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Ginzburg–Landau Theory of Phase Transitions in Compactified Spaces

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1. Introduction

In this chapter, we review some aspects of physical systems described by quantum fields defined on spaces with compactified dimensions. For a D-dimensional space, this means we are considering a space which has a topology of the type $\Gamma_D^d = \left(S^1\right)^d \times \mathbf{R}^{D-d}$, with d $(\leq D)$ being the number of compactified dimensions, each with the topology of a circle. This is the type of topology that emerges, for instance, in quantum field theory at finite temperature: the Matsubara formalism imposes that the time direction is compactified in a circle with length $\beta = 1/T$, where T is the temperature; its topology is then $\Gamma_4^1 = S^1 \times \mathbf{R}^3$ in the notation introduced above. Another important example involves spacetimes of dimensions D larger than four, with the "extra" or "hidden" dimensions being compactified and assumed to be very small, as in Kaluza–Klein and string theories. In any case, the topology Γ_D^d mentioned above corresponds to a generalized Matsubara formalism, in which imaginary-time and spatial coordinates may be simultaneously compactified.

In the last few decades, this generalized Matsubara formalism has been employed in many instances of condensed-matter and particle physics. Some of them are: (1) the Casimir effect, studied in various geometries, topologies, fields, and physical boundary conditions [Bordag et al. (2001); Milonni (1993); Mostepanenko & Trunov (1997)], in a diversity of subjects ranging from nanodevices to cosmological models [Bordag et al. (2001); Boyer (2003); Levin & Micha (1993); Milonni (1993); Mostepanenko & Trunov (1997); Seife (1997)]; (2) the confinement/deconfinement phase transition of hadronic matter, in the Gross–Neveu and Nambu–Jona-Lasinio models as effective theories for quantum chromodynamics [Abreu et al. (2009); Khanna et al. (2010); Malbouisson et al. (2002)]; (3) quantum electrodynamics with one extra compactified dimension, which leads to estimates of the size of extra dimensions compatible with present-day experimental data [Ccapa Tira et al. (2010)]; (4) the study of superconductors in the form of films, wires and grains [Abreu et al. (2003; 2005); Khanna et al. (2009); Linhares et al. (2006; 2007); Malbouisson (2002); Malbouisson et al. (2009)], in which the Ginzburg–Landau model for phase transitions is defined on a three-dimensional Euclidean space with one, two or three dimensions compactified.

When studying the compactification of spatial coordinates, however, it is argued in Khanna et al. (2009) from topological considerations, that we may have a quite different interpretation of the generalized Matsubara prescription: it provides a general and practical way to account for systems confined in limited regions of space at finite temperatures. Distinctly, we shall be concerned here with stationary field theories and employ the generalized Matsubara prescription to study bounded systems by implementing the compactification of spatial coordinates; no imaginary-time compactification will be done, temperature will be introduced through the mass parameter in the Ginzburg-Landau Hamiltonian. We will consider a topology of the type $\Gamma_D^d = \mathbf{R}^{D-d} \times (S^1)_1 \times (S^1)_2 \times \cdots \times (S^1)_d$, where $(S^1)_1, \ldots, (S^1)_d$ refer to the compactification of d *spatial* dimensions.

In the following, we shall concentrate on Euclidean scalar field theories defined on such spaces, with the Matsubara formalism applied to spatial coordinates. Our aim is to describe the influence of compactification on physical phenomena as phase transitions in which, for instance, the critical temperature depends on the parameters of compactification, that is, on the "size" of the system. This means that, for instance, superconductors inside spatially bound spaces such as films, wires and grains may have a critical temperature distinct from the same material in the bulk form.

In this chapter, the way in which the critical temperature for a second-order phase transition is affected by the presence of confining boundaries is investigated on general grounds. We consider that the system is a portion of material of some size, the behavior of which in the critical region is derived from a quantum field theory calculation of the dependence of the physical mass parameter on its size. We focus in particular on the mathematical aspects of the formalism, which furnish the tools to study boundary effects on the phase transition. We consider the D-dimensional Ginzburg–Landau model compactified in d ($\leq D$) of the spatial dimensions. The Ginzburg–Landau Hamiltonian, considering only the term $\lambda\varphi^4$, is known to lead to second-order transitions. In its version with N-components, in the large-N limit, we are able to take into account nonperturbatively corrections to the coupling constant. In this case, we shall obtain expressions for the transition temperature in the general situation. Particularizing for $D = 3$ and $d = 1$, $d = 2$ and $d = 3$, we have the critical temperature $T_c(L)$ for the system in the form of a film of thickness L, an infinitely long wire having a square cross-section L^2, and for a cubic grain of volume L^3, respectively. We show that $T_c(L)$ decreases as the size L is diminished and a minimal size for the suppression of the second-order transition is obtained.

We also consider the model which, besides the quartic scalar field self-interaction, a sextic one is present. The model with both interactions taken together leads to a renormalizable quantum field theory in three dimensions and it may describe *first-order* phase transitions. We consider this formalism in a general framework, taking the Euclidean D-dimensional $-\lambda\,|\varphi|^4 + \eta\,|\varphi|^6$ (λ, $\eta > 0$) model with $d = 1, 2, 3$ compactified dimensions. It is known that such potential *ensures* that the system undergoes a *first-order* transition. We obtain formulas for the dependence of the transition temperature on the parameters delimiting the spatial region within which the system is confined. Surely, there are other potentials which may be considered, for instance, the Halperin–Lubensky–Ma potential [Halperin et al. (1974)], which also engender first-order transitions in superconducting materials by effect of integration over the gauge field and takes the form $-\alpha\varphi^3 + \beta\varphi^4$.

We start from the effective potential, which is related to the physical mass and coupling constant through renormalization conditions. These conditions, however, reduce considerably the number of relevant contributing Feynman diagrams, if one wishes to be restricted to 1- or 2-loop approximations. For second-order transitions, we need to consider only the tadpole diagram to correct the mass and the 1-loop four-point function to correct the coupling constant. For first-order transitions, we will not, for simplicity, make corrections to the coupling constant. In this case, just two diagrams need to be considered: a tadpole graph with the φ^4 coupling (one loop) and a "shoestring" graph with the φ^6 coupling (two loops). No diagram with both couplings needs to be considered. The size dependence appears from the treatment of the loop integrals. The dimensions of finite extent are treated in momentum space using the formalism of Khanna et al. (2009).

It is worth noticing that for superconducting films with thickness L, a qualitative agreement of our theoretical L-dependent critical temperature is found with experiments. This occurs in particular for thin films (in the case of first-order transitions) and for a wide range of values of L for second-order transitions [Linhares et al. (2006)]. Moreover, available experimental data for superconducting wires are compatible with our theoretical prediction of the first-order critical temperature as a function of the transverse cross section of the wire.

Finally, we discuss the infrared behavior and the fixed-point structure for the N-component $\lambda\varphi^4$ model in the large-N limit, with a compactified subspace. We study the cases in which the system has no external influence and in which the system is submitted to the action of an external magnetic field. In both situations, with or without a magnetic field, we get the result that the existence of an infrared stable fixed-point depends only on the space dimension; it does not depend on the number of compactified dimensions.

2. Critical behavior of the compactified $\lambda\varphi^4$ model

We start by considering the complex scalar field model described by the Ginzburg–Landau Hamiltonian density in a Euclidean D-dimensional space, in the absence of any geometrical constraints, given by (in natural units, $\hbar = c = k_B = 1$)

$$\mathcal{H} = \frac{1}{2}\left|\partial_\mu \varphi\right| \left|\partial^\mu \varphi\right| + \frac{1}{2} m_0^2 \left|\varphi\right|^2 + \frac{\lambda}{4}\left|\varphi\right|^4 , \tag{1}$$

where $\lambda > 0$ is the physical coupling constant. As usual, near criticality, the bare mass is taken as $m_0^2 = \alpha(T - T_0)$, with $\alpha > 0$ and T_0 being a parameter with the dimension of temperature, which is interpreted as the bulk transition temperature.

Let us now take the system in D dimensions confined to a region of space delimited by $d \leq D$ pairs of parallel planes. Each plane of a pair j is at a distance L_j from the other member of the pair, $j = 1, 2, \ldots, d$, and is orthogonal to all other planes belonging to distinct pairs $\{i\}$, $i \neq j$. This may be pictured as a parallelepipedal box embedded in the D-dimensional space, whose parallel faces are separated by distances $L_1, L_2, \ldots, L_d$. To simplify matters, we shall take all $L_i = L$. Let us define Cartesian coordinates $r = (x_1, x_2, \ldots, x_d, z)$, where z is a $(D-d)$-dimensional vector, with corresponding momentum $k = (k_1, k_2, \ldots, k_d, q)$, q being a $(D-d)$-dimensional vector in momentum space. The generating functional of Schwinger functions is written in the form

$$Z = \int \mathcal{D}\varphi\, \mathcal{D}\varphi^* \exp\left(- \int_0^{L_1} dx_1 \cdots \int_0^{L_d} dx_d \int d^{D-d}z\, \mathcal{H}(\left|\varphi\right|, \left|\nabla\varphi\right|) \right), \tag{2}$$

with the field $\varphi(x_1, ..., x_d, z)$ satisfying the condition of confinement inside the box, $\varphi(x_i \leq 0, z) = \varphi(x_i \geq 0, z) = \text{const}$. Then, following the procedure developed in Khanna et al.

