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Quantizing with a Higher Time Derivative

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1. Introduction

It is commonplace in *Quantum Field Theory* (QFT) that a QFT with higher (time) derivatives is believed to be doomed from the point of view of physics, because of ghosts or states of negative norm, and thus it should be dismissed. The standard reference is the very old result (known in the literature as the Ostrogradski theorem (1)) claiming a linear instability in any Hamiltonian system associated with the Lagrangian having the higher (ie. more than one) time derivative that cannot be eliminated by partial integration.

The key point of the Ostrogradski method (1) is a canonical quantization of the clasically equivalent theory without higher derivatives via considering the higher derivatives of the initial coordinates as the *independent* variables.

The interest in the higher-derivative QFT was recently revived due to some novel developments in the gravitational theory, related to the so-called $f(R)$-gravity theories – see eg., ref. (2) for a review. The $f(R)$ gravity theories are defined by replacing the scalar curvature R in the Einstein action by a function $f(R)$. The $f(R)$ gravity theories give the self-consistent non-trivial alternative to the standard Λ-CDM Model of Cosmology, by providing the geometrical phenomenological description of inflation in the early universe and Dark Energy in the present universe. Despite of the apparent presence of the higher derivatives, a classical $f(R)$ gravity theory can be free of ghosts and tachyons. A supersymmetric extension of $f(R)$ gravity was recently constructed in superspace (3).

Already the simplest model of $(R + R^2)$ gravity (4) is known as the viable model of chaotic inflation, because it is consistent with the recent WMAP measurements of the Cosmic Macrowave Background (CMB) radiation (5). Its supersymmetric extension was recently constructed in refs. (6; 7).

On the one side, any quadratically generated (with respect to the curvature) quantum theory of gravity has ghosts in its perturbative quantum propagator (8). However, on the other side, any $f(R)$ gravity theory is known to be *classically* equivalent to the scalar-tensor gravity (ie. to the usual quintessence) (9–11), while the stability conditions in the $f(R)$ gravity ensure the ghost-and-tachyon-freedom of the classically equivalent quintessence theory (12; 13). It now appears that in some cases the presence of the higher derivatives may be harmless (14). It also

gives rise to the non-trivial natural question of how to make sense out of the quantized $f(R)$ gravity?

The $f(R)$ gravity theories are just the particular case of the higher-derivative quantum gravity theories which have been investigated in the past. They were found to be renormalizable (15) and asymptotically free (16). A generic higher-derivative gravity suffers, however, from the presence of ghosts and states of negative norm which apparently spoil those QFT from physical applications. However, the issue of ghosts and their physical interpretation deserves a more detailed study. The complexity of the higher-derivative gravity is the formidable technical obstacle for that. It is, therefore, of interest to consider simpler QFT as the toy-models.

Similar features (like renormalizability and asymptotic freedom) exhibit the quantum Non-Linear Sigma-Models with higher derivatives, which have striking similarities to the higher-derivative quantum gravity (17–19). However, even those QFT are too complicated because of their high degree of non-linearity.

Perhaps, the simplest toy-model is given by the *Pais-Uhlenbeck* (PU) quantum oscillator in Quantum Mechanics (20). As was demonstrated by Hawking and Hertog (21), it may be possible to give physical meaning to the Euclidean path integral of the PU oscillator, as the set of consistent rules for calculation of observables, even when "living with ghosts". The basic idea of ref. (21) is to abandom unitarity, while never producing and observing negative norm states.

The idea of Hawking and Hertog found further support in refs. (22; 23) where the physical propagator of the PU oscillator was calculated by using the van Vleck-Pauli approach (the saddle point method for the Euclidean path integral) and Forman's theorem (24). In this Chapter we systematically review the classical and quantum theory of the PU oscillator from the first principles, along the lines of refs. (14; 21–23).

2. Ostrogradski method with higher derivatives

Consider a one-dimensional mechanical system with the action

$$S[q] = \int dt\ L\left(q, Dq, \cdots, D^n q\right) \tag{2.1}$$

in terms of the Lagrange function L of $q(t)$ and its time derivatives, where $n \geq 2$ and $D = \frac{d}{dt}$. The Euler-Lagrange equation reads

$$\sum_{i=0}^{n} (-D)^i \frac{\partial L}{\partial (D^i q)} = 0 \tag{2.2}$$

The Ostrogradski method (1) gives the Hamiltonian formulation of the higher derivative Lagrange formulation by introducing more independent variables.

The independent generalized coordinates Q_i are defined by

$$Q_i = D^{i-1} q \qquad \left(i = 1, \cdots, n\right) \tag{2.3}$$

The generalized momentum P_n is defined by

$$\left.\frac{\partial L}{\partial(D^n q)}\right|_{\substack{D^{i-1}q=Q_i \\ D^n q=A}} = P_n \tag{2.4}$$

There are $n+1$ independent variables $\{Q_1, \cdots, Q_n, P_n\}$ that are in correspondence to the $n+1$ variables $\{D^0 q, \cdots, D^n q\}$ of the higher derivative action (2.1).

By solving eq.(2.4) with respect to $A = D^n q$ (assuming that it is possible), one gets

$$D^n q = A(Q_1, \cdots, Q_n, P_n) \tag{2.5}$$

Therefore, the Lagrange dynamics can be represented in terms of the $n+1$ independent variables $\{Q_1, \cdots, Q_n, P_n\}$ as

$$L = L\big(Q_1, \cdots, Q_n, A(Q_1, \cdots, Q_n, P_n)\big) \tag{2.6}$$

A Legendre transformation is used to pass from the Lagrange formulation to the Hamiltonian one. With the generalized coordinates $\{Q_1, \cdots, Q_n\}$ and the generalized momentum P_n as the independent variables, the total differential of the Lagrangian is given by

$$\begin{aligned} dL &= \sum_{j=1}^{n} \left.\frac{\partial L}{\partial(D^{j-1}q)}\right|_{\substack{D^{i-1}q=Q_i \\ D^n q=A}} dQ_j + P_n dA \\ &= \frac{\partial L}{\partial q} dQ_1 + \sum_{j=2}^{n} \frac{\partial L}{\partial(D^{j-1}q)} dQ_j + P_n dA \\ &= D\sum_{j=1}^{n} (-D)^{j-1} \frac{\partial L}{\partial(D^j q)} dQ_1 + \sum_{j=1}^{n-1} \frac{\partial L}{\partial(D^j q)} dQ_{j+1} + P_n dA \end{aligned} \tag{2.7}$$

where we have used eqs. (2.2) and (2.4), and

$$dA = \sum_{j=1}^{n} \frac{\partial A}{\partial Q_j} dQ_j + \frac{\partial A}{\partial P_n} dP_n \tag{2.8}$$

Let us now define the $n-1$ generalized momenta as

$$P_i = \sum_{j=i}^{n} (-D)^{j-i} \frac{\partial L}{\partial(D^j q)} \qquad \big(i = 1, \cdots, n-1\big) \tag{2.9}$$

They satisfy the relations

$$\frac{\partial L}{\partial(D^i q)} = P_i + DP_{i+1} \tag{2.10}$$

Therefore, eq. (2.7) can be rewritten to the form

$$d\left[\sum_{i=1}^{n-1} P_i(DQ_i) + P_n A - L\right] = -\sum_{i=1}^{n} (DP_i) dQ_i + \sum_{i=1}^{n} (DQ_i) dP_i \tag{2.11}$$

Equation (2.11) gives rise to the Hamiltonian in the form

$$H = \sum_{j=1}^{n-1} P_j (DQ_j) + P_n A - L \tag{2.12}$$

The Hamilton equations of motion are given by

$$DQ_i = \frac{\partial H}{\partial P_i} \qquad \text{and} \qquad DP_i = -\frac{\partial H}{\partial Q_i} \tag{2.13}$$

3. PU oscillator

The PU oscillator (20) is an extension of the harmonic oscillator with the higher time derivatives, and is the particular case of the higher-derivative theory introduced in Sec. 2. The special features of the PU opscilator are
(i) the equation of motion is *linear*:

$$F(D)q = 0 \tag{3.1}$$

where F is a linear differential operator;
(ii) the F is *polynomial* (with respect to D) with *constant* coefficients:

$$F(D) = \sum_{i=0}^{n} a_i D^i \tag{3.2}$$

where $a_0, \cdots, a_n$ are the real constants;
(iii) there is the time reversal invariance with respect to $t \to -t$. Hence, the polynomial F has only even powers of the time derivative D.

