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# Fusion Frames and Dynamics of Open Quantum Systems

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## 1. Introduction

Heisenberg's uncertainty principle is one of the manifestations of quantum complementarity. In particular, it states that upon measuring both the momentum and the position of a particle, the product of uncertainties has a fundamental lower bound proportional to Planck's constant. Hence, one cannot measure position and momentum simultaneously with a prescribed accuracy. In general, the quantum complementarity principle does not permit to identify a quantum state from measurements on a single copy of the system unless some extra knowledge is available.

One of the consequences of fundamental assumptions of quantum mechanics is the fact that determination of an unknown state can be achieved by appropriate measurements only if we have at our disposal a set of identically prepared copies of the system in question. Moreover, to devise a successful approach to the above problem of state reconstruction one has to identify a collection of observables, so-called *quorum*, such that their expectation values provide the complete information about the system state.

The problems of state determination have gained new relevance in recent years, following the realization that quantum systems and their evolutions can perform practical tasks such as teleportation, secure communication or dense coding. It is important to realize that if we identify the quorum of observables, then we also have a possibility to determine expectation values of physical quantities (observables) for which *no* measuring apparatuses are available.

Quantum tomography is a procedure of reconstructing the properties of a quantum object on the basis of experimentally accessible data. This means that quantum tomography can be classified by the type of object to be reconstructed:

1. *state tomography* treats density operators, which describe states of quantum systems;
2. *process tomography* discusses linear trace-preserving completely positive maps;
3. *device tomography* treats quantum instruments, and so on.

In what follows, we briefly describe the theory of quantum state tomography (cf. e.g. (Nielsen & Chuang, 2000; Weigert, 2000)).

The aim of quantum state tomography is to identify the density operator characterizing the state of a quantum system under consideration. Let  $\mathcal{H}$  and  $\mathcal{S}(\mathcal{H})$  denote the Hilbert space

corresponding to the system and the set of all density operators on  $\mathcal{H}$ , respectively. We assume that the dimension of  $\mathcal{H}$  is finite,  $\dim \mathcal{H} = N$ . According to the famous Born rule, if an observable corresponding to a Hermitian operator  $Q$  with discrete spectrum is measured in a system whose state is given by the vector  $|\psi\rangle$ , then 1) the measured result will be one of the eigenvalues  $\lambda$  of  $Q$ , and 2) the probability of measuring a given eigenvalue  $\lambda_i$  will be  $\langle \psi | P_i | \psi \rangle$ , where  $P_i$  denotes the projection onto the eigenspace of  $Q$  corresponding to  $\lambda_i$ . These statements are based on the existence of the spectral resolution for any observable  $Q$ . However, if  $Q$  is given as a square matrix of order  $N > 4$ , then it is well known that the problem of calculation of eigenvectors and eigenvalues of  $Q$  over the field  $\mathbb{C}$  of complex numbers is *not* solvable by radicals in the general case. Even more, it is not solvable by any finite procedure in the situation, where only arithmetic operations are allowed. This means that, in fact, for a given  $Q$  we are not able to find effectively the spectral decomposition  $Q = \sum \lambda_i P_i$ . Therefore, we will suppose that the information about the state  $\rho \in \mathcal{S}(\mathcal{H})$  is extracted from the expectation values of some observables  $Q_1, \dots, Q_r$ , i.e.,

$$q_i = \text{Tr}(\rho Q_i), \quad (1)$$

where  $q_i$  are real numbers inferred from the measurement and  $Q_i$  are self-adjoint operators on  $\mathcal{H}$ . (We do not assume the knowledge of spectral decompositions for  $Q_i$ .)

The question, how to construct a quorum of meaningful observables for a given quantum state is quite fundamental. Usually, one can identify only a small number of observables  $Q_1, \dots, Q_r$ , where  $r \ll N^2$ , with clear physical meaning, and their expectation values are not enough for the determination of a quantum state. As a natural remedy for this situation we can ask about the results of the measurements of these observables (their mean values) at different time instants  $t_1, \dots, t_s$  during the time evolution of the system in question (Jamiołkowski, 1982; 1983).

Summing up, as the fundamental objects in modern quantum theory one considers the set of states

$$\mathcal{S}(\mathcal{H}) := \{\rho : \mathcal{H} \rightarrow \mathcal{H}; \rho \geq 0, \text{Tr} \rho = 1\}, \quad (2)$$

and the set of bounded hermitean (self-adjoint) operators

$$\mathcal{B}_* := \{Q : \mathcal{H} \rightarrow \mathcal{H}; Q = Q^*\}. \quad (3)$$

Time evolutions of systems are governed by linear master equations of the form (in the so-called Schrödinger picture)

$$\frac{d\rho(t)}{dt} = \mathbb{K} \rho(t), \quad (4)$$

or in the dual form (in the so-called Heisenberg picture)

$$\frac{dQ(t)}{dt} = \mathbb{L} Q(t), \quad (5)$$

where superoperators  $\mathbb{K}$  and  $\mathbb{L}$  act on operators from the sets  $\mathcal{S}(\mathcal{H})$  and  $\mathcal{B}_*(\mathcal{H})$ , respectively. They represent dual forms of the same physical idea. Both sets  $\mathcal{S}(\mathcal{H})$  and  $\mathcal{B}_*(\mathcal{H})$  can be considered as subsets of the vector space  $\mathcal{B}(\mathcal{H})$  of all bounded linear operators on  $\mathcal{H}$  and

they can be treated as scenes on which problems of quantum mechanical systems should be discussed.

Since in this paper we will consider finite-dimensional Hilbert spaces, therefore in fact  $\mathcal{B}(\mathcal{H})$  denotes the set of *all* linear operators on  $\mathcal{H}$ . If we introduce in  $\mathcal{B}(\mathcal{H})$  the scalar product by the equality

$$\langle A, B \rangle := \text{Tr}(A^* B), \quad (6)$$

then  $\mathcal{B}(\mathcal{H})$  can be regarded as yet another inner product space, namely the so-called *Hilbert-Schmidt space*. It is not difficult to see that  $\mathcal{B}_*(\mathcal{H})$  with scalar product defined by (6) is a *real* vector space and  $\dim \mathcal{B}_*(\mathcal{H}) = N^2$ .

If one does not intend to describe the full dynamics but instead to give a “snapshot” of its effect at a particular time instant  $t$ , then one introduces the idea of a *quantum channel* which mathematically is represented by a completely positive trace preserving (CPTP) map. A completely positive map (a superoperator) is a transformation on density operators defined by the expression

$$\tilde{\rho} = \Phi(\rho(0)) = \sum_i A_i^* \rho(0) A_i, \quad (7)$$

where  $A_i \in \mathcal{B}(\mathcal{H})$  are called *Kraus operators* (Kraus, 1971) or *noise operators* of the map  $\Phi$ . The trace preservation condition implies that

$$\sum_i A_i A_i^* = \mathbb{I}. \quad (8)$$

Let us observe that a unitary evolution is a special case of the CPTP transformation, where there is only one unitary Kraus operator.

According to one of fundamental postulates of quantum theory one assumes that measurements change the state of the system in a way radically different from unitary evolution. The process of making a *von Neumann measurement* is formally described by an expression of the form (7) with the Kraus operators being some commuting self-adjoint idempotent operators  $P_i$  with the property  $\sum P_i = \mathbb{I}$ . A more general concept of measurement was introduced in the 1970-s by Davies and Lewis. This concept is formally expressed as a positive operator-valued measure (POVM) which is defined as a set of positive semidefinite operators  $\{M_k\}$  satisfying  $\sum M_k = \mathbb{I}$  and, obviously, every such  $M_k$  can be expressed in the form  $M_k = F_k F_k^*$  (cf. e.g. Nielsen & Chuang, 2000). The operators  $M_k$  need not commute, and the result of a particular measurement depends, in general, on the order in which the measurements of  $M_k$  are performed.

