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Meshless Heat Conduction Analysis by Triple-Reciprocity Boundary Element Method

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1. Introduction

The main advantage of the boundary element method (BEM) formulation for the solution of boundary value problems results from the localization of unknowns on the boundary of the analyzed domain. The necessary condition for a pure boundary formulation is the knowledge of the fundamental solution of the governing differential operator. In addition to the reduction of the dimensionality, other advantages of the BEM formulation include good conditioning of the discretized equations, high accuracy and the stability of numerical computations because of the utilization of fundamental solutions. Sometimes, domain integrals are also involved in integral equation formulations; in such cases, the advantageousness of the BEM formulation is partially decreased. The most frequent reasons for the occurrence of domain integrals are body sources, nonlinear constitutive laws and nonvanishing initial conditions in time-dependent problems (Partridge et al., 1992, Sladek and Sladek, 2003, Tanaka et al., 2003).

Since the fundamental solution for a diffusion operator is available in closed form, one can attempt to achieve a pure boundary integral formulation for transient heat conduction problems considered within the linear theory. This can be easily achieved provided that the initial temperature and/or heat sources are distributed uniformly. Then, one can convert the domain integrals of the fundamental solution into boundary integrals using the higher-order polyharmonic fundamental solutions (Nowak, 1989, 1994). As regards the discretization of the time variable, two time-marching schemes are appropriate in formulations with time-dependent fundamental solutions. In one of them, the integration is performed from the initial time to the current time, while in the second scheme the integration is considered within a single time step, taking the temperature at the end of the previous time step as the initial value (pseudo-initial) at the current time step (Ochiai, 2006). Although the domain integral of the uniform initial temperature can be avoided in the first time-marching scheme, the number of boundary integrals increases with increasing number of time steps even in this special case. On the other hand, the spatial integrations are performed only once and are used at each time step in the second scheme provided that a constant length of the time steps is used. The time-marching scheme with integration within a single time step increases the efficiency of numerical integration over boundary elements. The integral formulation as well as the triple-reciprocity approximation are derived in this chapter. The higher-order polyharmonic fundamental solutions and their time integrals are shown in the Appendies. The numerical examples given concern the investigation of the accuracy of the proposed BEM formulation using the triple-reciprocity approximation of either pseudo-initial temperatures or body heat sources.

In this chapter, the steady and unsteady problems in the one-, two- and three-dimensional cases are discussed. In the triple-reciprocity BEM, the distributions of heat generation and initial temperature are interpolated using two Poisson equations. These two Poisson equations are solved using boundary integral equations. This interpolation method is very important in the triple-reciprocity BEM. This numerical process is particularly focused on this chapter.

2. Basic equations

2.1 Steady heat conduction

Point and line heat sources can easily be treated by the conventional BEM. In this study an arbitrarily distributed heat source W_1^S is treated. In steady heat conduction problems, the temperature T under an arbitrarily distributed heat source W_1^S is obtained by solving the following equation (Carslaw, 1938):

$$\nabla^2 T = \frac{-W_1^s}{\lambda} \, , \tag{1}$$

where λ is thermal conductivity. Denoting heat generation by $W_1^S(q)$, the boundary integral equation for the temperature in the case of steady heat conduction is given by (Brebbia, 1984)

$$\mathrm{cT(P)} = \int_\Gamma \{\mathrm{T_1(P,Q)} \frac{\partial T(Q)}{\partial n} - \frac{\partial T_1(P,Q)}{\partial n} T(Q)\} d\Gamma(Q) + \lambda^{-1} \int_\Omega T_1(P,q) W_1^S(q) d\Omega \, , \tag{2}$$

where $c = 0.5$ on the smooth boundary and $c = 1$ in the domain. The notations Γ and Ω represent the boundary and domain, respectively. The notations p and q become P and Q on the boundary.

In one-dimensional problems, the fundamental solution $T_1(p,q)$ in Eq. (2) used for steady temperature analyses and its normal derivative are given by

$$T_1(p,q) = -\frac{1}{2} r \tag{3}$$

$$\frac{\partial T_1(p,q)}{\partial n} = -\frac{1}{2} \frac{\partial r}{\partial n} \, . \tag{4}$$

In two-dimensional problems,

$$T_1(p,q) = \frac{1}{2\pi} \ln(\frac{1}{r}) \tag{5}$$

$$\frac{\partial T_1(p,q)}{\partial n} = \frac{-1}{2\pi r}\frac{\partial r}{\partial n}, \tag{6}$$

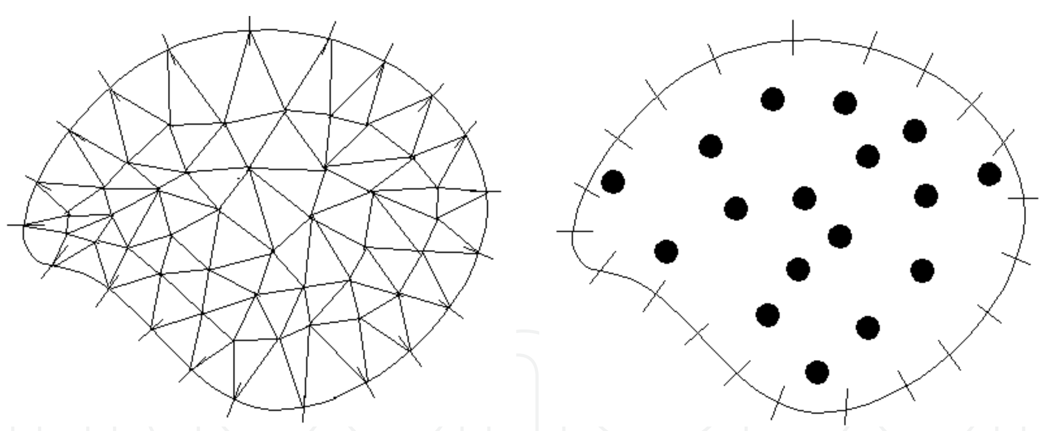
and in three-dimensional problems,

$$\mathrm{T}_1(\mathrm{p,q}) = \frac{1}{4\pi \mathrm{r}} \tag{7}$$

$$\frac{\partial T_1(p,q)}{\partial n} = \frac{-1}{4\pi r^2}\frac{\partial r}{\partial n}, \tag{8}$$

where r is the distance between the observation point p and the loading point q. As shown in Eq. (2),when arbitrary heat generation $W_1^S(q)$ exists in the domain, a domain integral is necessary.

In the triple-reciprocity BEM, the distribution of heat generation is interpolated using integral equations. Using these interpolated values, a heat conduction problem with arbitrary heat generation can be solved without internal cells by the triple-reciprocity BEM. The conventional BEM requires internal cells for the domain integral. The internal cells decrease the advantageousness of the BEM, in which the arrangement of data is simple. In the triple-reciprocity BEM, the fundamental solution of lower order is used. The triple-reciprocity BEM requires internal points similarly to the dual reciprocity method (DRM) (Partridge, 1992) as shown in Fig. 1, although the boundary values W_f need not be given analytically.



(a) Internal cells (b) Internal points

Fig. 1. Triple-reciprocity BEM.