The Lagrangian of the one-dimensional PU oscillator reads

$$L\left(q, Dq, \cdots, D^n q\right) = -\sum_{i=0}^{n} \frac{a_i}{2} (D^i q)^2 \qquad \left(a_0 \neq 0, a_n \neq 0\right) \tag{3.3}$$

where a_i $(i = 0, \cdots, n)$ are real constants. The Euler-Lagrange equation of motion is given by

$$\begin{aligned} 0 &= \sum_{i=0}^{n} (-D)^i \left[-a_i D^i q\right] \\ &= -a_0 \left[\sum_{i=0}^{n} (-1)^i \frac{a_i}{a_0} D^{2i}\right] q \end{aligned} \tag{3.4}$$

Accordingly, the differential operator $F(D)$ reads

$$F(D) = \sum_{i=0}^{n} (-1)^i \frac{a_i}{a_0} D^{2i} \tag{3.5}$$

The equation of motion can be rewritten to the form

$$F(D)q = 0 \tag{3.6}$$

The PU Lagrangian takes the form (up to a boundary term)

$$\bar{L} = -\frac{a_0}{2} q F(D) q \tag{3.7}$$

The differential operator $F(D)$ can be brought to the factorized form

$$F(D) = \prod_{i=1}^{n} \left(1 + \frac{D^2}{\omega_i^2}\right) \tag{3.8}$$

where the constants ω_i $(i = 1, \cdots, n)$ are the solutions (roots) of the equation $F(i\omega) = 0$. Let us introduce n new operators

$$G_i(D) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(1 + \frac{D^2}{\omega_j^2}\right) \qquad (i = 1, \cdots, n) \tag{3.9}$$

and define the n generalized coordinates as

$$Q_i = G_i(D) q \qquad (i = 1, \cdots, n) \tag{3.10}$$

Those generalized coordinates Q_j are called *harmonic* coordinates. By using the harmonic coordinates, the PU Euler-Lagrange eq. (3.6) can be rewitten to the n equations

$$\left[1 + \frac{D^2}{\omega_i^2}\right] Q_i = 0 \tag{3.11}$$

It means that the PU oscillator can be interpreted as n harmonic oscillators. Accordingly, the PU Lagrangian (3.7) can be rewritten to the form

$$\bar{L} = -\frac{a_0}{2} \sum_{i=1}^{n} \eta_i Q_i \left(1 + \frac{D^2}{\omega_i^2}\right) Q_i \tag{3.12}$$

where the n constants η_i have been introduced as

$$\eta_i = \left(\omega_i^2 \left.\frac{dF}{d(D^2)}\right|_{D^2 = -\omega_i^2}\right)^{-1} \tag{3.13}$$

To prove eq. (3.13), we first notice that it amounts to

$$\sum_{i=1}^{n} \eta_i G_i(D) = 1 \tag{3.14}$$

By the definiton of $G(D)$ in eq.(3.9) we have

$$\begin{aligned} G_i(D^2 = -\omega_j^2) &= \prod_{\substack{k=1 \\ k \neq i}}^{n} \left(1 - \frac{\omega_j^2}{\omega_k^2}\right) \\ &= \delta_{ij} \prod_{\substack{k=1 \\ k \neq j}}^{n} \left(1 - \frac{\omega_j^2}{\omega_k^2}\right) \end{aligned} \tag{3.15}$$

so that

$$\sum_{i=1}^{n} \eta_i G_i(D) = 1 \tag{3.16}$$

$$\sum_{i=1}^{n} \eta_i G_i(D^2 = -\omega_j^2) = \eta_j \prod_{\substack{k=1 \\ k\neq j}}^{n} \left(1 - \frac{\omega_j^2}{\omega_k^2}\right) = 1 \tag{3.17}$$

indeed. Therefore, the constants η_i are given by

$$\eta_i = \left[\prod_{\substack{k=1 \\ k\neq i}}^{n} \left(1 - \frac{\omega_i^2}{\omega_k^2}\right)\right]^{-1} \tag{3.18}$$

Next, we prove that

$$\omega_i^2 \frac{dF}{dD^2}\Big|_{D^2=-\omega_i^2} = \prod_{\substack{k=1 \\ k\neq i}}^{n} \left(1 - \frac{\omega_i^2}{\omega_k^2}\right) \tag{3.19}$$

By the use of eq.(3.8) we find

$$\begin{aligned} \frac{dF}{dD^2} &= \frac{d}{dD^2} \prod_{j=1}^{n} (1 + \frac{D^2}{\omega_j^2}) \\ &= \sum_{k=1}^{n} \frac{1}{\omega_k^2} \prod_{\substack{j=1 \\ j\neq k}}^{n} (1 + \frac{D^2}{\omega_j^2}) \\ &= \sum_{j=1}^{n} \frac{1}{\omega_j^2} G_j(D) \end{aligned} \tag{3.20}$$

so that

$$\begin{aligned} \frac{dF}{dD^2}\Big|_{D^2=-\omega_i^2} &= \sum_{j=1}^{n} \frac{1}{\omega_j^2} G_j(D^2 = -\omega_i^2) \\ &= \sum_{j=1}^{n} \frac{1}{\omega_j^2} \delta_{ji} \prod_{\substack{k=1 \\ k\neq i}}^{n} (1 - \frac{\omega_i^2}{\omega_k^2}) \\ &= \frac{1}{\omega_i^2} \prod_{\substack{k=1 \\ k\neq i}}^{n} (1 - \frac{\omega_i^2}{\omega_k^2}) \end{aligned} \tag{3.21}$$

Equation (3.19) is now confirmed and, hence, via eq. (3.18) also eq. (3.13) follows.

In terms of the harmonic coordinates (3.10), the Lagrangian $\bar{L}$,

$$\begin{aligned} \bar{L} &= -\frac{a_0}{2} q F(D) q \\ &= -\frac{a_0}{2} \sum_{i=1}^{n} \eta_i Q_i (1 + \frac{D^2}{\omega_i^2}) Q_i \end{aligned} \tag{3.22}$$

with the constants η_i given by eq. (3.13), can be rewritten to the form

$$\tilde{L} = \frac{a_0}{2}\sum_{i=1}^{n}\eta_i\Big(\frac{1}{\omega_i^2}(DQ_i)^2 - Q_i^2\Big) \tag{3.23}$$

up to a boundary term.

The Lagrangian (3.23) is just a sum of the Lagrangians of n harmonic oscillators. Hence, similarly to a free system of n particles, we can change the Lagrangian formulation into the Hamiltonian formulation. We define the generalized momenta P_i by taking the harmonic coordinates Q_i and the velocities DQ_i as the Lagrange variables,

$$\begin{aligned} P_i &= \frac{\partial\tilde{L}}{\partial(DQ_i)} \\ &= \frac{a_0\eta_i}{\omega_i^2}DQ_i \qquad (i = 1, \cdots, n) \end{aligned} \tag{3.24}$$

The system of n free particles does not have higher derivatives, so its Hamiltonian is

$$H = \sum_{i=1}^{n} P_i(DQ_i) - L \tag{3.25}$$

Equations (3.23) and (3.24) imply

$$H = \sum_{i=1}^{n}\left(\frac{\omega_i^2}{2a_0\eta_i}P_i^2 + \frac{a_0\eta_i}{2}Q_i^2\right) \tag{3.26}$$

By rescaling the harmonic coordinates and the generalized momenta as

$$Q_i \to \tilde{Q}_i = \frac{\sqrt{a_0|\eta_i|}}{\omega_i}Q_i \qquad \text{and} \qquad P_i \to \tilde{P}_i = \frac{\omega_i\sqrt{|\eta_i|}}{\eta_i\sqrt{a_0}}P_i \tag{3.27}$$

we get the final Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n}\frac{\eta_i}{|\eta_i|}\big(\tilde{P}_i{}^2 + \omega_i^2\tilde{Q}_i{}^2\big) \tag{3.28}$$

The presence of both positive and negative values of the constants η_i in the Hamiltonian implies both positive and negative values of energy. The constants η_i are given by eq. (3.18). If ω_i satisfy $i < j \Rightarrow \omega_i < \omega_j$, the constants η_i are positive for the odd number i, and are negative for the even number i. Therefore, the Hamiltonian is

$$H = \frac{1}{2}\sum_{i=1}^{n}(-1)^{i-1}\left(\tilde{P}_i{}^2 + \omega_i^2\tilde{Q}_i{}^2\right) \tag{3.29}$$

This Hamiltonian can be interpreted as that of n harmonic oscillators, with the positive and negative energy levels appearing alternatively. Because of that reason, the PU oscillator has an instability (for any interaction). It is related to a possible ghost state of negative norm in PU quantum theory (see Sec. 6). In what follows we consider the simplest case of PU oscillator with $n = 2$ only.

4. PU oscillator for $n = 2$: explicit results

Let us consider the Lagrangian

$$L = \frac{1}{2}\left(\frac{dq}{dt}\right)^2 - V(q) - \frac{\alpha^2}{2}\left(\frac{d^2q}{dt^2}\right)^2 \qquad \left(\text{where } \alpha \neq 0\right) \tag{4.1}$$

with a scalar potential $V(q)$. In the case of the PU oscillator, the potential $V(q)$ is a quadratic function of q. Since the (mass) dimension of time is -1 (in the natural units $\hbar = c = 1$), the dimension of the Lagrangian L is 1, the dimension of q is $-1/2$, and that of the constant α is -1.