(2009), we introduce a generalized Matsubara prescription, in which the Feynman rules are modified through the replacements

$$\int \frac{dk_i}{2\pi} \to \frac{1}{L} \sum_{n_i=-\infty}^{+\infty} ; \qquad k_i \to \frac{2n_i\pi}{L} , \quad i = 1, 2..., d. \tag{3}$$

Notice that compactification can be implemented in different ways, as for instance by imposing specific conditions on the fields at spatial boundaries. We here choose periodic boundary conditions.

In principle, the effective potential for systems with spontaneous symmetry breaking is obtained, following the Coleman–Weinberg analysis [Coleman & Weinberg (1973)], as an expansion in the number of loops in Feynman diagrams. Accordingly, to the free propagator and to the tree diagrams, radiative corrections are added, with increasing number of loops. Thus, at the 1-loop approximation, we get the infinite series of 1-loop diagrams with all numbers of insertions of the φ^4 vertex (two external legs in each vertex).

At the 1-loop approximation, the contribution of loops with only $|\varphi|^4$ vertices to the effective potential in unbounded space is

$$U_1(\varphi_0) = \sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{2s} \left[3\lambda|\varphi_0|^2\right]^s \int \frac{1}{(2\pi)^D} \frac{d^D k}{(k^2 + m^2)^s}, \tag{4}$$

where m is the *physical* mass and the parameter s counts the number of vertices on the loop.

In the following, to deal with dimensionless quantities in the regularization procedures, we introduce parameters $c^2 = m^2/4\pi^2$, $L^2 = a^{-1}$, $g = 3\lambda/8\pi^2$, where φ_0 is the normalized vacuum expectation value of the field (the classical field). In terms of these parameters and performing the Matsubara replacements (3), the one-loop contribution to the effective potential can be written in the form

$$\begin{aligned} U_1(\phi_0, a) &= a^{d/2} \sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{2s} g^s |\varphi_0|^{2s} \\ &\times \sum_{n_1,\ldots,n_d=-\infty}^{+\infty} \int \frac{d^{D-d}q}{\left[a\left(n_1^2 + \cdots + n_d^2\right) + c^2 + q^2\right]^s}. \end{aligned} \tag{5}$$

It is easily seen that only the $s = 1$ term contributes to the renormalization condition

$$\left.\frac{\partial^2 U(\varphi_0)}{\partial \varphi_0^2}\right|_{\varphi_0=0} = m^2. \tag{6}$$

It corresponds to the tadpole diagram. The integral over the $D - d$ noncompactified momentum variables is performed using a well-known dimensional regularization formula [Zinn-Justin (2002)] so that, for $s = 1$, we obtain

$$U_1(\phi_0, a) = \frac{1}{2} a^{d/2} \pi^{(D-d)/2} \Gamma\left(1 - \frac{D-d}{2}\right) g|\varphi_0|^2 Z_d^{c^2}\left(\frac{2 - D + d}{2}; a\right), \tag{7}$$

where $Z_d^{c^2}(\frac{2-D+d}{2}; a)$ is one of the Epstein–Hurwitz zeta functions, defined by

$$Z_d^{c^2}(\nu; a_1, ..., a_d) = \sum_{n_1,...,n_d=-\infty}^{+\infty} (a_1 n_1^2 + \cdots + a_d n_d^2 + c^2)^{-\nu}, \tag{8}$$

valid for $\mathrm{Re}(\nu) > 1$.

Next, we can use the generalization to several dimensions of the mode-sum regularization prescription described in Elizalde (1995). It results that the multidimensional Epstein–Hurwitz function has an analytic extension to the whole ν complex plane, which may be written as

$$\begin{aligned} Z_d^{c^2}(\nu; L) = \frac{2^{\nu-\frac{d}{2}+1}\pi^{2\nu-\frac{d}{2}}L^{d/2}}{\Gamma\left(1-\frac{D-d}{2}\right)\Gamma(\nu)} & \left[2^{\nu-\frac{d}{2}-1}m^{d-2\nu}\Gamma\left(\nu-\frac{d}{2}\right)\right. \\ & +2d\sum_{n=1}^{\infty}\left(\frac{m}{Ln_i}\right)^{\frac{d}{2}-\nu}K_{\nu-\frac{d}{2}}(mLn)+\cdots \\ & +2^d\sum_{n_1,...,n_d=1}^{\infty}\left(\frac{m}{L\sqrt{n_1^2+\cdots+n_d^2}}\right)^{\frac{d}{2}-\nu} \\ & \left.\times K_{\nu-\frac{d}{2}}\left(mL\sqrt{n_1^2+\cdots+n_d^2}\right)\right], \end{aligned} \tag{9}$$

where the $K_\nu(z)$ are modified Bessel functions of the second kind. Taking $\nu = (2-D+d)/2$ in Eq. (9), we obtain from Eq. (7) the effective potential in D dimensions with a compactified d-dimensional subspace:

$$\begin{aligned} U_1(\varphi_0, L) = \frac{3\lambda|\varphi_0|^2}{(2\pi)^{D/2}} & \left[2^{-D/2-1}m^{D-2}\Gamma\left(\frac{2-D}{2}\right)\right. \\ & +d\sum_{n=1}^{\infty}\left(\frac{m}{Ln}\right)^{D/2-1}K_{D/2-1}(mLn)+\cdots \\ & +2^{d-1}\sum_{n_1,...,n_d=1}^{\infty}\left(\frac{m}{L\sqrt{n_1^2+\cdots+n_d^2}}\right)^{D/2-1} \\ & \left.\times K_{D/2-1}\left(mL\sqrt{n_1^2+\cdots+n_d^2}\right)\right], \end{aligned} \tag{10}$$

where we have returned to the original variables, λ and L.

Notice that in Eq. (10) there is a term proportional to $\Gamma\left(\frac{2-D}{2}\right)$, which is divergent for even dimensions $D \geq 2$, and should be subtracted in order to obtain finite physical parameters. For odd D, the above gamma function is finite, but we also subtract this term (corresponding to a finite renormalization), for the sake of uniformity. We get

$$
\begin{aligned}
U_{1,R}(\varphi_0, L) = \frac{3\lambda|\varphi_0|^2}{(2\pi)^{D/2}} \Bigg[& d \sum_{n=1}^{\infty} \left(\frac{m}{Ln}\right)^{D/2-1} K_{D/2-1}(mLn) + \cdots \\
& + 2^{d-1} \sum_{n_1,\ldots,n_d=1}^{\infty} \left(\frac{m}{L\sqrt{n_1^2+\cdots+n_d^2}} \right)^{D/2-1} \\
& \times K_{D/2-1}\left(mL\sqrt{n_1^2+\cdots+n_d^2} \right) \Bigg].
\end{aligned} \tag{11}
$$

Then the physical mass is obtained from Eq. (6), using Eq. (11) and also taking into account the contribution at the tree level; it satisfies a generalized Dyson–Schwinger equation depending on the finite extension L of the confining box:

$$
\begin{aligned}
m^2(L) = m_0^2 + \frac{6\lambda}{(2\pi)^{D/2}} \Bigg[& d \sum_{n=1}^{\infty} \left(\frac{m}{Ln}\right)^{D/2-1} K_{D/2-1}(mLn) + \cdots \\
& + 2^{d-1} \sum_{n_1,\ldots,n_d=1}^{\infty} \left(\frac{m}{L\sqrt{n_1^2+\cdots+n_d^2}} \right)^{D/2-1} \\
& \times K_{D/2-1}\left(mL\sqrt{n_1^2+\cdots+n_d^2} \right) \Bigg].
\end{aligned} \tag{12}
$$

It is not envisageable to solve the above equation analytically for the mass. However, if we limit ourselves to the neighborhood of criticality, then we can put $m^2(L) \approx 0$, and we may also use an asymptotic formula for a Bessel function with a small argument, $K_\nu(z) \approx \frac{1}{2}\Gamma(\nu)(2/z)^\nu$ ($z \sim 0$). In this way, the coefficients and arguments of the Bessel functions cancel out and we rewrite (12) as

$$
\begin{aligned}
m^2(L) \approx m_0^2 + \frac{3\lambda}{\pi^{D/2}} \Gamma\left(\frac{D}{2}-1\right) \Bigg[& \frac{d}{2} E_1\left(\frac{D}{2}-1; L\right) \\
& + d(d-1) E_2\left(\frac{D}{2}-1; L\right) + \cdots + 2^{d-2} E_d\left(\frac{D}{2}-1; L\right) \Bigg],
\end{aligned} \tag{13}
$$

where the $E_p(\nu; L)$ are generalized Epstein–Hurwitz zeta functions defined by Kirsten (1994)

$$
E_p(\nu; L) = L^\nu \sum_{n_1=1}^{\infty} \cdots \sum_{n_p=1}^{\infty} \left(n_1^2 + \cdots + n_p^2 \right)^{-\nu}, \tag{14}
$$

[for details, see Malbouisson et al. (2002)]. Notice that, for $p = 1$, E_p reduces to the Riemann zeta function $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$.

