{BA} (\underline{0}-A) \wedge (A-B) \\ & - \int_{BA} (\underline{0}-B) \wedge (B-A) + \int_{BA} (0-B) \wedge (0-A) \\ =& - \int_{BA} (\underline{0}-A) \wedge (A-B) - \int_{BA} (\underline{0}-B) \wedge (B-A). \end{aligned}$$

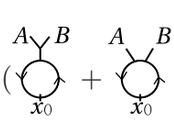
Observe that $(\underline{0}-A)$ and $(\underline{0}-B)$ are both functions in A and B which do not involve any differential forms in (dA) or (dB) respectively; so to carry out the above two integrations we may regard $(\underline{0}-A)$ and $(\underline{0}-B)$ as weight functions by computing first $\int_{BA}(A-B)$ and $\int_{BA}(B-A)$ to get Gauss signs of crossings of $L_A \cap L_B$ when $L = \{L_0, L_1, \ldots, L_n\}$ is in a generic position in $\mathbb{R}^2$ with pairwise crossings specified. Also observe that

$$-\int_{BA} (\underline{0}-A)(A-B) = -\sum_{L_A \cap L_B} (\underline{0}-A)(A,B)$$

and

$$\int_{BA} (\underline{0}-B)(B-A) = +\sum_{L_A \cap L_B} (\underline{0}-B)(A,B)$$

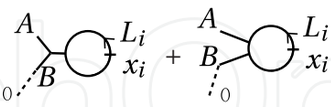
where (A,B), for the moment, stands for the Gaussian signs of crossings $L_A \cap L_B$ which is a finite sum supported on $L_A \cap L_B$; and this concludes the proof. □

Lemma 4. $d_0($  $) = 0$, *if* A, B *are two sharp brackets.*

Proof. The proof is exactly the same as that of Lemma 3. By integration-by-parts and by Lemma 1, we could regard both A and B as d-closed since both of them are sharp brackets. □

From now on we assume that $L = \{L_0, L_1, \ldots, L_n\}$ is in a generic position in $\mathbb{R}^2$ with pairwise crossings specified.

First we introduce some more notations: $\overline{(2,1)^*} = 2 \wedge 1$, $\overline{(3,2,1)^*} = \overline{\overline{(3,2)^*} \wedge 1} + 3 \wedge \overline{(2,1)^*}$, and in general, $\overline{(n,n-1,\ldots,1)^*} = \overline{(n,\ldots,2)^*} \wedge 1 + \overline{(n,\ldots,3)^*} \wedge \overline{(2,1)^*} + \cdots + n \wedge \overline{(n-1,\ldots,1)^*}$.

Lemma 5. *The outer edge contractions of* 

at L_i of the variation cancel each other, where the graph components containing vertex B connect L_i to L_0.

Proof. Starting with the exterior differentiation at L_0, d_0 the integration by parts as in Lemma 2 up to x in the first graph, which is a dummy variable on L_i to get the same resulting integrals for the above graph of two components. Then do the integrations by part again at x and at y, to get the same outer edge-contraction at L_i except for a difference of $(-)$ sign, so these two outer edge contractions at L_i cancel each other. This concludes the poof. □

Lemmas 1 to 5 are essentially preparatory computation for the calculus relevant to $L_{n,n-1,\ldots,1,0}$—the Chern-Simons-Witten invariants; next we do some preparatory computation for $L^*_{n,n-1,\ldots,1,0}$—the Massey-Milnor linking theory.

Lemma 6. *Denote(* 

, 

respectively) as $(+)$ ($(-)$, respectively) for crossings of $L = \{L_0, L_1\}$, then we have $\int_{(+)}(0-1) = \frac{1}{2}$ and $\int_{(-)}(0-1) = -\frac{1}{2}$ where to ease the notation we keep the convention and notation in (1) of Definition 3.

Proof. Without loss of generality, we need only to prove

$$\frac{1}{4\pi}\int_{(+)} \begin{pmatrix} x-y \\ dx \\ dy \end{pmatrix} \frac{1}{|x-y|^3} = \frac{1}{2} \times \lim_{\substack{\epsilon\to 0 \\ \delta\to 0-\epsilon \\ \delta \ll \epsilon}} \int_{-\epsilon}^{\epsilon} \frac{\delta dx dy}{(x^2+y^2+\delta^2)} = \frac{1}{2}$$

□

Note 2. *As our link $L = \{L_0, L_1, \ldots, L_n\}$ is represented by a link diagram in the plane $\mathbb{R}^2$ with arbitrarily small "germs" of pairwise crossings specified, it is easy to see that the only contribution of Gauss linking—which is equal to both $L^*_{i,j}$ and $L_{i,j}$—comes from the crossing part of L_i and L_j, and is denoted as (i,j) or (j,i).*

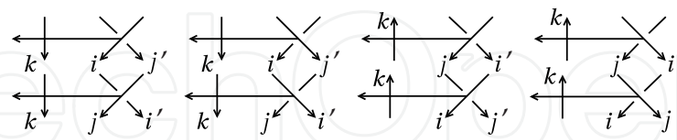
Lemma 7. *For two knot components L_i and L_j in $\{L_i, L_j, L_k\}$ with the associated 1-form $i(x)$, $j(x)$ as above the 1-form*

$$\overline{i \wedge j}(x) = \int_{-\infty}^{x} i(t) \wedge j(t)$$

$x \in \mathbb{R}^2$ is a Dirac-like 1-form supported on the horizontal positive $\frac{\partial}{\partial x}$-ray through the crossing of L_i and L_j in the link diagram of $\{L_i, L_j, L_k\}$. Moreover,

$$\int_{L_k} \overline{i \wedge j}(x) \stackrel{def}{=} (i,j)_k$$

$$= \begin{cases} \frac{1}{4} \text{ — if } \\ -\frac{1}{4} \text{ — if } \end{cases}$$



Proof. By the very defintion of integration along positive $\frac{\partial}{\partial x}$-direction in the plane $\mathbb{R}^2$, we have $\int_{-\infty}^{x} i(t) \wedge j(t) \equiv (\int_{-\infty}^{x} i(t)) \wedge j(x) - i(x) \wedge \int_{-\infty}^{x} j(t))$. And also notice that $\int_{-\infty}^{x} i(t) \wedge j(t)$ as a 1-form in $x \in \mathbb{R}^2$, is supported on the horizontal $\frac{\partial}{\partial x}$-ray passing through the crossing of the link diagram of L_i and L_j by a simple localization computation. Another simple localized estimate shows that $\int_{L_k} \overline{i \wedge j} = \int_{L_k} (\int_{-\infty}^{x} i(t)) \wedge j(x) - \int_{L_k} i(x) \wedge (\int_{-\infty}^{x} j(t)) = (i,j)_k$ as claimed. □

6. Main theorem

In this section we will state the main theorem of this paper and some proofs for the case of low degrees: $L_{1,0} = L^*_{1,0}$, $L_{2,1,0} = L^*_{2,1,0}$ and $L_{3,2,1,0} = L^*_{3,2,1,0}$. We will come back to the proof of the main theorem in Section 7 in full generality.