Let the trajectory q be a sum of the classical trajectory q_{cl} and the displacement $\tilde{q}$, ie. $q = q_{cl} + \tilde{q}$, where the classical trajectory q_{cl} is a solution to the *equation of motion* (EOM) with the boundary conditions (21)

$$\mathscr{A}: \quad q(0) = q_0, \quad q(T) = q_T, \quad \dot{q}(0) = \dot{q}_0, \quad \dot{q}(T) = \dot{q}_T \tag{4.2}$$

where the dots above stand for the time derivatives.

With the boundary conditions (4.2), the boundary condition of $\tilde{q}$ is

$$\tilde{\mathscr{A}}: \quad \tilde{q}(0) = 0, \quad \tilde{q}(T) = 0, \quad \dot{\tilde{q}}(0) = 0, \quad \dot{\tilde{q}}(T) = 0 \tag{4.3}$$

The action of $q_{cl} + \tilde{q}$ is given by

$$S[q_{cl} + \tilde{q}] = S[q_{cl}] + \int_0^T dt \left(\frac{1}{2}\dot{\tilde{q}}^2 - V(q_{cl} + \tilde{q}) + V(q_{cl}) + \tilde{q}V'(q_{cl}) - \frac{\alpha^2}{2}\ddot{\tilde{q}}^2\right) \tag{4.4}$$

where we have introduced the notation

$$V'(q_{cl}) = \left.\frac{dV}{dq}\right|_{q=q_{cl}} \tag{4.5}$$

In eq.(4.4) the term $V(q_{cl} + \tilde{q}) - V(q_{cl}) - \tilde{q}V'(q_{cl})$ represents the *gap* between the full action $S[q]$ and the classical action $S[q_{cl}]$, which generically depends on both the classical trajectory q_{cl} and the displacement $\tilde{q}$. After expanding the scalar potential V in Taylor series,

$$V(q_{cl} + \tilde{q}) = V(q_{cl}) + \tilde{q}V'(q_{cl}) + \frac{1}{2!}\tilde{q}^2 V''(q_{cl}) + \cdots \tag{4.6}$$

we find that, when the second derivative V'' is constant, the gap $V(q_{cl} + \tilde{q}) - V(q_{cl}) - \tilde{q}V'(q_{cl})$ does *not* depend on the classical trajectory q_{cl}. It is the case when the potential V is a quadratic function of q, like the PU oscillator.

In the path integral quantization (sec. 7), the gap between the full action and the classical action is a quantum effect. When the potential is a quadratic function (like that of the PU oscillator), that quantum effect does depend on $\tilde{q}$, but does not depend on the classical trajectory. In what follows, we only consider a quadratic function for the scalar potential in the form

$$V(q) = \frac{m^2}{2}q^2 \tag{4.7}$$

ie. the scalar potential of a harmonic oscillator with the mass $m > 0$, The Lagrangian is given by

$$L_{PU} = \frac{1}{2}\dot{q}^2 - \frac{m^2}{2}q^2 - \frac{\alpha^2}{2}\ddot{q}^2 \tag{4.8}$$

The parameter α measures a contribution of the second derivative to the harmonic oscillator. Therefore, we can expect the classical trajectory to behave just like that of the harmonic oscillator when α is small.

The Euler-Lagrange EOM of the Lagrangian (4.8) are given by eq.(2.2),

$$\begin{aligned} 0 &= \sum_{i=0}^{2}(-D)^i \frac{\partial L}{\partial(D^i q)} \\ &= -m^2 q - \ddot{q} - \alpha^2 \dddot{q} \end{aligned} \tag{4.9}$$

or, equivalently,

$$\left(m^2 + D^2 + \alpha^2 D^4\right) q = 0 \tag{4.10}$$

It is not difficult to find clasical solutions to the EOM in eq. (4.10). When searching for the classical trajectory in the oscillatory form $q_{cl} = \exp(i\lambda t)$, the EOM reads

$$\left(m^2 - \lambda^2 + \alpha^2\lambda^4\right) e^{i\lambda t} = 0 \tag{4.11}$$

and, therefore, we have

$$\lambda^2 = \frac{1 \pm \sqrt{1 - 4\alpha^2 m^2}}{2\alpha^2} \tag{4.12}$$

When λ is real, the Lagrangian $L_(PU)$ is an extension of the harmonic oscillator indeed. Hence, we need the condition

$$0 < \alpha m < \frac{1}{2} \tag{4.13}$$

It means that the Lagrangina L_{PU} has the oscillating solution which is similar to the trajectory of the harmonic oscillator. A general solution reads

$$q(t) = A_+ \cos(\lambda_+ t) + B_+ \sin(\lambda_+ t) + A_- \cos(\lambda_- t) + B_- \sin(\lambda_- t) \tag{4.14}$$

where A_+, B_+, A_-, B_- are the integration constants, and

$$\lambda_\pm = \sqrt{\frac{1 \mp \sqrt{1 - 4\alpha^2 m^2}}{2\alpha^2}} \tag{4.15}$$

The values of the constants (A_+, B_+, A_-, B_-) are determined by the boundary conditions.

The Hamiltonian formulation for the Lagrangian (4.8) can be obtained by the Ostrogradski method. The generalized coodinates and momenta are given in Sec. 2, ie.

$$\begin{aligned} Q_1 &= q \qquad \text{and} \qquad P_1 = \frac{\partial L}{\partial \dot{q}} - D\frac{\partial L}{\partial \ddot{q}} \\ Q_2 &= \dot{q} \qquad \text{and} \qquad P_2 = \frac{\partial L}{\partial \ddot{q}} \end{aligned} \tag{4.16}$$

which imply

$$\begin{aligned} P_1 &= \dot{q} + \alpha^2 \dddot{q} \\ P_2 &= -\alpha^2 \ddot{q} \end{aligned} \tag{4.17}$$

The Hamiltonian is given by eq.(2.12). ie.

$$\begin{aligned} H &= P_1(DQ_1) + P_2 A - L \\ &= P_1 Q_2 - \frac{1}{2\alpha^2} P_2^2 - \frac{1}{2} Q_2^2 + \frac{m^2}{2} Q_1^2 \end{aligned} \tag{4.18}$$

or, equivalently,

$$H = \alpha^2 \dot{q} \dddot{q} - \frac{\alpha^2}{2} \ddot{q}^2 + \frac{1}{2} \dot{q}^2 + \frac{m^2}{2} q^2 \tag{4.19}$$

Since the Hamiltonian does not evolve with time, we can find the energy by substituting $q(t)$ of eq. (4.14) at $t = 0$ into eq. (4.19), as well as $q, \dot{q}, \ddot{q}$ and $\dddot{q}$ at $t = 0$, ie.

$$\begin{aligned} q(0) &= A_+ + A_- \\ \dot{q}(0) &= B_+ \lambda_+ + B_- \lambda_- \\ \ddot{q}(0) &= -A_+ \lambda_+^2 - A_- \lambda_-^2 \\ \dddot{q}(0) &= -B_+ \lambda_+^3 - B_- \lambda_-^3 \end{aligned} \tag{4.20}$$

It is now straightforward to calculate the Hamiltonian (4.19). We find

$$\begin{aligned} H &= \alpha^2 \dot{q}(0) \dddot{q}(0) - \frac{\alpha^2}{2} \ddot{q}(0)^2 + \frac{1}{2} \dot{q}(0)^2 + \frac{m^2}{2} q(0)^2 \\ &= \frac{1}{2} \lambda_+^2 \sqrt{1 - 4\alpha^2 m^2} (A_+^2 + B_+^2) - \frac{1}{2} \lambda_-^2 \sqrt{1 - 4\alpha^2 m^2} (A_-^2 + B_-^2) \end{aligned} \tag{4.21}$$

To get the Hamiltonian formulation in the harmonic coordinates, we begin with the EOM in the form (4.10), whose differential operator $F(D)$ is defined by

$$F(D) = 1 + \frac{D^2}{m^2} + \frac{\alpha^2 D^4}{m^2} \tag{4.22}$$

It can be factorized as

$$F(D) = \left(1 + \frac{D^2}{\lambda_+^2}\right) \left(1 + \frac{D^2}{\lambda_-^2}\right) \tag{4.23}$$

where $\lambda_\pm$ are given by eq. (4.15). Therefore, the harmonic coodinates are given by

$$Q_+ = \left(1 + \frac{D^2}{\lambda_-^2}\right) q \qquad \text{and} \qquad Q_- = \left(1 + \frac{D^2}{\lambda_+^2}\right) q \tag{4.24}$$

