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Planar Stokes Flows with Free Boundary

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1. Introduction

The quasi-stationary Stokes approximation (Frenkel, 1945; Happel & Brenner, 1965) is used to describe viscous flows with small Reynolds numbers. Two-dimensional Stokes flow with free boundary attracted the attention of many researches. In particular, an analogy is drawn (Ionesku, 1965) between the equations of the theory of elasticity (Muskeleshvili, 1966) and the equations of hydrodynamics in the Stokes approximation. This idea allowed (Antanovskii, 1988) to study the relaxation of a simply connected cylinder under the effect of capillary forces. Hopper (1984) proposed to describe the dynamics of the free boundary through a family of conformal mappings. This approach was later used in (Jeong & Moffatt, 1992; Tanveer & Vasconcelos, 1994) for analysis of free-surface cusps and bubble breakup.
We have developed a method of flow calculation, which is based on the expansion of pressure in a complete system of harmonic functions. The structure of this system depends on the topology of the region. Using the pressure distribution, we calculate the velocity on the boundary and investigate the motion of the boundary. In case of capillary forces the pressure is the projection of a generalized function with the carrier on the boundary on the subspace of harmonic functions (Chivilikhin, 1992).
We show that in the 2D case there exists a non-trivial variation of pressure and velocity which keeps the Reynolds stress tensor unchanged. The correspondent variations of pressure give us the basis for pressure presentation in form of a series. Using this fact and the variation formulation of the Stokes problem we obtain a system of equations for the coefficients of this series. The variations of velocity give us the basis for the vortical part of velocity presentation in the form of a serial expansion with the same coefficients as for the pressure series.
We obtain the potential part of velocity on the boundary directly from the boundary conditions - known external stress applied to the boundary. After calculating velocity on the boundary with given shape we calculate the boundary deformation during a small time step.
Based on this theory we have developed a method for calculation of the planar Stokes flows driven by arbitrary surface forces and potential volume forces. We can apply this method for investigating boundary deformation due to capillary forces, external pressure, centrifugal forces, etc.

Taking into account the capillary forces and external pressure, the strict limitations for motion of the free boundary are obtained. In particular, the lifetime of the configurations with given number of bubbles was predicted.

2. General equations

2.1 The quasi-stationary Stokes approximation

The equations of viscous fluid motion in the quasi-stationary Stokes approximation due to arbitrary surface force f_α and the continuity equation in the region $G \subset R^2$ with boundary Γ have the form

$$\frac{\partial p_{\alpha\beta}}{\partial x_\beta} = 0, \tag{1}$$

$$\frac{\partial v_\beta}{\partial x_\beta} = 0, \tag{2}$$

where $p_{\alpha\beta} = -p\delta_{\alpha\beta} + \mu\left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right)$ is the Newtonian stress tensor; v_α are the components of the velocity; p is the pressure; μ is the coefficient of the dynamical viscosity, which is assumed to be constant. The indices α, β take the values 1, 2. Summation over repeated indices is expected. The boundary conditions have the form

$$p_{\alpha\beta} n_\beta = f_\alpha, \quad \mathbf{x} \in \Gamma \tag{3}$$

where n_α and f_α are the components of the vector of outer normal to the boundary and the surface force. Let Γ_0 be the outer boundary of the region; Γ_k($k = 1,2,...,m$) - the inner boundaries (boundaries of bubbles); $\Gamma = \bigcup_{k=0}^{m} \Gamma_k$ - see Fig.1.

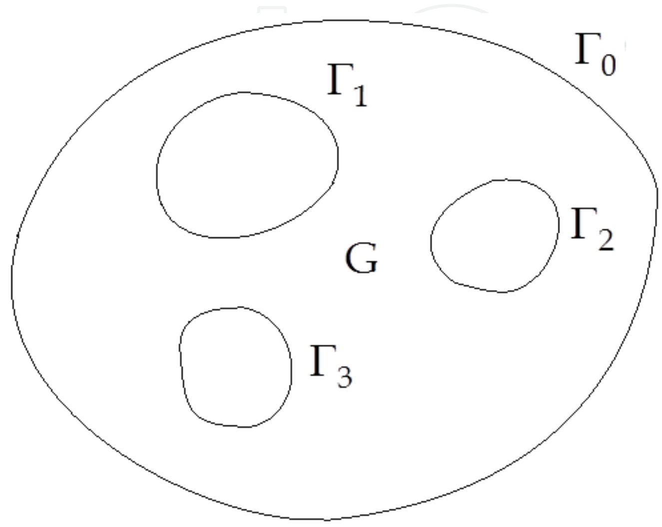

Fig. 1. Region G with multiply connected boundary Γ

The free boundary evolution is determined from the condition of equality of the normal velocity V_n of the boundary and the normal component of the velocity of the fluid at the boundary:

$$V_n = v_\beta n_\beta, \quad \mathbf{x} \in \Gamma \tag{4}$$

In case of a volume force F_α acting on G, the equation of motion takes the form

$$\frac{\partial p_{\alpha\beta}}{\partial x_\beta} = -F_\alpha \tag{5}$$

If the volume force is potential $F_\alpha = -\dfrac{\partial U}{\partial x_\alpha}$ one can renormalize the pressure $p \to p + U$ and present (3), (5) in the form

$$\frac{\partial p_{\alpha\beta}}{\partial x_\beta} = 0 \tag{6}$$

$$p_{\alpha\beta} n_\beta = f'_\alpha, \quad \mathbf{x} \in \Gamma \tag{7}$$

where $f'_\alpha = f_\alpha + U n_\alpha$ is the renormalized surface force.