Here is the main theorem and as usual we use numerals $i, j, k, \ldots$, etc. for the corresponding knot components $L_i, L_j, L_k, \ldots$, etc. to ease the notation.

Theorem 1. *Given a link $L = \{L_0, L_1, \ldots, L_n\}$ of $(n+1)$ components arranged diagrammatically as in the setup for which all Chern-Simons-Witten graphs $L_{m,m-1,\ldots,1,0} = 0$, and all Massey-Milnor linkings $L^*_{m,m-1,\ldots,1,0} = 0$, where the sublink $\{L^*_0, L^*_1, \ldots, L^*_m\} \subseteq \{L_0, L_1, \ldots, L_n\}$ is any ordered subset of $(m+1)$ components, $m \leq n-1$, then for the first non-vanishing invariants $L_{n,n-1,\ldots,1,0}$ and $^*_{n,n-1,\ldots,1,0}$ we have*

(1) $L_{n,n-1,\ldots,1,0} = L^*_{n,n-1,\ldots,1,0}$, and

(2) Both $L_{n,n-1,\ldots,1,0}$ and $L^*_{n,n-1,\ldots,1,0}$ are independent of the base point $x_j \in L_j$, for $j = 0, 1, 2, \ldots, n$.

We will prove this theorem in full generality in Section 7. Here to make the presentation smoother and to show the idea, we do the "detailed" proof of: $L_{1,0} = L^*_{1,0}$, $L_{2,1,0} = L^*_{2,1,0}$ and $L_{3,2,1,0} = L^*_{3,2,1,0}$ as follows.

Example 6. *For $n = 1$ and $L = \{L_0, L_1\}$, by the very definition of $L_{1,0}$ and $L^*_{1,0}$ it is obvious that both $L_{1,0}$ and $L^*_{1,0}$ are exactly the Gauss linking and are formulated combinatorically as $L_{1,0} = L^*_{1,0} = \sum_{L_1 \cap L_0} (1,0)$. Here we follow the convention and notation of the **Note 2** after Lemma 6: $(1,0) = (\pm\frac{1}{2})$ according to the arrangement of L_1 and L_0 around the crossing "$L_1 \cap L_0$" in the link diagram of $\{L_0, L_1\}$.*

Example 7. *For $n = 2$ and $L = \{L_0, L_1, L_2\}$, we assume that all pairwise invariants $L_{i,j} = 0 = L^*_{i,j}$ then for the first non-vanishing invariants: $L_{2,1,0} = L^*_{2,1,0}$ and they can be computed combinatorically as follows. First recall that invariants $L_{2,1,0}$—the Chern-Simons-Witten graph defined in Section 4—is the sum of 4 configuration space integrals defined in Example 1; and $L^*_{2,1,0}$—the Massey-Milnor linking defined in Definition 4—is*

$$L^*_{2,1,0} = \int_0 \overline{2 \wedge 1} + \int_0 \overline{\overline{2} \wedge d1} - \int_0 \overline{\overline{1} d2} + \int_0 2\overline{1}$$

So to prove $L_{2,1,0} = L^*_{2,1,0}$, we need only to prove that the configuration space integral of "Y-graph"—which is nothing but the first integral in Example 4—in Example 1 is $\int_0 \overline{2 \wedge 1}$ in $L^*_{2,1,0}$.

Claim 1. *If we set the* 1*-form*

$$\phi(0) \stackrel{def}{=} \int_{\mathbb{R}^3} \int_{L_1} \int_{L_2} (0-x) \wedge (x-1) \wedge (x-2)$$

which is a part of the configuration space integrals of the Chern-Simons-Witten graph "Y", then $d\phi(0) = (2 \wedge 1)(0)$.

Proof. With the link $L = \{L_0, L_1, L_2\}$ represented by a link diagram in the plane $\mathbb{R}^2$ and in the spirit of localization computation of Lemma 6 and Lemma 7, a direct computation concludes the proof by using the integration by part in Lemma 1.

And so,

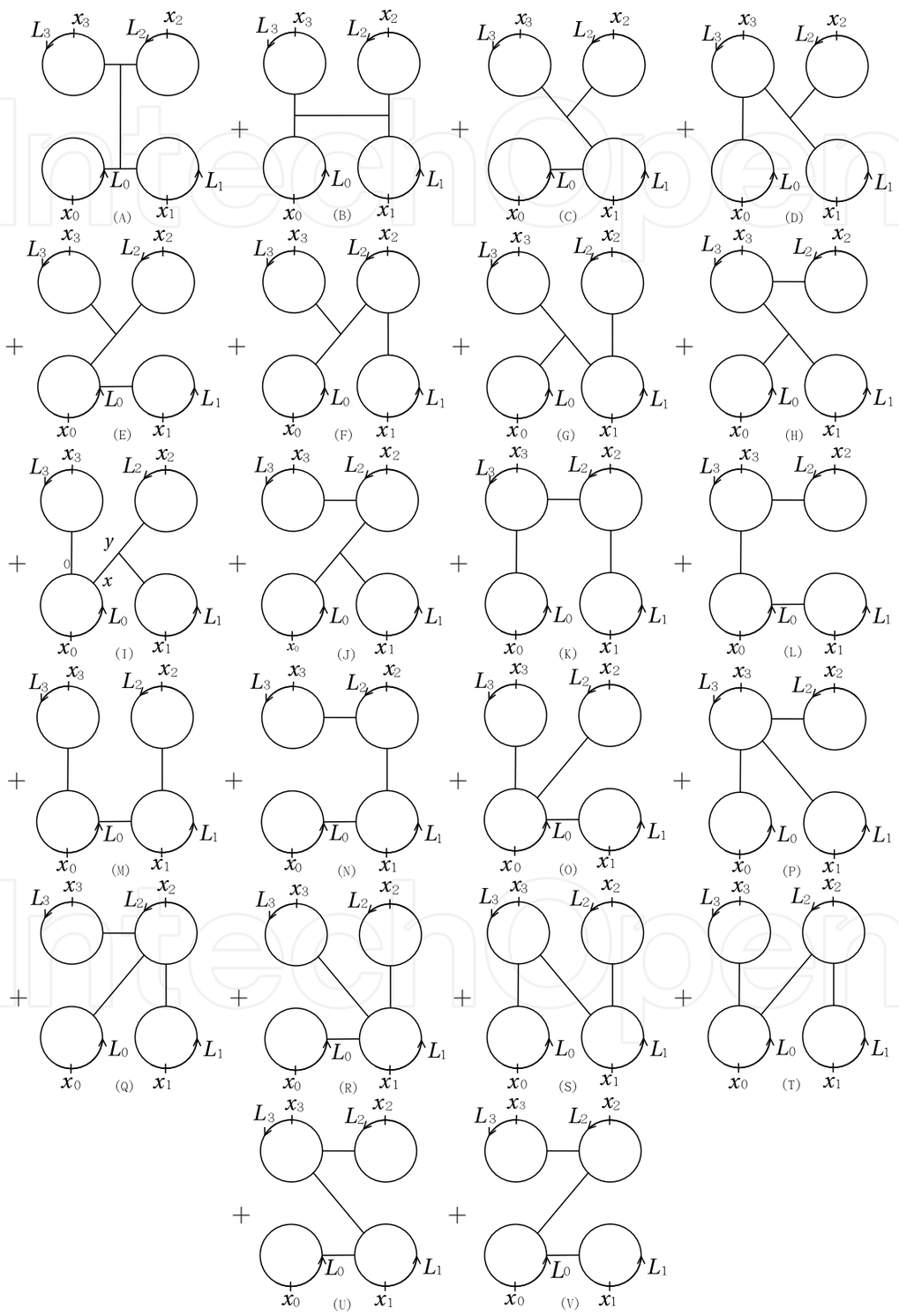
$$\int_0 \overline{2 \wedge 1} = \int_0 \overline{d\phi} = \int_0 \overline{d\phi} + \int_0 d\overline{\phi} = \int_0 \phi = \text{ the configuration space integral of "}Y\text{"}.$$