The constants η_i of eq. (3.13) can be computed as follows. We have

$$\frac{dF}{dD^2} = \frac{1}{m^2} + \frac{2\alpha^2 D^2}{m^2} \tag{4.25}$$

so that

$$\begin{aligned}\eta_{\pm} &= \left(\lambda_{\pm}^2 \left.\frac{dF}{dD^2}\right|_{D^2=-\lambda_{\pm}^2}\right)^{-1} \\ &= \left(\frac{\lambda_{\pm}^2}{m^2}(1-2\alpha^2\lambda_{\pm}^2)\right)^{-1} \\ &= \left(\pm\frac{\lambda_{\pm}^2}{m^2}\sqrt{1-4\alpha^2m^2}\right)^{-1} \\ &= \pm\frac{m^2}{\lambda_{\pm}^2\sqrt{1-4\alpha^2m^2}} \end{aligned} \tag{4.26}$$

Therefore, the generalized momenta in eq. (3.24) are

$$\begin{aligned}P_{\pm} &= \frac{m^2\eta_{\pm}}{\lambda_{\pm}^2}DQ_{\pm} \\ &= \pm\frac{m^4}{\lambda_{\pm}^4\sqrt{1-4\alpha^2m^2}}DQ_{\pm} \end{aligned} \tag{4.27}$$

and the Hamiltonian is given by

$$\begin{aligned}H &= \sum_{j=\pm}\left(\frac{\lambda_j^2}{2m^2\eta_j}P_j^2 + \frac{m^2\eta_j}{2}Q_j^2\right) \\ &= \sum_{j=\pm} j\frac{m^4}{2\lambda_j^4\sqrt{1-4\alpha^2m^2}}\left((DQ_j)^2 + \lambda_j Q_j^2\right) \end{aligned} \tag{4.28}$$

where we have substituted the classical solution (4.14).

The harmonic coodinates (4.24) read

$$Q_+ = A_+\left(1-\frac{\lambda_+^2}{\lambda_-^2}\right)\cos(\lambda_+ t) + B_+\left(1-\frac{\lambda_+^2}{\lambda_-^2}\right)\sin(\lambda_+ t) \tag{4.29}$$

$$Q_- = A_-\left(1-\frac{\lambda_-^2}{\lambda_+^2}\right)\cos(\lambda_- t) + B_-\left(1-\frac{\lambda_-^2}{\lambda_+^2}\right)\sin(\lambda_- t) \tag{4.30}$$

where

$$\begin{aligned}1-\frac{\lambda_{\pm}^2}{\lambda_{\mp}^2} &= \lambda_{\pm}^2\left(\frac{1}{\lambda_{\pm}^2}-\frac{1}{\lambda_{\mp}^2}\right) \\ &= \pm\frac{\lambda_{\pm}^2}{m^2}\sqrt{1-4\alpha^2m^2} \end{aligned} \tag{4.31}$$

Hence, we find

$$Q_{\pm} = \pm\frac{\lambda_{\pm}^2}{m^2}\sqrt{1-4\alpha^2m^2}\left(A_{\pm}\cos(\lambda_{\pm}t) + B_{\pm}\sin(\lambda_{\pm}t)\right) \tag{4.32}$$

Substituting them into the Hamiltonian (4.28), we get

$$H = \frac{1}{2}\lambda_+^2\sqrt{1-4\alpha^2m^2}(A_+^2+B_+^2) - \frac{1}{2}\lambda_-^2\sqrt{1-4\alpha^2m^2}(A_-^2+B_-^2) \tag{4.33}$$

Equations (4.22) and (4.33) are *the same.* Therefore, we conclude that the Hamiltonian formulation by the Ostrogradski method is consistent with the Hamiltonian formulation in the harmonic coordinates, as they should.

The integration constants (A_+, B_+) correspond to the harmonic oscillator with positive energy, while the integration constants (A_-, B_-) correspond to the harmonic oscillator with negative energy.

5. Boundary conditions and spectrum

Going back to the Lagrangian (4.8), let us consider its action over a finite time period T,

$$S[q] = \int_0^T dt\ L_{PU} \tag{5.1}$$

with the trajectory q being a sum of the classical trajectory q_{cl} and the displacement $\tilde{q}$, $q = q_{cl} + \tilde{q}$. In quantum theory, the displacement $\tilde{q}$ is a quantum coordinate. The action can be rewritten as

$$S[q] = S[q_{cl}] + S[\tilde{q}] - \int_0^T dt\ \left(\ddot{q}_{cl} + m^2 q_{cl} + \alpha^2 \ddot{\ddot{q}}_{cl}\right)\tilde{q} + \left[\dot{q}_{cl}\tilde{q} - \alpha^2\ddot{q}_{cl}\dot{\tilde{q}} + \alpha^2\dddot{q}_{cl}\tilde{q}\right]_0^T \tag{5.2}$$

Here the first term is the action of the classical trajectory q_{cl}, and the second term is the action of the quantum part $\tilde{q}$. The integrand of the third term vanishes because the classical trajectory is a solution of the (Euler-Lagrange) EOM. The fourth term depends on the boundary. However, if the boundary condition on $\tilde{q}$ is given by

$$\tilde{\mathscr{A}}: \quad \tilde{q}(0) = 0, \quad \tilde{q}(T) = 0, \quad \dot{\tilde{q}}(0) = 0, \quad \dot{\tilde{q}}(T) = 0 \tag{5.3}$$

the fourth term in eq. (5.2) also vanishes. That boundary condition is the same as that of

$$\mathscr{A}: \quad q(0) = q_0, \quad q(T) = q_T, \quad \dot{q}(0) = \dot{q}_0, \quad \dot{q}(T) = \dot{q}_T \tag{5.4}$$

which was proposed in ref. (21). The quantum action now takes the form

$$\begin{aligned} S[\tilde{q}] &= \int_0^T dt\ \left(\frac{1}{2}\dot{\tilde{q}}^2 - \frac{m^2}{2}\tilde{q}^2 - \frac{\alpha^2}{2}\ddot{\tilde{q}}^2\right) \\ &= -\frac{1}{2}\int_0^T dt\ \tilde{q}\left(D^2 + m^2 + \alpha^2 D^4\right)\tilde{q} + \frac{1}{2}\left[\tilde{q}\dot{\tilde{q}} - \alpha^2\dot{\tilde{q}}\ddot{\tilde{q}} + \alpha^2\tilde{q}\dddot{\tilde{q}}\right]_0^T \end{aligned} \tag{5.5}$$

where the (last) boundary term vanishes due to the boundary condition (5.3).

The boundary term in eq. (5.5) also vanishes by another boundary condition,

$$\tilde{\mathscr{A}}': \quad \tilde{q}(0) = 0, \quad \tilde{q}(T) = 0, \quad \ddot{\tilde{q}}(0) = 0, \quad \ddot{\tilde{q}}(T) = 0 \tag{5.6}$$

As a result, the action (5.5) takes the Gaussian form, which is quite appropriate for a path integral quantization with the Gaussian functional

$$-\frac{1}{2}\int_0^T dt\ \tilde{q}\Big(D^2+m^2+\alpha^2D^4\Big)\tilde{q} \tag{5.7}$$

Let us now compute the *spectrum* of the operator $D^2+m^2+\alpha^2D^4$. For this purpose, we need to find the solutions u_k to the eigenvalue equation

$$\big(D^2+m^2+\alpha^2D^4\big)u_k(t)=ku_k(t) \tag{5.8}$$

with the eigenvalues k. A general solution is

$$\begin{aligned} u_k(t) &= A_1\cos\big(\omega_+t\big)+A_2\sin\big(\omega_+t\big)+A_3\cos\big(\omega_-t\big)+A_4\sin\big(\omega_-t\big)\\ \omega_\pm &= \sqrt{\frac{1\mp\sqrt{1-4\alpha^2(m^2-k)}}{2\alpha^2}} \end{aligned} \tag{5.9}$$

where A_1, A_2, A_3, A_4 is the constants of integration. The function $\tilde{q}$ can be expanded in terms of u_k,

$$\tilde{q}=\int dk\ u_k(t) \tag{5.10}$$

The spectrum of k is now determined by appying the physical boundary conditions (5.3) or (5.6) to u_k in the form of eq. (5.9). Applying the boundary condition (5.3) at $t=0$ yields

$$\tilde{q}(0)=A_1+A_3=0, \qquad \dot{\tilde{q}}(0)=A_2\omega_++A_4\omega_-=0 \tag{5.11}$$

The boundary condition (5.3) at $t=T$ then takes the form

$$\begin{aligned} \tilde{q}(T) &= A_1\cos\big(\omega_+T\big)+A_2\sin\big(\omega_+T\big)-A_1\cos\big(\omega_-T\big)\\ &\quad -A_2\frac{\omega_+}{\omega_-}\sin\big(\omega_-T\big)=0\\ \dot{\tilde{q}}(T) &= -A_1\omega_+\sin\big(\omega_+T\big)+A_2\omega_+\cos\big(\omega_+T\big)\\ &\quad +A_1\omega_-\sin\big(\omega_-T\big)-A_2\omega_+\cos\big(\omega_-T\big)=0 \end{aligned}$$