The idea of stroboscopic tomography for open quantum systems appeared for the first time in the beginning of 1980's (although expressed in different terms (Jamiołkowski, 1982; 1983; 1986)). The main motivation came from quantum optics and the theory of lasers. In particular, using the concept of observability, in (Jamiołkowski, 1983) and (Jamiołkowski, 1986) the question of the minimal number of observables  $Q_1, \dots, Q_\eta$  for which the quantum systems can be  $(Q_1, \dots, Q_\eta)$ -reconstructible was discussed.

On the other hand, theory of *frames*, which are collections of vectors that provide robust and usually non-unique representations of vectors, has been the subject of research in last decades and has been applied in these disciplines where redundancy played a vital and useful role.

However, in some applications it is natural to model and describe considered systems by collections of families of subspaces, and to split a large (global) frame system into a set of much smaller frame systems in these subspaces. This has led to the development of a suitable theory based on *fusion frames* (families of subspaces), which provides the framework to model these more complex applications (Casazza & Kutyniok, 2004; Casazza et al., 2008). In particular, a sequence of the so-called *k-order Krylov subspaces* which appear naturally in stroboscopic tomography (Jamiołkowski, 1986) and are defined by (see also the next Section)

$$\mathcal{K}_k(\mathbb{L}, Q) := \text{Span}_{\mathbb{R}^1} \{Q, \mathbb{L}Q, \dots, \mathbb{L}^{k-1}Q\}, \quad (9)$$

where  $Q$  is a fixed observable and  $\mathbb{L}$  is a generator of time evolution of the system in question, constitutes a fusion frame in the Hilbert-Schmidt space  $\mathcal{B}_*(\mathcal{H})$  if (Jamiołkowski, 2000)

$$\boxplus_{i=1}^r \mathcal{K}_\mu(\mathbb{L}, Q_i) = \mathcal{B}_*(\mathcal{H}). \quad (10)$$

In the above equality  $\mu$  denotes the degree of the minimal polynomial of the superoperator  $\mathbb{L}$  and  $Q_1, \dots, Q_r$  represent fixed observables. The symbol  $\boxplus$  denotes Minkowski sum of subspaces (10) (see (Hauseholder, 2009; Jamiołkowski, 2010)). We recall that for two subspaces  $K_1$  and  $K_2$  of the vector space  $\mathcal{H}$ , by  $K_1 \boxplus K_2$  one understands the smallest subspace of  $\mathcal{H}$  which contains  $K_1$  and  $K_2$ .

It is well known that the Krylov subspaces  $\mathcal{K}_k(\mathbb{L}, Q)$  for  $k = 1, 2, \dots$  form a nested sequence of subspaces of increasing dimensions that eventually become invariant under  $\mathbb{L}$ . Hence for a given  $Q$ , there exists an index  $\mu = \mu(Q)$ , often called the *grade of  $Q$  with respect to  $\mathbb{L}$*  for which

$$\mathcal{K}_1(\mathbb{L}, Q) \subsetneq \dots \subsetneq \mathcal{K}_\mu(\mathbb{L}, Q) = \mathcal{K}_{\mu+1}(\mathbb{L}, Q) = \mathcal{K}_{\mu+2}(\mathbb{L}, Q) \dots \quad (11)$$

It is easy to see, that for a given operator  $Q$ , the natural number  $\mu(Q)$  is equal to the degree of the *minimal polynomial of  $\mathbb{L}$  with respect of  $Q$* . Clearly,  $\mu(Q) \leq \mu(\mathbb{L})$ , where  $\mu(\mathbb{L})$  denotes the degree of the minimal polynomial of superoperator  $\mathbb{L}$  (cf. e.g. (Jamiołkowski, 2000)). Now, let us observe that even if observables  $Q_1, \dots, Q_r$  are linearly independent, the Krylov subspaces  $\mathcal{K}_k(\mathbb{L}, Q_i)$  for  $i = 1, \dots, r$  can have nonempty intersections. At the same time they can constitute a fusion frame for the space of all observables  $\mathcal{B}_*(\mathcal{H})$ .

In the statistical description of physical systems the main role of observables is to statistically identify states, or some of their properties. A typical goal of an experiment can be to decide among various alternatives or hypothesis about states. As a very good reference on such type of problems we recommend the review book (Paris & Rehacek, 2004). The details of a particular identification problem depend on our prior knowledge and the properties we want to discuss. One can say that owing to both the a priori knowledge about states and the knowledge of our technical possibilities we define the alternatives that we should experimentally verify.

In general, depending whether the set of alternatives is finite or not, one makes a distinction between discrimination and estimation problems. One can introduce three different types of problems:

1. *State estimation problem.* In its most general form, one wants to identify the state of a system assuming that no additional (prior) knowledge is available. In other words, the whole state space of a system constitutes the set of possible hypotheses.
2. *Sufficient statistics for families of states.* In this case we are interested in considering only a subset of the whole set of states. We encode prior knowledge about the preparation of states in a multiparameter family of states and consider them as a possible set of hypotheses. For example, we can assume that one considers states which are pure states or have a particular block-diagonal form.
3. *State discrimination problem.* A particular case of the problem 2). One assumes that we want to identify the state which belongs to a finite set  $\{\rho_1, \dots, \rho_p\}$  and our aim is to distinguish among these  $p$  possibilities. It is an obvious observation that in this case the set of observables used for identification can be restricted in an essential way.

All above problems create very interesting particular questions and we will discuss them in separate publications. A general description and some results concerning the problems 2 and 3 based on the idea of fusion frames are discussed in the present paper.

The organization of the paper is as follows: In Section 2, we summarize some concepts and results of the theory of frames; Section 3 presents the main ideas of stroboscopic tomography. We conclude the paper in Section 4 by discussing some applications of the notions of frames and fusion frames to problems of open quantum systems and we discuss some examples of algebraic methods in low-dimensional quantum systems.

## 2. Frames and fusion frames

Frames were first introduced by Duffin and Schaeffer in 1952 as a natural concept that appeared during their research in nonharmonic Fourier analysis (Duffin & Schaeffer, 1952). After more than three decades Daubechies, Grossman and Meyer (Daubechies et al., 1986) initiated the use of frame theory in the description of signal processing. Today, frame theory plays an important role in dozens of applied areas, cf. e.g. (Christensen, 2008; Heil, 2006; Kovacevic & Chebira, 2008).

Let us consider a Hilbert space  $\mathcal{H}$  ( $\dim \mathcal{H} = N < \infty$ ) with scalar product  $\langle \cdot | \cdot \rangle$  which is linear in the second argument. A collection of vectors  $\mathcal{F} = \{|f_i\rangle : i \in \mathcal{I}\}$ ,  $|f_i\rangle \in \mathcal{H}$ , is called a *frame* if there are two positive constants  $\alpha, \beta > 0$  such that for every vector  $x \in \mathcal{H}$

$$\alpha \|x\|^2 \leq \sum_{i \in \mathcal{I}} |\langle f_i | x \rangle|^2 \leq \beta \|x\|^2. \quad (12)$$

One assumes that the number of vectors  $|f_i\rangle$  is greater or equal to  $N$ . The frame is *tight* when the constants  $\alpha$  and  $\beta$  are equal,  $\alpha = \beta$ . If  $\alpha = \beta = 1$ , then  $\mathcal{F}$  is called a *Parseval frame*. The numbers  $\langle f_i | x \rangle$  are called *frame coefficients*.