And this concludes the proof that $L_{2,1,0} = L^*_{2,1,0}$. And by using the combinatoric formulae of $L^*_{2,1,0}$—that in Lemma 6 and Lemma 7—we derive the explicit formulae of $L_{2,1,0} = L^*_{2,1,0}$

$$L_{2,1,0} = (2,1)_0 + \sum_{\substack{1<2 \\ 0}} (1,0)(2,0) + \sum_{\substack{2<0 \\ 1}} (2,1)(0,1) + \sum_{\substack{0<1 \\ 2}} (0,2)(1,2)$$

□

Example 8. *For $n = 3$, $L = \{L_0, L_1, L_2, L_3\}$, for which all pairwise $L_{i,j} = L^*_{i,j} = 0$ and all triple-wise $L_{i,j,k} = L^*_{i,j,k} = 0$, then for the first non-vanishing invariants: we have $L_{3,2,1,0} = L^*_{3,2,1,0}$ and also they can be computed as listed below correspondingly, both graphically and combinatorically as follows:*


(A)
(B)
(C)
(D)
(E)
(F)
(G)
(H)
(I)
(J)
(K)
(L)
(M)
(N)
(O)
(P)
(Q)
(R)
(S)
(T)
(U)
(V)

and

$$\sum_{(2,1)\underset{0}{<}3}(2,1)_0(3,0)+\sum_{1\underset{0}{<}(3,2)}(3,2)_0(1,0)+\sum_{(3,2)\underset{1}{<}0}(3,2)_1(0,1)$$

$$+\sum_{2\underset{1}{<}(0,3)}(0,3)_1(2,1)+\sum_{(0,3)\underset{2}{<}1}(0,3)_2(1,2)+\sum_{3\underset{2}{<}(1,0)}(1,0)_2(3,2)$$

$$+\sum_{(1,0)\underset{3}{<}2}(1,0)_3(2,3)+\sum_{0\underset{3}{<}(2,1)}(2,1)_3(0,3)+\sum_{1\underset{0}{<}2\underset{0}{<}3}(0,1)(0,2)(0,3)$$

$$+\sum_{2\underset{1}{<}3\underset{1}{<}0}(1,2)(1,3)(1,0)+\sum_{3\underset{2}{<}0\underset{2}{<}1}(2,3)(2,0)(2,1)+\sum_{0\underset{3}{<}1\underset{3}{<}2}(3,0)(3,1)(3,2)$$

$$+\sum_{\substack{1\underset{0}{<}3\\ 2<0\\ \ \ 1}}(1,0)(3,0)(2,1)+\sum_{\substack{2\underset{1}{<}0\\ 3<1\\ \ \ 2}}(2,1)(0,1)(3,2)+\sum_{\substack{3\underset{2}{<}1\\ 0<2\\ \ \ 3}}(3,2)(1,2)(0,3)$$

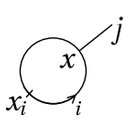
$$+\sum_{\substack{0\underset{3}{<}2\\ 1<3\\ \ \ 0}}(0,3)(2,3)(1,0)+\sum_{\substack{2\underset{0}{<}3\\ 0<1\\ \ \ 2}}(2,0)(3,0)(1,2)+\sum_{\substack{2\underset{1}{<}3\\ 0<1\\ \ \ 3}}(2,1)(3,1)(0,3)$$

$$+\sum_{\substack{3\underset{1}{<}0\\ 1<2\\ \ \ 3}}(3,1)(0,1)(2,3)+\sum_{\substack{1\underset{0}{<}2\\ 3<0\\ \ \ 2}}(1,0)(2,0)(3,2)\ .$$

For this example we try to be as explicit as possible to show the general scheme of the proof of the main theorem.

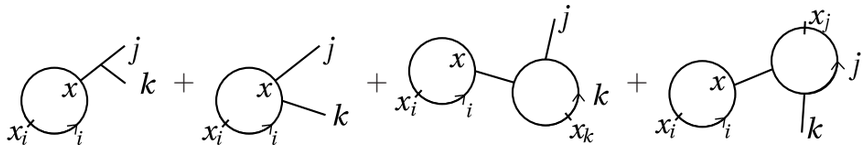
First we derive the related combinatorical formulae of the Massey-Milnor linkings $L^*_{n,n-1,\ldots,1,0}$ as defined in Section 4 from which we do use the vanishing of Massey-Milnor linkings of strictly lower degrees. Also to ease the notation we use: $\overline{j}(x)=\int_{x_i}^{x} j(t)$, where x_i, $x\in L_i$ and x_i is the base point of L_i, whenever $L_{i,j}=0$. And $\overline{<j,k>}(x)=\int_{x_i}^{x}<j,k>(t)$ where x_i, $x\in L_i$ and x_i is the base point of L_i, whenever $L_{j,k,i}=0$. And obviously by induction they could be represented graphically as follows:

$$\overline{j}(x)=$$



and

$$\overline{<j,k>}(x)=$$



Recalling that:

$$\begin{aligned}
L^{*}_{3,2,1,0} &= \int_0 < 3,2,1 >= \int_0 \overline{(3,2,1)} + \int_0 \overline{(3,2)}\overline{1} + \int_0 3\overline{< 2,1 >} \\
&= \int_0 \overline{\overline{(3,2)} \wedge 1} + \int_0 \overline{3 \wedge \overline{(2,1)}} + \int_0 \overline{\overline{< 3,2 >} d1} \\
&\quad - \int_0 \overline{\overline{3}\overline{1}d2} + \int_0 \overline{\overline{< 1,2 >} d3} + \int_0 \overline{(3 \wedge 2)}\overline{1} \\
&\quad + \int_0 \overline{\overline{3}d2}\overline{1} - \int_0 \overline{\overline{2}d3}\overline{1} + \int_0 3\overline{< 2,1 >} \\
&= \int_0 \overline{\overline{(3 \wedge 2)} \wedge 1} + \int_0 \overline{(\overline{3}d2) \wedge 1} - \int_0 \overline{\overline{(\overline{3}d2)} \wedge 1} \\
&\quad + \int_0 \overline{3 \wedge \overline{(2 \wedge 1)}} + \int_0 \overline{3 \wedge \overline{(\overline{2}d1)}} - \int_0 \overline{3 \wedge \overline{(\overline{1}d2)}} \\
&\quad + \int_0 \overline{\overline{< 3,2 >} d1} - \int_0 \overline{\overline{3}\overline{1}d2} + \int_0 \overline{\overline{< 1,2 >} d3} + \int_0 \overline{(3 \wedge 2)}\overline{1} \\
&\quad + \int_0 \overline{\overline{3}d2}\overline{1} - \int_0 \overline{\overline{2}d3}\overline{1} + \int_0 3\overline{< 2,1 >};
\end{aligned}$$

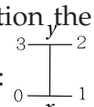
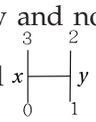
and using the above inductive scheme and by induction we have combinatorically,

$$\begin{aligned}
&L^{*}_{3,2,1,0} \\
&= \int_0 \overline{\overline{(3 \wedge 2} \wedge 1} + \int_0 \overline{3 \wedge \overline{(2 \wedge 1)}} + \sum_{\substack{3 < (1,0) \\ \scriptstyle 2}} (1,0)_2(3,2) \\
&\quad + \sum_{\substack{(1,0) < 2 \\ \scriptstyle 3}} (1,0)_3(2,3) + \sum_{\substack{2 < (0,3) \\ \scriptstyle 1}} (0,3)_1(2,1) + \sum_{\substack{(0,3) < 1 \\ \scriptstyle 2}} (0,3)_2(1,2) \\
&\quad + \sum_{\substack{(3,2) < 0 \\ \scriptstyle 1}} (3,2)_1(0,1) + \sum_{\substack{2 < 3 < 0 \\ \scriptstyle 1 \quad 1}} (2,1)(3,1)(0,1) + \sum_{\substack{2 < 0 \\ \scriptstyle 1 \\ \scriptstyle 3 < 1 \\ \scriptstyle 2}} (2,1)(0,1)(3,2) \\
&\quad + \sum_{\substack{3 < 0 \\ \scriptstyle 1 \\ \scriptstyle 1 < 2 \\ \scriptstyle 3}} (3,1)(0,1)(2,3) + \sum_{\substack{3 < 0 < 1 \\ \scriptstyle 2 \quad 2}} (3,2)(0,2)(1,2) + \sum_{\substack{1 < (2,1) \\ \scriptstyle 3}} (2,1)_3(0,3) \\
&\quad + \sum_{\substack{0 < 1 < 2 \\ \scriptstyle 3 \quad 3}} (0,3)(1,3)(2,3) + \sum_{\substack{3 < 1 \\ \scriptstyle 2 \\ \scriptstyle 0 < 2 \\ \scriptstyle 3}} (3,2)(1,2)(0,3) + \sum_{\substack{2 < 3 \\ \scriptstyle 1 \\ \scriptstyle 0 < 1 \\ \scriptstyle 3}} (2,1)(3,1)(0,3) \\
&\quad + \sum_{\substack{1 < (3,2) \\ \scriptstyle 0}} (3,2)_0(1,0) + \sum_{\substack{1 < 2 \\ \scriptstyle 0 \\ \scriptstyle 3 < 0 \\ \scriptstyle 2}} (1,0)(2,0)(3,2) + \sum_{\substack{1 < 3 \\ \scriptstyle 0 \\ \scriptstyle 0 < 2 \\ \scriptstyle 3}} (1,0)(3,0)(2,3) \\
&\quad + \sum_{\substack{(2,1) < 3 \\ \scriptstyle 0}} (2,1)_0(3,0) + \sum_{\substack{1 < 2 < 3 \\ \scriptstyle 0 \quad 0}} (1,0)(2,0)(3,0) + \sum_{\substack{2 < 0 \\ \scriptstyle 1 \\ \scriptstyle 1 < 3 \\ \scriptstyle 0}} (2,1)(0,1)(3,0) \\
&\quad + \sum_{\substack{2 < 3 \\ \scriptstyle 0 \\ \scriptstyle 0 < 1 \\ \scriptstyle 2}} (2,0)(3,0)(1,2)
\end{aligned}$$

For the reamining two integrals above we have

Claim 2. $\int_0 \overline{\overline{(3\wedge 2)}\wedge 1} = 0$ *and* $\int_0 \overline{3\wedge\overline{(2\wedge 1)}} = 0$.

Proof. Firstly we do the Massey-Milnor linking: $L^*_{3,2,1,0}$. By notation for a given over-barred 2-form $\overline{\phi}$ which is 1-form after doing the horizontal $\frac{\partial}{\partial x}$-integration which is the inverse of exterior differentiation, (after gauge-fixing) evaluated on the plane $\mathbb{R}^2$—as all integrations defined in the Massey-Milnor linkings $L^*_{n,n-1,\ldots,1,0}$ are carried out in the plane $\mathbb{R}^2$. We need to define the suitable d^{-1} as the average of horizontal $\frac{\partial}{\partial x}$-integrations, both infinitesimally above $\mathbb{R}^2$ and infinitesimally below $\mathbb{R}^2$. And so for each wedge product 2-form like $\overline{(3\wedge 2)}\wedge 1$, once we do the horizontal $\frac{\partial}{\partial x}$-integration again (the average of two, one above $\mathbb{R}^2$ infinitesimally and one below $\mathbb{R}^2$ infinitesimally) we always get zero by simply observing that $\overline{\overline{(3\wedge 2)}\wedge 1} = (-)\overline{(3\wedge 2)}\overline{1} = 0$ □

Next we do the Chern-Simons-Witten graph: $L_{3,2,1,0}$. By induction the new and nontrivial graphs appearing in the Chern-Simons-Witten graphs $L_{3,2,1,0}$ are:  and  of which the Chern-Simons-Witten configuration space integrals are defined in Example 5. Now by shifting gear we regard these two configuration space integrals as two suitable 1-forms in $d0$ before carrying out the "final" integration over L_0; namely the two relevant 1-forms in $d0$ are:

$$\phi(0) = \int_{\mathbb{R}^3}\int_1\int_{\mathbb{R}^3}\int_2\int_3 (0-x)\wedge(x-1)\wedge(x-y)\wedge(y-2)\wedge(y-3),$$