In particular, the determinant of the matrix on the left side of this equation,

$$\begin{aligned} &\det\begin{pmatrix} \omega_-\Big[\cos\big(\omega_+T\big)-\cos\big(\omega_-T\big)\Big] & \omega_-\sin\big(\omega_+T\big)-\omega_+\sin\big(\omega_-T\big)\\ -\omega_+\sin\big(\omega_+T\big)+\omega_-\sin\big(\omega_+T\big) & \omega_+\Big[\cos\big(\omega_+T\big)-\cos\big(\omega_-T\big)\Big] \end{pmatrix}\\ =&\omega_+\omega_-\Big[\cos\big(\omega_+T\big)-\cos\big(\omega_-T\big)\Big]^2\\ &+\omega_+\omega_-\Big[\sin^2\big(\omega_+T\big)+\sin^2\big(\omega_-T\big)\Big]-(\omega_+^2+\omega_-^2)\sin\big(\omega_+T\big)\sin\big(\omega_-T\big)\\ =&2\omega_+\omega_-\Big[1-\cos\big(\omega_+T\big)\cos\big(\omega_-T\big)\Big]-(\omega_+^2+\omega_-^2)\sin\big(\omega_+T\big)\sin\big(\omega_-T\big) \end{aligned} \tag{5.12}$$

must vanish. We find

$$\begin{aligned} 2\omega_+\omega_-\Big[1-\cos\big(\omega_+T\big)\cos\big(\omega_-T\big)\Big] &= (\omega_+^2+\omega_-^2)\sin\big(\omega_+T\big)\sin\big(\omega_-T\big) \\ \frac{2\sqrt{m^2-k}}{\alpha}\Big[1-\cos\big(\omega_+T\big)\cos\big(\omega_-T\big)\Big] &= \frac{1}{\alpha^2}\sin\big(\omega_+T\big)\sin\big(\omega_-T\big) \\ 1-\cos\big(\omega_+T\big)\cos\big(\omega_-T\big) &= \frac{1}{\alpha\sqrt{m^2-k}}\sin\big(\omega_+T\big)\sin\big(\omega_-T\big) \end{aligned} \tag{5.13}$$

where $\omega_\pm(k)$ ar given by eq. (5.9). Apparently, there is no simple solution here.

When employing the boundary conditions (5.6) with eq. (5.9) on u_k, the boundary condition in $t=0$ yields

$$\tilde{q}(0)=A_1+A_3=0, \qquad \ddot{\tilde{q}}(0)=-A_1\omega_+^2-A_3\omega_-^2=0 \tag{5.14}$$

so that we find $A_1=A_3=0$ when $\omega_+\neq\omega_-$. Now the boundary condition at $t=T$ reads

$$\begin{aligned} \tilde{q}(T) &= A_2\sin\big(\omega_+T\big)+A_4\sin\big(\omega_-T\big)=0 \\ \ddot{\tilde{q}}(T) &= -A_2\omega_+^2\sin\big(\omega_+T\big)-A_4\omega_-^2\sin\big(\omega_-T\big)=0 \end{aligned} \tag{5.15}$$

To get a nontrivial solution, the correspending determinant must vanish, which yields the condition

$$(\omega_+^2-\omega_-^2)\sin\big(\omega_+T\big)\sin\big(\omega_-T\big)=0 \tag{5.16}$$

Since $\omega_+\neq\omega_-$, we find

$$\sin\big(\omega_+T\big)=0 \qquad \text{or} \qquad \sin\big(\omega_-T\big)=0 \tag{5.17}$$

It means

$$\omega_+=\frac{n\pi}{T} \qquad \text{or} \qquad \omega_-=\frac{n\pi}{T} \qquad \big(\text{where } n \text{ is an integer}\big) \tag{5.18}$$

and $\omega_\pm$ are the solutions to the equation

$$x^2+m^2+\alpha^2x^4=k \tag{5.19}$$

Therefore, the spectrum of k with the boundary condition $\tilde{\mathscr{A}}'$ has the simple form

$$k=\left(\frac{n\pi}{T}\right)^2+m^2+\alpha^2\left(\frac{n\pi}{T}\right)^4 \tag{5.20}$$

6. Canonical quantization and instabilities

In this section we recall about istabilities and ghosts in the quantum PU oscillator (14). The most straightforward way is based on identifying the energy rasing and lowering operators (14). The classical solution (4.14) can be rewritten to the form

$$\begin{aligned} q(t) = &\frac{1}{2}(A_+-iB_+)e^{i\lambda_+t}+\frac{1}{2}(A_++iB_+)e^{-i\lambda_+t} \\ &+\frac{1}{2}(A_--iB_-)e^{i\lambda_-t}+\frac{1}{2}(A_-+iB_-)e^{-i\lambda_+t} \end{aligned} \tag{6.1}$$

Since the λ_- modes have negative energy, the lowering operator must be proportional to the $(A_- - iB_-)$ amplitude. Similarly, since the λ_+ modes have negative energy, the raising operator must be proportional to the $(A_+ + iB_+)$ amplitude, ie.

$$\begin{aligned}\alpha_\pm &\sim A_\pm \pm iB_\pm \\ &\sim \frac{\lambda_\pm}{2}(1 \pm \sqrt{1-4\alpha^2m^2})Q_1 \pm iP_1 \mp \frac{i}{2}(1 \mp \sqrt{1-4\alpha^2m^2}) - \lambda_\pm P_2\end{aligned} \tag{6.2}$$

where we have used

$$A_\pm = \frac{\ddot{q}_0 + \lambda_\mp^2 q_0}{\lambda_\mp^2 - \lambda_\pm^2} \tag{6.3}$$

and

$$B_\pm = \frac{\dddot{q}_0 + \lambda_\mp^2 \dot{q}_0}{\lambda_\pm(\lambda_\mp^2 - \lambda_\pm^2)} \tag{6.4}$$

as well as [1]

$$Q_1 = q_0 \tag{6.5}$$

$$Q_2 = \dot{q}_0 \tag{6.6}$$

$$P_1 = \dot{q}_0 + \alpha^2 \dddot{q}_0 \tag{6.7}$$

$$P_2 = -\alpha^2 \ddot{q}_0 \tag{6.8}$$

It is now straightforward to derive the commutation relations,

$$[\alpha_\pm, \alpha_\pm^\dagger] = 1 \tag{6.9}$$

The next step depends upon physical interpretation (14).

(I) The 'empty' (or 'ground') state may be defined by the condition

$$\alpha_+ \left|\bar{\Omega}\right\rangle = \alpha_-^\dagger \left|\bar{\Omega}\right\rangle = 0 \tag{6.10}$$

Then the 'empty' state wave function $\bar{\Omega}(Q_1, Q_2)$ (in the Q-representation, with $P = -i\partial/\partial Q$) reads

$$\bar{\Omega}(Q_1, Q_2) = N \exp\left[-\frac{\sqrt{1-4\alpha^2m^2}}{2(\lambda_- - \lambda_+)}(\lambda_+\lambda_- Q_1^2 - Q_2^2) - im\alpha Q_1 Q_2\right] \tag{6.11}$$

and is *infinite or not normalizable*, because the size of the wave function gets bigger with the increase of Q_2, so that the integral over the whole space diverges.

In addition, when the eigenstate $|\bar{N}_+, \bar{N}_-\rangle$ with the eigenvalues $\bar{N} = (\bar{N}_+, \bar{N}_-)$ is defined by

$$|\bar{N}_+, \bar{N}_-\rangle = \frac{a_+^\dagger}{\sqrt{N_+!}} \frac{a_-}{\sqrt{N_-!}} \left|\bar{\Omega}\right\rangle \tag{6.12}$$

[1] The canonical variables were calculated at the initial time value because the operators in Schrodinger picture do not depend upon time.

the norm of the $(0,1)$ state is given by

$$\begin{aligned} < 0,\bar{1}|0,\bar{1} > &= \left\langle \bar{\Omega} \right| \alpha_-^\dagger \alpha_- \left| \bar{\Omega} \right\rangle \\ &= \left\langle \bar{\Omega} \right| \left(-1 + \alpha_- \alpha_-^\dagger\right) \left| \bar{\Omega} \right\rangle \\ &= - < \bar{\Omega}|\bar{\Omega} > \\ &= -1 \end{aligned} \tag{6.13}$$

which is a ghost. The non-normalizable quantum 'states' are physically unacceptable, so the interpretation (I) should be dismissed (14).

(II) It is, however, possible to treat all particles (with positive or negative energy) as the truly ones by defining the 'empty' state Ω differently, namely, as

$$\alpha_\pm \left|\Omega\right\rangle = 0 \tag{6.14}$$

In this interpretation the negative energy can arbitrarily decrease and the Hamiltian is unbounded from below. The 'empty' state solution $\Omega(Q_1, Q_2)$ in the Q representation is now given by

$$\Omega(Q_1, Q_2) = N \exp\left[-\frac{\sqrt{1 - 4\alpha^2 m^2}}{2(\lambda_- + \lambda_+)} (\lambda_+ \lambda_- Q_1^2 + Q_2^2) + im\alpha Q_1 Q_2 \right] \tag{6.15}$$

and is apparently *finite or normalizable*, because the first term in the exponential is negative.