For a given frame  $\mathcal{F}$  we can introduce the *analysis*  $\Theta$  and *synthesis*  $\Theta^*$  operators. They are defined by the equality

$$\Theta(x) = \sum_{i \in \mathcal{I}} \langle f_i | x \rangle |e_i\rangle, \quad (13)$$

where  $|e_i\rangle$  stands for the standard basis in  $\mathbb{C}^m$  (we will consider only finite dimensional frames, so that  $\mathcal{I} = \{1, \dots, m\}$  and  $m \geq N$ ). Composing  $\Theta$  with its adjoint operator  $\Theta^*$ , we obtain the *frame operator*

$$F : \mathcal{H} \rightarrow \mathcal{H}, \quad (14)$$

defined by

$$Fx := \Theta^* \Theta x = \sum_{i=1}^m \langle f_i | x \rangle |f_i\rangle. \quad (15)$$

It is not difficult to see that any collection of vectors  $\{|f_i\rangle\}_{i=1}^m$  constitutes a frame for the vector space  $\mathcal{N} := \text{span}\{|f_i\rangle\}_{i=1}^m$ ,  $\mathcal{N} \subseteq \mathcal{H}$ . On the other hand a family of elements  $\{|f_i\rangle\}_{i=1}^m$  in  $\mathcal{H}$  is a frame for  $\mathcal{H}$  if and only if  $\text{span}\{|f_i\rangle\}_{i=1}^m = \mathcal{H}$ . This means that a frame may contain more elements than it is necessary for it to be a basis. In particular, if  $\{|f_i\rangle\}_{i=1}^m$  is a frame for  $\mathcal{H}$  and  $\{|g_i\rangle\}_{i=1}^n$  is an arbitrary finite collection of elements in  $\mathcal{H}$ , then the set  $\{|f_1\rangle, \dots, |f_m\rangle, |g_1\rangle, \dots, |g_n\rangle\}$  is also a frame for  $\mathcal{H}$ .

Generally speaking, frame theory is the study of how  $\{|f_i\rangle\}_{i=1}^m$  should be chosen in order to guarantee that the frame operator  $\Theta^* \Theta$  is well-conditioned. In particular,  $\{|f_i\rangle\}_{i=1}^m$  is a frame for  $\mathcal{H}$  if there exist frame bounds  $\alpha, \beta$  such that

$$\alpha \mathbb{I} \leq \Theta^* \Theta \leq \beta \mathbb{I}, \quad (16)$$

and is a tight frame iff  $\Theta^* \Theta = \alpha \mathbb{I}$ . It is an obvious observation that  $F = \Theta^* \Theta$  is a self-adjoint and invertible operator.

*Fusion frame theory* (theory of frames of subspaces) is an emerging mathematical theory that provides a natural setting for performing distributed data processing in many fields Casazza & Kutyniok (2004); Casazza et al. (2008). In particular, one can apply these ideas in quantum state tomography. The notion of fusion frame was introduced in Casazza & Kutyniok (2004) and further developed by Casazza et al. (2008). A fusion frame in a Hilbert space  $\mathcal{H} \cong \mathbb{C}^N$  is a finite collection of subspaces  $\{W_i\}_{i=1}^m$  of  $\mathcal{H}$ , such that there exist constants  $0 < \alpha < \beta < \infty$  satisfying, for any  $|\varphi\rangle \in \mathcal{H}$ , the two inequalities

$$\alpha \|\varphi\|^2 \leq \sum_{i=1}^m \|\mathbb{P}_i \varphi\|^2 \leq \beta \|\varphi\|^2, \quad (17)$$

where  $\mathbb{P}_i$  denotes the non-orthogonal projection on  $W_i$ . In other words, a collection  $\{W_i\}_{i=1}^m$  is a fusion frame if and only if

$$\alpha \mathbb{I} \leq \sum_{i=1}^m \mathbb{P}_i \leq \beta \mathbb{I}. \quad (18)$$

The constants  $\alpha$  and  $\beta$  are called fusion frame bounds. An important class of fusion frames is the class of tight fusion frames, for which  $\alpha = \beta$ . This equality leads to the operator relation  $\sum_{i=1}^m \mathbb{P}_i = \alpha \mathbb{I}$ . Let us note that definition given in (Casazza & Kutyniok, 2004; Casazza et al., 2008) for fusion frames applies to weighted subspaces in any Hilbert space as well. However, since the scope of this paper is limited to non-weighted subspaces only, the definition of a fusion frame is presented for this restricted situation. If we compare the definition of a quantum channel and that of a tight fusion frame, it becomes evident that every quantum channel can be considered a special case of a fusion frame (18) with  $\alpha = \beta = 1$ .

Now, let us recall that for a given operator  $M : \mathcal{H} \rightarrow \mathcal{H}$  and a given fixed nonzero vector  $|x\rangle \in \mathcal{H}$ , one introduces the  $k$ th-order Krylov subspace of  $\mathcal{H}$  by the equality

$$\mathcal{K}_k(M, x) := \text{span}\{|x\rangle, M|x\rangle, \dots, M^{k-1}|x\rangle\}. \quad (19)$$

The above definition can also be written as

$$\mathcal{K}_k(M, x) := \text{span}\{p(M)|x\rangle; \deg(p) \leq k - 1\}, \quad (20)$$

where  $p$  denotes an arbitrary polynomial and  $\deg(p)$  is its degree. It is an obvious observation that the size of a Krylov subspace depends on both  $M$  and  $|x\rangle$ . Note also that there exists such  $k$  that  $\mathcal{K}_k(M, x) = \mathcal{K}_{k+1}(M, x)$  and this  $k$  is the degree of the minimal polynomial of  $M$  with respect to  $|x\rangle$ . If by  $\mu(\lambda, M)$  we denote the minimal polynomial of the operator  $M$ , then the minimal polynomial of  $M$  with respect to any vector  $|x\rangle \in \mathcal{H}$  divides  $\mu(\lambda, M)$ .

For a given operator  $M : \mathcal{H} \rightarrow \mathcal{H}$  Krylov subspaces generated by a fixed set of vectors  $|x_1\rangle, \dots, |x_r\rangle$  constitute a fusion frame in  $\mathcal{H}$  if and only if the following equality is satisfied

$$\bigoplus_{i=1}^r \mathcal{K}_{\mu}(M, x_i) = \mathcal{H}. \quad (21)$$

### 3. Stroboscopic tomography of open quantum systems

Quantum theory — as a description of properties of microsystems — was born more than a hundred years ago. But for a long time it was merely a theory of isolated systems. Only around fifty years ago the theory of quantum systems was generalized. The so-called *theory of open quantum systems* (systems interacting with their environments) was established, and the main sources of inspiration for it were quantum optics and the theory of lasers. This led to the generalization of states (now density operators are considered to be a natural representation of quantum states), and to generalized description of their time evolution. At that time the concept of so-called *quantum master equations* — which preserve positive semi-definiteness of density operators — and the idea of a *quantum communication channel* were born, cf. e.g. (Gorini et al., 1976; Kossakowski, 1972; Kraus, 1971; Lindblad, 1976). On the mathematical level, this approach initiated the study of semigroups of completely positive maps and their generators. Now, for the convenience of the readers, we summarize the main ideas and methods of description of open quantum systems and the so-called stroboscopic tomography.