Having developed the general case of a d-dimensional compactified subspace, we consider an illustrative example. We choose $d = 1$, the compactification of just one dimension, along the

x_1-axis, say, meaning that we are considering that the system is confined between two planes, separated by a distance L (film of thickness L). Then, Eq. (13) simplifies to

$$m^2(L) \approx m_0^2(L) + \frac{3\lambda}{2\pi^{D/2}L^{D-2}}\Gamma\left(\frac{D}{2}-1\right)\zeta\left(D-2\right), \tag{15}$$

where $\zeta(z)$ is the Riemann zeta function. This equation is well defined for $D > 3$, but not for $D = 3$, due to the pole of the zeta function. However, we can assign it a meaning for the significative dimension $D = 3$ by adopting a regularization procedure: we use the well-known formula

$$\lim_{z\to 1}\zeta(z) = \frac{1}{z-1} + \gamma, \tag{16}$$

where $\gamma \approx 0.5772$ is the Euler–Mascheroni constant, for $\zeta\left(D-2\right)$ in Eq. (15) and afterwards we suppress the pole term at $D = 3$ ($z = 1$). Then, remembering that $m_0^2 = \alpha(T - T_0)$, we get the L-dependent critical temperature,

$$\begin{aligned} T_c^{\mathrm{film}}(L) &= T_0 - C_1\frac{\lambda}{\alpha L}, \\ \text{with } C_1 &= \frac{3\gamma}{2\pi}. \end{aligned} \tag{17}$$

We see that, for $L < (3\gamma/2\pi)(\lambda/\alpha T_0)$, the critical temperature becomes negative, meaning that the transition does not occur.

With analogous steps, we can take the cases of $d = 2$ and $d = 3$, in which the system is confined within an infinite wire of rectangular cross section $L^2 \equiv A$ and a grain of volume $L^3 \equiv V$, respectively. In those cases, it is not necessary to renormalized the bare mass, as we have done for a film, as the divergences coming from the zeta and gamma functions completely cancel out algebraically. One obtains [Abreu et al. (2005)]

$$\begin{aligned} T_c^{\mathrm{wire}}(A) &= T_0 - C_2\frac{\lambda}{\alpha A^{1/2}}, \\ T_c^{\mathrm{grain}}(V) &= T_0 - C_3\frac{\lambda}{\alpha V^{1/3}}, \end{aligned} \tag{18}$$

where C_2 and C_3 are numerical constants. We note that, in all cases, it is found that the boundary-dependent critical temperature decreases linearly with the inverse of the linear dimension L, $T_c(L) = T_0 - C_d\lambda/\alpha L$, where α and λ are the Ginzburg–Landau parameters, T_0 is the bulk transition temperature and C_d is a constant depending on the number of compactified dimensions. This is in accordance with arguments raised from finite-size scaling [Zinn-Justin (2002)].

Such behavior suggests the existence of a minimal size of the system, below which the transition is suppressed. It seems to be in qualitative agreement with experimental results which indicate a minimal thickness of a film for the disapearance of superconductivity [Abreu et al. (2004); Kodama et al. (1983)]; also, the behavior of nanowires and nanograins have been studied [Shanenko et al. (2006); Zgirski et al. (2005)], searching for a limit on its size for the material while retaining its superconducting character.

3. First-order phase transitions

In the previous section, we have studied the Ginzburg–Landau Hamiltonian density, solely containing the interaction term $\lambda\varphi^4$, with $\lambda > 0$, which describes second-order phase transitions. Here we pass to consider the Ginzburg–Landau model in a Euclidean D-dimensional space, including both φ^4 and φ^6 interactions, in the absence of external fields; its Hamiltonian is given by (again, in natural units, $\hbar = c = k_B = 1$)

$$\mathcal{H} = \frac{1}{2}\left|\partial_\mu\varphi\right|\left|\partial^\mu\varphi\right| + \frac{1}{2}m_0^2\left|\varphi\right|^2 - \frac{\lambda}{4}\left|\varphi\right|^4 + \frac{\eta}{6}\left|\varphi\right|^6, \tag{19}$$

where $\lambda > 0$ and $\eta > 0$ are the physical quartic and sextic coupling constants. Near criticality, the bare mass is given by $m_0^2 = \alpha(T/T_0 - 1)$, with $\alpha > 0$ and T_0 being a parameter with the dimension of temperature. A potential of this type, with the minus sign in the quartic term, ensures that the system undergoes a first-order transition. Recall that the critical temperature for a first-order transition described by the Hamiltonian above is higher than T_0. This will be explicitly stated in Eq. (25) below. Our purpose will be to develop the general case of compactifying a d-dimensional subspace, in order to compare results for films, wires and grains with the second-order ones given above.

We thus consider the system in D dimensions confined to a region of space delimited by $d \leq D$ pairs of parallel planes, as was done in the previous section, and introduce a generalized Matsubara prescription as in Eq. (3), with periodic boundary conditions. We again start from establishing the effective potential, related to the physical mass through a renormalization condition, Eq. (6). This condition, however, reduces considerably the number of relevant Feynman diagrams contributing to the mass, if we restrict ourselves to first-order terms in both coupling constants: in fact, just two diagrams need to be considered in this approximation, a tadpole graph with the φ^4 coupling (1 loop) and a "shoestring" graph with the φ^6 coupling (2 loops).

Within our approximation, we do not take into account the renormalization conditions for the interaction coupling constants, i.e., they are considered as already renormalized when they are written in the Hamiltonian (the same was assumed in the previous section).

At the 1-loop approximation, the contribution of loops with only $|\varphi|^4$ vertices to the effective potential is obtained directly from the previous section, Eq. (5). As before, we see that only the $s = 1$ term contributes to the renormalization condition in Eq. (6). It corresponds to the tadpole diagram. It is then also clear that all $|\varphi_0|^6$-vertex and mixed $|\varphi_0|^4$- and $|\varphi_0|^6$-vertex insertions on the 1-loop diagrams do not contribute when one computes the second derivative of similar expressions with respect to the field at zero field: only diagrams with two external legs should survive. This is impossible for a $|\varphi_0|^6$-vertex insertion at the 1-loop approximation. Therefore, the first contribution from the $|\varphi_0|^6$ coupling must come from a higher-order term in the loop expansion. Two-loop diagrams with two external legs and only $|\varphi_0|^4$ vertices are of second order in its coupling constant, and we neglect them, as well as all possible diagrams with vertices of mixed type. However, the 2-loop shoestring diagram, with only one $|\varphi_0|^6$ vertex and two external legs is a first-order (in η) contribution to the effective potential, according to our approximation.

The tadpole contribution to the effective potential was treated in the previous section, through dimensional and Epstein–Hurwitz zeta-function regularizations and subtraction of a polar term, resulting in the expression $U_{1,R}$ of Eq. (11), in terms of modified Bessel functions. Now, proceeding analogously for the 2-loop shoestring diagram contribution, we arrive at

$$U_{2,R}(\varphi_0, L_1, \ldots, L_d) = \frac{\eta|\varphi_0|^2}{4(2\pi)^D}\left[d\sum_{n=1}^{\infty}\left(\frac{m}{Ln}\right)^{D/2-1}K_{D/2-1}(mLn) + \cdots \right.$$
$$+2^{d-1}\sum_{n_1,\ldots,n_d=1}^{\infty}\left(\frac{m}{L\sqrt{n_1^2+\cdots+n_d^2}}\right)^{D/2-1}$$
$$\left.\times K_{D/2-1}\left(mL\sqrt{n_1^2+\cdots+n_d^2}\right)\right]^2. \tag{20}$$

Then the physical mass $m^2(L)$ with both contributions is obtained from Eq. (6), using Eqs. (11), (20) and also taking into account the contribution at the tree level; it satisfies a generalized Dyson–Schwinger equation depending on the extensions L of each dimension of the confining box, as in Eq. (12). We should remember that the tadpole part has a change of sign with respect to (12), reflecting the sign of λ in the Hamiltonian (19).