2.2 Interpolation of heat generation

The distribution of heat generation W is interpolated using integral equations to transform the domain integral into the boundary integral. The deformation of a thin plate is utilized to interpolate the distribution of the heat source W_1^S, where superscript S indicates a surface distribution. The following equations can be used for interpolation (Ochiai, 1995a-c, 1996a, b):

$$\nabla^2 W_1^S = -W_2^S, \tag{9}$$

$$\nabla^2 W_2^S = -\sum_{m=1}^{M} W_3^P(q_m), \tag{10}$$

where W_3^P is a Dirac-type function, which has a value at only one point. The term W_2^S in Eq. (9) corresponds to the sum of curvatures $\partial^2 W_1^S / \partial x^2$ and $\partial^2 W_1^S / \partial y^2$. From Eqs. (9) and (10), the following equation can be obtained:

$$\nabla^4 W_1^S = \sum_{m=1}^{M} W_3^P(q_m). \tag{11}$$

This equation is the same type of equation as that for the deformation w_1^S of a thin plate with point load P, which is

$$\nabla^4 w_1^S = \sum_{m=1}^{M} \frac{P_m}{D}, \tag{12}$$

where the Poisson's ratio is ν =0 and the flexural rigidity is D=1. A natural spline originates from the deformation of a thin beam, which is used to interpolate the one-dimensional distribution, as shown in Fig. 2. In this chapter, the deformation of a thin plate is utilized to interpolate the two-dimensional distribution W_1^S. The deformation w_1^S is given, and the force of point load P is unknown and is obtained inversely from the deformation of the fictitious thin plate, as shown in Fig. 3. The term W_2^S corresponds to the moment of the beam.

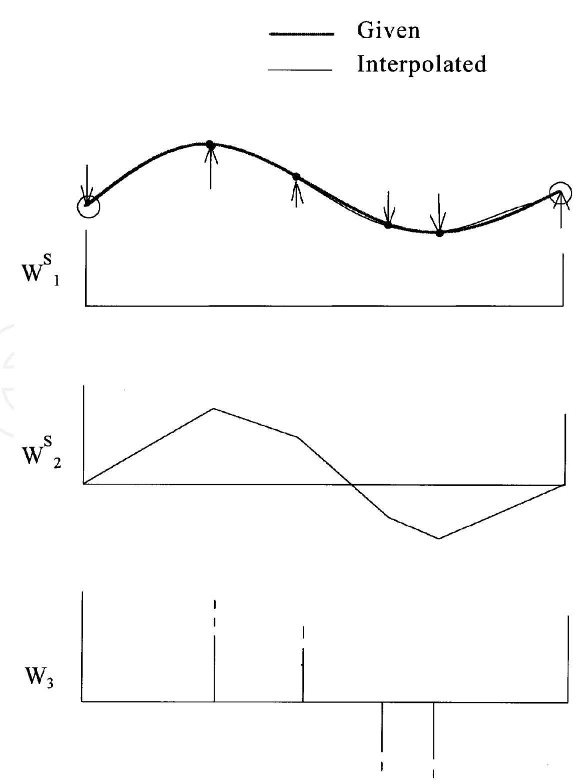
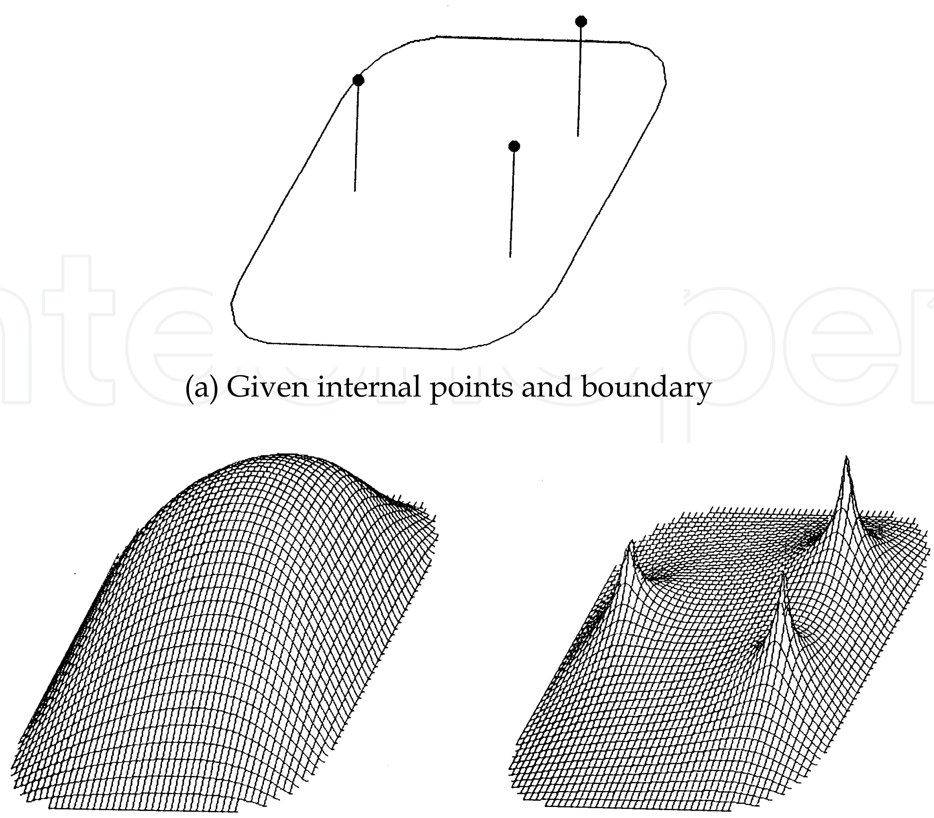


Fig. 2. Interpolation using thin beam with unknown point load.



(a) Given internal points and boundary

(b) Obtained shape W_1^S (c) Sum of curvatures W_2^S

Fig. 3. Interpolation using fictitious thin plate with unknown point load.

The moment W_2^S on the boundary is assumed to be 0, the same as in the case of the natural spline. This means that the thin plate is simply supported. In this method, the distribution of heat generation is assumed to be that for a freeform surface (Ochiai, 1995c). Equations (9) and (10) are similar to the equation used to generate a freeform surface using integral equations.

2.3 Representation of heat generation by integral equations

The distribution of heat generation is represented by an integral equation. The following harmonic function $T_1(p,q)$ and biharmonic function $T_2(p,q)$ are used for interpolation (Ochiai, 1999-2003).

$$T_1(p,q) = \frac{1}{2\pi}[\ln(\frac{1}{r}) + B] \tag{13}$$

$$T_2(p,q) = \frac{r^2}{8\pi}[\ln(\frac{1}{r}) + B + 1] \tag{14}$$

B is an arbitrary constant. $T_1(p,q)$ and $T_2(p,q)$ have the relationship

$$\nabla^2 T_2(p,q) = T_1(p,q) . \tag{15}$$

Let the number of W_3^P be M. The heat generation W_1^S is given by Green's theorem and Eqs. (9), (10) and (15) as

$$\begin{aligned} cW_1^S(P) &= \int_\Gamma \{T_1(P,Q)\frac{\partial W_1^S(Q)}{\partial n} - \frac{\partial T_1(P,Q)}{\partial n}W_f^S(Q)\}d\Gamma(Q) + \sum_{m=1}^{M} T_2(P,q)W_3^P(q_m) \\ &= \sum_{f=1}^{2}(-1)^{f+1}\int_\Gamma \left\{T_f(P,Q)\frac{\partial W_f^S(Q)}{\partial n} - \frac{\partial T_f(P,Q)}{\partial n}W_f^S(Q)\right\}d\Gamma - \sum_{m=1}^{M} T_2(P,q_m)W_3^P(q_m), \end{aligned} \tag{16}$$

where $c = 0.5$ on the smooth boundary and $c = 1$ in the domain. Moreover, W_2^S in Eq. (10) is similarly given by

$$cW_2{}^S(P) = \int_\Gamma \left\{T_1(P,Q)\frac{\partial W_2^S(Q)}{\partial n} - \frac{\partial T_1(P,Q)}{\partial n}W_2^S(Q)\right\}d\Gamma + \sum_{m=1}^{M} T_1(P,q_m)W_3^P(q_m)\,. \tag{17}$$

The integral equations (16) and (17) are used to interpolate the distribution. The thin plate spline $F(p, q)$ used to make a freeform surface is defined as (Dyn, 1987, Micchelli, 1986)

$$F(p,q)=r^2\ln(r). \tag{18}$$

Equations (14) and (18) include the same type of function. Assuming $W_1^S(Q) = 0$, the values of W_3^P and $\partial W_f^S / \partial n$ are obtained using Eqs. (16) and (17).