The time evolution of a quantum system of finitely many degrees of freedom (a qudit) coupled with an infinite quantum system, usually called a reservoir, can be described, under certain limiting conditions, by a one-parameter semigroup of maps (cf. e.g. (Gorini et al., 1976; Jamiołkowski, 1974; Kossakowski, 1972)). Let  $\mathcal{H}$  be the Hilbert space of the first system ( $\dim \mathcal{H} = N$ ) and let

$$\Phi(t) : \mathcal{B}_*(\mathcal{H}) \rightarrow \mathcal{B}_*(\mathcal{H}), \quad t \in \mathbb{R}_+^1, \quad (22)$$

be a dynamical semigroup, where  $\mathcal{B}_*(\mathcal{H})$  denotes the real vector space of all self-adjoint operators on  $\mathcal{H}$ . If one introduces the scalar product of operators  $A, B$  by the formula  $\langle A, B \rangle = \text{Tr}(A^*B)$ , then  $\mathcal{B}_*(\mathcal{H})$  can be considered as yet another inner product space, namely the so-called Hilbert-Schmidt space with the norm defined by  $\|\rho\|^2 = \text{Tr}(\rho^*\rho)$ . States of the

system are described by density operators  $\rho \in S(\mathcal{H})$ , where

$$S(\mathcal{H}) := \{\rho \in \mathcal{B}_*(\mathcal{H}); \rho \geq 0, \text{Tr} \rho = 1\}. \quad (23)$$

Usually one assumes that the family of linear superoperators  $\Phi(t)$  satisfies

1.  $\Phi(t)$  is trace preserving,  $t \in \mathbb{R}_+^1$ ,
2.  $\|\Phi(t)\rho\| \leq \|\rho\|$  for all  $\rho \in \mathcal{B}_*(\mathcal{H})$ ,
3.  $\Phi(t_1) \circ \Phi(t_2) = \Phi(t_1 + t_2)$ ,

for all  $t_1, t_2$  in  $\mathbb{R}_+^1$ , and if  $t \rightarrow 0$ , then  $\lim \Phi(t) = \mathbb{I}$ . Since such defined  $\Phi(t)$  is a contraction, it follows from the Hille-Yosida theorem that there exists a linear superoperator  $\mathbb{K} : \mathcal{B}_*(\mathcal{H}) \rightarrow \mathcal{B}_*(\mathcal{H})$  such that  $\Phi(t) = \exp(t\mathbb{K})$  for all  $t \geq 0$  and

$$\frac{d\rho(t)}{dt} = \mathbb{K}\rho(t), \quad (24)$$

where  $\rho(t) = \Phi(t)\rho(0)$ . One should stress that the above conditions for semigroup  $\Phi(t)$  imply preservation of positivity of density operators,  $\rho(0) \geq 0 \Rightarrow \rho(t) = \Phi(t)\rho(0) \geq 0$  for all  $t \in \mathbb{R}_+^1$ . Now, the above equation (usually called the *master equation*) defines an assignment (the trajectory of  $\rho(0)$ )

$$\mathbb{R}_+^1 \ni t \mapsto \rho(t) \in S(\mathcal{H}), \quad (25)$$

provided that we know the initial state of the system  $\rho(0) \in S(\mathcal{H})$ . The fundamental question of the stroboscopic tomography reads: What can we say about the trajectories (initial state  $\rho(0)$ ) if the only information about the system in question is given by the mean values

$$E_i(t_j) = \text{Tr}(Q_i\rho(t_j)), \quad (26)$$

of, say,  $r$  linearly independent self-adjoint operators  $Q_1, \dots, Q_r$  at some instants  $t_1, \dots, t_p$ , where  $r < N^2 - 1$  and  $t_j \in [0, T]$  for  $j = 1, \dots, p$ ,  $T > 0$ . In other words, the problem of the stroboscopic tomography consists in the reconstruction of the initial state  $\rho(0)$ , or a current state  $\rho(t)$  for any  $t \in \mathbb{R}_+^1$ , from known expectation values (26). To be more precise we introduce the following description. Suppose that we can prepare a quantum system repeatedly in the same initial state and we make a series of experiments such that we know the expectation values  $E_Q(t_j) = \text{Tr}(Q\rho(t_j))$  for a fixed set of observables  $Q_1, \dots, Q_r$  at different time instants  $t_1 < t_2 < \dots < t_p$ . The basic question is: can we find the expectation value of any other operator  $Q \in \mathcal{B}_*(\mathcal{H})$ , that is any other observable from  $\mathcal{B}_*(\mathcal{H})$ , knowing the set of measured outcomes of a given set  $Q_1, \dots, Q_r$  at  $t_1, \dots, t_p$ , i.e. knowing  $E_j(t_k)$  for  $j = 1, \dots, r$  and where  $0 \leq t_1 < t_2 < \dots < t_p \leq T$ , for an interval  $[0, T]$ ?

If the problem under consideration is static, then the state of a  $N$ -level open quantum system (a qudit) can be uniquely determined only if  $r = N^2 - 1$  expectation values of linearly independent observables are at our disposal. However, if we assume that we know the dynamics of our system i.e. we know the generator  $\mathbb{K}$  or  $\mathbb{L} := (\mathbb{K})^*$  (in the Heisenberg picture) of the time evolution, then we can use the stroboscopic approach based on a discrete set of times  $t_1, \dots, t_p$ . In general, we use the term "state-tomography" to denote any kind of state-reconstruction method.

With reference to the terminology used in system theory, we introduce the following definition: An  $N$ -level open quantum system  $S$  is said to be  $(Q_1, \dots, Q_r)$ -reconstructible on the interval  $[0, T]$ , if for every two trajectories defined by the equation (24) there exists at least one instant  $\hat{t} \in [0, T]$  and at least one operator  $Q_k \in \{Q_1, \dots, Q_r\}$  such that

$$\text{Tr}(Q_k \rho_1(\hat{t})) \neq \text{Tr}(Q_k \rho_2(\hat{t})). \quad (27)$$

The above definition is equivalent to the following statement. An  $N$ -level open quantum system  $S$  is  $(Q_1, \dots, Q_r)$ -reconstructible on the interval  $[0, T]$  iff there exists at least one set of time instants  $0 < t_1 < \dots < t_p \leq T$  such that the state trajectory can be uniquely determined by the correspondence

$$[0, T] \ni t_j \mapsto E_i(t_j) = \text{Tr}(Q_i \rho(t_j)), \quad (28)$$

for  $i = 1, \dots, r$  and  $j = 1, \dots, p$ .

Let us observe that in the above definition of reconstructibility we discuss the problem of verifying whether the accessible information about the system is sufficient to determine the state uniquely and we do not insist on determining it explicitly.