A first-order transition occurs when all the three minima of the potential

$$U(\varphi_0) = \frac{1}{2}m^2(L)|\varphi_0|^2 - \frac{\lambda}{4}|\varphi_0|^4 + \frac{\eta}{6}|\varphi_0|^6, \tag{21}$$

where $m(L)$ is the renormalized mass defined above, are simultaneously on the line $U(\varphi_0) = 0$. This gives the condition

$$m^2(L) = \frac{3\lambda^2}{16\eta}. \tag{22}$$

For $D = 3$, the Bessel functions have an explicit form, $K_{1/2}(z) = \sqrt{\pi}e^{-z}/\sqrt{2z}$, which is to be replaced in the expression for the renormalized mass. Performing the resulting sums, and remembering that $m_0^2 = \alpha(T/T_0 - 1)$, we get

$$m^2(L) = \alpha\left(\frac{T}{T_0} - 1\right) + \frac{3\lambda}{4\pi}\left[\frac{d}{L}\ln\left(1 - e^{-m(L)L}\right) + \cdots \right.$$
$$\left. +2^{d-1}\sum_{n_1,\ldots,n_d=1}^{\infty}\frac{e^{-m(L)L\sqrt{n_1^2+\cdots+n_d^2}}}{\sqrt{n_1^2+\cdots+n_d^2}}\right]$$
$$+\frac{\eta\pi}{8\,(2\pi)^3}\left[\frac{d}{L}\ln\left(1 - e^{-m(L)L}\right) + \cdots \right.$$
$$\left. +2^{d-1}\sum_{n_1,\ldots,n_d=1}^{\infty}\frac{e^{-m(L)L\sqrt{n_1^2+\cdots+n_d^2}}}{L\sqrt{n_1^2+\cdots+n_d^2}}\right]^2. \tag{23}$$

Then, introducing the value of the mass, Eq. (22), in Eq. (23), one obtains the critical temperature

$$T_c(L) = T_c \left\{ 1 - \left(1 + \frac{3\lambda^2}{16\eta\alpha}\right)^{-1} \left\{ \frac{3\lambda}{4\pi\alpha} \left[\frac{d}{L} \ln\left(1 - e^{-L\sqrt{\frac{3\lambda^2}{16\eta}}}\right) + \cdots \right. \right. \right.$$
$$\left. + 2^{d-1} \sum_{n_1,\ldots,n_d=1}^{\infty} \frac{e^{-L\sqrt{\frac{3\lambda^2}{16\eta}}\sqrt{n_1^2+\cdots+n_d^2}}}{L\sqrt{n_1^2+\cdots+n_d^2}} \right]$$
$$- \frac{\eta}{64\pi^2\alpha} \left[\frac{d}{L} \ln\left(1 - e^{-L\sqrt{\frac{3\lambda^2}{16\eta}}}\right) + \cdots \right.$$
$$\left. \left. \left. + 2^{d-1} \sum_{n_1,\ldots,n_d=1}^{\infty} \frac{e^{-L\sqrt{\frac{3\lambda^2}{16\eta}}\sqrt{n_1^2+\cdots+n_d^2}}}{L\sqrt{n_1^2+\cdots+n_d^2}} \right]^2 \right\} \right\}, \tag{24}$$

where

$$T_c = T_0 \left(1 + \frac{3\lambda}{16\eta\alpha}\right) \tag{25}$$

is the bulk ($L \to \infty$) critical temperature for the first-order phase transition.

Specific formulas for particular values of d are now given. If we choose $d = 1$, this corresponds physically to a film of superconducting material, and we have that the transition occurs at the critical temperature $T_c^{\text{film}}(L)$ given by

$$T_c^{\text{film}}(L) = T_c \left\{ 1 - \left(1 + \frac{3\lambda^2}{16\eta\alpha}\right)^{-1} \left[\frac{3\lambda}{4\pi\alpha L} \ln\left(1 - e^{-L\sqrt{\frac{3\lambda^2}{16\eta}}}\right) \right. \right.$$
$$\left. \left. - \frac{\eta}{64\pi^2\alpha L^2} \left(\ln(1 - e^{-L\sqrt{\frac{3\lambda^2}{16\eta}}}) \right)^2 \right] \right\}. \tag{26}$$

In the case of a wire, $d = 2$, the critical temperature is written in terms of L as

$$T_c^{\text{wire}}(L) = T_c \left\{ 1 - \left(1 + \frac{3\lambda^2}{16\eta\alpha}\right)^{-1} \right.$$
$$\times \left[\frac{3\lambda}{2\pi\alpha L} \left[\ln\left(1 - e^{-L\sqrt{\frac{3\lambda^2}{16\eta}}}\right) + \ln\left(1 - e^{-L\sqrt{\frac{3\lambda^2}{16\eta}}}\right) \right. \right.$$
$$\left. + 2 \sum_{n_1,n_2=1}^{\infty} \frac{e^{-L\sqrt{\frac{3\lambda^2}{16\eta}}\sqrt{n_1^2+n_2^2}}}{\sqrt{n_1^2+n_2^2}} \right]$$
$$- \frac{\eta}{32\pi^2\alpha L^2} \left(\ln\left(1 - e^{-L\sqrt{\frac{3\lambda^2}{16\eta}}}\right) + \ln\left(1 - e^{-L\sqrt{\frac{3\lambda^2}{16\eta}}}\right) \right.$$
$$\left. \left. \left. + 2 \sum_{n_1,n_2=1}^{\infty} \frac{e^{-L\sqrt{\frac{3\lambda^2}{16\eta}}\sqrt{n_1^2+n_2^2}}}{\sqrt{n_1^2+n_2^2}} \right)^2 \right] \right\}. \tag{27}$$

Finally, if we compactify all three dimensions ($d = 3$), which leaves us with a system in the form of a cubic "grain" of some material, the dependence of the critical temperature on its linear dimension L is given by

$$\begin{aligned} T_c^{\mathrm{grain}}(L) = T_c \Bigg\{ 1 - \left(1 + \frac{3\lambda^2}{16\eta\alpha}\right)^{-1} \Bigg\{ \frac{3\lambda}{2\pi\alpha L} \\ \times \Bigg[3\ln(1 - e^{-L\sqrt{\frac{3\lambda^2}{16\eta}}}) + \cdots \\ + 4 \sum_{n_1,\ldots,n_3=1}^{\infty} \frac{e^{-L\sqrt{\frac{3\lambda^2}{16\eta}}\sqrt{n_1^2+n_2^2+n_3^2}}}{\sqrt{n_1^2+n_2^2+n_3^2}} \Bigg] \\ - \frac{\eta}{32\pi^2\alpha L^2} \Bigg[3\ln(1 - e^{-L\sqrt{\frac{3\lambda^2}{16\eta}}}) + \cdots \\ + 4 \sum_{n_1,\ldots,n_3=1}^{\infty} \frac{e^{-L\sqrt{\frac{3\lambda^2}{16\eta}}\sqrt{n_1^2+n_2^2+n_3^2}}}{\sqrt{n_1^2+n_2^2+n_3^2}} \Bigg]^2 \Bigg\} \Bigg\}. \end{aligned} \tag{28}$$

Comparing Eqs. (26)-(28) with the general behavior of the critical temperature obtained in the previous section, we see that in all cases (film, wire or grain), there is a sharp contrast between the simple inverse linear behavior of $T_c(L)$ for second-order transitions and the rather involved dependence on L of the critical temperature for first-order transitions.

In Linhares et al. (2006; 2007), we have shown that our general formalism could be not of a purely academic interest, but that it could be used to describe some experimentally observable situations. Experimental data on the critical temperature obtained from superconducting films and wires can be compared with our theoretical expressions. In Linhares et al. (2006), the coupling constants λ and η have been determined as functions of the microscopic parameters of the material, which was done generalizing Gorkov's [Kleinert (1989)] microscopic derivation done for the $\lambda\varphi^4$ model, in order to include the additional interaction term $\eta\varphi^6$ in the free energy. See Linhares et al. (2006; 2007) for details.

As described in Linhares et al. (2006), the transition temperature as a function of the thickness for a film grows from zero at a nonnull minimal allowed film thickness above the bulk transition temperature T_c as the thickness is enlarged, reaching a maximum and afterwards starting to decrease, going asymptotically to T_c as $L \to \infty$. Our theoretical curve is in qualitatively good agreement with measurements, especially for thin films [Strongin et al. (1970)]. This is illustrated in Figure 1. This behavior can be contrasted with the one shown by the critical temperature for a second-order transition. As one can see in Figure 2, in this case, the critical temperature increases monotonically from zero, again corresponding to a finite minimal film thickness, going asymptotically to the bulk transition temperature as $L \to \infty$ [Abreu et al. (2004)]. Such behavior has been experimentally found for a variety of transition-metal materials [Kodama et al. (1983); Minhaj et al. (1994); Pogrebnyakov et al. (2003); Raffy et al. (1983)]. Since in this section a first-order transition is explicitly assumed, it is tempting to infer that the transition described in the experiments of Strongin et al. (1970) is first order. In other words, one could say that an experimentally observed behavior of the critical temperature as a function of the film thickness may serve as a possible criterion

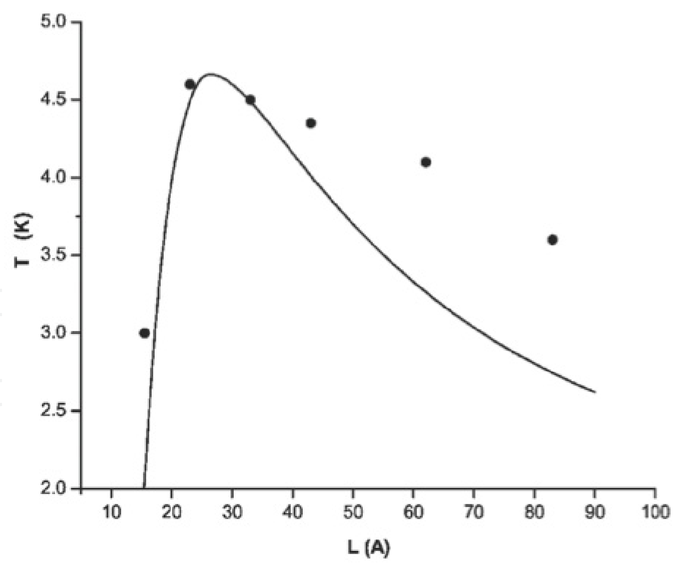


Fig. 1. Critical temperature $T_c^{\mathrm{film}}(K)$ as a function of the thickness $L(\mathrm{\AA})$, with data from Strongin et al. (1970) for a superconducting film made from aluminum.