2.2 The transformational invariance of the Stokes equations

Let's point out a specificity of the quasi-stationary Stokes approximation (1), (2). This system is invariant under the transformation

$$v_\alpha \to v_\alpha + V_\alpha + e_{\alpha\beta} x_\beta \omega \tag{8}$$

where V_α and ω are constants, $e_{\alpha\beta}$ is the unit antisymmetric tensor. Therefore, for this approximation the total linear momentum and the total angular momentum are indefinite. These values should be determined from the initial conditions.

2.3 The conditions of the quasi-stationary Stokes approximation applicability

The Navier-Stokes equations

$$\rho\left(\frac{\partial v_\alpha}{\partial t} + v_\beta \frac{\partial v_\alpha}{\partial x_\beta}\right) = \frac{\partial p_{\alpha\beta}}{\partial x_\beta} + F_\alpha, \tag{9}$$

where ρ is the density of liquid, lead to the quasi-stationary Stokes equations (5) if the convective and non-stationary terms in (9) can be neglected. The neglection of the convective term leads to the requirement of a small Reynolds number $\mathrm{Re} = VL/\nu$, where V is the characteristic velocity, L is the spatial scale of the region G, and ν is the kinematic viscosity. The non-stationary term in the equation (9) can be omitted if during the velocity field relaxation time $T = L^2/\nu$ the shape of the boundary changes insignificantly, namely $VT << L$ which again leads to the condition $\mathrm{Re} << 1$. The change of the volume force F_α and the surface force f_α during the time T should also be small:

$$\frac{\delta F_\alpha}{\delta t}T << F_a, \quad \frac{\delta f_\alpha}{\delta t}T << f_a, \tag{10}$$

For the forces determined by the region shape (like capillary force or centrifugal force) the conditions (10) lead to $\mathrm{Re} << 1$ again.
The neglection of the non-stationary term is a singular perturbation of the motion equation in respect of the time variable. It leads to the formation of a time boundary layer of duration T, during which the initial velocity field relaxates to a quasi-steady state. The condition of a small deformation of the region during this time interval $V^0T << L^0$ is ensured by the requirement of a small Reynolds number Re^0 constructed from the characteristic initial velocity V^0 and the initial region scale L^0.
Let's integrate the motion equation (5) over the region G and use the boundary condition (3). As a result we obtain the condition

$$\Phi_\alpha = \int F_\alpha dG + \int f_\alpha d\Gamma = 0. \tag{11}$$

The equations of viscous fluid motion in the quasi-stationary Stokes approximation (5) have the form of local equilibrium conditions. Correspondingly, the total force Φ_α which acts on the system should be zero. The same way, using (5) and (3) one can obtain the condition

$$M = \int e_{\alpha\beta} x_\alpha F_\beta dG + \int e_{\alpha\beta} x_\alpha f_\beta d\Gamma = 0. \tag{12}$$

where $e_{\alpha\beta}$ is the unit antisymmetric tensor. Therefore, the total moment of force M acting on the system should be zero.

2.4 The Stokes equations in the special noninertial system of reference

Conditions (11) and (12) are the classical conditions of solubility of system (2), (5) with boundary conditions (3). Let's show that these conditions are too restrictive. For example, for a small drop of high viscous liquid falling in the gravitation field the total force is not zero, but equal to the weight of the drop. Therefore, we cannot use the quasi-stationary Stokes approximation to describe the evolution of the drop's shape due to capillary forces. But in a noninertial system of reference which falls together with the drop with the same acceleration, the total force is equal to zero.
In a general case, the total force Φ_α and total moment of force M acting on the system are not equal to zero. The Newton's second law for translational motion has the form

$$\rho S \frac{d\langle v_\alpha \rangle}{dt} = \Phi_\alpha, \tag{13}$$

where S is the area of the region, $\langle v_\alpha \rangle = \frac{1}{S}\int v_\alpha dG$ is the average velocity of the system, and Φ_α is the total force. Let's choose the center-of-mass reference system K' instead of the initial laboratory system K. The velocity and coordinate transformations have the form

$$v'_\alpha = v_\alpha - \langle v_\alpha \rangle, \qquad x'_\alpha = x_\alpha - \langle x_\alpha \rangle, \tag{14}$$

where $\langle x_\alpha \rangle = \frac{1}{S}\int x_\alpha dG$ is the coordinate of the center of mass in the initial system K, $\langle v_\alpha \rangle = \frac{d\langle x_\alpha \rangle}{dt}$. In the new system the surface force is the same as in the initial system $f'_\alpha = f_\alpha$, but the volume force transforms to $F'_\alpha = F_\alpha - \Phi_\alpha$ and total force is equal to zero: $\Phi'_\alpha = 0$. So, we eliminated the total force Φ_α using a noninertial center-of-mass reference system K'.