The positive dynamical semigroup  $\{\Phi(t), t \in \mathbb{R}_+^1\}$  is determined by the generator  $\mathbb{K} : \mathcal{B}_*(\mathcal{H}) \rightarrow \mathcal{B}_*(\mathcal{H})$  (the Schrödinger picture) and it is related to the generator  $\mathbb{L}$  of the semigroup in the Heisenberg picture by the duality relation

$$\text{Tr}[Q(\mathbb{K}\rho)] = \text{Tr}[(\mathbb{L}Q)\rho]. \quad (29)$$

For a given set of observables  $Q_1, \dots, Q_r$ , the subspace spanned on the operators

$$Q_i, \mathbb{L}Q_i, \dots, (\mathbb{L})^{k-1}Q_i,$$

will be denoted by

$$\mathcal{K}_k(\mathbb{L}, Q_i) := \text{Span}_{\mathbb{R}^1} \{Q_i, \mathbb{L}Q_i, \dots, \mathbb{L}^{k-1}Q_i\}, \quad (30)$$

as the Krylov subspace in the Hilbert-Schmidt space  $\mathcal{B}_*(\mathcal{H})$ . If  $k = \mu$ , where  $\mu$  is the degree of the minimal polynomial of the generator  $\mathbb{L}$ , then the subspace  $\mathcal{K}_\mu(\mathbb{L}, Q_i)$  is an invariant subspace of the superoperator  $\mathbb{L}$  with respect to  $Q_i$ . It can be easily seen that the subspace  $\mathcal{K}_\mu(\mathbb{L}, Q_i)$  is essentially spanned on all operators of the form  $(\mathbb{L})^k Q_i$ , where  $k = 0, 1, \dots$ . Furthermore, it is the smallest invariant subspace of the superoperator  $\mathbb{L}$  containing  $Q_i$  (i.e. the common part of all invariant subspaces of the operator  $\mathbb{L}$  containing  $Q_i$ ).

One can now formulate the sufficient conditions for the reconstructibility of an  $N$ -level open quantum system (c.f. Jamiołkowski (1983; 2000)).

Let  $S$  be an  $N$ -level open quantum system with the evolution governed by an equation of the form  $\dot{Q}(t) = \mathbb{L}Q(t)$  (the Heisenberg picture), where  $\mathbb{L}$  is the generator of the dynamical semigroup  $\Psi(t) = \exp(t\mathbb{L})$ . Suppose that, by performing measurements, the correspondence

$$[0, T] \ni t_j \mapsto E_i(t_j) = \text{Tr}(\rho(0)Q_i(t_j)) \quad (31)$$

can be established for fixed observables  $Q_1, \dots, Q_r$  at selected time instants  $t_1, \dots, t_p$ . The system  $S$  is  $(Q_1, \dots, Q_r)$ -reconstructible if

$$\bigoplus_{i=1}^r \mathcal{K}_\mu(\mathbb{L}, Q_i) = \mathcal{B}_*(\mathcal{H}). \quad (32)$$

The above condition has been obtained by using the polynomial representation of the semigroup  $\Psi(t)$ . Indeed, if  $\mu(\lambda, \mathbb{L})$  denotes the minimal polynomial of the generator  $\mathbb{L}$  and  $\mu = \deg \mu(\lambda, \mathbb{L})$ , then  $\Psi(t) = \exp(t\mathbb{L})$  can be represented in the form

$$\Psi(t) = \sum_{k=0}^{\mu-1} \alpha_k(t) \mathbb{L}^k, \quad (33)$$

where the functions  $\alpha_k(t)$  for  $k = 0, \dots, \mu - 1$  are particular solutions of the scalar linear differential equation with *characteristic polynomial*  $\mu(\lambda, \mathbb{L})$ . Since the functions  $\alpha_k(t)$  are mutually independent, therefore for arbitrary  $T > 0$  there exists at least one set of moments  $t_1, \dots, t_\mu$  ( $\mu = \deg \mu(\lambda, \mathbb{L})$ ) such that

$$0 \leq t_1 < t_2 < \dots < t_\mu \leq T, \quad (34)$$

and  $\det[\alpha_k(t_j)] \neq 0$ . Taking into account these conditions one finds that the state  $\rho(0)$  can be determined uniquely if operators of the form

$$f_{kl} := (\mathbb{L})^k Q_l \quad (35)$$

for  $l = 1, \dots, r$  and  $k = 0, 1, \dots$  span the space  $\mathcal{B}_*(\mathcal{H})$ . In other words, we can say that  $\rho(0)$  can be determined if vectors (35) constitute a frame in Hilbert-Schmidt space  $\mathcal{B}_*(\mathcal{H})$  or, equivalently, if Krylov subspaces  $\mathcal{K}_\mu(\mathbb{L}, Q_l)$  for  $l = 1, \dots, r$  constitute a fusion frame in  $\mathcal{B}_*(\mathcal{H})$ .

It should be noted that almost all the above considerations can be generalized to infinite dimensional Hilbert spaces (Lindblad, 1976, Jamiołkowski, 1982). Such approach is also discussed in a recent literature on infinite dimensional Kraus operators describing amplitude-damping channels and laser processes. For instance, the above techniques are used in the description of such situations in which beamsplitters allow photons to be coupled to another optical modes representing the environment (cf. e.g. Fan & Hu).

### 3.1 Minimal number of observables

The question of an obvious physical interest is to find the minimal number of observables  $Q_1, \dots, Q_\eta$  for which an  $N$ -level quantum system  $S$  with a fixed generator  $\mathbb{L}$  can be  $(Q_1, \dots, Q_\eta)$ -reconstructible. It can be shown that for an  $N$ -level generator there always exists a set of observables  $Q_1, \dots, Q_\eta$ , where

$$\eta := \max_{\lambda \in \sigma(\mathbb{L})} \{\dim \text{Ker}(\lambda \mathbb{I} - \mathbb{L})\}, \quad (36)$$

such that the system is  $(Q_1, \dots, Q_\eta)$ -reconstructible (Jamiołkowski, 2000). Moreover, if we have another set of observables  $\tilde{Q}_1, \dots, \tilde{Q}_{\tilde{\eta}}$  such that the system is  $(\tilde{Q}_1, \dots, \tilde{Q}_{\tilde{\eta}})$ -reconstructible, then  $\tilde{\eta} \geq \eta$ . The number  $\eta$  defined by (36) is called the

*index of cyclicity* of the quantum open system  $S$  (Jamiołkowski, 2000). The symbol  $\sigma(\mathbb{L})$  in (36) denotes the spectrum of the superoperator  $\mathbb{L}$ .

In particular, if we consider an isolated quantum system characterized by Hamiltonian  $H_0$ , then the minimal number of observables  $Q_1, \dots, Q_\eta$  for which the system is  $(Q_1, \dots, Q_\eta)$ -reconstructible is given by

$$\eta = n_1^2 + n_2^2 + \dots + n_m^2, \quad (37)$$

where  $n_i = \dim \text{Ker}(\lambda_i I - H_0)$  for all  $\lambda_i \in \sigma(H_0)$ ,  $i = 1, \dots, m$  (for details cf. Jamiołkowski (1982; 2000)).