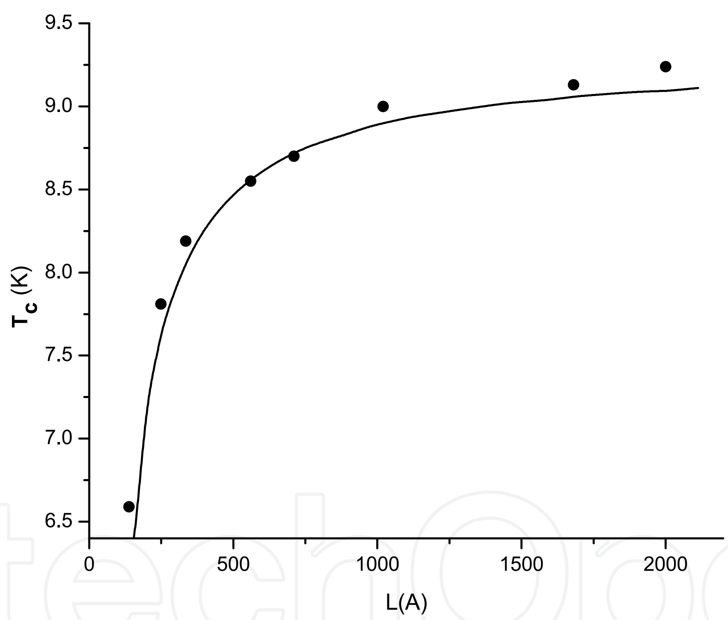


Fig. 2. Critical temperature $T_c^{\mathrm{film}}(K)$ as a function of the thickness $L(\mathrm{\AA})$ for a second-order transition, with data from Kodama et al. (1983) for a superconducting film made from niobium.

to decide about the order of the superconductivity transition: a monotonically increasing critical temperature as L grows would indicate that the system undergoes a second-order transition, whereas if the critical temperature presents a maximum for a value of L larger than the minimal allowed one, this would be signaling the occurrence of a first-order transition. If we consider a sample of superconducting material in the form of an infinitely long wire with a cross section L^2, the same arguments and rescaling procedures used for films apply. In this case, the theoretical curve T_c^{wire} vs. L, together with Al data from Shanenko et al. (2006); Zgirski et al. (2005) agree quite well, for not extremely thin wires. One may conclude that the phase transition of these superconducting aluminum wires is first order, just as for aluminum films. The interested reader will find details in Linhares et al. (2006; 2007).

4. Coupling-constant corrections for second-order transitions

We have so far discussed the critical properties of confined superconducting matter under the assumption that the coupling constants, as they appear in the Hamiltonian, are the physical ones. It is however expected that the compactification of spatial dimensions as we have described also has an influence on the coupling constants and consequently on the behavior of the transition temperature with respect to the size of the compactified space. To undertake such study, we shall consider the four-point function at zero external momenta, which is the basic object for our definition of the renormalized coupling constant. We shall analyze it in the O(N)-symmetric version of the D-dimensional Ginzburg–Landau model, described by the Hamiltonian density

$$\mathcal{H} = \partial_\mu \varphi_a \partial^\mu \varphi_a + m_0^2(T)\varphi_a\varphi_a + \frac{\lambda}{N}\left(\varphi_a\varphi_a\right)^2, \tag{29}$$

and take the large-N limit. In Eq. (29), λ is the coupling constant and $m_0^2(T) = \alpha(T - T_0)$ is the bare mass, as before. The compactification procedure is the same as that implemented in section 2 and we look for the 1-loop contribution from φ^4 vertices for the effective potential after compactification of d dimensions. We may use directly Eq. (10), taking care that the convention for the coupling constant has changed: $\lambda/4 \to \lambda$. The mass is obtained from the normalization condition (6) and the coupling constant from

$$\left.\frac{\partial^4}{\partial\varphi_0^2}U(\varphi_0)\right|_{\varphi_0=0} = \frac{\lambda}{N}, \tag{30}$$

where U is the sum of the tree-level and 1-loop contributions to the effective potential.

The coupling constant is defined in terms of the 4-point function for zero external momenta, which, at leading order in $1/N$, is given by the sum of all chains of 1-loop diagrams, which has the formal expression

$$\Gamma_D^{(4)}(p=0, m, L) = \frac{\lambda/N}{1+\lambda\Pi(m,L)}, \tag{31}$$

where $\Pi(m,L) \equiv \Pi(p = 0, m, L)$ corresponds to the one-loop four-point diagram, after compactification. Next, we use the renormalization condition (30), from which we deduce formally that the one-loop four-point function $\Pi(m,L)$ is obtained from the coefficient of the fourth power of the field ($s = 2$) in Eq. (10). A divergent (for even dimensions) term is subtracted to give the finite one-loop four-point function $\Pi_R(m,L)$, which corresponds to (11). Such subtraction is performed even in the case of odd dimensions, where no pole singularity occurs (finite renormalization). From the properties of Bessel functions, we see that $\Pi_R(m,L) \to 0$ as $L \to \infty$, whereas it diverges when $L \to 0$. We conclude that the renormalized one-loop four-point function is positive for all values of D and L.

Let us define the the L-dependent renormalized coupling constant $\lambda_R(m, L)$, at leading order in $1/N$, as

$$N\,\Gamma^{(4)}_{D,R}(p=0,m,L) \equiv \lambda_R(m,L) = \frac{\lambda}{1+\lambda\Pi_R(m,L)}. \tag{32}$$

In the absence of constraints, the $L \to \infty$ limit of $\Gamma^{(4)}_{D,R}(p=0,m,L)$ defines the corresponding renormalized coupling constant $\lambda_R(m)$. We get simply that $\lambda_R(m) = \lambda$. This means that a renormalization scheme has been chosen so that the constant λ appearing in the Hamiltonian corresponds to the renormalized coupling constant in the absence of boundaries.

The physical mass is obtained at 1-loop from (12), with $\lambda/4 \to \lambda$, and (6), after also changing $\lambda \to \lambda_R(m, L)$, given by (32). One should remember, however, that $\lambda_R(m, L)$ is itself a function of $m = m(T, L)$. Therefore, $m(T, L)$ is given by a complicated set of coupled equations. Just like in the situation in section 2, without the corrections in λ, it has no analytical solution in general. Nevertheless, as before, if we limit ourselves to the neighborhood of criticality, $m^2(T, L) \approx 0$, the behavior of the system can be studied by using the approximation $K_\nu(z) \approx \frac{1}{2}\Gamma(\nu)(2/z)^\nu$, for $z \sim 0$. The same kind of simplifications occurs and we regain Eq. (13), with $\lambda \to \lambda_R(D, L)$ given by

$$\begin{aligned}\lambda_R(D,L) \approx \lambda \Big\{1+\lambda C(D)L^{4-D}\,\big[d\zeta(D-4)+2d(d-1)E_2\left(D/2-2,1\right)\\ +\cdots+2^{d-1}E_d\left(D/2-2,1\right)\big]\Big\}^{-1},\end{aligned} \tag{33}$$

where $C(D) = \frac{1}{8\pi^{D/2}}\Gamma\left(\frac{D}{2}-2\right)$. It then ensues that we obtain the critical temperature as a function of L. Taking $D = 3$, we have a similar situation as that of section 2. We find modified

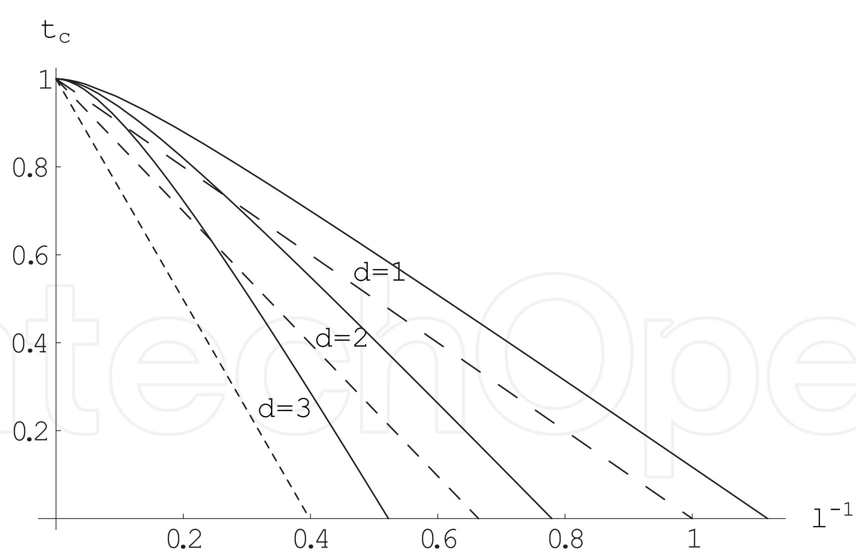


Fig. 3. Reduced transition temperature (t_c) as a function of the inverse of the reduced compactification length (l), for films ($d = 1$), square wires ($d = 2$) and cubic grains ($d = 3$). The full and dashed lines correspond to results with and without correction of the coupling constant, respectively.