2.4 Polyharmonic functions

The polyharmonic function $T_f(p,q)$ is defined by

$$\nabla^2 T_{f+1} = T_f\,. \tag{19}$$

Therefore, $T_f(p,q)$ for the Kth dimensional case can be obtained using the next equation

$$T_f = \int \frac{1}{r^{K-1}}[\int r^{K-1}T_{f-1}dr]dr\,. \tag{20}$$

From Eq. (20), $T_f(p,q)$ for the two-dimensional case can be obtained using the next equation

$$T_f = \int \frac{1}{r}[\int rT_{f-1}dr]dr\,. \tag{21}$$

The function $T_f(p,q)$ and its normal derivative for the two-dimensional case are explicitly expressed as

$$T_f(p,q) = \frac{r^{2(f-1)}}{2\pi[(2f-1)!!]^2}[\ln(\frac{1}{r}) + B + \mathrm{sgn}(f-1)\sum_{e=1}^{f-1}\frac{1}{e}], \tag{22}$$

$$\frac{\partial T_f(p,q)}{\partial n} = \frac{r^{2f-3}}{2\pi[(2f-2)!!]^2}[2(f-1)\{\ln(\frac{1}{r})+B\} \;\; -1+2(f-1)\sum_{e=1}^{f-1}\frac{1}{e}]\frac{\partial r}{\partial n} \;, \tag{23}$$

where $(2f-1)!!=(2f-1)(2f-3)(2f-5)....1$, and sgn() is the sign function.
For the one-dimensional case,

$$T_f(p,q) = -\frac{1}{2}\frac{r^{2f-1}}{(2f-1)!} \,, \tag{24}$$

$$\frac{\partial T_f(p,q)}{\partial n} = \frac{1}{2}\frac{r^{2f-2}}{(2f-2)!}\frac{\partial r}{\partial n} \,. \tag{25}$$

For the three-dimensional case,

$$T_f(p,q) = \frac{r^{2f-3}}{4\pi(2f-2)!} \,, \tag{26}$$

$$\frac{\partial T_f(p,q)}{\partial n} = \frac{(2f-3)r^{2f-4}}{4\pi(2f-2)!}\frac{\partial r}{\partial n} \,. \tag{27}$$

Equations (16) and (17) are similar to the equation used to generate a freeform surface using integral equations (Ochiai, 1995c).
Using Green's theorem three times and Eqs. (9), (10) and (19), Eq. (2) becomes

$$CT(P) = \int_\Gamma \{T_1(P,Q)\frac{\partial T(Q)}{\partial n} - \frac{\partial T_1(P,Q)}{\partial n}T(Q)\}d\Gamma(Q)$$

$$-\lambda^{-1}\int_\Gamma \{T_2(P,Q)\frac{\partial W_1^S(Q)}{\partial n} - \frac{\partial T_2(P,Q)}{\partial n}W_1^S(Q)\}d\Gamma(Q) \;+\lambda^{-1}\int_\Omega T_2(P,q)\nabla^2 W_1^S(q)d\Omega$$

$$= \int_\Gamma \{T_1(P,Q)\frac{\partial T(Q)}{\partial n} - \frac{\partial T_1(P,Q)}{\partial n}T(Q)\}d\Gamma(Q)$$

$$-\lambda^{-1}\int_\Gamma \{T_2(P,Q)\frac{\partial W_1^S(Q)}{\partial n} - \frac{\partial T_2(P,Q)}{\partial n}W_1^S(Q)\}d\Gamma(Q) \;-\lambda^{-1}\int_\Omega \nabla^2 T_3(P,q)W_2^S(q)d\Omega$$

$$= \int_\Gamma \{T_1(P,Q)\frac{\partial T(Q)}{\partial n} - \frac{\partial T_1(P,Q)}{\partial n}T(Q)\}d\Gamma(Q)$$

$$+\lambda^{-1}\sum_{f=1}^{2}(-1)^f\int_\Gamma \{T_{f+1}(P,Q)\frac{\partial W_f^S(Q)}{\partial n} - \frac{\partial T_{f+1}(P,Q)}{\partial n}W_f^S(Q)\}d\Gamma(Q) \;+\lambda^{-1}\sum_{m=1}^{M}T_3(P,q_m)W_3^P(q_m) \tag{28}$$

2.5 Interpolation for 3D case

In the three-dimensional case, the following equations are used for smooth interpolation:

$$\nabla^2 W_1^S(q) = -W_2^S(q) , \tag{29}$$

$$\nabla^2 W_2^S(q) = -\sum_{m=1}^{M} W_{3A}^P(q_m) , \tag{30}$$

where the function W_{3A}^P expresses the state of a uniformly distributed polyharmonic function in a spherical region with radius A. Figure 4 shows the shape of the polyharmonic functions; the biharmonic function T_2 is not smooth at $r = 0$. In the three-dimensional case, smooth interpolation cannot be obtained solely by using the biharmonic function T_2. To obtain smooth interpolation, a polyharmonic function with volume distribution T_{2A} is introduced. The function T_{fA} shown in Fig. 5 is defined as (Ochiai, 2005)

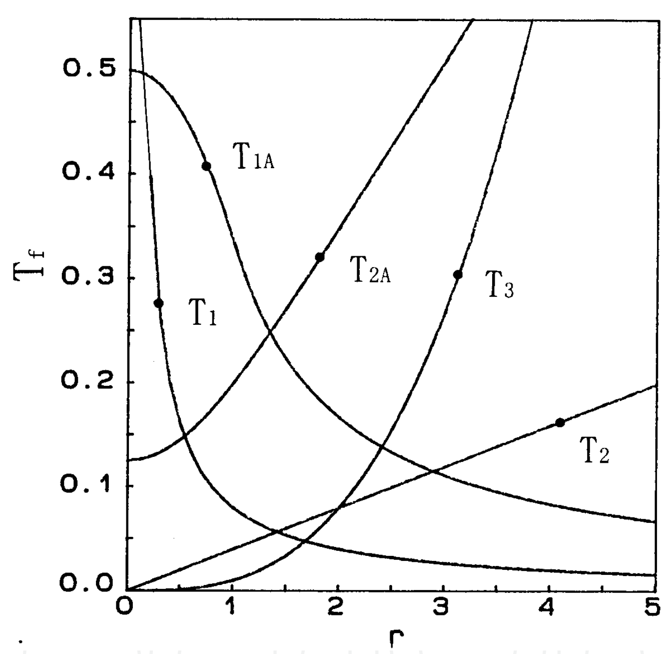


Fig. 4. Polyharmonic functions (T_f, T_{fA})

$$T_{fA}(p,q) = \int_0^A [\int_0^{2\pi} \{\int_0^{\pi} T_f(p,q) a^2 \sin\theta d\theta\} d\varphi] da , \tag{31}$$

where Ω_A is a spherical region with radius A, and S is the surface of a spherical shell with radius a. The function T_{fA} can be easily obtained using the relationships $r^2 = R^2 + a^2 - 2aR\cos\theta$ and $dr = aR\sin\theta d\theta$ as shown in Fig. 5. Therefore,

$$\sin\theta d\theta = \frac{r}{aR} dr . \tag{32}$$

This function is written using r instead of R, similarly to Eqs. (26) and (27), although the function obtained from Eq. (31) is a function of R. The newly defined function T_{fA} can be explicitly written as

$$T_{fA}(p,q) = \frac{1}{2r(2f+1)!}\{(2fA-r)(r+A)^{2f} + (2fA+r)(r-A)^{2f}\} \quad r > A \tag{33}$$

$$T_{fA}(p,q) = \frac{1}{2r(2f+1)!}\{(2fA-r)(A+r)^{2f} - (2fA+r)(A-r)^{2f}\} \quad r \le A\,. \tag{34}$$