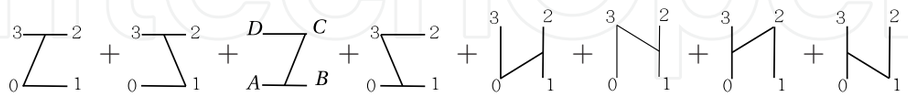
and

$$\psi(0) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_1\int_2\int_3 (0-x)\wedge(x-y)\wedge(y-1)\wedge(y-2)\wedge(y-3).$$

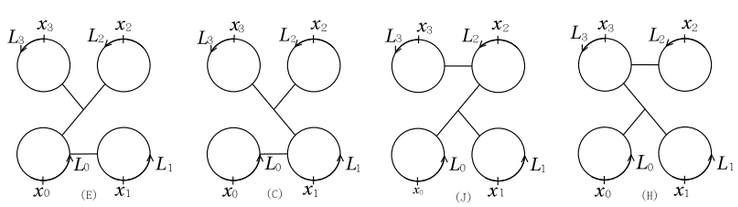
By stoke's theorem:

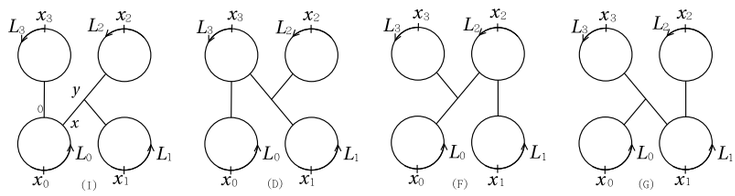
$$\int_0 \phi(0)+\psi(0) = \int_0 \overline{d\phi(0)} + \overline{d\psi(0)} = \text{ sum of following eight integrals}$$

(which will be made explicit in lemma 10 in Section 7):

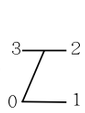
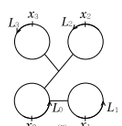


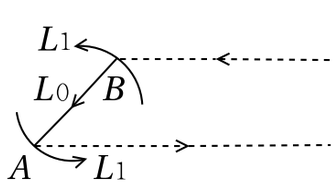
In $L_{3,2,1,0}$, corresponding to the above eight contracted graphs we have eight lower Chern-Simons-Witten linking together with an extra 1-chord:

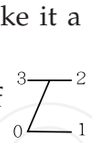
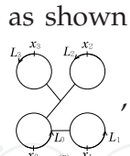
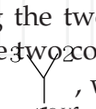




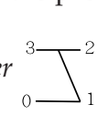
But to compute the latter 8 Chern-Simons-Witten configuration space integrals aiming at the combinatorial formulae we need to take into account the boundary terms for each related Chern-Simons-Witten graphs of degree 2—say the first graph along L_0. For example by induction $L_{2,1,0} = L^*_{2,1,0}$, but not now; as we don't have full cycles as before (as L_0 is cut into finitely many pieces of arcs by L_3). And the beauty of computing the extra correction due to the end points of the set of arcs along L_0 is that: the correction is compensated by those 8 integrals in $\int_0 \phi(0) + \psi(0)$ which is just a discrete sum on the corresponding end points of the pieces of arcs—cut by the attached chord in the contracted Chern-Simons-Witten graphs—of the configuration space integrals of Y-graphs thereof.

In greater details: take  and  as an example to show the details proper. For simplicity we assume that a piece of arc lying on L_0 is cut out by L_1 as shown.



By induction: applying linking of degree 2 to this piece of arc as shown we need to artificially attach two horizontal rays to both A and B to make it a cycle as shown above. Then on this full cycle, we could compute both integrals of  and  , by replacing the original sub-arc $\overline{AB}$ with the full "cycle" made up of two horizontal rays and $\overline{AB}$ to get two extra corrections of opposite signs along the two added horizontal rays at A and B in the above two integrals. More precisely these two corrections—up to signs—are nothing but the configuration space integrals of Y-graph:  , where ray at the root denotes the added ray and numeral 2 (respectively numeral 3) stands for L_2 (respectively L_3).

With the same computation for the other 7 pairs of Chern-Simons-Witten graphs listed above, we are done with the proof of $L^*_{3,2,1,0} = L_{3,2,1,0}$ if we have proved the following.

Claim 3. *Consider*  *as either a* 2*-form in d*0*:* $\eta(0) = \int (\underline{0} - 1)(1 - x)(x - 2)(x - 3)$ *or a* 2*-form in d*1*:* $\eta(1) = \int (0 - 1)(1 - x)(x - 2)(x - 3)$ *where we carry out all related integrations except that over* L_0 *in* $\eta(0)$ *and except that over* L_1 *in* $\eta(1)$*, then we have* $\int_{L_0} \overline{\eta(0)} = \int_{L_1} \overline{\eta(1)}$.

Note 3. *Before giving the proof we notice that this scheme of proof works fine for the other contracted Chern-Simons-Witten graphs in $L_{3,2,1,0}$. And this proves that $L_{3,2,1,0} = L^*_{3,2,1,0}$, and also derives the combinatorial formulae of $L^*_{3,2,1,0}$.*

Note 4. *We will repeat the same trick for connected Chern-Simons-Witten graphs in $L_{n,n-1,..,1,0}$ in Lemma 10, and will be more explicit in detail in this aspect. More precisely there are three aspects in this trick: firstly, we do contraction on the set of connected Chern-Simons-Witten graphs to get a bunch of 2-forms in $d0$ (namely those ones corresponding to the connected Chern-Simons-Witten graphs contracted at uni-valent vertices); secondly, for a connected Chern-Simons-Witten graph contracted at a vertex L_j with $j \neq 0$, we repeat the proof of the above claim to show: $\int_{L_0} \overline{\eta(0)} = \int_{L_j} \overline{\eta(j)}$; thirdly, we compute explicitly $\int_{L_j} \overline{\eta(j)}$ and prove that it compensates the corresponding correction coming from the disconnected Chern-Simons-Witten graph which has the same contracted graph after being contracted at vertex L_j.*

Now we come to the proof of the claim.

Proof. Recall that the over-bars on 2-forms, by definition, are horizontal $\frac{\partial}{\partial x}$-integrations from $x = -\infty$ up to points of interests; and that if the point of interests is a planar point, the horizontal $\frac{\partial}{\partial x}$-integration should be defined as the average of the one infinitesimally above $\mathbb{R}^2$ and the one infinitesimally below $\mathbb{R}^2$.

(A) First we define and state contracted Chern-Simons-Witten graphs and the corresponding 2-forms—either in $d0$ or $d1$ explicitly. But this is nothing but the content of Lemma 1 and Lemma 2 in Section 5.

(B) Next we compute: $\int_{L_1} \overline{\eta(1)} = \sum_{L_0 \cap L_1} (0,1) \int_{\text{hori}} \int_2 \int_3 \int_{\mathbb{R}^3} (h-x)(x-2)(x-3)$, where hori stands for horizontal $\frac{\partial}{\partial x}$-ray starting/ending at the crossings of L_1 and L_0, $(0,1)$ is the Gauss linking of L_1 and L_0, and h denotes the dummy variable on the horizontal $\frac{\partial}{\partial x}$-rays stated above.