The total moment of force in the new system stays unchanged: $M' = M$.To eliminate the total moment of force M we switch from the system K' to the rotating reference system K'':

$$v''_\alpha \to v'_\alpha - e_{\alpha\beta} x'_\beta \Omega, \tag{15}$$

where Ω is the angular velocity of the rigid-body rotation

$$I \frac{d\Omega}{dt} = M, \tag{16}$$

where $I = \rho \int x'_\alpha x'_\alpha dG$ is the moment of inertia of our system. In the new system the surface force is the same as in the initial system $f''_\alpha = f'_\alpha$, but the volume force transforms to:

$$F''_\alpha \to F'_\alpha + \rho \left(e_{\alpha\beta} x'_\beta \dot{\Omega} + 2 e_{\alpha\beta} v'_\beta \Omega + \Omega^2 x'_\alpha \right), \tag{17}$$

and the total moment of force is equal zero: $M'' = 0$. In case of a small Reynolds number, the Coriolis force $2\rho e_{\alpha\beta} v'_\beta \Omega$ is small compared with the viscous force.

So in case of the total force Φ_α and total moment of force M not equal to zero we can eliminate them using the noninertial reference system with the rigid-body motion due to the force and moment of force.

3. Pressure calculation

Let χ_α and ψ be smooth fields in the region G related by

$$\frac{\partial \chi_\alpha}{\partial x_\beta} + \frac{\partial \chi_\beta}{\partial x_\alpha} = 2\psi \delta_{\alpha\beta}. \tag{18}$$

Multiplying the equation of motion (1) by χ_α, integrating over G, and using (2), (3), (18), we obtain

$$\int p\psi dG = -\frac{1}{2}\int f_\alpha \chi_\alpha d\Gamma \tag{19}$$

In the special case when $\psi = 1$ the expression (18) gives us $\chi_\alpha = x_\alpha$ and, according with (19),

$$\int p dG = -\frac{1}{2}\int f_\alpha x_\alpha d\Gamma \tag{20}$$

see (Landau & Lifshitz, 1986). In a general case, according with (18), ψ is an arbitrary harmonic function and $\chi = \chi_1 + i\chi_2$ is the analytical function associated with ψ as

$$d\chi = (\psi + i\omega)dz \tag{21}$$

where ω is a harmonic function conjugate to ψ.
The expressions (18) and (19) are basic in our theory. There is also an alternative way to derive them. The equations of motion (1), continuity (2) and the boundary conditions (3) can be obtained from the variation principle (Berdichevsky, 2009).

$$\delta\left[\frac{1}{4\mu}\int\left(p_{\alpha\beta}p_{\alpha\beta} - 2p^2\right)dG - \int f_\alpha v_\alpha d\Gamma\right] = 0 \tag{22}$$

or

$$\frac{1}{2\mu}\int\left(p_{\alpha\beta}\delta p_{\alpha\beta} - 2p\delta p\right)dG - \int f_\alpha \delta v_\alpha d\Gamma = 0 \tag{23}$$

Since (23) is valid for arbitrary variations of pressure δp and velocity δv_α we choose them such that $p_{\alpha\beta}$ is left unchanged:

$$\delta p_{\alpha\beta} = -\delta p \cdot \delta_{\alpha\beta} + \mu\left(\frac{\partial \delta v_\alpha}{\partial x_\beta} + \frac{\partial \delta v_\beta}{\partial x_\alpha}\right) = 0. \tag{24}$$

In this case (23) gives us

$$\frac{1}{\mu}\int p\delta p dG + \int f_\alpha \delta v_\alpha d\Gamma = 0. \tag{25}$$

We introduce the one-parameter family of variations $\delta v_\alpha = \frac{\chi_\alpha}{2\mu}\delta\varepsilon, \delta p = \psi\delta\varepsilon$. Then (24) and (25) take the form (18) and (19).
Suppose $x \in R^N$. Then it follows from (18) that

$$(N-2)\frac{\partial^2 \psi}{\partial x_\alpha \partial x_\beta} = 0. \tag{26}$$

Therefore, in the three-dimensional case ψ is a linear function. Only in the two-dimensional case ψ can be an arbitrary harmonic function. Formulating in terms of (3.5), only in the two-dimensional space there exists a non-trivial system of pressure and velocity variations providing zero stress tensor variation.
The complete set of analytical functions ζ_k in the region G with the multiply connected boundary Γ consists of functions of the form z_k, $\left(z - z_m^o\right)^{-k}$, where z_m^o are fixed points, each situated in one bubble. The complete set of harmonic functions ψ_k can be obtained in the form of $\mathrm{Re}\,\zeta_k$ and $\mathrm{Im}\,\zeta_k$.