L-dependent transition temperatures, which are given by

$$T_c^{\text{film}}(L) = T_0 - \frac{48\pi C_1 \lambda}{48\pi\alpha L + \lambda\alpha L^2};$$
$$T_c^{\text{wire}}(A) = T_0 - \frac{48\pi C_2 \lambda}{48\pi\alpha\sqrt{A} + \mathcal{E}_2\lambda\alpha A}; \tag{34}$$
$$T_c^{\text{grain}}(L) = T_0 - \frac{48\pi C_1 \lambda}{48\pi\alpha V^{1/3} + \mathcal{E}_3\lambda\alpha V^{2/3}},$$

with C_1, C_2 and C_3 as before and where $\mathcal{E}_2$ and $\mathcal{E}_3$ are constants, resulting from sums involving the Bessel functions [Malbouisson et al. (2009)]. We see that the critical temperature has the same kind of dependence on the size extension L for $d = 1, 2, 3$, only constants differ in each case. The functional behavior does not depend on the number of compactified dimensions, only on the dimension of the Euclidean space, which we have computed for $D = 3$. One can also notice that the minimal size of the compact superconductor has lesser values than those computed without taking into account corrections to the coupling constant. This can be seen in Figure 3, where we have plotted the reduced transition temperature $t_c = T_c/T_0$ as a function of the inverse of the reduced compactification length $l = L/L_{\text{min}}$, where L_{min} is the corresponding minimal allowed linear extension without coupling constant boundary corrections.

5. Infrared fixed-point structure for the $\lambda\varphi^4$ model

5.1 The system in the absence of an external magnetic field

In this subsection, we study the fixed-point structure of the compactified model described by the Hamiltonian density in Eq. (29) in the large-N limit. We start from the four-point function at the critical point ($m = 0$) and for small external momenta, before compactification, which is given by

$$\Gamma_{\text{cr}}^{(4)}(p) = \frac{\lambda/N}{1 + \lambda\Pi_{\text{cr}}(p)}\,. \tag{35}$$

In the equation above, $\Pi_{\text{cr}}(p)$ is the one-loop four-point function at the critical point; introducing a Feynman parameter x, it is written in the form

$$\Pi_{\text{cr}}(p) = \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left[k^2 + p^2 x(1-x)\right]^2}. \tag{36}$$

Performing the Matsubara replacements (3) for d dimensions, Eq. (36) becomes ($\omega_i = 2\pi n_i/L$)

$$\Pi_{\text{cr}}(p, L) = \frac{1}{L^d} \sum_{n_1,\ldots,n_d=-\infty}^{\infty} \int_0^1 dx \int \frac{d^{D-d}q}{(2\pi)^{D-d}} \times \frac{1}{\left[q^2 + \omega_{n_1}^2 + \cdots + \omega_{n_d}^2 + p^2 x(1-x)\right]^2}, \tag{37}$$

and we define the effective L-dependent coupling constant in the large-N limit as

$$\lambda(p, L) \equiv \lim_{N\to\infty} N\Gamma_D^{(4)}(p, L) = \frac{\lambda}{1 + \lambda\Pi(p, L)}. \tag{38}$$

The sum over the n_i and the integral over q above can be treated using the formalism developed in Khanna et al. (2009) and described in section 2. We obtain

$$\Pi_{\text{cr}}(p,L) = (2\pi)^{-D/2}\int_0^1 dx \left[2^{-D/2}\left(\frac{1}{(2\pi)^2}p^2x(1-x)\right)^{D/2-2}\Gamma\left(2-\frac{D}{2}\right)\right.$$
$$+d\sum_{n=1}^{\infty}\left(\frac{\sqrt{p^2x(1-x)}}{2\pi Ln}\right)^{D/2-2} K_{D/2-2}\left(\frac{Ln}{2\pi}\sqrt{p^2x(1-x)}\right)+\cdots$$
$$+2^{d-1}\sum_{n_1,\ldots,n_d=1}^{\infty}\left(\frac{\sqrt{p^2x(1-x)}}{2\pi L\sqrt{n_1^2+\cdots+n_d^2}}\right)^{D/2-2}$$
$$\left.\times K_{D/2-2}\left(\frac{L}{2\pi}\sqrt{p^2x(1-x)}\sqrt{n_1^2+\cdots+n_d^2}\right)\right], \tag{39}$$

which, replaced in Eq. (38), gives the boundary-dependent four-point function in the large-N limit. We can write Eq. (39) in the form

$$\Pi(p,L) = A(D)|p|^{D-4} + B_d(D,L), \tag{40}$$

with the d-independent coefficient of the $|p|$-term being

$$A(D) = (2\pi)^{4-3D/2}\,2^{-D/2}b(D)\Gamma\left(2-\frac{D}{2}\right), \tag{41}$$

and where we have defined

$$b(D) = \int_0^1 dx\,[x(1-x)]^{D/2-2} = 2^{3-D}\sqrt{\pi}\frac{\Gamma\left(\frac{D}{2}-1\right)}{\Gamma\left(\frac{D-1}{2}\right)}, \qquad \text{for } \mathrm{Re}(D) > 2. \tag{42}$$

We remark that, for the physically interesting dimension $D = 3$, $b(3) = \pi$. This implies that $A(3) = \pi/4$.

If an infrared-stable fixed point exists for any of the models with d compactified dimensions, it is possible to determine it by a study of the infrared behavior of the Callan–Symanzik β function. Therefore, we investigate the above equations for $|p| \approx 0$. With this restriction, we may use the asymptotic formula for small values of the argument of the Bessel functions, and the expressions for B_d simplify considerably [see the reasoning leading to Eq. (13)]. The result is expressed in terms of one of the multidimensional Epstein–Hurwitz zeta functions of Eq. (14). In this limit, the p^2-dependence of the Bessel functions exactly compensates the one coming from the accompanying factors. Thus, the remaining p^2-dependence is only that of the first term of (39), which is the same for all number of compactified dimensions d. For general and detailed expressions, see Linhares et al. (2011). One can also construct analytical continuations and recurrence relations for the multidimensional Epstein functions, which permit to write them in terms of modified Bessel and Riemann zeta functions [Khanna et al. (2009); Kirsten (1994)]. We thus are able to derive expressions for each particular value of

d, from 1 to D, in the $|p| \approx 0$ limit, but let us restrict ourselves to the most expressive values, corresponding to materials in the form of a film, a wire, or a grain.

Therefore, for a film, we obtain

$$B_{d=1}(D, L) \sim (2\pi)^{-D/2} 2^{D/2-3} L^{4-D} \Gamma\left(\frac{D}{2} - 2\right) \zeta(D-4). \tag{43}$$

The above expression is valid for all *odd* dimensions $D > 5$, due to the poles of the Γ and ζ functions. We can obtain an expression for smaller values of D by performing an analytic continuation of the Riemann zeta function $\zeta(D-4)$ by means of its reflection property,

$$\zeta(z) = \frac{1}{\Gamma(z/2)} \Gamma\left(\frac{1-z}{2}\right) \pi^{z-1/2} \zeta(1-z). \tag{44}$$

Then Eq. (43) leads to an expression valid for $2 < D < 4$ given by

$$B_{d=1}(D, L) = 2^{-3} \pi^{(D-9)/2} L^{4-D} \Gamma\left(\frac{5-D}{2}\right) \zeta(5-D). \tag{45}$$

For $D = 3$, we have $B_{d=1}(3, L) = L/48\pi$. For $d = 2$ and $d = 3$, similar expressions are obtained. An analysis of the singularity structure of the quantities B_d shows that their domain of existence can be extended to $2 < D < 4$ [Linhares et al. (2011)].

To discuss infrared properties of these compactified models, we insert Eq. (40) in Eq. (38) and we get the (p, L)-dependent coupling constant

$$\lambda(|p| \approx 0, D, L) \approx \frac{\lambda}{1 + \lambda\left[A(D)|p|^{D-4} + B_d(D, L)\right]}. \tag{46}$$

Let us take $|p|$ as a running scale, and define the dimensionless coupling constant

$$g = \lambda(p, D, L)|p|^{D-4}. \tag{47}$$

We recall that in these expressions p is a D-dimensional vector. The Callan-Symanzik β function controls the rate of the renormalization-group flow of the running coupling constant and a (nontrivial) fixed point of this flow is given by a (nontrivial) zero of the β function. For $|p| \approx 0$, it is obtained straightforwardly from Eq. (47),

$$\beta(g) = |p| \frac{\partial g}{\partial |p|} \approx (D-4)\left[g - A(D) g^2\right], \tag{48}$$

from which we get the infrared-stable fixed point

$$g_*(D) = \frac{1}{A(D)}. \tag{49}$$

We see that the L-dependent B_d-part of the subdiagram Π_{cr} does not play any role in this expression and, as remarked before, $A(D)$ is the same for all number of compactified dimensions, so is g_* only dependent on the space dimension.