In Fig. 5, A=1. The newly defined functions T_{fA} used in the chapter can be explicitly written as

$$T_{1A} = \frac{A^3}{3r} \qquad r>A \tag{35}$$

$$T_{1A} = \frac{3A^2 - r^2}{6} \quad r \le A \tag{36}$$

$$T_{2A} = \frac{A^3}{6r}(r^2 + \frac{A^2}{5}) \quad r>A \tag{37}$$

$$T_{2A} = -\frac{r^4 - 10r^2A^2 - 15A^4}{120} \quad r \le A \tag{38}$$

$$T_{3A} = \frac{A^3(35r^4 + 42r^2A^2 + 3A^4)}{2520r} \quad r>A \tag{39}$$

$$T_{3A} = \frac{-r^6 + 21r^4A^2 + 105r^2A^4 + 35A^6}{5040} \quad r \le A\,. \tag{40}$$

The heat generation W_1^S is given by Green's theorem and Eqs. (29)-(31) as

$$cW_1^S(P) = \sum_{f=1}^{2}(-1)^{f+1}\int_\Gamma \{T_f(P,Q)\frac{\partial W_f^S(Q)}{\partial n} - \frac{\partial T_f(P,Q)}{\partial n}W_f^S(Q)\}d\Gamma(Q) - \sum_{m=1}^{M} T_{2A}(P,q_m)W_{3A}^P(q_m) \tag{41}$$

Moreover, W_2^S in Eq. (30) is similarly given by

$$cW_2^S(P) = \int_\Gamma \{T_1(P,Q)\frac{\partial W_2^S(Q)}{\partial n} - \frac{\partial T_1(P,Q)}{\partial n}W_2^S(Q)\}d\Gamma(Q) + \sum_{m=1}^{M} T_{1A}(P,q_m)W_{3A}^P(q_m)\,. \tag{42}$$

Equations (41) and (42) are similar to the equation used to generate the freeform surface using integral equations. Using Green's theorem three times and Eqs. (29), (30) and (15), Eq. (2) becomes

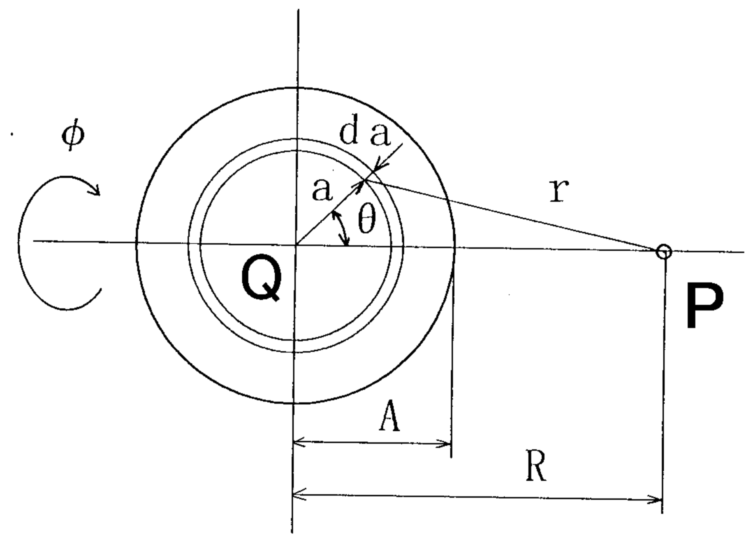


Fig. 5. Notations in three-dimensional problem

$$cT(P) = \int_{\Gamma} \{T_1(P,Q) \frac{\partial T(Q)}{\partial n} - \frac{\partial T_1(P,Q)}{\partial n} T(Q)\} d\Gamma(Q)$$

$$-\lambda^{-1} \sum_{f=1}^{2} (-1)^f \int_{\Gamma} \{T_{f+1}(P,Q) \frac{\partial W_f^S(Q)}{\partial n} - \frac{\partial T_{f+1}(P,Q)}{\partial n} W_f^S(Q)\} d\Gamma(Q) + \lambda^{-1} \sum_{m=1}^{M} T_{3A}(P,q_m) W_{3A}^P(q_m) \quad (43)$$

In the same manner, a polyharmonic function with surface distribution T_{fB} is defined as (Ochiai, 2009)

$$T_{Bf}(p,q) = \int_0^{2\pi} (\int_0^{\pi} T_f(p,q) A^2 \sin\theta \, d\theta) d\varphi \, . \quad (44)$$

The newly defined function T_{fB} can be explicitly written as

$$T_{fB}(p,q) = \frac{A\{(r+A)^{2f-1} - (r-A)^{2f-1}\}}{2(2f-1)!r} \quad r > A \ , \quad (45)$$

$$T_{fB}(p,q) = \frac{A\{(A+r)^{2f-1} - (A-r)^{2f-1}\}}{2(2f-1)!r} \quad r \le A \, . \quad (46)$$

Additionally, the temperature gradient is given by differentiating Equation (28), and expressed as:

$$\frac{\partial T(P)}{\partial x_i} = \int_{\Gamma} \{\frac{\partial T_1(P,Q)}{\partial x_i} \frac{\partial T(Q)}{\partial n} - \frac{\partial^2 T_1(P,Q)}{\partial x_i \partial n} T(Q)\} d\Gamma(Q)$$

$$+\lambda^{-1} \sum_{f=1}^{2} (-1)^f \int_{\Gamma} \{\frac{\partial T_{f+1}(P,Q)}{\partial x_i} \frac{\partial W_f^S(Q)}{\partial n} - \frac{\partial^2 T_{f+1}(P,Q)}{\partial x_i \partial n} W_f^S(Q)\} d\Gamma(Q)$$

$$+\lambda^{-1}\sum_{m=1}^{M}\frac{\partial T_3(P,q_m)}{\partial x_i}W_3^P(q_m) \tag{47}$$

The function $\partial T_f(p,q)/\partial x_i$ and its normal derivative for the two-dimensional case are explicitly expressed as

$$\frac{\partial T_f(p,q)}{\partial x_i}=\frac{r^{2f-3}}{2\pi[(2f-2)!!]^2}[2(f-1)\{\ln(\frac{1}{r})+B\}\ -1+2(f-1)\sum_{e=1}^{f-1}\frac{1}{e}]\frac{\partial r}{\partial x_i}, \tag{48}$$

$$\frac{\partial^2 T_f(p,q)}{\partial x_i\partial n}=\frac{r^{2f-4}}{2\pi[(2f-2)!!]^2}\{[n_i+2(f-2)r_{,i}r_{,i}n_j][2(f-1)\{\ln(\frac{1}{r})+B\}$$

$$-1+2(f-1)\sum_{e=1}^{f-1}\frac{1}{e}]-2(f-1)r_{,i}r_{,i}n_j\} \tag{49}$$

2.6 Basic equations for unsteady heat conduction

In unsteady heat conduction problems with heat generation $W_1^S(q,t)$, the temperature T is obtained by solving

$$\nabla^2 T+\frac{W_1^S}{\lambda}=\kappa^{-1}\frac{\partial T}{\partial t}\ , \tag{50}$$

where κ and t are the thermal diffusivity and time, respectively. Denoting an arbitrary time and the pseudo-initial temperature by τ and $T^0(q,0)$, respectively, the boundary integral equation for the temperature in the case of unsteady heat conduction is expressed as (Wrobel, 2002)

$$cT(P,t)=-\kappa\int_0^t\int_\Gamma[T(Q,\tau)\frac{\partial T_1^*(P,Q,t,\tau)}{\partial n}\ -\frac{\partial T(P,Q)}{\partial n}T_1^*(P,Q,t,\tau)]d\Gamma d\tau$$

$$+\int_\Omega T_1^*(P,q,t,0)T^0(q,0)d\Omega\ +\frac{\kappa}{\lambda}\int_0^t\ \int_\Omega T_1^*(P,q,t,\tau)W_1^S(q,\tau)d\Omega d\tau\ , \tag{51}$$

where c=0.5 on the smooth boundary and c=1 in the domain. The notations Γ and Ω represent the boundary and domain, respectively. The notations p and q become P and Q on the boundary. In the case of K-dimensional problems, the time-dependent fundamental solution $T_1^*(p,q,t,\tau)$ in Eq. (51) used for the unsteady temperature analyses and its normal derivative are given by

$$T_1^*(p,q,t,\tau)=\frac{1}{4\pi\kappa(t-\tau)^{K/2}}\exp(-a)\,, \tag{52}$$

$$\frac{\partial T_1^*(p,q,t,\tau)}{\partial n} = \frac{-r}{8\pi\kappa^2(t-\tau)^{K/2+1}}\frac{\partial r}{\partial n}\exp(-a)\,, \tag{53}$$

where

$$a = \frac{r^2}{4\kappa(t-\tau)}\,. \tag{54}$$

Here, r is the distance between the observation point p and the loading point q. As shown in Eq. (51), when an arbitrary pseudo-initial temperature distribution $T^0(q,0)$ exists in the domain, a domain integral is necessary. Therefore, the triple-reciprocity BEM (Ochiai, 2001) is used to avoid internal cells.