According with (1), (2) the pressure p is a harmonic function. We present it in the form

$$p = \sum_k p_k \psi_k. \tag{27}$$

Using the expression (19) we obtain the algebraic system for coefficients p_k:

$$\sum_k \left(\int \psi_k \psi_n dG \right) p_k = -\frac{1}{2} \int f_\alpha \chi_{\alpha n} d\Gamma, \quad n = 0,1,... \tag{28}$$

4. Velocity calculation

The stress tensor, expressed in terms of the Airy function φ,

$$p_{\alpha\beta} = \frac{\partial^2 \varphi}{\partial x_\alpha \partial x_\beta} - \frac{\partial^2 \varphi}{\partial x_\gamma \partial x_\gamma} \delta_{\alpha\beta}, \tag{29}$$

satisfies the equation of motion (1) identically. The boundary conditions (3) take the form

$$e_{\alpha\beta} \tau_\gamma \frac{\partial^2 \varphi}{\partial x_\beta \partial x_\gamma} = -f_\alpha, \quad x \in \Gamma, \tag{30}$$

where τ_γ are the components of the unit tangential vector to the boundary, its direction being matched to the direction of circulation. Integrating (30) along the component boundary Γ_k from a fixed point to an arbitrary one we obtain

$$\frac{\partial \varphi}{\partial x_\alpha} = e_{\alpha\beta} \int f_\alpha d\Gamma_k, \quad x \in \Gamma_k. \tag{31}$$

Using (1), (29) and the explicit form of the stress tensor, we get

$$d\left(\frac{\partial \varphi}{\partial x_\alpha} \right) = 2\mu dv_\alpha + d\Phi_\alpha, \quad x \in G, \tag{32}$$

where

$$d(\Phi_1 + i\Phi_2) = (p + i\Omega), \quad \Omega = \mu \left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \right), \tag{33}$$

Ω is a harmonic function conjugate to p,

$$\frac{\partial \Phi_\alpha}{\partial x_\beta} + \frac{\partial \Phi_\beta}{\partial x_\alpha} = 2p\delta_{\alpha\beta}. \tag{34}$$

Therefore

$$\Phi_\alpha = \sum_n p_n \chi_{\alpha n}, \tag{35}$$

where p_k are the coefficients of the pressure expansion (27). These coefficients are the solution of the system (28). According with (32) the velocity in the region G can be presented in the form

$$v_\alpha = \frac{1}{2\mu}\left(\frac{\partial\varphi}{\partial x_\alpha} - \Phi_\alpha\right), \quad x \in G. \tag{36}$$

The first term in the right-hand part of (36) is the potential part of velocity; the second term is the vortex part.
The gradient of the Airy function on the boundary was calculated in (31). Then we can calculate the velocity on the boundary as

$$v_\alpha = \frac{1}{2\mu}\left(e_{\alpha\beta}\int f_\alpha d\Gamma_k - \Phi_\alpha\right), \quad x \in \Gamma_k. \tag{37}$$

The expression (37) gives us the explicit presentation of the velocity on the boundary.

5. Limitations for the motion of the boundary

5.1 The rate of change of region perimeter

The strong limitation for the motion of the boundary is based on a general expression regarding the rate of change of perimeter L. To obtain this expression we use the fact (Dubrovin at al, 1984) that

$$\frac{d|\Gamma|}{dt} = \int v_\alpha n_\alpha H d\Gamma, \tag{38}$$

where $H = \frac{\partial n_\beta}{\partial x_\beta}$ is the mean curvature of the boundary. In the 2D case $|\Gamma|$ is the perimeter of the region, and in the 3D case $|\Gamma|$ is the area of the boundary. We introduce the operator of differentiation along the boundary $D_{\alpha\beta} = n_\alpha \frac{\partial}{\partial x_\beta} - n_\beta \frac{\partial}{\partial x_\alpha}$. Then we can write (38) in the form

$$\frac{dL}{dt} = \int v_\alpha D_{\alpha\beta} n_\beta d\Gamma. \tag{39}$$

Using the identity

$$\int D_{\alpha\beta}\Lambda d\Gamma = 0, \tag{40}$$

where Λ is an arbitrary field which is continuous on the boundary, and also the equation of continuity (2) and the boundary conditions (3) we can write (39) in the final form

$$\frac{d|\Gamma|}{dt} = -\int \frac{p + f_\alpha n_\alpha}{2\mu} d\Gamma. \tag{41}$$

This expression is valid for any flow of incompressible Newtonian liquid (without Stokes approximation), generally speaking, with variable viscosity. We will use it for a 2D flow ($|\Gamma|$ =L is the perimeter of region), in case of constant viscosity:

$$\frac{dL}{dt} = -\frac{1}{2\mu}\int (p + f_\alpha n_\alpha) d\Gamma. \tag{42}$$