Now let us assume that the time evolution of an  $N$ -level quantum system  $S$  is described by the generator  $\mathbb{L}$  given by

$$\mathbb{L}\rho = \frac{1}{2}\{[R\rho, R] + [R, \rho R]\} = -\frac{1}{2}[R, [R, \rho]], \quad (38)$$

that is, we consider the so-called Gaussian semigroup. The symbol  $R$  in (38) denotes a self-adjoint operator with the spectrum

$$\sigma(R) = \{\lambda_1, \dots, \lambda_m\}. \quad (39)$$

In the sequel  $n_i$  stands for the multiplicity of the eigenvalue  $\lambda_i$  for  $i = 1, \dots, m$ . One can assume that the elements of the spectrum of  $R$  are numbered in such a way that the inequalities  $\lambda_1 < \lambda_2 < \dots < \lambda_m$  are fulfilled. The following theorem holds:

The index of cyclicity of the Gaussian semigroup with a generator  $\mathbb{L}$  given by (38) is expressed by the formula

$$\eta = \max\{\kappa, \gamma_1, \dots, \gamma_r\}, \quad (40)$$

where  $r = (m - 1)/2$  if  $m$  is odd or  $r = (m - 2)/2$  if  $m$  is even, and

$$\kappa := n_1^2 + n_2^2 + \dots + n_m^2, \quad (41)$$

$$\gamma_k := 2 \sum_{i=1}^{m-k} n_i n_{i+k}. \quad (42)$$

In order to prove the above theorem and to determine the value of  $\eta$  for the generator  $\mathbb{L}$  defined by (38) we must find the number of nontrivial invariant factors of the operator  $\mathbb{L}$ . Let us observe that if  $\sigma(N) = \{\lambda_1, \dots, \lambda_m\}$ , then the spectrum of the operator  $\mathbb{L}$  is given by

$$\sigma(\mathbb{L}) = \left\{ v_{ij} \in \mathbb{R}; v_{ij} = (\lambda_i - \lambda_j)^2, i, j = 1, \dots, m \right\}. \quad (43)$$

The above statement follows from the fact that the operator  $\mathbb{L}$  can also be represented as

$$\mathbb{L} = R^2 \otimes I + I \otimes R^2 - 2R \otimes R, \quad (44)$$

where  $I$  denotes the identity in the space  $\mathcal{H}$ . Since  $R$  is self-adjoint therefore the algebraic multiplicity of  $\lambda_i$ , i.e. the multiplicity of  $\lambda_i$  as the root of the characteristic polynomial of  $R$ ,

is equal to the geometric multiplicity of  $\lambda_i$ ,  $n_i = \dim \text{Ker} (\lambda_i I - R)$ . Of course, we have  $n_1 + \dots + n_m = \dim \mathcal{H}$ .

From (44) we can see that the multiplicities of the eigenvalues of the operator  $\mathbb{L}$  are not determined uniquely by the multiplicities of  $\lambda_i \in \sigma(R)$ . But if we assume that  $\lambda_1 < \dots < \lambda_m$  and  $\lambda_k = (k-1)c + \lambda_1$ , where  $k = 1, \dots, m$ , and  $c = \text{const} > 0$ , then the multiplicities of all eigenvalues of  $\mathbb{L}$  are given by

$$\gamma_{|i-j|} = \dim \text{Ker} [(\lambda_i - \lambda_j)^2 \mathbb{I} - \mathbb{L}] \quad (45)$$

for  $i \neq j$  and

$$\dim \text{Ker} (\mathbb{L}) = n_1^2 + \dots + n_m^2 = \kappa \quad (46)$$

when  $i = j$ . Now, as we know, the minimal number of observables  $Q_1, \dots, Q_\eta$  for which the qudit  $\mathcal{S}$  can be  $(Q_1, \dots, Q_\eta)$ -reconstructible is given by (36), so in our case

$$\eta = \max_{i,j=1,\dots,m} \left\{ \dim \text{Ker} [(\lambda_i - \lambda_j)^2 \mathbb{I} - \mathbb{L}] \right\}, \quad (47)$$

where  $\lambda_i \in \sigma(R)$ . Using the above formulae and the inequality  $\gamma_k < \kappa$  for  $k > r$ , where  $r$  is given by  $(m-1)/2$  if  $m$  is odd and  $(m-2)/2$  if  $m$  is even, we can observe that also without the assumption  $\lambda_k = (k-1)c + \lambda_1$  one obtains

$$\eta = \max\{\kappa, \gamma_1, \dots, \gamma_r\}. \quad (48)$$

This completes the proof.

### 3.2 The choice of moments of observations

Another natural question arises: what are the criteria governing the choice of time instants  $t_1, \dots, t_\mu$ ? The following theorem holds:

Let us assume that  $0 \leq t_1 < t_2 < \dots < t_\mu \leq T$ . Suppose that the mutual distribution of time instants  $t_1, \dots, t_\mu$  is fixed, i.e. a set of nonnegative numbers  $c_1 < \dots < c_\mu$  is given and  $t_j := c_j t$  for  $j = 1, \dots, \mu$ , and  $t \in \mathbb{R}_+$ . Then for  $T > 0$  the set

$$\tau(T) := \left\{ (t_1, \dots, t_\mu) : t_j = c_j t, \quad 0 \leq t \leq \frac{T}{c_\mu} \right\}$$

contains almost all sequences of time instants  $t_1, \dots, t_\mu$ , i.e. all of them except a finite number.

As one can check, the expectation values  $E_i(t_j)$  and the operators  $(\mathbb{L})^k Q_i$  are related by the equality

$$E_i(t_j) = \sum_{k=0}^{\mu-1} \alpha_k(c_j t) \left( (\mathbb{L})^k Q_i, \rho_0 \right), \quad (49)$$

where we assume that  $t_j = c_j t$  and the bracket  $(\cdot, \cdot)$  denotes the Hilbert-Schmidt product in  $\mathcal{B}_*(\mathcal{H})$ . One can determine  $\rho_0$  from (49) for all those values  $t \in \mathbb{R}_+$  for which the determinant  $\Omega(t)$  is different from zero, i.e.

$$\Omega(t) := \det[\alpha_k(c_j t)] \neq 0. \quad (50)$$

One can prove that the range of the parameter  $t \in \mathbb{R}_+$  for which  $\Omega(t) = 0$  consists only of isolated points on the semiaxis  $\mathbb{R}_+$ , i.e. does not possess any accumulation points on  $\mathbb{R}_+$ . To this end let us note that since the functions  $t \rightarrow \alpha_k(t)$  for  $k = 0, 1, \dots, \mu - 1$ , are analytic on  $\mathbb{R}$ , the determinant  $\Omega(t)$  defined by (50) is also an analytic function of  $t \in \mathbb{R}$ . If  $\Omega(t)$  can be proved to be nonvanishing identically on  $\mathbb{R}$ , then, making use of its analyticity, we shall be in position to conclude that the values of  $t$ , for which  $\Omega(t) = 0$ , are isolated points on the axis  $\mathbb{R}$ .

It is easy to check that for  $k = \mu(\mu - 1)/2$

$$\left. \frac{d^k \Omega(t)}{dt^k} \right|_{t=0} = \prod_{1 \leq j < i \leq \mu} (c_i - c_j). \quad (51)$$

According to the assumption  $c_1 < c_2 < \dots < c_\mu$ , we have  $\Omega^{(k)}(0) \neq 0$  if  $k = \mu(\mu - 1)/2$ . This means that the analytic function  $t \rightarrow \Omega(t)$  does not vanish identically on  $\mathbb{R}$  and the set of values of  $t$  for which  $\Omega(t) = 0$  cannot contain accumulation points. In other words, if we limit ourselves to an arbitrary finite interval  $[0, T]$ , then  $\Omega(t)$  can vanish only on a finite number of points belonging to  $[0, T]$ . This completes the proof.