5.2 The system with an external magnetic field

We now take the N-component Ginzburg–Landau model of the previous subsection to describe the behavior of d-confined systems, now in the presence of an external magnetic field, at leading order in $1/N$. The Hamiltonian density (29) is then modified to

$$\mathcal{H} = \left[\left(\partial_\mu - ieA_\mu^{\text{ext}}\right)\varphi_a\right]\left[\left(\partial^\mu - ieA^{\text{ext},\mu}\right)\varphi_a\right] + m^2\varphi_a\varphi_a + \frac{\lambda}{N}\left(\varphi_a\varphi_a\right)^2, \tag{50}$$

where $m^2 = \alpha(T - T_c)$, with $\alpha > 0$. For $D = 3$, from a physical point of view, such Hamiltonian is supposed to describe type-II superconductors. In this case, we assume that the external magnetic field H is parallel to the z-axis and we choose the gauge $A^{\text{ext}} = (0, xH, 0)$. In the present D-dimensional case, we assume analogously a gauge $A^{\text{ext}} = (0, x_1 H, 0, 0, \ldots, 0)$, with $\{x_i\} = x_1, x_2, \ldots, x_D$, meaning that the applied external magnetic field lies on a fixed direction along one of the coordinate axis; for simplicity, in the calculations that follow, we have adopted the notation $x_1 \equiv x$, $x_2 \equiv y$. If we consider the system in unlimited space, the field φ should be written in terms of the well-known Landau-level basis,

$$\varphi(r) = \sum_{\ell=0}^{\infty}\int\frac{dp_y}{2\pi}\int\frac{d^{D-2}p}{(2\pi)^{D-2}}\tilde{\varphi}_{\ell,p_y,p}\chi_{\ell,p_y,p}(r), \tag{51}$$

where $\chi_{\ell,p_y,p}(r)$ are the Landau-level eigenfunctions given in terms of Hermite polynomials H_ℓ by

$$\chi_{\ell,p_y,p}(r) = \frac{1}{\sqrt{2^\ell \ell!}}\left(\frac{\omega}{\pi}\right)^{1/4} e^{i\left(p\cdot r + p_y y\right)} e^{-\omega(x - p_y/\omega)^2/2} H_\ell\left(\sqrt{\omega}x - \frac{p_y}{\sqrt{\omega}}\right), \tag{52}$$

with energy eigenvalues $E_\ell\left(|p|\right) = |p|^2 + (2\ell+1)\,\omega + m^2$ and $\omega = eH$ is the so-called cyclotron frequency. In the above equation, p and r are $(D-2)$-dimensional vectors.

In the following, we consider only the lowest Landau level $\ell = 0$. For $D = 3$, this assumption usually corresponds to the description of superconductors in the extreme type-II limit. Under this assumption, we obtain that the effective $|\varphi|^4$ interaction in momentum space and at the critical point ($m = 0$) is written as

$$\lambda(p, L; \omega) = \frac{\lambda}{1 + \lambda\omega e^{-(1/2\omega)(p_1^2 + p_2^2)}\Pi(p, L; \omega)}, \tag{53}$$

where the single 1-loop four-point function, $\Pi(p, L; \omega)$, is given by

$$\Pi(p, L; \omega) = \frac{1}{L^d}\sum_{i=1}^{d}\sum_{n_i=-\infty}^{\infty}\int_0^1 dx\int\frac{d^{D-d-2}q}{(2\pi)^{D-d-2}} \times \left[q^2 + \omega_{n_1}^2 + \cdots + \omega_{n_d}^2 + p^2 x(1-x)\right]^{-2}. \tag{54}$$

This is the same kind of expression that is encountered in the previous subsection, Eq. (37), with the only modification that $D \to D-2$. The analysis is then performed along the same lines and we obtain, analogously,

$$\Pi(p,L;\omega) = (2\pi)^{1-D/2}\left[2^{1-D/2}\frac{1}{(2\pi)^2}c(D)\Gamma\left(3-\frac{D}{2}\right)\left(p^2\right)^{D/2-3}\right.$$
$$+\int_0^1 dx\, d\sum_{n=1}^{\infty}\left(\frac{\sqrt{p^2x(1-x)}}{2\pi Ln}\right)^{D/2-3}K_{D/2-3}\left(\frac{Ln}{2\pi}\sqrt{p^2x(1-x)}\right)$$
$$+\cdots+2^{d-1}\int_0^1 dx\sum_{n_1,\ldots,n_d=1}^{\infty}\left(\frac{\sqrt{p^2x(1-x)}}{2\pi L\sqrt{n_1^2+\cdots+n_d^2}}\right)^{D/2-3}$$
$$\left.\times K_{D/2-3}\left(\frac{1}{2\pi}\sqrt{p^2x(1-x)}\sqrt{n_1^2+\cdots+n_d^2}\right)\right], \tag{55}$$

where

$$c(D) = \int_0^1 dx\,(x(1-x))^{D/2-3} = 2^{5-D}\sqrt{\pi}\frac{\Gamma\left(\frac{D}{2}-2\right)}{\Gamma\left(\frac{D-3}{2}\right)}, \qquad \text{for } \mathrm{Re}(D) > 4. \tag{56}$$

As for the infrared behavior of the β function, it suffices to study it in the neighborhood of $|p| = 0$, so that we can again use the asymptotic formula for Bessel functions for small values of the argument, as before. It turns out that in the $|p| \approx 0$ limit, the bubble Π_{cr} is written in the form

$$\Pi_{\mathrm{cr}}(|p| \approx 0, L;\omega) = A_1(D)\,|p|^{D-6} + C_d(D,L), \tag{57}$$

with

$$A_1(D) = (2\pi)^{-D/2-1}\,2^{1-D/2}c(D)\Gamma\left(3-\frac{D}{2}\right), \tag{58}$$

and where the quantity $C_d(D,L)$ is obtained by simply making the change $D \to D-2$ in the formula for $B_d(D,L)$ in the preceding subsection.

Let us remind Eq. (53) and define the dimensionless coupling constant

$$g^{(1)} = \omega\lambda(p_1 = p_2 = 0, D, L)|p|^{D-6}, \tag{59}$$

where we remember that in this context p is a $(D-2)$-dimensional vector. As before, we take as a running scale $|p|$ and after performing manipulations entirely analogous to those in the previous subsection and recalling Eq. (56), we have the extended domain of validity $4 < D < 6$ for the quantities $C_{d=1}(D;L)$, for all $d = 1,2,3$. We then get the β function for $|p| \approx 0$,

$$\beta(g) = |p|\frac{\partial g^{(1)}}{\partial |p|} \approx (D-6)\left[g^{(1)} - A_1(D)\left(g^{(1)}\right)^2\right], \tag{60}$$

from which the infrared-stable fixed point is obtained:

$$g_*^{(1)}(D) = \frac{1}{A_1(D)}. \tag{61}$$

6. Concluding remarks

Investigations on the dependence of the critical temperature for films with its thickness have been done in other contexts and approaches, different from the one we adopt. For instance, in Zinn-Justin (2002), an analysis of the renormalization group in finite-size geometries can be found and scaling laws have been studied. Also, such a dependence has been investigated in Asamitsu et al. (1994); Minhaj et al. (1994); Quateman (1986); Raffy et al. (1983) from both experimental and theoretical points of view, explaining this effect in terms of proximity, localization and Coulomb interaction. In particular, Quateman (1986) predicts, as our model also does, a suppression of the superconducting transition for thicknesses below a minimal value. More recently, in Shanenko et al. (2006) the thickness dependence of the critical temperature is explained in terms of a shape-dependent superconducting resonance, but no suppression of the transition is predicted or exhibited.

In this chapter, we have adopted a phenomenological approach, discussing the $\left(\lambda|\varphi|^4\right)_D$ and $\left(-\lambda|\varphi|^4+\eta|\varphi|^6\right)_D$ theories compactified in $d \leq D$ Euclidean dimensions. We have presented a general formalism which, in the framework of the Ginzburg–Landau model, is able to describe phase transitions for systems defined in spaces of arbitrary dimensions, some of them being compactified. We have focused in particular on the situations with $D=3$ and $d=1,2,3$, corresponding (in the context of condensed-matter systems) to films, wires and grains, respectively, undergoing phase transitions which may be described by Ginzburg–Landau models. This generalizes previous works dealing with first- and second-order transitions in low-dimensional systems [Abreu et al. (2005); Linhares et al. (2006); Malbouisson et al. (2002)].

We have observed the contrasting behavior of the critical temperature on the size of the system, whether the transition is first- or second-order. This may indicate that from this shape dependence one can infer the order of the transition the system undergoes.

In what a renormalization group approach is concerned, we have discussed the infrared behavior and the fixed-point structure of the compactified O(N) $\lambda\varphi^4$ in the large-N limit. We have shown that, whether in the absence or presence of an external magnetic field, the existence of an infrared-stable fixed point depends only on the space dimension D, not on the number of compactified dimensions.