5.2 The dynamics of bubbles due to capillarity and air pressure

Let's take into account the capillary forces on the boundary, the external pressure p_0 and the pressure inside of the bubbles $p_k = p_b$, $k = 1,2,...,m$, equal in every bubble. Then the boundary force has the form

$$f_\alpha = -\sigma n_\alpha \frac{\partial n_\beta}{\partial x_\beta} - p_k n_\alpha, \quad x \in \Gamma_k, \tag{43}$$

where σ is the coefficient of surface tension. Using (42), (43) we get

$$\frac{dL}{dt} = -\frac{1}{2\mu}\left[\int p d\Gamma - (p_0 L_0 + p_b L_b) + 2\pi\sigma(m-1)\right], \tag{44}$$

where L_0 and L_b are the perimeter of external boundary and the total perimeter of the bubbles correspondingly.
Using (20) we obtain

$$\int p dG = p_0 S + (p_0 - p_b) S_b + \frac{\sigma}{2} L, \tag{45}$$

where S and S_b are the area of region and the total area of the bubbles.
For $\psi = p$, $\chi_\alpha = \Phi_\alpha$, the expressions (19), (34), (37) give us

$$\begin{aligned}\int p^2 dG = \frac{\sigma}{2}\left(p_0 L_0 + p_b L_b + \int p d\gamma\right) + p_0^2 S + \left(p_0^2 - p_b^2\right) S_b - \\ -\mu(p_0 - p_b)\frac{dS_b}{dt}.\end{aligned} \tag{46}$$

Using (44) - (46) and the inequality $\int p^2 dG \geq \frac{1}{S}\left(\int p dG\right)^2$ we obtain the differential inequality

$$\begin{aligned}&\mu\left[\sigma\frac{dL}{dt} + (p_0 - p_b)\frac{dS_b}{dt}\right] \leq \\ &\leq -(p_0 - p_b)\left[\sigma L_b + (p_0 - p_b) S_b\right] - \\ &-\frac{1}{S}\left[(p_0 - p_b) S_b + \frac{\sigma}{2} L\right]^2 - \pi\sigma^2(m-1).\end{aligned} \tag{47}$$

This expression gives us the possibility to obtain the strict limitations for the motion of the free boundary in some special cases.

5.3 The influence of capillary forces only

In this case the inequality (47) may be simplified:

$$\frac{dL}{dt} \le -\frac{\sigma}{2\mu}\left[\frac{L^2}{2S} + 2\pi(m-1)\right]. \tag{48}$$

where m is the number of bubbles. Let $L_\infty = 2\sqrt{\pi S}$ be the asymptotic value of the perimeter and let $\tau = \frac{\sigma t}{2\mu}\sqrt{\frac{\pi}{S}}$ be the dimensionless time. Then, according with (48), $L(\tau) \le L_{up}(\tau)$,

$$\begin{aligned} L_{up} &= \frac{L_0 + L_\infty th(\tau)}{L_\infty + L_0 th(\tau)}, \qquad m = 0, \\ L_{up} &= \frac{L_0 L_\infty}{L_\infty + L_0\tau}(L_\infty + L_0\tau), \qquad m = 1, \\ L_{up} &= \frac{\sqrt{m-1}\left(L_0 - L_\infty\sqrt{m-1}\, tg\left(\sqrt{m-1}\,\tau\right)\right)}{\sqrt{m-1} + \lambda_0 tg\left(\sqrt{m-1}\,\tau\right)}, \quad m \ge 2. \end{aligned} \tag{49}$$

where $L_{up}(\tau)$ is the upper limitation for time dependence of the perimeter - see Fig.2. The perimeter of system L lies in the interval $L_\infty \le L \le L_{up}(\tau)$.

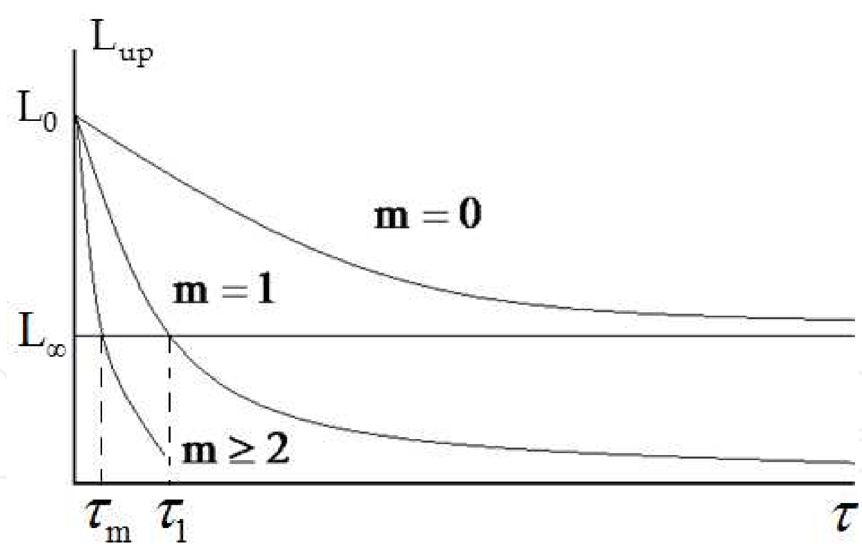

Fig. 2. The upper limitation for the time dependence of the perimeter for various number of bubbles m.