(C) Next we compute $\int_{L_0} \overline{\eta(0)}$. To apply the calculus Lemma 1 and Lemma 2 to an sub-arc on L_1 cut-off by crossing with L_0, we need to artificially add two horizontal $\frac{\partial}{\partial x}$-rays at the ends of this arc to make it a full cycle. Now carry out the fundamental theorem of calculus as stated in Lemma 1 and Lemma 2 for this sub-arc of L_1 to get exactly the extra boundary evaluation at the ends of the configuration space integrals of Y-graph 

, and then identify Y-graph with symbols $(3,2)_1 = (2,1)_3 = (1,3)_2$ for L_2, L_3 and the artificial cycle L_1 (which is the union of the above sub-arc and two horizontal $\frac{\partial}{\partial x}$-rays) to conclude the proof of the claim. □

Note 5. *The proof of the claim applies also to any double round brackets in Definition 2, the so-called connected Chern-Simons-Witten graphs. In short in the Massey-Milnor linkings $L^*_{n,n-1,..,1,0}$, connected Chern-Simons-Witten graphs always contribute nothing except those connected Y-graphs of degree 2.*

Note 6. *We notice that we could only do the trick of contraction for connected graphs as above, but not for non-connected ones.*

Note 7. *In next section we will repeat the same trick of contracting connected Chern-Simons-Witten graphs to compensate the corrections on the "cut-points"—due to lower linking functions (by induction, a discrete sum) along a specific knot component of interests—of suitable configuration space integrals involving artificial horizontal rays starting/ending at the cut-points, and other knot components proper.*

Note 8. *In essence the proof and the computation for $L^*_{3,2,1,0} = L_{3,2,1,0}$ and the combinatorial formulae thereof prevail in general case, because we assume the vanishing of linkings; and so we could compute the related lower linking functions along some specific knot component—which by induction are just some discrete sums along that knot component.*

7. Proof of the main theorem—general case

In this section we prove the main theorem of this paper. Here are some preparatory lemmas.

Lemma 8. *In the Massey-Milnor linking $L^*_{n,n-1,\ldots,1,0}$ or the associated 1-form $< n, n-1, \ldots, 2, 1 >$ on L_0, the Dirac-like singular one-form when evaluated at $x \in \mathbb{R}^2$, $\overline{dj}(x)$ is nothing but the one-form $j(x) = \int_{L_j}(x-t)$ where $(x-t) = \frac{1}{4\pi}\frac{1}{|x-t|^3} det \begin{pmatrix} x-t \\ dt \\ dx \end{pmatrix}$, and the over-bar denotes the horizontal $\frac{\partial}{\partial x}$-integration as defined in Definition 3.*

Proof. By de Rham theory for any 1-form $j(x)$ in $\mathbb{R}^3$ we have $j(x) = \overline{dj} + d\overline{j}$, where as usual the over-bar stands for the inverse of exterior differentiation—after gauge-fixing, which is the horizontal $\frac{\partial}{\partial x}$-integration from $x = -\infty$ up to the point of interests. But in Massey-Milnor linking the integrations of either $j(x)$ or $\overline{dj}(x)$ are carried out in $\mathbb{R}^2$, so the suitable horizontal $\frac{\partial}{\partial x}$-integration (namely the over-bar) in $\overline{dj}$ should be defined as the average of the two natural ones—either infinitesimally above $\mathbb{R}^2$ or infinitesimally below $\mathbb{R}^2$. Hence $\overline{j}(x) = 0$ when restricted to the plane $\mathbb{R}^2$, and this concludes the proof. □

Note 9. *From Lemma 8 when restricted to the plane $\mathbb{R}^2$ $\overline{dj} \wedge i$ is nothing but $j(x) \wedge i(x)$, so in particular $\int_0 \overline{\overline{dj} \wedge j} = \int_0 \overline{j \wedge i} = (j,i)_0$, in the notation of Lemma 7.*

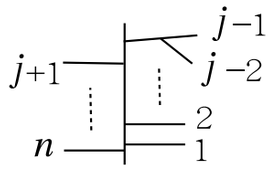
Lemma 9. *In Massey-Milnor linking $L^*_{n,n-1,\ldots,1,0}$, all connected Chern-Simons-Witten graphs of degree larger than 2 contribute nothing.*

Proof. For connected Chern-Simons-Witten graphs of degree larger than 2 we have double round brackets of length larger than 2. Take a double round bracket of length 3 such as $\int_0 \overline{\overline{(3 \wedge 2} \wedge 1} = + \int_0 \overline{\overline{(3 \wedge 2)}} \wedge 1 - \int_0 \overline{(3 \wedge 2)} \wedge \overline{1} = 0$, we get nothing simply because the term $\overline{\overline{(3 \wedge 2)}}$ has repeated $\frac{\partial}{\partial x}$-integrations and $\overline{1}(x)$ vanishes by the proof of Lemma 8. So any double round bracket of length larger than 2 also contributes nothing as it contains double round brackets of length 3. □

For the Chern-Simons-Witten perspective $L_{n,n-1,\ldots,1,0}$, we have the following.

Lemma 10. *(A) Regard the set of connected Chern-Simons-Witten graphs in $L_{n,n-1,..,1,0}$—namely those corresponding to double round brackets in Definition 2—as 1-forms in $d0$ after doing all related integrations except the one over the component L_0. And if we take the exterior differentiation once with respect to $d0$ of the sum of these 1-forms, then we are left with only those Chern-Simons-Witten graphs contracted at the uni-vertices.*

(B) For any contracted Chern-Simons-Witten graph in (A), if the contracted vertex—only one such for each contracted Chern-Simons-Witten graph—is L_j, $j \neq 0$, then it is the same as the contracted Chern-Simons-Witten graph when considered as taking the exterior differential with respect to the coordinate j(the dummy variable for L_j). More precisely, for example if the contracted connected Chern-Simon-Witten graph Γ looks like



Then $\int\int_{D_0} \Gamma = \int\int_{D_j} \Gamma$, where the Γ inside the integral sign on the left hand side is considered as a 2-form in $d0$, say $\eta(0)$—as we take an exterior differentiation with respect to $d0$ once; and where the Γ in the integral sign on the right hand side is considered as a 2-form in dj, say $\eta(j)$—as, this time we take an exterior differentiation with respect to djonce. In short, $\int\int_{D_0} \eta(0) = \int_{L_0} \overline{\eta(0)} = \int\int_{D_j} \eta(j) = int_{L_j}\overline{\eta(j)}$.

Proof. (A) The content (A) is nothing but a corollary of Lemma 1 and Lemma 2 in Section 5.