This study reveals that the problem of unsteady heat conduction with many time steps can be solved effectively by the triple-reciprocity BEM. Two different numerical procedures can be employed for the numerical solution of Eq. (51). One method requires internal cells. At the end of each time step, the temperature at a sufficient number of internal points must be computed for use as the initial temperature in the next time step. The other method uses the history of boundary values, making internal cells unnecessary, if the initial temperature can be assumed to be 0. However, the CPU time required for calculation increases rapidly with increasing number of time steps. In the presented method, the temperature distributions in some time steps are assumed to be pseudo-initial and are interpolated using integral equations and internal points.

2.7 Interpolation of time-dependent value

Heat generation $W_1^S(q,t)$ is assumed to vary within each time step in accordance with the time interpolation function such thatφηγϕδσαζξχϖβνμ

$$W_1^S(q,t) = \mathbf{W}_1^S\ \phi\ \ , \tag{55}$$

where ϕ is the time interpolation function. Let us now assume a linear variation of $W_1^S(q,t)$,

$$\psi_1 = \frac{t_f - t}{\Delta t_f}\,,\ \psi_2 = \frac{t - t_{f-1}}{\Delta t_f}\,, \tag{56}$$

where $\Delta t_f = t_f - t_{f-1}$.

The following equations can be used to obtain time-dependent values of heat generation $W_1\,(q,t_f)$:

$$\nabla^2 W_1^S(q,t_f) = -W_2^S(q,t_f)\,, \tag{57}$$

$$\nabla^2 W_2^S(q,t_f) = -\sum_{m=1}^{M} W_3^P(q_m,t_f)\,. \tag{58}$$

An interpolation method for the pseudo-initial temperature distribution using the boundary integral equations that avoids the use of internal cells is next shown. The pseudo-initial temperature $T^0(q,0)$ in Eq. (51) is represented as $T^{0S}(q,0)$.
The following equations can be used for the two-dimensional interpolation (Ochiai, 2001):

$$\nabla^2 T_1^{0S}(q,0) = -T_2^{0S}(q,0)\,, \tag{59}$$

$$\nabla^2 T_2^{0S}(q,0) = -\sum_{m=1}^{M} T_3^{0P}(q_m,0)\,. \tag{60}$$

The term T_2^{0S} in Eq. (59) corresponds to the sum of the curvatures $\partial^2 T_1^{0S} / \partial x^2$ and $\partial^2 T_1^{0S} / \partial y^2$. The term T_2^{0S} is the unknown strength of a Dirac-type function. From Eqs. (59) and (60), the following equation can be obtained.

$$\nabla^4 T_1^{0S}(q,0) = \sum_{m=1}^{M} T_3^{0P}(q,0) \tag{61}$$

In this study, the deformation of an imaginary thin plate is utilized to interpolate the two-dimensional distribution T_1^{0S}. The deformation T_1^{0S} is given, but the force of the point load T_3^{0P} is unknown. T_3^{0P} is obtained inversely from the deformation T_1^{0S} of the fictitious thin plate, as shown in Fig. 3. T_2^{0S} corresponds to the moment of the thin plate. The moment T_2^{0S} on the boundary is assumed to be 0, which is the same as that in the natural spline. This indicates that the fictitious thin plate is simply supported.
Using Green's second identity and Eqs. (59), (60) and (15), we obtain (Ochiai, 2001)

$$cT_1^{0S}(P,0) = \sum_{f=1}^{2} (-1)^{f+1} \int_\Gamma \{T_f(P,Q) \frac{\partial T_f^{0S}(Q,0)}{\partial n} - \frac{\partial T_f(P,Q)}{\partial n} T_f^{0S}(Q,0)\} d\Gamma - \sum_{m=1}^{M} T_2(P,q_m) T_3^{0P}(q_m,0)\,, \tag{62}$$

where M is the number of T_3^{0P}. Moreover, T_2^{0S} in Eq. (60) is similarly given by

$$cT_2^{0S}(P,0) = \int_\Gamma \{T_1(P,Q) \frac{\partial T_2^{0S}(Q,0)}{\partial n} - \frac{\partial T_1(P,Q)}{\partial n} T_2^{0S}(Q,0)\} d\Gamma + \sum_{m=1}^{M} T_1(P,q_m) T_3^{0P}(q_m,0)\,. \tag{63}$$

The integral equations (62) and (63) are used to interpolate the pseudo-initial temperature distribution T_1^{0S}. On the other hand, the polyharmonic function $T_f^*(p,q,t,\tau)$ in the unsteady heat conduction problem is defined by

$$\nabla^2 T^*_{f+1}(p,q,t,\tau) = T^*_f(p,q,t,\tau)\ . \tag{64}$$

Using Green's theorem twice and Eqs. (54)- (57) and (61), Eq. (51) becomes

$$T_3^{0P}\ T^*_{f+1}(p,q,t,\tau) = \int \frac{1}{r}\Big[\int r T^*_f(p,q,t,\tau)dr\Big]dr$$

$$+\frac{\kappa}{\lambda}\sum_{f=1}^{2}(-1)^f \int_0^t \int_\Gamma [T^*_{f+1}(P,Q,t,\tau)\frac{\partial W_f^S(Q,\tau)}{\partial n} - \frac{\partial T^*_{f+1}(P,Q,t,\tau)}{\partial n} W_f^S(Q,\tau)]d\Gamma d\tau$$

$$+\frac{\kappa}{\lambda}\sum_{m=1}^{M}\int_0^t W^P_{3(m)}(q,\tau)T_3^{*A}(P,q,t,\tau)d\tau \ +\sum_{f=1}^{2}(-1)^f \int_\Gamma [T^*_{f+1}(P,Q,t,0)\frac{\partial T_f^{0S}(Q,0)}{\partial n}$$

$$-\frac{\partial T^*_{f+1}(P,Q,t,0)}{\partial n}T_f^{0S}(Q,0)]d\Gamma \ +\sum_{m=1}^{M} T_3^{0P}(q_m,0)T_3^*(P,q_m,t,0)\ . \tag{65}$$

Therefore, it is clear that temperature analysis without the use of a domain integral is possible, provided that the initial temperature T_1^0 is interpolated using Eqs. (62) and (63). In practice, T_1^{0S} and $\partial T_2^S / \partial n$ are obtained using results from the previous time step; however, T_2^{0S}, $\partial T_2^S / \partial n$ and T_3^{0P} in Eq. (65) are not obtained in this way.