#### 4. Frames and fusion frames in stroboscopic tomography. Generalizations to subalgebras

As we have seen the concepts of frames and fusion frames appear in stroboscopic tomography in natural way. The conclusion is based on the discussed above polynomial representations of semigroups which describe evolutions of open systems. The possibility to represent the semigroup  $\Phi(t) = \exp(t\mathbb{L})$  in the form

$$\Phi(t) = \sum_{k=0}^{\mu-1} \alpha_k(t) \mathbb{L}^k, \quad (52)$$

where  $\mu$  stands for the degree of the minimal polynomial of the superoperator  $\mathbb{L}$  and  $\alpha_k(t)$ ,  $k = 0, \dots, \mu - 1$ , denote some functions of the eigenvalues of  $\mathbb{L}$  gives the equality (32) as a sufficient condition for stroboscopic tomography. On the other hand, this equality means that the Krylov subspaces  $\mathcal{K}_\mu(\mathbb{L}, Q_i)$ ,  $i = 1, \dots, r$ , constitute a fusion frame in the Hilbert-Schmidt space  $\mathcal{B}_*(\mathcal{H})$  of all observables. Moreover, this also means that the collection of vectors

$$f_{jk} := \mathbb{L}^k Q_j, \quad (53)$$

for  $j = 1, \dots, r$  and  $k = 0, 1, \dots, \mu - 1$ , constitute a frame in  $\mathcal{B}_*(\mathcal{H})$  and the system in question is  $(Q_1, \dots, Q_r)$ -reconstructible. In this case every element  $Q$  of the space  $\mathcal{B}_*(\mathcal{H})$  can be represented as

$$Q = \sum_{j,k} \langle F^{-1} f_{jk} | Q \rangle f_{jk} = \sum_{j,k} \langle f_{jk} | Q \rangle F^{-1} f_{jk}, \quad (54)$$

where  $F$  denotes the frame operator of the collection of vectors (53). One can say even more. If  $Q \in \mathcal{B}_*(\mathcal{H})$  also has another representation  $Q = \sum_{j,k} c_{jk} f_{jk}$  for some scalar coefficients  $c_{jk}$ ,

$j = 1, \dots, r$  and  $k = 0, 1, \dots, \mu - 1$ , then

$$\sum_{jk} |c_{jk}|^2 = \sum_{jk} |\langle F^{-1} f_{jk} | Q \rangle|^2 + \sum_{jk} |c_{jk} - \langle F^{-1} f_{jk} | Q \rangle|^2. \quad (55)$$

It is obvious that every frame in finite-dimensional space contains a subset that is a basis. As a conclusion we can say that if  $\{f_{jk}\}$  is a frame but not a basis, then there exists a set of scalars  $\{d_{jk}\}$  such that  $\sum_{j,k} d_{jk} f_{jk} = 0$ . Therefore, any fixed element  $Q$  of  $\mathcal{B}_*(\mathcal{H})$  can also be represented as

$$Q = \sum_{j,k} \left( \langle F^{-1} f_{jk} | Q \rangle + d_{jk} \right) f_{jk}. \quad (56)$$

The above equality means that every  $Q \in \mathcal{B}_*(\mathcal{H})$  has many representations as superpositions of elements from the set (53). But according to equality (55) among all scalar coefficients  $\{c_{jk}\}$  for which

$$Q = \sum_{j,k} c_{jk} f_{jk}, \quad (57)$$

the sequence  $\{\langle F^{-1} f_{jk} | Q \rangle\}$  has minimal norm. This is a general method in frame theory (Christensen, 2008) and at the same time the main observation connected with the idea of stroboscopic tomography.

In conclusion, one can say that the Krylov subspaces  $\mathcal{K}_\mu(\mathbb{L}, Q_i)$  in the space  $\mathcal{B}_*(\mathcal{H})$  generated by the superoperator  $\mathbb{L}$  can be used in an effective way for procedures of stroboscopic tomography if they constitute appropriate fusion frames in this space.

#### 4.1 Generalizations to subalgebras

Now, we will discuss some problems of reconstruction of quantum states when the Krylov subspaces playing such important role in the stroboscopic tomography are replaced by some subalgebras of the Hilbert-Schmidt space  $\mathcal{B}_*(\mathcal{H})$ . Just as the fundamental theorem of algebra ensures that every linear operator acting on a finite dimensional complex Hilbert space has a nontrivial invariant subspace, the *fundamental theorem of noncommutative algebra* asserts the existence of invariant subspaces of  $\mathcal{H}$  for some families of operators from  $\mathcal{B}(\mathcal{H})$ . It is an obvious observation that an algebra generated by any fixed operator  $Q$  and the identity on  $\mathcal{H}$  can not be equal to  $\mathcal{B}_*(\mathcal{H})$ . This statement is based on the Hamilton-Cayley theorem. However, already for two operators  $Q_1, Q_2$  and the identity we can have  $\text{Alg}(I, Q_1, Q_2) = \mathcal{B}(\mathcal{H})$  (for details cf. below).

In general, the famous Burnside's theorem states (cf. e.g. (Farenick, 2001)) that an operator algebra on a finite-dimensional vector space with no nontrivial subspaces must be the algebra of *all* linear operators. In the sequel we will use the following version of this theorem:

*Fundamental theorem of noncommutative algebras.* If  $\mathcal{A}$  is a proper subalgebra of  $\mathcal{B}(\mathcal{H})$  containing identity, and the dimension of the Hilbert space  $\mathcal{H}$  is greater or equal to 2, then  $\mathcal{A}$  has a proper nonzero invariant subspace in  $\mathcal{H}$  (i.e., the subspace is invariant for all members  $Q$  of the algebra  $\mathcal{A}$ ).

We will apply the above theorem for the following problem. Given a set  $\mathcal{F} = \{Q_1, \dots, Q_r\}$  of observables, we would like to establish conditions, when the operators  $Q_1, \dots, Q_r$  generate the

whole algebra  $\mathcal{B}(\mathcal{H})$ . In other words, we want to determine whether every element in  $\mathcal{B}(\mathcal{H})$  can be represented in the form  $\pi(Q_1, \dots, Q_r)$ , where  $\pi$  is a polynomial in noncommutative variables.

Let us observe that according to the *fundamental theorem* if  $\mathcal{A}$  is a subalgebra of the full complex algebra  $\mathcal{B}(\mathcal{H})$ , then a nontrivial invariant subspace in  $\mathcal{H}$  exists if and only if

$$\dim \mathcal{A} < \dim \mathcal{B}(\mathcal{H}). \quad (58)$$

If a set of generators of  $\mathcal{A}$  is known, then the above inequality can be verified by a finite number of arithmetic operations. The procedures possessing such property are called *effective*. A very important example of an effective procedure can be formulated when we discuss the problem of the existence a common one-dimensional invariant subspace for a pair of operators  $Q_1, Q_2$ . In other words, we ask about a common eigenvector for two operators  $Q_1, Q_2$ . An answer to this question is given by the following procedure. Let the symbol  $[Q_1, Q_2]$  denote, as usual, the commutator of the operators  $Q_1, Q_2$ . Then a common eigenvector for  $Q_1$  and  $Q_2$  exists if and only if the subspace  $\mathcal{K}$  of  $\mathcal{H}$  defined by

$$\mathcal{K} := \bigcap_{\substack{j=1 \\ k=1}}^{N-1} \text{Ker}[Q_1^j, Q_2^k], \quad (59)$$

where  $N = \dim \mathcal{H}$ , satisfies the condition  $\dim \mathcal{K} > 0$  (this is the so-called Shemesh criterion (Shemesh, 1984)). A short proof of this condition is possible.