Therefore, if we have no bubbles in the region, the characteristic dimensionless time of relaxation of the boundary to the circle $\tau_0 \le 1$. In case of one bubble $(m=1)$, $L_{up}(\tau) \ge L_\infty$ at the time $\tau \le \tau_1 = 1 - L_\infty/L_0$. The system with this topology can exist in this time period only. The bubble must collapse or break into two bubbles in time $\tau_* \le \tau_1$. In case of $m > 2$ bubbles, such configuration will exist during the time

$$\tau \le \tau_m = \frac{1}{\sqrt{m-1}} arctg\left(\frac{\sqrt{m-1}\left(L_0 - L_\infty\right)}{L_0 + (m-1)L_\infty}\right). \tag{50}$$

5.4 Bubbles in an infinite region

The outer boundary of the region is a circle with a large radius R. The bubbles are localized around the center of the circle. Using the expressions $\pi R^2 - S_b = S, \quad L = 2\pi R + L_b$, we can see that the inequality (47) in the limit $R \to \infty$ takes the form

$$\mu \frac{dW}{dt} \le -\left(p_0 - p_b\right)W - \pi\sigma^2 m\,, \tag{51}$$

where $W = \sigma L_b + \left(p_0 - p_b\right)S_b$. Therefore, at $p_0 - p_b > 0$

$$W + \frac{\pi\sigma^2 m}{p_0 - p_b} \le \left(W(0) + \frac{\pi\sigma^2 m}{p_0 - p_b}\right)\exp\left(-\frac{p_0 - p_b}{\mu}t\right). \tag{52}$$

Because $W \ge 0$, this configuration exists without change of the number of bubbles during the time

$$t \le \frac{\mu}{p_0 - p_b}\ln\left[1 + \frac{p_0 - p_b}{\pi\sigma^2 m}\left(\sigma L_b(0) + \left(p_0 - p_b\right)S_b(0)\right)\right]. \tag{53}$$

6. Motion of the boundary due to capillary forces

6.1 Calculation of pressure and velocity

In case of capillary forces action

$$f_\alpha = -\sigma n_\alpha \frac{\partial n_\beta}{\partial x_\beta}, \quad x \in \Gamma \tag{54}$$

and expression (19) takes the form

$$\int p\psi\, dG = \frac{\sigma}{2}\int \psi\, d\Gamma\,, \tag{55}$$

or

$$\left\langle p\psi \right\rangle_G = \left\langle p \right\rangle_G \left\langle \psi \right\rangle_\Gamma\,, \tag{56}$$

where

$$\left\langle f \right\rangle_G = \frac{1}{S}\int f dG, \quad \left\langle f \right\rangle_\Gamma = \frac{1}{L}\int f d\Gamma, \quad \left\langle P \right\rangle_G = \frac{\sigma}{2S}. \tag{57}$$

The expression (56) is valid for any harmonic function ψ. Let's apply $\psi = p$. Then we obtain

$$\left\langle p^2 \right\rangle_G = \left\langle p \right\rangle_G \left\langle p \right\rangle_\Gamma\,, \tag{58}$$

It can be seen from (58) that

$$\langle p\rangle_{\Gamma} \geq \langle p\rangle_{\Gamma}. \tag{59}$$

Introducing the generalized function (simple layer)

$$\delta_s(\mathbf{x}) = \int \delta(\mathbf{x}-\mathbf{y})dl_{\mathbf{y}}, \tag{60}$$

we see that p is the projection of δ_s onto the subspace of harmonic functions.
Introducing in G a complete system of orthonormal harmonic functions $\{\Xi_k\}_{k=0}^{\infty}$ which obey the orthogonality condition $\langle \Xi_k \Xi_n\rangle_G = \delta_{kn}$, we obtain from (56) the following expression for the pressure

$$p = \langle p\rangle_G \sum_{k=0}^{\infty} \Xi_k \langle \Xi_k\rangle_{\Gamma}. \tag{61}$$

In case of capillary forces the expression (37) takes the form

$$v_\alpha = \frac{1}{2\mu}\left(\sigma n_\alpha - \Phi_\alpha\right), \quad x \in \Gamma. \tag{62}$$

6.2 Relaxation of a small perturbation of a circular cylinder

Consider a small perturbation of the circular cylinder boundary, given by $r = R + h(\varphi, t)$, $|h| << R$. Then we have from (62)

$$\frac{\partial h}{\partial t} = -\frac{\sigma}{2\mu R}\sum_{k=-\infty}^{\infty} |k| \exp(ik\varphi) h_k, \tag{63}$$

$$h_k(t) \equiv \int_0^{2\pi} \exp(-ik\varphi) h(\varphi, t)\frac{d\varphi}{2\pi} = h_k(0)\exp\left(-\frac{\sigma|k|t}{2\mu}\right), \tag{64}$$

in agreement with (Levich, 1962). According with (64), a small boundary perturbation of characteristic with $a << R$ and amplitude $H << a$ has a characteristic decay time $\tau \sim \frac{\mu a}{\sigma}$.