2.8 Polyharmonic function for unsteady state

The two-dimensional polyharmonic function $T^*_f(p,q,t,\tau)$ in Eq. (65) is determined as

$$T^*_{f+1}(p,q,t,\tau) = \int \frac{1}{r}\Big[\int r T^*_f(p,q,t,\tau)dr\Big]dr\ . \tag{66}$$

$T^*_f(p,q,t,\tau)$ in the unsteady state and its normal derivative are concretely given by

$$T_2^*(p,q,t,\tau) = \frac{1}{4\pi}\big[E_1(a) + \ln(a) + C\big], \tag{67}$$

$$\frac{\partial T_2^*(p,q,t,\tau)}{\partial n} = \frac{1}{2\pi r}\frac{\partial r}{\partial n}\big[1 - \exp(-a)\big]\ , \tag{68}$$

$$T_3^*(p,q,t,\tau) = \frac{r^2}{16\pi}\big[E_1(a) + \ln(a) + C \ + \frac{1-\exp(-a)}{a} + \frac{E_1(a)+\ln(a)+C}{a} - 2\big], \tag{69}$$

$$\frac{\partial T_3^*(p,q,t,\tau)}{\partial n} = \frac{r}{8\pi}\frac{\partial r}{\partial n}\big[E_1(a) + \ln(a) + C - 1 \ + \frac{1-\exp(-a)}{a}\big], \tag{70}$$

where E 1 () is the exponential integral function and C is Euler's constant.

Numerical solutions are obtained using the interpolation functions for time and space. If a constant time interpolation and time step $(t_k - t_{k-1})$ are used, the time integral can be treated analytically. The time integrals for $T_f^*(p,q,t,\tau)$ are given as follows:

$$\int_{t_f}^{t_F} T_1^*(p,q,t,\tau)d\tau = \frac{1}{4\kappa\pi}E_1(a_f)\,, \tag{71}$$

$$\int_{t_f}^{t_F} \frac{\partial T_1^*(p,q,t,\tau)}{\partial n}d\tau = \frac{1}{2\kappa\pi r}\frac{\partial r}{\partial n}\exp(-a_f)\,, \tag{72}$$

where

$$a_f = \frac{r^2}{4\kappa(t_F - t_f)}\,. \tag{73}$$

Assuming that functions $T(Q,\tau)$ and $\partial T(Q,\tau)/\partial n$ remain constant over time in each time step, Eq. (65) can be written in matrix form. Replacing $T(Q,\tau)$ and $\partial T(Q,\tau)/\partial n$ with vectors **T** f and **Q** f , respectively, and discretizing Eq. (65), we obtain (Brebbia ,1984)

$$\sum_{f=1}^{F} \mathbf{H_{fF}T_f} = \sum_{f=1}^{F} \mathbf{G_{fF}Q_f} + \mathbf{B_0}\,, \tag{74}$$

where $\mathbf{B}_0$ represents the effect of the pseudo-initial temperature. Adopting a constant time step throughout the analysis, the coefficients of the matrix at several time steps need to be computed and stored only once.
If there is heat generation, the following time integrals are used (Ochiai, 2001).

$$\int_{t_f}^{t_F} T_2^*(p,q,t,\tau)d\tau = \frac{r^2}{16\kappa\pi}\{E_1(a_f) + \frac{1}{a_f}[E_1(a_f) + \ln(a_f) + C + 1 - \exp(-a_f)]\} \tag{75}$$

$$\int_{t_f}^{t_F} \frac{\partial T_2^*(p,q,t,\tau)}{\partial n}d\tau = \frac{r}{8\kappa\pi}\frac{\partial r}{\partial n}[\frac{1-\exp(a_f)}{a_f} + E_1(a_f)] \tag{76}$$

$$\int_{t_f}^{t_F} T_3^*(p,q,t,\tau)d\tau = \frac{r^4}{256\kappa\pi}\{E_1(a_f) + \frac{1}{a_f}[4E_1(a_f) + 4\ln(a_f) + 4C + 1 - \exp(-a_f)] + \frac{1}{a_f^2}[2E(a_f)$$
$$+ 2\ln(a_f) + 2C - 2a_f + 3 - 3\exp(-a_f) - 5a_f]\} \tag{77}$$

$$\int_{t_f}^{t_F} \frac{\partial T_3^*(p,q,t,\tau)}{\partial n}d\tau = \frac{r^3}{64\kappa\pi}\frac{\partial r}{\partial n}\{\frac{1-\exp(a_f)-a_f}{a_f^2} + E_1(a_f)$$
$$+ \frac{1}{a_f}[2E_1(a_f) + 2\ln(a_f) + 2C + 1 - \exp(a_f)]\} \tag{78}$$

Additionally, the temperature gradient is given by differentiating Equation (65), and expressed as:

$$\frac{\partial T(p,t)}{\partial x_i} = -\kappa\int_0^t\int_\Gamma [T(Q,\tau)\frac{\partial^2 T_1^*(p,Q,t,\tau)}{\partial x_i \partial n} - \frac{\partial T(Q,\tau)}{\partial n}\frac{\partial T_1^*(p,Q,t,\tau)}{\partial x_i}]d\Gamma d\tau$$

$$+\kappa\sum_{f=1}^{2}(-1)^f\int_0^t\int_\Gamma [\frac{\partial T_{f+1}^*(p,Q,t,\tau)}{\partial x_i}\frac{\partial W_f(Q,\tau)}{\partial n} - \frac{\partial^2 T_{f+1}^*(p,Q,t,\tau)}{\partial x_i \partial n}W_f(Q,\tau)]d\Gamma d\tau$$

$$+\kappa\sum_{m=1}^{M}\int_0^t W_3^P(q_m,\tau)\frac{\partial T_3^*(p,q_m,t,\tau)}{\partial x_i}d\tau + \sum_{f=1}^{2}(-1)^f\int_\Gamma [\frac{\partial T_{f+1}^*(p,Q,t,0)}{\partial x_i}\frac{\partial T_f^0(Q,0)}{\partial n}$$

$$-\frac{\partial^2 T_{f+1}^*(p,Q,t,0)}{\partial x_i \partial n}T_f^0(Q,0)]d\Gamma d\tau + \sum_{m=1}^{M} T_3^{0P}(q_m,0)\frac{\partial T_3^*(p,q_m,t,0)}{\partial x_i}d\tau \quad (79)$$

The derivative of the polyharmonic function $T_f^*(P,q,t,\tau)$ and the normal derivative with respect to x_i in Eq.(79) are explicitly given by

$$\frac{\partial T_1^*(p,q,t,\tau)}{\partial x_i} = \frac{-rr,_i}{8\pi\kappa^2(t-\tau)^2}\exp(-a) \ , \quad (80)$$

$$\frac{\partial^2 T_1^*(p,q,t,\tau)}{\partial x_i \partial n} = \frac{-1}{8\pi\kappa^2(t-\tau)^2}[n_i\exp(-a) + 2ar,_i\frac{\partial r}{\partial n}\exp(-a)] \ , \quad (81)$$

$$\frac{\partial T_2^*(p,q,t,\tau)}{\partial x_i} = \frac{r,_i}{2\pi r}\left[1-\exp(-a)\right] , \quad (82)$$

$$\frac{\partial^2 T_2^*(p,q,t,\tau)}{\partial x_i \partial n} = \frac{1}{2\pi r^2}\{n_i[1-\exp(-a)] - 2r,_i\frac{\partial r}{\partial n}[1-\exp(-a) - a\cdot\exp(-a)]\} , \quad (83)$$

$$\frac{\partial T_3^*(p,q,t,\tau)}{\partial x_i} = \frac{rr,_i}{8\pi}\left\{E_1(a) + \ln(a) + C - 1 + \frac{1-\exp(-a)}{a}\right\} \ , \quad (84)$$

$$\frac{\partial^2 T_3^*(p,q,t,\tau)}{\partial x_i \partial n} = \frac{1}{8\pi}\{n_i[E_1(a) + \ln(a) + C + 1 - \frac{1-\exp(-a)}{a}] + 2r,_i\frac{\partial r}{\partial n}[1 - \frac{1-\exp(-a)}{a}]\} \ , \quad (85)$$

where $r,_i = \partial r / \partial x_i$. The time integrals for $\partial T_f^* / \partial x_i$ and $\partial^2 T_f^*(P,q,t,\tau) / \partial x_i \partial n$ in Eq. (79) are given as follows:

$$\int_{t_f}^{t_F}\frac{\partial T_1^*(p,q,t,\tau)}{\partial x_i}d\tau = \frac{-r,_i}{2\kappa\pi r}\exp(-a_f) \ , \quad (86)$$

$$\int_{t_f}^{t_F} \frac{\partial^2 T_1^*(p,q,t,\tau)}{\partial x_i \partial n} d\tau = \frac{-1}{2\pi\kappa r^2}[n_i - 2r,_i \frac{\partial r}{\partial n}(1+a_f)]\exp(-a_f) \ , \tag{87}$$

$$\int_{t_f}^{t_F} \frac{\partial T_2^*(p,q,t,\tau)}{\partial x_i} d\tau = \frac{r}{8\kappa\pi}\frac{\partial r}{\partial x_i}[\frac{1-\exp(-a_f)}{a_f} + E_1(a_f)] \ , \tag{88}$$

$$\int_{t_f}^{t_F} \frac{\partial^2 T_2^*(p,q,t,\tau)}{\partial x_i \partial n} d\tau = \frac{1}{8\kappa\pi}\left\langle n_i\{E_1(a_f) + \frac{1}{a_f}[1-\exp(-a_f)]\} - 2r,_i \frac{\partial r}{\partial n}\frac{1}{a_f}[1-\exp(-a_f)]\right\rangle, \tag{89}$$

$$\begin{aligned}\int_{t_f}^{t_F} \frac{\partial T_3^*(p,q,t,\tau)}{\partial x_i} d\tau = \frac{r^3 r,_i}{64\kappa\pi}\{\frac{1-\exp(-a_f)-a_f}{a_f^2} + \\ E_1(a_f) + \frac{1}{a_f}[2E_1(a_f) + 2\ln(a_f) + 2C + 1 - \exp(-a_f)]\}\end{aligned} \ , \tag{90}$$

$$\begin{aligned}\int_{t_f}^{t_F} \frac{\partial^2 T_3^*(p,q,t,\tau)}{\partial x_i \partial n} d\tau = \frac{r^2}{64\kappa\pi}\Big\langle n_i\{E_1(a_f) + \frac{2}{a_f}[E_1(a_f) + \ln(a_f) + C] \\ + \frac{1}{a_f{}^2}[1-\exp(-a_f) - a_f\exp(-a_f)]\} + 2r,_i \frac{\partial r}{\partial n}\{E_1(a_f) \\ + \frac{2}{a_f}[1-\exp(-a_f)] - \frac{1}{a_f{}^2}[1 - a_f - a_f\exp(-a_f)]\}\Big\rangle.\end{aligned} \tag{91}$$

3. Numerical examples

To verify the accuracy of the present method, unsteady heat conduction in a circular region with radius *a*, as shown in Fig. 6, is treated with a boundary temperature given by

$$T = T_H[1-\cos(\omega t)] \ . \tag{92}$$

We assume an initial temperature $T0 = 0\,^\circ\mathrm{C}$, and R denotes the distance from the center of the circular region. A two-dimensional state, in which there is no heat flow in the direction perpendicular to the plane of the domain, is assumed. Figure 6 also shows the internal points used for interpolation. A thermal diffusivity of $\kappa = 16$ mm2/s and a radius of a=10 mm are assumed. $T_H = 10\,^\circ\mathrm{C}$ in Eq. (92) and a frequency of $\omega = \pi/2$ rad/s are also assumed. The BEM results at R=0 and R=8 mm and the exact values are compared in Fig. 7. The exact solution for the temperature distribution is given by (Carslaw, 1938)

$$T(R,t) = T_H[1 - \frac{ber\sqrt{\frac{\omega}{\kappa}}R\,ber\sqrt{\frac{\omega}{\kappa}}a + bei\sqrt{\frac{\omega}{\kappa}}R\,bei\sqrt{\frac{\omega}{\kappa}}a}{ber^2\sqrt{\frac{\omega}{\kappa}}a + bei^2\sqrt{\frac{\omega}{\kappa}}a}\cos\omega t$$

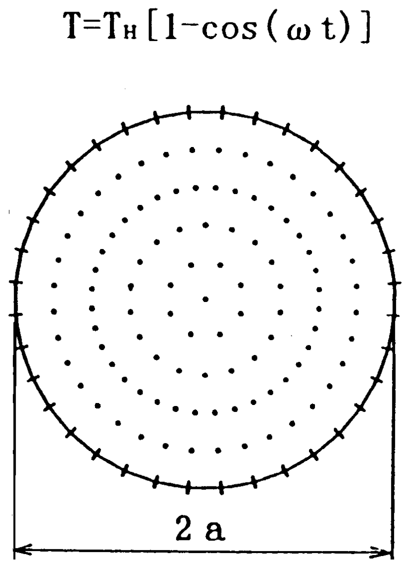


Fig. 6. Circular region with temperature change at the boundary.

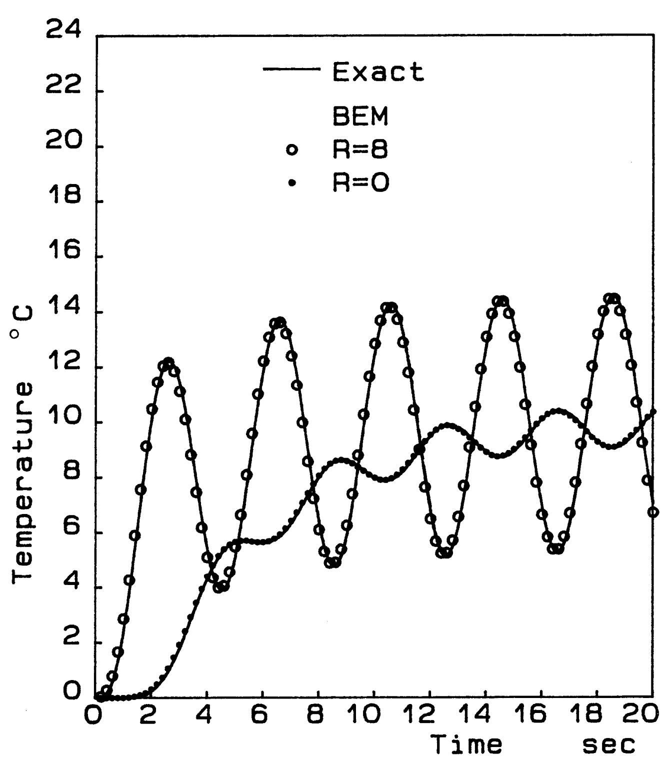


Fig. 7. Temperature history in circular region.

$$+\frac{ber\sqrt{\frac{\omega}{\kappa}}R\,bei\sqrt{\frac{\omega}{\kappa}}a-ber\sqrt{\frac{\omega}{\kappa}}a\,bei\sqrt{\frac{\omega}{\kappa}}R}{ber^2\sqrt{\frac{\omega}{\kappa}}a+bei^2\sqrt{\frac{\omega}{\kappa}}a}\sin\omega t-\frac{2\kappa^2}{a}\sum_{s=1}^{\infty}\exp(-\kappa\alpha_s^2 t)\frac{J_0(\alpha_s R)\alpha_s^3}{J_0^{'}(\alpha_s a)(\kappa^2\alpha_s^4+\omega^2)}] \quad (93)$$

where *ber*( ) and *bei*( ) are Kelvin functions, and α_s (s=1, 2, ...) are the roots of $J_0(a\alpha)=0$. Constant elements are used for boundary and time interpolation.