The eigenstate $|\bar{N}_+, \bar{N}_-\rangle$ of the eigenvalues $\bar{N} = (\bar{N}_+, \bar{N}_-)$ is now given by

$$\left|N_+, N_-\right\rangle = \frac{a_+^\dagger}{\sqrt{N_+!}} \frac{a_-^\dagger}{\sqrt{N_-!}} \left|\Omega\right\rangle \tag{6.16}$$

while the norm of the $(0,1)$ state is

$$\begin{aligned} < 0,1|0,1 > &= \left\langle \Omega \right| \alpha_- \alpha_-^\dagger \left| \Omega \right\rangle \\ &= \left\langle \Omega \right| \left(1 - \alpha_-^\dagger \alpha_-\right) \left| \Omega \right\rangle \\ &= < \Omega|\Omega > \\ &= 1 \end{aligned} \tag{6.17}$$

ie. it is not a ghost.

In the correct physical interpretation (II) the correspondence principle between the classical and quantum states is preserved, but the system has indefinite energy. When interactions are switched on, mixing the negative and positive energy states would lead to instabilities in the classical theory, and the exponentially growing and decaying states in quantum theory (25; 26). Excluding the negative energy states would lead to the loss of unitarity (21).

7. Path integral quantization and Forman theorem

The idea of ref. (21) is to define the quantum theory of the PU oscillator as the *Euclidean* path integral and then Wick rotate it back to Minkowski case. It makes sense since the Euclidean action of the PU oscillator — see eq. (8.3) below — is positively definite. It can also make the difference to the canonical quantization and the Ostrogradski method (Sec. 2) when one integrates over the path *only*, but not over its derivatives.

Let us first recall some basic facts about a path integral in QFT, according to the standard textbooks in Quantum Field Theory – see, for example, ref. (27).

The definition of the probability amplitude for a one-dimensional quantum particle by Feynman path integral is given by

$$Z(q_b,t_b;q_a,t_a)=\int_{q_a}^{q_b}\mathscr{D}q\exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b}dtL\right] \tag{7.1}$$

where the integration goes over all paths $q(t)$ between q_a and q_b. After Wick rotation

$$t\to t=-i\tau \tag{7.2}$$

the path integral takes the form[2]

$$Z(q_b,t_b;q_a,t_a)=\int_{q_a}^{q_b}\mathscr{D}q\exp\left[-\frac{1}{\hbar}\int_{\tau_a}^{\tau_b}d\tau L_E\right] \tag{7.3}$$

It is called the Euclidean path integral. In the case of the PU oscillator the Euclidean path integral is *Gaussian*. Let us recall some basic properties of the Gaussian integrals.

The simplest Gaussian integral reads

$$\int_{-\infty}^{\infty}dxe^{-ax^2}=\sqrt{\frac{\pi}{a}}\qquad a>0 \tag{7.4}$$

It can be easily extended to a quadratic form in the exponential as

$$\int_{-\infty}^{\infty}dxe^{-ax^2-bx}=\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}\right) \tag{7.5}$$

It can also be easily extended to the case of several variables with the diagonal quadratic form as

$$\int_{-\infty}^{\infty}[d^nx]\exp\left(-\sum_{i=1}^{n}a_ix_i^2\right)=\frac{1}{\prod_{i=1}^{n}a_i^{\frac{1}{2}}} \tag{7.6}$$

where we have introduced the normalized measure $[dx]=dx/\sqrt{\pi}$.

By diagonalizing a generic (non-degenerate) quadratic form, one can prove a general finite-dimensional formula,

$$\begin{aligned}\int_{\infty}^{\infty}[d^nx]\exp\left(-x^tAx-b^tx\right)&=\frac{1}{\prod_{i=1}^{n}\lambda_i}\exp\left(\frac{1}{4}b^tA^{-1}b\right)\\&=\frac{1}{detA}\exp\left(\frac{1}{4}b^tA^{-1}b\right)\end{aligned} \tag{7.7}$$

Finally, when formally sending the number of integrations to infinity, one gets the Gaussian path integral,

$$\begin{aligned}&\int_{q_a}^{q_b}\mathscr{D}q\exp\left[-\int_{t_a}^{t_b}dt(q(t)F(D)q(t)+q(t)J(t))\right]\\&=\frac{1}{\sqrt{DetF(D)}}\exp\left[-\frac{1}{4}\int_{t_a}^{t_b}dtJ(t)F^{-1}(D)J(t)\right]\end{aligned} \tag{7.8}$$

[2] The sign factor in the Wick rotation is chosen to make the path integral converging.

where $DetF(D)$ is now the functional determinant.

A generic functional determinant diverges since it is defined as the product of all the eigenvalues in the spectrum of a differential operator. Therefore, one needs a regularization. It is most convenient to use the zeta function regularization in our case — see, for example, ref. (28) for a comprehensive account. The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{7.9}$$

in the convergence area of the series. It is then expanded for $\mathrm{Re}(s) > 1$ by analytic continuation. It is often useful to employ an integral representation of the zeta function in the form

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt\, t^{s-1} \sum_{n=1}^{\infty} e^{-nt} \tag{7.10}$$

where the (Euler) gamma function has been introduced,

$$\Gamma(s) = \int_0^{\infty} dt\, t^{s-1} e^{-t} \tag{7.11}$$

Equation (7.10) allows one to define the zeta function for an elliptic operator L as

$$\zeta(s|L) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt\, t^{s-1} \mathrm{tr} e^{-tL} \tag{7.12}$$

where $\mathrm{tr}e^{-tL}$ is given by

$$\mathrm{tr}e^{-tL} = \mathrm{tr} \begin{pmatrix} e^{-\lambda_1 t} & & \\ & e^{-\lambda_2 t} & \\ & & \ddots \end{pmatrix} = \sum_{n=1}^{\infty} e^{-\lambda_n t} \tag{7.13}$$

in terms of the positive eigenvalues λ_n of L. One easily finds

$$\zeta(s|L) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} = \sum_{n=0}^{\infty} e^{-s \ln \lambda_n} \tag{7.14}$$

Differentiating both sides of this equation with respect to s at $s = 0$, one finds

$$\begin{aligned} \left. \frac{d\zeta(s|L)}{ds} \right|_{s=0} &= -\sum_{n=1}^{\infty} \ln \lambda_n \\ &= -\ln \prod_{n=1}^{\infty} \lambda_n \\ &= -\ln DetL \end{aligned} \tag{7.15}$$

so that the functional determinant of an elliptic operator L is given by

$$DetL = e^{-\zeta'|_{s=0}} \tag{7.16}$$

The zeta function regularization of the right hand side of this equation is

$$\begin{aligned}\ln Det\frac{L(\epsilon)}{\mu^2} &= -\frac{1}{\epsilon}\zeta\left(\epsilon\left|\frac{L}{\mu^2}\right.\right) = -\frac{1}{\Gamma(\epsilon+1)}\int_0^\infty dt\, t^{\epsilon-1}\mathrm{tr}e^{-t\frac{L}{\mu^2}} \\ &= -\frac{1}{\epsilon}\sum_{n=1}^{\infty}\frac{1}{\left(\frac{\lambda_n}{\mu^2}\right)^\epsilon} = -\frac{\mu^{2\epsilon}}{\epsilon}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^\epsilon} = -\frac{\mu^{2\epsilon}}{\epsilon}\zeta(\epsilon|L) \\ &= -\frac{1}{\epsilon}(1+\epsilon\ln\mu^2)(\zeta(0|L)+\epsilon\zeta'(0|L))+O(\epsilon^2) \\ &= -\frac{1}{\epsilon}\zeta(0|L)-\zeta'(0|L)-\ln\mu^2\zeta(0|L)+O(\epsilon^2) \end{aligned} \tag{7.17}$$

where we have introduced the regularization parameter ϵ and the dimension parameter μ.

The zeta-function renormalization amounts to deleting the first term in eq. (7.17), since it UV-diverges in the limit $\epsilon \to 0$, as well as the third term since it IR-diverges in the limit $\mu \to 0$.