6.3 The capillary relaxation of an ellipse

Let's test our theory on an example of a large amplitude perturbation. We calculate the capillary relaxation of boundary with initial shape $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$ in two ways - using the numerical calculation based on (6.4) and the finite-element software ANSYS POLYFLOW (see Fig. 3 and Fig.4). These methods of calculation give us the same results with discrepancy about 1%.

6.4 The collapse of a cavity

Let's now consider a large amplitude perturbation in the shape of a cavity (Fig. 5). By symmetry, the pressure must be an even function with respect to x_2, i.e. $p(x_1, -x_2) = p(x_1, x_2)$.

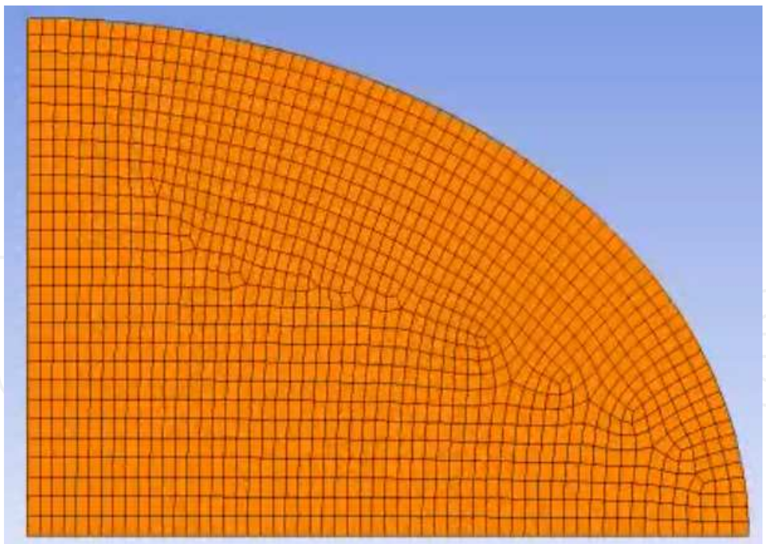

Fig. 3. Computational domain used in finite-element calculation of ellipse relaxation.

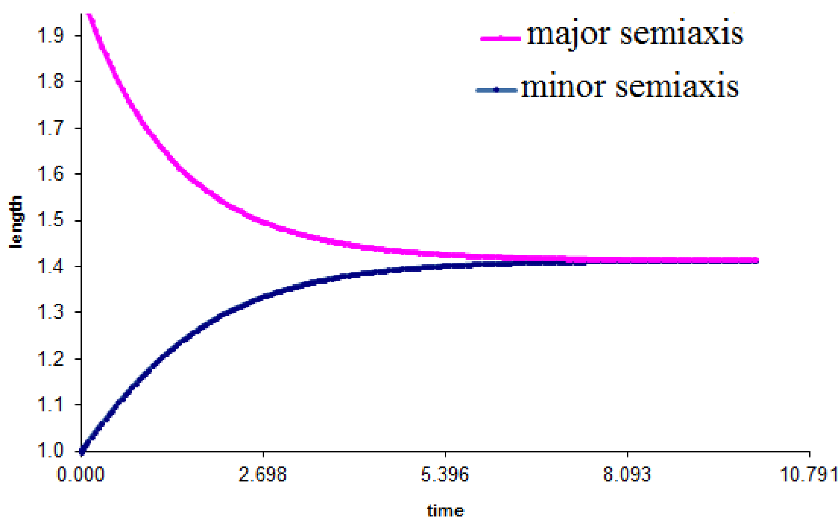

Fig. 4. Relaxation from ellipse to a circle in finite-element calculation.

We introduce a space of two-variable harmonic functions which are even with respect to the second argument, and choose in it the complete system of functions in the form $\psi_n = r^n \cos(n\varphi)$ (r and φ are the polar coordinates in the x_1, x_2 plane). Since the width δ is small $\langle \psi_m \psi_n \rangle_g = R^{2n} \frac{\delta_{mn}}{2(n+1)}$. Then the complete system of orthogonal harmonic functions in this space is

$$\Xi_n = \sqrt{2(n+1)} \left(\frac{r}{R} \right)^n \cos(n\varphi). \tag{65}$$

Inserting (65) in (61) and summing the series yields

$$p = \sigma \left[\frac{1}{R} - \frac{H}{\pi R^2} - \frac{2}{\pi} \mathrm{Re} \left(\frac{1}{R-z} - \frac{R-H}{R^2 - (R-H)R} \right) \right], \tag{66}$$