(B) We repeat the trick and the proof of the claims in Example 8. That is to each connected Chern-Simons-Witten graph contracted at a vertex L_j, with $j \neq 0$, we associated two 2-forms (one in $d0 \wedge d0$, and one in $dj \wedge dj$), $\eta(0)$ and $\eta(j)$. We will proceed by induction: $\int_{L_0} \overline{\eta(0)} = \int_{L_j} \overline{\eta(j)}$, where the over-bars are the average of horizontal $\dfrac{\partial}{\partial x}$-integration: one infinitesimally above $\mathbb{R}^2$ and one infinitesimally below $\mathbb{R}^2$ from $x = -\infty$ up to the points of interests in $\mathbb{R}^2$.

(B-1) *We treat $\int_{L_j} \overline{\eta(j)}$ first.* The connected Chern-Simons-Witten graph contracted at the vertex L_j is considered now in another perspective, as the contraction of two distinct connected graphs of strictly lower degree at the vertex L_j. By induction a connected component of the above two distinct graphs could be regarded as a part of linking function of lower degree at L_j, and hence would cut off L_j into finitely many sub-arcs. Repeat the trick of the claims in Example 8 for the set of knot components attached to the other graph compoment and a specific subarc of L_j to get the horizontal $\dfrac{\partial}{\partial x}$-rays start/end at the ends of the sub-arc of L_j cut-off by the combinatorial formulae (a part of the lower linking function associated to that graph component).

Repeat the same computaton with the roles of these two graph components switched. That is obvious as $\overline{\eta(j)} = \overline{\alpha(j) \wedge \beta(j)} = \overline{\alpha(j)}\beta(j) - \alpha(j)\overline{\beta(j)}$ which is a nontrivial and key formulae

in our discrete computation of horizontal $\frac{\partial}{\partial x}$-integration, but it is easy to see in the following set-up: cut the knot component L_j by the discrete contributions due to these two connected Chern-Simons-Witten graphs (which give rise to two 1-forms $\alpha(j)$ and $\beta(j)$ above); and as above we add artificially horizontal $\frac{\partial}{\partial x}$-rays strating/ending at these cut-point so that we could apply induction scheme for these two subgraphs of lower degree to suitable knot components attached thereon and the artificial cycles, each of which consists of a subarc of L_j and the two horizontal $\frac{\partial}{\partial x}$-rays added onto the boundary points of the subarc. Finally do the obvious discrete sum along L_j to get the net contribution as claimed.

(B-2) *Next we treat* $\int_{L_0} \overline{\eta(0)}$.

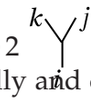
As the graph component containing L_0 is of strictly lower degree than n, by induction and after carrying out all the integrations of the configuration space integral of that graph component we are left with an honest function (not a differential form) on L_j. This also can be regarded as the boundary evaluation of the associated configuration space integral when adding the artificial horizontal $\frac{\partial}{\partial x}$-ray to the sub-arc of L_j to make a full cycle so that we could apply Lemma 1 and Lemma 2. And this boundary evaluation is exactly the horizontal $\frac{\partial}{\partial x}$-integration of the configuration space integral by the trick of Example 8.

Similarly switch the roles of these two graph components to get the other horizontal $\frac{\partial}{\partial x}$-integration of the configuration space integral of one graph component; while the other graph component plays the role of lower linking function and so is just a discrete evaluation there; and hence cuts off L_j into sub-arcs whose boundary points support the artificially horizontal $\frac{\partial}{\partial x}$-rays as above. In short the boundary evaluation at the ends of a specific subarc of L_j of the configuration space integral of the subgraph containing L_0, by Lemma 1 and Lemma 2 is exactly the horizontal $\frac{\partial}{\partial x}$-integration of the artificial rays as above simply by computing the Chern-Simons-Witten configuration space integrals in two ways—one, we integrate over just that specific sub-arc to get the extra boundary evaluation; and the other, we integrate over the full cycle consisting of the above sub-arc and two added horizontal $\frac{\partial}{\partial x}$-rays. □

With all the preparatory lemmas we come to the proof of Theorem 1 in Section 6.

Step 1. *Massey-Milnor linking* $L^*_{n,n-1,\ldots,1,0}$.

By assumption all Massey-Milnor linkings of lower degree vanish and by Lemma 9 all connected Chern-Simons-Witten graphs of degree larger than 2 contribute nothing to $L^*_{n,n-1,\ldots,1,0}$. The combinatorial formulae of $L^*_{n,n-1,\ldots,1,0}$ could be read out directly from the expansion of the sharp bracket $< n, n-1, \ldots, 2, 1 >$. More precisely, regard lower linkings as linking functions supported on some knot component L_j which corresponds to some dj in $< n, n-1, \ldots, 1 >$. By induction only those Chern-Simons-Witten connected graphs of degree one and two contribute to the combinatorial formulae of $L^*_{n,n-1,\ldots,1,0}$. Also for a

connected Chern-Simons-Witten graph of degree 2  which is not symmetric with respect to its vertices $\{i, j, k\}$, the symbol $(k, j)_i$ is naturally and canonically assigned to this Y-graph without ambiguity simply by inspecting the related linking function therewith.

Step 2. *Chern-Simons-Witten graph* $L_n, n-1, \ldots, 1, 0$.

(A) Consider the set of connected Chern-Simons-Witten graphs first. Mimicking the trick of contraction of connected Chern-Simons-Witten graphs in $L_{3,2,1,0}$ in Section 6, we regard this set of contracted connected Chern-Simons-Witten graphs a sum of 2-forms with only one contracted vertex for each, along which we will construct artificially horizontal $\frac{\partial}{\partial x}$-rays as before. And as in the Section 6 this kind of Chern-Simons-Witten configuration space integrals that involves the artificial horizontal rays will compensate those boundary evaluation of the associated lower linking functions at the contracted vertex (equivalently the knot component corresponding to the vertex numeral). In short even in Chern-Simons-Witten graphs, the connected graphs of degree larger than two do not contribute anything to the combinatorial formulae thereof. This is obvious by the above argument and by induction.

(B) For Chern-Simons-Witten graphs of degree 3 $L_{3,2,1,0}$, we have explicit combinatorial formulae as given in Example 8. And so by induction only connected Chern-Simons-Witten graphs of degree one and two contribute to the combinatorial formulae of $L_{n,n-1,\ldots,1,0}$. And for Y-graph  , joining 3 knot components $\{L_i, L_j, L_k\}$ we need to apply the identification of Massey-Milnor linkings and Chern-Simons-Witten graphs of strictly lower degrees to determine the suitable symble $(k, j)_i$ for this Y-graph.

Step 3. *Massey-Milnor linkings = Chern-Simons-Witten graphs; that is* $L^*_{n,n-1,\ldots,1,0} = L_{n,n-1,\ldots,1,0}$.

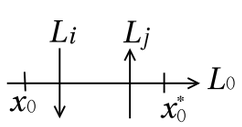
(A) We have done this for $n = 1, 2$ in Section 6.

(B) By induction.