To put equation (7.17) into a more explicit form, without resorting to the spectrum of the differential operator, it is convenient to use Forman's theorem (24): [3]

Let $K_{\mathscr{A}}$ and $\bar{K}_{\bar{\mathscr{A}}}$ are the differential operators defined by

$$\begin{cases} K = P_0(\tau)\frac{d^n}{d\tau^n} + O(\frac{d^{n-1}}{d\tau^{n-1}}) \\ \bar{K} = P_0(\tau)\frac{d^n}{d\tau^n} + O(\frac{d^{n-1}}{d\tau^{n-1}}) \end{cases} \tag{7.18}$$

over the domain $[0, T]$. Consider a linear differential equation

$$Kh(\tau) = 0 \tag{7.19}$$

with a boundary condition

$$\mathscr{A} : M\begin{pmatrix} h(0) \\ h^{(1)}(0) \\ \vdots \\ h^{(n-1)}(0) \end{pmatrix} + N\begin{pmatrix} h(0) \\ h^{(1)}(0) \\ \vdots \\ h^{(n-1)}(0) \end{pmatrix} = 0 \tag{7.20}$$

and take the boundary condition $\bar{\mathscr{A}}$ to be *smoothly* connected to $\mathscr{A}$. The time evolution operator $Y_K(\tau)$ is introduced as

$$\begin{pmatrix} h(\tau) \\ \vdots \\ h^{(n-1)}(\tau) \end{pmatrix} = Y_K(\tau)\begin{pmatrix} h(0) \\ \vdots \\ h^{(n-1)}(0) \end{pmatrix} \tag{7.21}$$

so that the boundary condition can be written to

$$(M + NY_K(T))\begin{pmatrix} h(0) \\ \vdots \\ h^{(n-1)}(0) \end{pmatrix} = 0 \tag{7.22}$$

[3] Forman theorem is an extension of the Gel'fand-Yaglom theorem.

The Forman theorem is given by the statement:

$$\frac{DetK_{\mathscr{A}}}{Det\bar{K}_{\bar{\mathscr{A}}}} = \frac{det\left(M + NY_K(T)\right)}{det\left(\bar{M} + \bar{N}Y_{\bar{K}}(T)\right)} \tag{7.23}$$

This theorem is effective for finding the functional determinant of the operator K with unknown spectrum by connecting it to the one with a simple spectrum via changing the boundary conditions.

8. Path integral of PU oscillator

The Euclidean path integral of the PU oscillator over a domain $[0, T]$ was calculated in refs. (21–23). Here we confirm the results of ref. (22) by our calculation.

The path integral of PU oscillator with the action

$$S_{PU} = \int_0^T dt \left(\frac{1}{2}\dot{q}(t)^2 - \frac{m^2}{2}q(t)^2 - \frac{\alpha^2}{2}\ddot{q}(t)^2\right) \tag{8.1}$$

after the Wick rotation ($t \to it$) takes the form

$$Z(q_T, T; q_0, 0) = \int_{q_0}^{q_T} \mathscr{D}q \exp\left(-S_E\right) \tag{8.2}$$

where the Euclidean PU action is given by

$$S_E = \int_0^T dt \left(\frac{1}{2}\dot{q}(t)^2 + \frac{m^2}{2}q(t)^2 + \frac{\alpha^2}{2}\ddot{q}(t)^2\right) \tag{8.3}$$

This S_E is *positively definite*, so that the Euclidean path integral is well defined.

Since our discussion of the classical theory (Sec. 4), the integral trajectory is a sum of a classical trajectory q_{cl} and quantum fluctuations $\hat{q}$, $q = q_{cl} + \hat{q}$. Accordingly, the action can be also written down as a sum,

$$S_E[q] = S_{cl} + S[\hat{q}] \tag{8.4}$$

and the path integral of the PU oscillator takes the form

$$Z(q_T, T; q_0, 0) = e^{-S_{cl}} \int_0^0 \mathscr{D}\hat{q} \exp\left(-S[\hat{q}]\right) \tag{8.5}$$

where the quantum action $S[\hat{q}]$ is given by

$$S[\hat{q}] = \frac{1}{2}\int_0^T dt\, \hat{q}\left(\alpha^2\frac{d^4}{dt^4} - \frac{d^2}{dt^2} + m^2\right)\hat{q} \tag{8.6}$$

after integration by parts.

Let us denote the differential operator $\alpha^2\frac{d^4}{dt^4} - \frac{d^2}{dt^2} + m^2$ with the boundary condition $\mathscr{A}$ as $K_{\mathscr{A}}$. Then the path integral can be written down in the form

$$Z(q_T, T; q_0, 0) = e^{-S_{cl}} \int_0^0 \mathscr{D}\hat{q} \exp\left(-\frac{1}{2}\int_0^T dt\, \hat{q}K_{\mathscr{A}}\hat{q}\right) \tag{8.7}$$

The path integral of the PU oscillator is Gaussian and, therefore, can be computed along the lines of Sec. 7 as

$$Z(q_T, T; q_0, 0) = \frac{N}{\sqrt{DetK_{\mathscr{A}}}} \exp\left(-S_{cl}\right) \tag{8.8}$$

where N is the normalization constant. The classical part S_{cl} was found in ref. (21), and it is quite involved. The functional determinant is the key part of a quantum propagator of PU oscillator, which is of primary physical interest. It can be computed by the use of Forman theorem (Sec. 7).

First, one calculates the time evolution operator Y_K. It is given by

$$Y_K(t) = \begin{pmatrix} u_1(t) & u_2(t) & u_3(t) & u_4(t) \\ \dot{u_1}(t) & \dot{u_2}(t) & \dot{u_3}(t) & \dot{u_4}(t) \\ \ddot{u_1}(t) & \ddot{u_2}(t) & \ddot{u_3}(t) & \ddot{u_4}(t) \\ \dddot{u_1}(t) & \dddot{u_2}(t) & \dddot{u_3}(t) & \dddot{u_4}(t) \end{pmatrix} \tag{8.9}$$

where

$$Ku_i(t) = 0 \, (i = 1, \ldots, 4) \tag{8.10}$$

and the inital condition is

$$Y_k(0) = 1 \tag{8.11}$$

The operator $K_{\bar{\mathscr{A}}}$ is equal to $K_{\mathscr{A}}$, so they have $Y_K(t)$ is common.

By solving the equation $Ku_i = 0$ for u_i with

$$K = \alpha^2 \frac{d^4}{dt^4} - \frac{d^2}{dt^2} + m^2 \tag{8.12}$$

one gets its general solution in the form

$$u_i(t) = A_i \sinh(\lambda_+ t) + B_i \cosh(\lambda_+ t) + C_i \sinh(\lambda_- t) + D_i \cosh(\lambda_- t) \tag{8.13}$$

The boundary condition $Y_K(0) = 1$ amounts to the relations

$$B_i + D_i = \delta_{1i} \tag{8.14}$$

$$\lambda_+ A_i + \lambda_- C_i = \delta_{2i} \tag{8.15}$$

$$\lambda_+^2 B_i + \lambda_-^2 D_i = \delta_{3i} \tag{8.16}$$

$$\lambda_+^3 A_i + \lambda_-^3 C_i = \delta_{4i} \tag{8.17}$$

Therefore, the solutions are

$$u_1 = \frac{\lambda_-^2}{\lambda_-^2 - \lambda_+^2} \cosh(\lambda_+ t) + \frac{\lambda_+^2}{\lambda_+^2 - \lambda_-^2} \cosh(\lambda_- t) \tag{8.18}$$

$$u_2 = \frac{\lambda_-^2}{\lambda_+(\lambda_-^2 - \lambda_+^2)} \sinh(\lambda_+ t) + \frac{\lambda_+^2}{\lambda_2(\lambda_+^2 - \lambda_-^2)} \sinh(\lambda_- t) \tag{8.19}$$

$$u_3 = -\frac{1}{\lambda_-^2 - \lambda_+^2} \cosh(\lambda_+ t) - \frac{1}{\lambda_+^2 - \lambda_-^2} \cosh(\lambda_- t) \tag{8.20}$$

$$u_4 = -\frac{1}{\lambda_+(\lambda_-^2 - \lambda_+^2)} \sinh(\lambda_+ t) + \frac{1}{\lambda_2(\lambda_+^2 - \lambda_-^2)} \sinh(\lambda_- t) \tag{8.21}$$

Next, one writes down the boundary conditions $\mathscr{A}$ and $\bar{\mathscr{A}}$ in terms of the matrices M and N appearing in the Forman theorem. The boundary condition $\mathscr{A}$ is

$$\mathscr{A} \,:\, \hat{q}(0) = 0\,,\, \hat{q}(T) = 0\,,\, \dot{\hat{q}}(0) = 0\,,\, \dot{\hat{q}}(T) = 0 \tag{8.22}$$

so that its matrices M and N are given by

$$M = \begin{pmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{pmatrix} \tag{8.23}$$

and

$$N = \begin{pmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 1\,0\,0\,0 \\ 0\,1\,0\,0 \end{pmatrix} \tag{8.24}$$

In the same way, the boundary condition $\bar{\mathscr{A}}$ is

$$\bar{\mathscr{A}} \,:\, \hat{q}(0) = 0\,,\, \hat{q}(T) = 0\,,\, \ddot{\hat{q}}(0) = 0\,,\, \ddot{\hat{q}}(T) = 0 \tag{8.25}$$

so that its matrices M and N are given by

$$\bar{M} = \begin{pmatrix} 1\,0 & 0 & 0 \\ 0\,0 & -1 & 0 \\ 0\,0 & 0 & 0 \\ 0\,0 & 0 & 0 \end{pmatrix} \tag{8.26}$$

and

$$\bar{N} = \begin{pmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 1\,0\,0\,0 \\ 0\,0\,1\,0 \end{pmatrix} \tag{8.27}$$