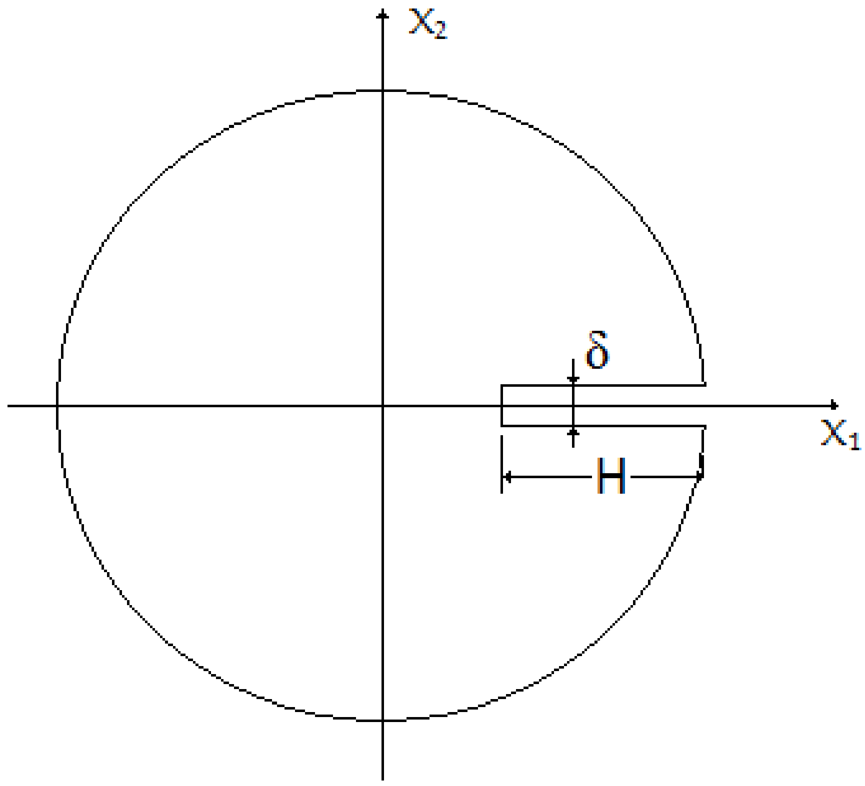

Fig. 5. Cavity perturbation.

whence, using (35), we have

$$\Phi = \sigma\left[\left(1-\frac{H}{\pi R}\right)\frac{z}{R}+\frac{2}{\pi}\ln\left(\frac{R^2-(R-H)z}{(R-H)R}\right)\right]. \tag{67}$$

In spite of the logarithm, (67) is a single-valued analytical function in G, because the boundary perturbation constitutes a branch cut. If we insert (67) in (62), we find that the normal velocity of the cut edges $V=\frac{\sigma}{2\mu}$ (in the zero approximation with respect to the small parameter $\frac{\delta}{H}$). The edges close up after a time $\tau=\frac{\mu\delta}{\sigma}$. Although capillary forces generally tend to flatten the boundary perturbation, in this case they produce the opposite effect. Acting to reduce the length of the cut, the capillary forces generate a flow of scale H in the region. The velocities along x_1 and x_2 have the scales $\dot{H}$ and $\dot{\delta}$, respectively. If we equate the work of surface-tension force with the rate of energy dissipation by viscous forces, we find that $\sigma\dot{H}\approx-\mu\left(\frac{\dot{H}}{H}\right)^2 H^2$ or $\dot{H}\approx\dot{\delta}\approx-\frac{\sigma}{\mu}$; this conforms to the rigorous result we obtained before.

7. Conclusion

We presented a method to calculate two-dimensional Stokes flow with free boundary, based on the expansion of pressure in a complete system of harmonic functions. The theory forms the basis for strict analytical results and numerical approximations. Using this approach we analyse the collapse of bubbles and relaxation of boundary perturbation. The results obtained by this method are correlating well with numerical calculations performed using commercial FEM software.

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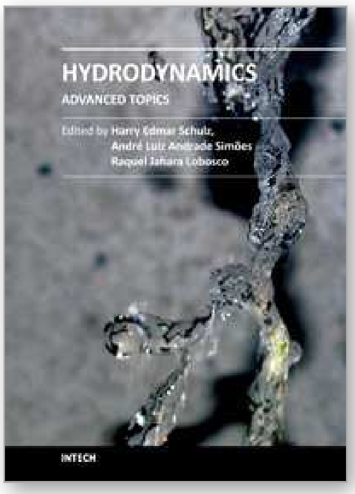

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The phenomena related to the flow of fluids are generally complex, and difficult to quantify. New approaches - considering points of view still not explored - may introduce useful tools in the study of Hydrodynamics and the related transport phenomena. The details of the flows and the properties of the fluids must be considered on a very small scale perspective. Consequently, new concepts and tools are generated to better describe the fluids and their properties. This volume presents conclusions about advanced topics of calculated and observed flows. It contains eighteen chapters, organized in five sections: 1) Mathematical Models in Fluid Mechanics, 2) Biological Applications and Biohydrodynamics, 3) Detailed Experimental Analyses of Fluids and Flows, 4) Radiation-, Electro-, Magnetohydrodynamics, and Magnetorheology, 5) Special Topics on Simulations and Experimental Data. These chapters present new points of view about methods and tools used in Hydrodynamics.