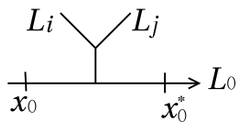
Assume that we are done for all such of degrees less than n, that is we have identified all those non-connected Chern-Simons-Witten graphs once they are grouped together in the manner of lower linking function which correspond to those over-barred sharp brackets with some dj attached besides, in $L^*_{n,n-1,\ldots,1,0}$. And obviously, we are left with only those connected Chern-Simons-Witten graphs of degree n in both Massey-Milnor linking and Chern-Simons-Witten graph set-ups. In the former, connected Chern-Simons-Witten graphs correspond to double round brackets, and so contribute nothing by Lemma 10 to $L^*_{n,n-1,\ldots,1,0}$; in the latter, the connected Chern-Simons-Witten graphs will compensate the artificial horizontal $\frac{\partial}{\partial x}$-rays derived from the lower linkings functions and the suitable sub-graphs contracted there on.

Step 4. *Both* $L_{n,n-1,\ldots,1,0}$ *and* $L^*_{n,n-1,\ldots,1,0}$ *are independent of the base points* $x_j \in L_j$, $j = 0, 1, 2, \ldots, n$.

Due to cyclic symmetry of $L^*_{n,n-1,\ldots,1,0}$—as is obvious from the Chern-Simons-Witten perspective—we need only to prove that $L^*_{n,n-1,\ldots,1,0}$ is independent of the choice of $x_0 \in L_0$. And we denote the change or the variation due to the change of base point $x_0 \in L_0$ by the notation δ. For example, δ may stand for changing the old base point $x_0 \in L_0$ to the new base point $x_0^* \in L_0$; and within $[x_0, x_0^*] \subseteq L_0$, we may have



or



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By definition, $L^*_{n,n-1,\ldots,1,0} = \int_0 \overline{(n,n-1,\ldots,1} + \int_0 \overline{(n,\ldots,2)1} + \int_0 \overline{(n,\ldots,3)} \cdot \overline{<2,1>} + \cdots + \int_0 n\overline{<n-1,\ldots,1>}$, and if for some k such that $\delta\overline{<k,k-1,\ldots,1>} = c \neq 0$ and $\delta\overline{<k-1,\ldots,1>} = 0 = \delta\overline{<k-2,\ldots,1>} = \cdots = \delta\overline{<3,2,1>} = \delta\overline{<2,1>} = \delta\overline{1} = 0$, then we have

Claim 4.

$$\delta\overline{<k+1,k,\ldots,1>} = C\overline{(k+1)},$$
$$\delta <k+2,k+1,\ldots,1> = C\overline{<k+2,k+1>},$$
$$\delta <k+3,k+2,\ldots,1> = C\overline{<k+3,k+2,k+1>},$$
$$\cdots \quad \cdots \quad \cdots$$
$$\delta\overline{<n-1,n-2,\ldots,1>} = C\overline{<n-1,n-2,\ldots,k+1>}.$$

Proof. By the very definition of $\overline{<k+1,\ldots,1>}, \overline{<k+2,\ldots,1>}, \cdots, \overline{<n-1,n-2,\ldots,1>}$, for $x \in L_0$ we have (being the dumming variable of L_0), $\delta\overline{<k+1,\ldots,1>}(x) = \delta\int_{x_0}^{x}\overline{(k+1,\ldots,1)} + \delta\int_{x_0}^{x}\overline{(k+1,\ldots,2)1} + \delta\int_{x_0}^{x}\overline{(k+1,\ldots,3)}\overline{<21>} + \cdots + \delta\int_{x_0}^{x}(k+1)\overline{<k,\ldots,1>} = C\int_{x_0}^{x}(k+1) = C\overline{(k+1)}(x)$, where as usual, for $j \in \{1,2,\ldots,n\}$, $j(x)$ is regarded as 1-form.

Similarly, $\delta\overline{<k+2,k+1,\ldots,1>} = \delta\int_{x_0}^{x}\overline{(k+2,\ldots,1)} + \delta\int_{x_0}^{x}\overline{(k+2,\ldots,2)1} + \cdots + \delta\int_{x_0}^{x}(k+2)\overline{<k+1,\ldots,1>} = C\int_{x_0}^{x}\overline{(k+2,k+1)} + C\int_{x_0}^{x}(k+2)\overline{<k+1>} = C\overline{<k+2,k+1>}(x)$.

In general, $\delta < l,l-1,\ldots,1 >= \delta\int_{x_0}^{x}\overline{(l,l-1,\ldots,1)} + \delta\int_{x_0}^{x}\overline{(l,l-1,\ldots,2)1} + \cdots + \delta\int_{x_0}^{x} l\overline{<l-1,l-2,\ldots,1>} = C\int_{x_0}^{x}\overline{(l,l-1,\ldots,k+1)} + C\int_{x_0}^{x}\overline{(l,l-1,\ldots,k+2)(k+1)} + \cdots + C\int_{x_0}^{x} l\overline{<l-1,\ldots,k+1>} = C\overline{<l,l-1,\ldots,k+1>}(x)$. And this concludes the proof of the claim.

Now, we come back to the proof of independence of the choice of base point $x_0 \in L_0$.

By the very definition of $L^*_{n,n-1,..,1,0}$, we have

$$\begin{aligned}
&\delta L^*_{n,n-1,..,1,0}\\
=&\delta\int_0\overline{(n,n-1,\ldots,1)}+\delta\int_0\overline{(n,n-1,\ldots,2)1}\\
&\quad+\delta\int_0\overline{(n,n-1,\ldots,3)<2,1>}+\cdots\\
&\quad+\delta\int_0 n\overline{<n-1,n-2,\ldots,1>}\\
=&C\int_0\overline{(n,n-1,\ldots,k+1)}\\
&\quad+C\int_0\overline{(n,n-1,\ldots,k+2)<k+1>}+\cdots\\
&\quad+C\int_0\overline{(n,n-1)<n-2,\ldots,k+1>}\\
&\quad+C\int_0 n\overline{<n-1,n-2,\ldots,k+1>}\\
=&L^*_{n,n-1,..,k+2,k+1,0}\\
=&0,
\end{aligned}$$

by the assumption that all Massey-Milnor linkings of lower degrees vanish.

And this concludes the proof of Theorem 1. □

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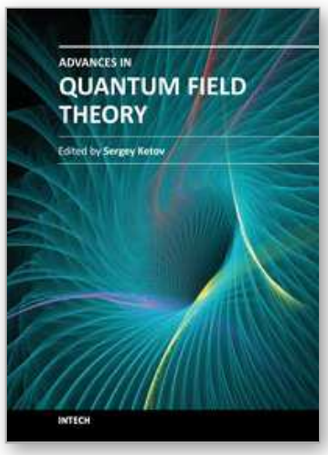

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Quantum Field Theory is now well recognized as a powerful tool not only in Particle Physics but also in Nuclear Physics, Condensed Matter Physics, Solid State Physics and even in Mathematics. In this book some current applications of Quantum Field Theory to those areas of modern physics and mathematics are collected, in order to offer a deeper understanding of known facts and unsolved problems.

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