Having found M, N and Y_K, as well as $\bar{M}$, $\bar{N}$ and $Y_{\bar{K}}$, we calculate

$$\begin{aligned} &det(M + NY_K(T)) \\ &= \frac{\alpha^3}{m}\left[\frac{1}{1+2m\alpha}\sinh^2\left(\frac{\sqrt{1+2m\alpha}}{2\alpha}T\right) - \frac{1}{1-2m\alpha}\sinh^2\left(\frac{\sqrt{1-2m\alpha}}{2\alpha}T\right)\right] \end{aligned} \tag{8.28}$$

and

$$\begin{aligned} &det(\bar{M} + \bar{N}Y_{\bar{K}}(T)) \\ &= \frac{\alpha}{m}\left[\sinh^2\left(\frac{\sqrt{1+2m\alpha}}{2\alpha}T\right) - \sinh^2\left(\frac{\sqrt{1-2m\alpha}}{2\alpha}T\right)\right] \end{aligned} \tag{8.29}$$

A calculation of $DetK_{\mathscr{J}}$ goes along the standard lines (21–23),

$$DetK_{\mathscr{J}} = \prod_{n=1}^{\infty} k_n = \prod_{n=1}^{\infty} \left(\alpha^2 \left(\frac{n\pi}{T} \right)^4 + \left(\frac{n\pi}{T} \right)^2 + m^2 \right) \tag{8.30}$$

$$= \frac{\alpha}{mT^2} \left[\sinh^2 \left(\frac{\sqrt{1+2m\alpha}}{2\alpha} T \right) - \sinh^2 \left(\frac{\sqrt{1-2m\alpha}}{2\alpha} T \right) \right] \tag{8.31}$$

By using the Forman formula, one gets the final answer:

$$DetK_{\mathscr{A}} = \frac{det\left(M + NY_K(T)\right)}{det\left(\bar{M} + \bar{N}Y_{\bar{K}}(T)\right)} DetK_{\mathscr{J}}$$

$$= \frac{\alpha^3}{mT^2} \left[(1+2\alpha m)^{-1} \sinh^2 \left(\frac{\sqrt{1+2m\alpha}}{2\alpha} T \right) \right.$$

$$\left. -(1-2\alpha m)^{-1} \sinh^2 \left(\frac{\sqrt{1-2m\alpha}}{2\alpha} T \right) \right] \tag{8.32}$$

in full agreement with ref. (22) in its last (v2) version. In the large T limit one finds

$$DetK_{\mathscr{A}} \approx \frac{\alpha}{m} \left[(1+2\alpha m)^{-1} \frac{\exp\left(\sqrt{1+2m\alpha} \frac{T}{\alpha} \right)}{\left(\frac{2T}{\alpha} \right)^2} - (1-2\alpha m)^{-1} \frac{\exp\left(\sqrt{1-2m\alpha} \frac{T}{\alpha} \right)}{\left(\frac{2T}{\alpha} \right)^2} \right] \ , \tag{8.33}$$

and in the small T limit one gets

$$DetK_{\mathscr{A}} \approx \frac{T^2}{12} + \mathcal{O}(T^4) \tag{8.34}$$

The ground state probability amplitude (or the Euclidean quantum propagator) of PU oscillator, is given by

$$< q_T, \dot{q}_T; \tau = T | q_0, \dot{q}_0; \tau = 0 >= \sqrt{\frac{2\pi}{DetK_{\mathscr{A}}}} \exp\left(-S_E[q_{cl}] \right) \ , \tag{8.35}$$

The classical Euclidean action $S_E[q_{cl}]$ was calculated in Appendix of ref. (21). It is finite for large $T \gg 1$ and behaves like $\frac{1}{2T}$ for small $T \ll 1$. Hence, the transition amplitude (or the quantum Euclidean propagator) is exponentially suppressed both for small and large T, ie. the transition amplitude is normalizable and the Euclidean path integral is well defined indeed.

9. Conclusion

The procedure of calculating Euclidean transition probabilities (for observables) in the quantum PU theory was outlined in ref. (21). The probabilities in the Minkowski space can be obtained by analytic continuation. It is, therefore, possible to make physical sense out of the quantum PU theory.

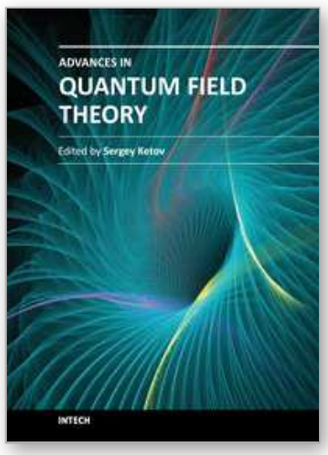
In classical PU theory with interactions, even at a very small value of the parameter $\alpha > 0$, one gets runaway production of states with negative and positive energy. However, as was suggested in ref. (21), the Euclidean formulation of the quantum theory implicitly

imposes certain restrictions that can remove classical instabilities. The price of removing the instabilities is given by an apparent violation of unitarity (21). Indeed, integrating over the basic trajectory, and not over its derivatives in the Euclidean path integral formulation of the quantum PU oscillator given above is not in line with the canonical quantization and the Ostrogradski method. By doing it, one looses some information and, hence, one loses unitarity. As was argued in ref. (21), one can, nevertheless, never produce a negative norm state or get a negative probability, so that the departure from unitarity may be very small at the low energies (say, in the present universe), but important at the very high energies (say, in the early universe). Of course, it is debateable whether the 'price' of loosing unitarity is too high or not.

Apparently, the $f(R)$ gravity theories are special in the sense that for each of them there exist the classically equivalent scalar-tensor field theory without higher derivatives, under the physical stability conditions. Still, as the quantum field theories, they may be very different. It may be possible to quantise $f(R)$ gravity without loosing unitarity. Figuring out details is still a challenge.

10. References

[1] M. Ostrogradski, Mem. Ac. St. Petersbourg VI 4 (1850) 385
[2] A. De Felice and S. Tsujikawa, Living Rev. Rel. 13 (2010) 3
[3] S. James Gates, Jr., and S.V. Ketov, Phys. Lett. B674 (2009) 59
[4] A.A. Starobinsky, Phys. Lett. B91 (1980) 99
[5] E. Komatsu et al., Astrophys. J. Suppl. 192 (2011) 18
[6] S.V. Ketov and A.A. Starobinsky, Phys. Rev. D83 (2011) 063512
[7] S.V. Ketov and N. Watanabe, Phys. Lett. B705 (2011) 410
[8] S.W. Hawking, in "Quantum Field Theory and Quantum Statistics: Essays in Honer of the 60th Birthday of E.S. Fradkin", A. Batalin et al. (Eds.), Hilger, Bristol, UK, 1987
[9] B. Whitt, Phys. Lett. B145 (1984) 176
[10] J.D. Barrow and S. Cotsakis, Phys. Lett. B214 (1988) 515
[11] K.-I. Maeda, Phys. Rev. D39 (1989) 3159
[12] S. Kaneda, S.V. Ketov and N. Watanabe, Mod. Phys. Lett. A25 (2010) 2753
[13] S.V. Ketov and N. Watanabe, JCAP 1103 (2011) 011
[14] R. P. Woodard, Lect. Notes Phys. 720 (2007) 403
[15] K.S. Stelle, Phys. Rev. D16 (1977) 953
[16] E.S. Fradkin and A.A. Tseytlin, Nucl. Phys. B201 (1982) 469
[17] I.L. Buchbinder and S.V. Ketov, Teor. Mat. Fiz. 77 (1988) 42, Fortschr. Phys. 39 (1991) 1
[18] R. Percacci and O. Zanusso, Phys. Rev. D81 (2010) 065012
[19] S.V. Ketov, *Quantum Non-linear Sigma-models*, Springer-Verlag, 2000
[20] A. Pais and G.E. Uhlenbeck, Phys. Rev. 79 (1950) 145
[21] S.W. Hawking and T. Hertog, Phys. Rev. D 65 (2002) 103515
[22] R. Di Criscienzo and S. Zerbini, J. Math. Phys. 50 (2009) 103517; arXiv:0907.4265 v2 [hep-th]
[23] K. Andrzejewski, J. Gonera and P. Maslanka, Prog. Theor.Phys. 125 (2011) 247
[24] R. Forman, Invent. Math. 88 (1987) 447
[25] E. Tomboulis, Phys. Lett. B70 (1977) 361
[26] V.V. Nesterenko, Phys. Rev. D75 (2007) 087703
[27] L.H. Ryder, *Quantum Field Theory*, Cambridge University Press, 1985
[28] E. Elizalde, *Ten Physical Applications of Spectral Zeta Functions*, Springer-Verlag, 1995.



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