7. References

Abreu, L.M.; Malbouisson, A.P.C.; Malbouisson, J.M.C. & Santana, A.E. (2003). Large-N transition temperature for superconducting films in a magnetic field, Phys. Rev. B, Vol. 67, No. 21, June 2003, 212502-1–212502-4, ISSN 0163-1829

Abreu, L.M.; Malbouisson, A.P.C. & Roditi, I. (2004). Gauge fluctuations in superconducting films, Eur. Phys. J. B, Vol. 37, No. 4, April 2004, 515–522, ISSN 1434-6028

Abreu, L.M.; de Calan, C.; Malbouisson, A.P.C.; Malbouisson,J.M.C. & Santana, A.E. (2005). Critical behavior of the compactified $\lambda\varphi^4$, J. Math. Phys., Vol. 46, No. 1, January 2005, 012304-1–012304-12, ISSN 0022-2488

Abreu, L.M.; Malbouisson, A.P.C.; Malbouisson, J.M.C. & Santana, A.E. (2009). Finite-size effects on the chiral phase diagram of four-fermion models in four dimensions, Nucl. Phys. B, Vol. 819, Nos. 1-2, September 2009, 127–138, ISSN 0550-3213

Asamitsu, A.; Iguchi, M.; Ichikawa, A. & Nishida, N. (1994). Disappearance of superconductivity in ultra thin amorphous Nb films, Physica B, Vols. 194-196, Part 2, February 1994, 1649–1650, ISSN 0163-1829

Bordag, M.; Mohideed U. & Mostepanenko, V.M. (2001). New developments in the Casimir effect, Phys. Rep., Vol. 353, No. 1-3, October 2001, 1–206, ISSN 0370-1573

Boyer, T.H. (2003). Casimir forces and boundary conditions in one dimension: Attraction, repulsion, Planck spectrum, and entropy, Am. J. Phys., Vol. 71, No. 10, October 2003, 990–998, ISSN 0002-9505

Ccapa Tira, C.; Fosco, C.D.; Malbouisson, A.P.C. & Roditi, I. (2010). Vacuum polarization for compactified QED_{4+1} in a magnetic flux background, Phys. Rev. A, Vol. 81, No. 3, March 2010, 032116-1–032116-7, ISSN 1050-2947

Coleman, S. & Weinberg, E. (1973). Radiative corrections as the origin of spontaneous symmetry breaking, Phys. Rev. D, Vol. 7, No. 6, March 1973, 1888–1910, ISSN 0556-2821

Elizalde, A. (1995). *Ten Physical Applications of Spectral Zeta-Functions*, Springer, ISBN 3-540-60230-5, Berlin

Halperin, L.; Lubensky, T.C. & Ma, S.-K. (1974). First-order phase transitions in superconductors and smectic-A liquid crystals, Phys. Rev. Lett., Vol. 32, No. 6, February 1974, 292–295, ISSN 0031-9007

Khanna, F.C.; Malbouisson, A.P.C.; Malbouisson, J.M.C. & Santana, A.E. (2009). *Thermal Quantum Field Theory: Algebraic Aspects and Applications*, World Scientific, ISBN-13 978-981-281-887-4, Singapore

Khanna, F.C.; Malbouisson, A.P.C.; Malbouisson, J.M.C. & Santana, A.E. (2010). Phase transition in the 3D massive Gross-Neveu model, EPL, Vol. 92, No. 1, October 2010, 11001-p1–11001-p6, ISSN 0295-5075

Kirsten, K. (1994). Generalized multidimensional Epstein zeta functions, J. Math. Phys., Vol. 35, No. 1, January 1994, 459–470, ISSN 0022-2488

Kleinert, H. (1989). *Gauge Fields in Condensed Matter, Vol. 1: Superflow and Vortex Lines*, World Scientific, ISBN-10 9971502100, Singapore

Kodama, J.; Itoh, M. & Hirai, H. (1983). Superconducting transition temperature versus thickness of Nb film on various substrates, J. Appl. Phys., Vol. 54, No. 7, July 1983, 4050–4053, ISSN 0021-8979

Levin, F.S. & Micha, D.A. (Eds.) (1993). *Long Range Casimir Forces*, Plenum, ISBN-10 0306443856, New York

Linhares, C.A.; Malbouisson, A.P.C.; Milla, Y.W. & Roditi, I. (2006). First-order phase transitions in superconducting films: A Euclidean model, Phys. Rev. B, Vol. 73, No. 21, June 2006, 214525-1–214525-7, ISSN 1098-0121

Linhares, C.A.; Malbouisson, A.P.C.; Milla, Y.W. & Roditi, I. (2007). Critical temperature for first-order phase transitions in critical systems, Eur. Phys. J. B, Vol. 60, No. 3, December 2007, 353–362, ISSN 1434-6028

Linhares, C.A.; Malbouisson, A.P.C. & Souza, M.L. (2011). A note on the infrared behavior of the compactified Ginzburg–Landau model in a magnetic field, EPL, Vol. 96, No. 3, November 2011, 31002-p1–31002-p6, ISSN 0295-5075

Malbouisson, A.P.C. (2002). Large-N β function for superconducting films in a magnetic field, Phys. Rev. B, Vol. 66, No. 9, September 2002, 092502-1–092502-3, ISSN 0163-1829

Malbouisson, A.P.C.; Malbouisson, J.M.C. & Santana, A.E. (2002). Spontaneous symmetry breaking in compactified $\lambda\varphi^4$ theory, Nucl. Phys. B, Vol. 631, Nos. 1-2, June 2002, 83–94, ISSN 0550-3213

Malbouisson, A.P.C.; Malbouisson, J.M.C. & Pereira, R.C. (2009). Boundary effects on the mass and coupling constant in the compactified Ginzburg–Landau model: The boundary dependent critical temperature, J. Math. Phys., Vol. 50, No. 8, August 2009, 083304-1–083304-16, ISSN 0022-2488

Milonni, P.W. (1993). *The Quantum Vacuum*, Academic, ISBN-10 0124980805, Boston

Minhaj, M.S.M.; Meepagala, S.; Chen, J.T. & Wenger, L.E. (1994). Thickness dependence on the superconducting properties of thin Nb films, Phys. Rev. B, Vol. 49, No. 21, June 1994, 15235–15240, ISSN 0163-1829

Mostepanenko, V.M. & Trunov, N.N. *The Casimir Effect and Its Applications*, Clarendon, ISBN-10 0124980805, Oxford

Pogrebnyakov, A.V.; Redwing, J.M.; Jones, J.E.; Xi, X.X.; Xu, S.Y.; Qi Li; Vaithyanathan, V. & Schlom, D.G. (2003). Thickness dependence of the properties of epitaxial MgB_2 thin films grown by hybrid physical chemical vapor deposition, Appl. Phys. Lett., Vol. 82, No. 24, June 2003, 4319–4321, ISSN 0003-6951

Quateman, J. (1986). T_c suppression and critical fields in thin superconducting Nb films, Phys. Rev. B, Vol. 34, No. 3, August 1986, 1948–1951, ISSN 0163-1829

Raffy, H.; Laibowitz, R.B.; Chaudhari, P. & Maekawa, S. (1983). Localization and interaction effects in two-dimensional W-Re films, Phys. Rev. B, Vol. 28, No. 11, December 1983, 6607–6609, ISSN 0163-1829

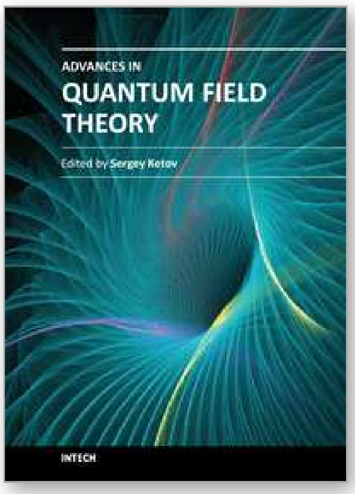
Seife, C. (1997). The subtle pull of emptiness, Science, Vol. 275, No. 5297, October 1997, 158, ISSN 0036-8075

Shanenko, A.A.; Croitoru, M.D.; Zgirski, M.; Peeters, J.M. & Arutyunov, K. (2006). Size-dependent enhancement of superconductivity in Al and Sn nanowires: Shape-resonance effect, Phys. Rev. B, Vol. 74, No. 5, August 2006, 052502-1–052502-4, ISSN 1098-0121

Strongin, M.; Thompson, R.; Kammerer, O.F. & Crow, J.E. (1970). Destruction of superconductivity in disordered near-monolayer films, Phys. Rev. B, Vol. 1, No. 3, February 1970, 1078–1091, ISSN 0556-2805

Zinn-Justin, J. (2002). *Quantum Field Theory and Critical Phenomena*, Clarendon, ISBN-10 0198509235, Oxford

Zgirski, M.; Riikonen, K.-P.; Touboltsev, V.; Arutyunov, K. (2005). Size Dependent Breakdown of Superconductivity in Ultranarrow Nanowires, Nano Lett., Vol. 5, No. 6, May 2005, 1029–1033, ISSN 1530-6984



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