First of all, let us observe that if  $|\psi\rangle$  is a common eigenvector of the operators  $Q_1$  and  $Q_2$ , i.e.,

$$Q_1|\psi\rangle = \alpha|\psi\rangle \quad \text{and} \quad Q_2|\psi\rangle = \beta|\psi\rangle, \quad (60)$$

then  $|\psi\rangle$  belongs to  $\text{Ker}[Q_1^j, Q_2^k]$  for all  $j, k$  greater than 1. This fact and the inequality  $\dim \mathcal{K} > 0$  means that the gist of the Shemesh condition is in observation that the subspace  $\mathcal{K}$  is invariant under  $Q_1$  and  $Q_2$ . Indeed, if  $|\psi\rangle$  belongs to  $\mathcal{K}$ , then by the definition of subspaces  $\text{Ker}[Q_1^j, Q_2^k]$  one can check that  $Q_1|\psi\rangle \in \mathcal{K}$  and  $Q_2|\psi\rangle \in \mathcal{K}$ . Now, let us choose a basis for  $\mathcal{K}$  and extend it to a basis in  $\mathcal{H}$ . We then observe that there exists a nonsingular matrix  $S$  such that matrices  $SQ_1S^{-1}$  and  $SQ_2S^{-1}$  have block-triangular forms and the submatrices which correspond to subspace  $\mathcal{K}$  commute. This means that these submatrices have a common eigenvector and therefore the same is true for  $Q_1$  and  $Q_2$ . D. Shemesh observed that the condition  $\dim \mathcal{K} > 0$  is equivalent to the singularity of the matrix

$$\mathbb{M} := \sum_{\substack{j=1 \\ k=1}}^{N-1} [Q_1^j, Q_2^k]^* [Q_1^j, Q_2^k], \quad (61)$$

where  $*$  denotes complex conjugate transpose. For our purposes, on the basis of Burnside's theorem, more interesting is the case when matrices  $Q_1, Q_2$  do not have common eigenvectors and the algebra  $A(Q_1, Q_2)$  generated by them coincides with  $\mathcal{B}(\mathcal{H})$ . This situation may be

expressed by the following inequality

$$\det \mathbb{M} > 0, \quad (62)$$

which can be checked by an effective procedure, that is, by a finite number of arithmetic operations. It is obvious, that the matrix  $\mathbb{M}$  is in general semipositive definite, and the above condition means the strict positivity of  $\mathbb{M}$ .

#### 4.2 Examples

In order to illustrate algebraic methods in reconstruction problems, we will discuss some algebraic procedures in low dimensional cases. For quantum systems of qubits and qutrits one can formulate an explicit form of some conditions in a matrix form which is sometimes more transparent than the general operator form. We will use the so-called *vec operator* procedure which transforms a matrix into a vector by stacking its columns one underneath the other. It is well known, that the tensor product of matrices and the *vec operator* are intimately connected. If  $A$  denotes a  $N \times N$  matrix and  $a_j$  its  $j$ -th column, then  $\text{vec } A$  is the  $N^2$ -dimensional vector constructed from  $a_1, \dots, a_N$ . Moreover if  $A, B, C$  are three matrices such that the matrix product  $ABC$  is well defined, then

$$\text{vec}(ABC) = (C^T \otimes A) \text{vec } B. \quad (63)$$

In the above formula  $C^T$  denotes the transposition of the matrix  $C$ . In particular we have

$$\text{vec } A = (\mathbb{I} \otimes A) \text{vec } \mathbb{I} = (A^T \otimes \mathbb{I}) \text{vec } \mathbb{I}. \quad (64)$$

Let us agree that when we say that a set of matrices generates the set  $\mathcal{B}(\mathcal{H})$ , we are thinking about  $\mathcal{B}(\mathcal{H})$  as an algebra, while when we say that a set of matrices forms a basis for  $\mathcal{B}(\mathcal{H})$ , we are talking about  $\mathcal{B}(\mathcal{H})$  as a vector space (here we identify  $\mathcal{B}(\mathcal{H})$  with the set of all matrices on  $\mathcal{H} = \mathbb{C}^N$ ).

For qubits, that is for two-dimensional Hilbert space, one can show by a direct computation that

$$\det(\text{vec } \mathbb{I}, \text{vec } Q_1, \text{vec } Q_2, \text{vec}(Q_1 Q_2)) = \det([Q_1, Q_2]) \quad (65)$$

and

$$\det(\text{vec } \mathbb{I}, \text{vec } Q_1, \text{vec } Q_2, \text{vec}[Q_1, Q_2]) = 2 \det([Q_1, Q_2]), \quad (66)$$

where on the left hand side we have the determinants of the  $4 \times 4$  matrices and on the right hand sides  $[Q_1, Q_2]$  denotes the commutator of the two  $2 \times 2$  matrices.

From the last equality it follows, that if matrices  $\mathbb{I}, Q_1, Q_2$  and  $[Q_1, Q_2]$  are linearly independent, then the algebra which is spanned by them has the dimension 4, so  $Q_1, Q_2$  and  $\mathbb{I}$  generate  $\mathcal{B}(\mathcal{H})$ . In other words, two operators  $Q_1, Q_2$  and the identity generate  $\mathcal{B}(\mathcal{H})$  if and only if the matrix  $[Q_1, Q_2]$  has the determinant different from zero. In a similar way one can show that the matrices  $Q_1, Q_2, Q_3$ , such that *no* two of them generate  $\mathcal{B}(\mathcal{H})$ , can generate  $\mathcal{B}(\mathcal{H})$  if and only if the double commutator  $[Q_1, [Q_2, Q_3]]$  is invertible. In general, the matrices  $Q_1, \dots, Q_r$  generate  $\mathcal{B}(\mathcal{H})$  iff at least one of the commutators  $[Q_i, Q_j]$  or double commutators  $[Q_i, [Q_j, Q_k]]$  is invertible (Aslaksen & Sletsj e, 2009).

In the case of qutrits, that is for a three-dimensional Hilbert space, one can show by direct calculation that if  $[Q_1, Q_2]$  is invertible and  $\omega([Q_1, Q_2]) \neq 0$ , where for  $Q \in \mathcal{B}(\mathcal{H})$  the symbol  $\omega(Q)$  denotes the linear term in the characteristic polynomial of  $Q$ , then one can construct an explicit basis for  $\mathcal{B}(\mathcal{H})$ . Indeed, if  $Q_1, Q_2$  belong to  $\mathcal{B}(\mathcal{H})$ , and  $(\dim \mathcal{H}) = 3$ , then the determinant of the 9-dimensional matrix  $\Omega$  build from vec transformations of  $\mathbb{I}, Q_1, Q_2, Q_1^2, Q_2^2, Q_1 Q_2, Q_2 Q_1, [Q_1, [Q_1, Q_2]], [Q_2, [Q_2, Q_1]]$  satisfies the equality

$$\det \Omega = 9 \det([Q_1, Q_2]) \omega([Q_1, Q_2]). \quad (67)$$

That is, if  $\det([Q_1, Q_2]) \neq 0$  and  $\omega(Q) \neq 0$ , then the columns of the matrix  $\Omega$  correspond to a basis for  $\mathcal{B}(\mathcal{H})$ .

Of course, one can also use the Shemesh criterion to characterize pairs of generators for  $\mathcal{B}(\mathcal{H})$ , where  $\dim \mathcal{H} = 3$ .

## 5. Conclusions

Papers written by mathematicians are usually focused on characterization of various properties of discussed objects and search for necessary and sufficient conditions for desired conclusion to hold. Concrete constructions often play a minor role. The problems of frames and fusion frames are no exceptions. The main purpose of this paper was to discuss properties of some Krylov subspaces in a given Hilbert space as a natural examples of fusion frames and their applications in reconstruction of trajectories of open quantum systems.

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