Appendix A (3D)

The higher-order functions for 3D unsteady heat conduction are

$$T_2^*(p,q,t,\tau) = \frac{1}{2\pi^{3/2} r}\{-\gamma(1.5,a) + a^{1/2}[1-\exp(-a)]\} = \frac{-1}{2\pi^{3/2} r}\gamma(0.5,a) \tag{A-1}$$

$$\frac{\partial T_2^*(p,q,t,\tau)}{\partial n} = \frac{1}{2\pi^{3/2} r^2}\gamma(1.5,a)\frac{\partial r}{\partial n} \tag{A-2}$$

$$T_3^* = \frac{r}{12\pi^{3/2}}\{-3\gamma(1.5,a) + 6a^{-1/2}\gamma(2,a) - 3a^{-1}\gamma(2.5,a) + a^{1/2} + 3\gamma(1.5,a)\frac{1}{a} - 3a^{-1/2}[1-\exp(-a)]\}$$
$$= \frac{r}{4\pi^{3/2}}\{-\gamma(0.5,a) + 2\gamma(1.5,a)\frac{1}{a} - 2a^{-1/2}[1-\exp(-a)]\} \tag{A-3}$$

$$.\frac{\partial T_3^*}{\partial n} = \frac{1}{4\pi^{3/2}}[\gamma(0.5,a) - \frac{1}{a}\gamma(1.5,a)]\frac{\partial r}{\partial n}. \tag{A-4}$$

where $\gamma(,)$ is an incomplete gamma function of the first kind (Abramowitz, 1970) and $r_{,i} = \partial r / \partial x_i$. Using Eqs. (44) and (A-3), the polyharmonic function with a surface distribution is obtained as follows:

$$T_{3B}^* = \frac{2A(kt)^{3/2}}{3\pi^{1/2} r}\{-2u_2^{3/2}\gamma(1.5,u_2) + 2u_1^{3/2}\gamma(1.5,u_1) + 2\gamma(3,u_2) - 2\gamma(3,u_1) + 6u_2\gamma(2,u_2) - 6u_1\gamma(2,u_1)$$
$$-6u_2^{1/2}\gamma(2.5,u_2) + 6u_1^{1/2}\gamma(2.5,u_1) + \frac{1}{2}u_2^2 - \frac{1}{2}u_1^2$$
$$+6u_2^{1/2}\gamma(1.5,u_2) - 6u_1^{1/2}\gamma(1.5,u_1) - 6\gamma(2,u_2) + 6\gamma(2,u_1) - 3u_2 + 3u_1 - 3\exp(-u_2) + 3\exp(-u_1)\}, \tag{A-5}$$

where

$$u_1 = \frac{(r-A)^2}{4\kappa(t-\tau)} \tag{A-6}$$

$$u_2 = \frac{(r+A)^2}{4\kappa(t-\tau)}. \tag{A-7}$$

The time integral of Eq. (62) can be obtained as follows:

$$\int_{t_f}^{t_F} T_1^*(p,q,t,\tau)d\tau = \frac{1}{4\kappa\pi^{3/2} r}\Gamma(0.5,a_f) \tag{A-8}$$

$$\int_{t_f}^{t_F} \frac{\partial T_1^*(p,q,t,\tau)}{\partial n}d\tau = \frac{1}{2\kappa\pi^{3/2} r^2}\frac{\partial r}{\partial n}\Gamma(1.5,a_f) \tag{A-9}$$

$$\int_{t_f}^{t_F} T_2^*(p,q,t,\tau)d\tau = \frac{r}{8\kappa\pi^{3/2}}[\gamma(1.5,a_f)\ \frac{-1}{a_f} - \Gamma(0.5,a_f) + \frac{2}{a_f^{1/2}} - \Gamma(-0.5,a_f)]$$

$$= \frac{r}{8\kappa\pi^{3/2}}[\gamma(0.5, a_f)\ \frac{1}{a_f} - \Gamma(-0.5, a_f)] \tag{A-10}$$

$$\int_{t_f}^{t_F} \frac{\partial T_2^*(p,q,t,\tau)}{\partial n} d\tau = \frac{1}{8\kappa\pi^{3/2}} \frac{\partial r}{\partial n}[\gamma(1.5, a_f)\frac{1}{a_f} + \Gamma(0.5, a_f)] \tag{A-11}$$

$$\int_{t_f}^{t_F} T_3^*(p,q,t,\tau) d\tau = \frac{r^3}{96\kappa\pi^{3/2}}[-6\gamma\ (1.5, a_f)\frac{1}{a_f} - \Gamma(0.5, a_f)\ + 8\gamma\ (2, a_f)\frac{1}{a_f^{\ 3/2}}\ - 3\gamma(2.5, a_f)\ \frac{1}{a_f^{\ 2}}$$
$$+ \frac{4}{a_f^{\ 1/2}}\ + 3\gamma(1.5, a_f)\frac{1}{a_f^{\ 2}} + 3\Gamma(-0.5, a_f)\ - \frac{4}{a_f^{\ 3/2}} + 6\Gamma(-1.5, a_f)] \tag{A-12}$$

$$\int_{t_f}^{t_F} \frac{\partial T_3^*(p,q,t,\tau)}{\partial n} d\tau = \frac{r^2}{96\kappa\pi^{3/2}} \frac{\partial r}{\partial n}[-6\gamma\ (1.5, a_f)\frac{1}{a_f} - 3\Gamma(0.5, a_f)\ + 3\gamma\ (2.5, a_f)\frac{1}{a_f^{\ 2}}\ + \frac{8}{a_f^{\ 1/2}}$$
$$- \gamma(1.5, a_f)\frac{3}{a_f^{\ 2}} - 3\Gamma(-0.5, a_f)], \tag{A-13}$$

where

$$a_f = \frac{r^2}{4\kappa(t_F - t_f)} \tag{A-14}$$

and $\Gamma(,)$ is an incomplete gamma function of the second kind (Abramowitz, 1970). The time integral of Eq. (A-5) can be obtained as follows:

$$\int_{t_f}^{t_F} T_{3B}^*(p,q,t,\tau) d\tau = \frac{A}{48\pi^{1/2} r} \frac{(r-A)^5}{\kappa}\{2\gamma(1.5, a_{1f})\ \frac{1}{a_f} + \frac{1}{5}\Gamma(0.5, a_{1f})\ - \frac{4}{5}\gamma(3, a_{1f})\ \frac{1}{a_{1f}^{\ 5/2}}$$
$$- 4\gamma(2, a_{1f})\ \frac{1}{a_{1f}^{\ 3/2}}\ + 3\gamma(2.5, a_{1f})\ \frac{1}{a_{1f}^{\ 2}} - \frac{1}{a_{1f}^{\ 1/2}}\ - 3\gamma(1.5, a_{1f})\ \frac{1}{a_{1f}^{\ 2}} - \frac{3}{5}\Gamma(-0.5, a_{1f})$$
$$+ \frac{12}{5}\gamma(2, a_{1f})\ \frac{1}{a_{1f}^{\ 5/2}}\ + \frac{2}{a_{1f}^{\ 3/2}}\ + 3\Gamma(-2.5, a_{1f})\}, \tag{A-15}$$

where

$$a_{1f} = \frac{(r-A)^2}{4\kappa(t_F - t_f)}\ . \tag{A-16}$$

For the sake of conciseness, the terms involving u_2 in Eq. (A-5) are omitted. The derivative of the polyharmonic function $T_f^*(P,q,t,\tau)$ and the normal derivative with respect to x_i are explicitly given by

$$\frac{\partial T_1^*}{\partial x_i} = \frac{-r}{16\pi^{3/2}[k(t-\tau)]^{5/2}} \exp(-a) \frac{\partial r}{\partial x_i} \tag{A-17}$$

$$\frac{\partial}{\partial n}\frac{\partial T_1^*}{\partial x_i} = \frac{1}{16\pi^{3/2}[k(t-\tau)]^{5/2}}\left(-n_i + 2ur,_j n_j r,_i\right)\exp(-a) \tag{A-18}$$

$$\frac{\partial T_2^{\,*}}{\partial x_i} = \frac{1}{2\pi^{3/2}r^2}\gamma\left(\frac{3}{2},a\right)\frac{\partial r}{\partial x_i} \tag{A-19}$$

$$\frac{\partial}{\partial n}\frac{\partial T_2^*}{\partial x_i} = \frac{1}{2\pi^{3/2}r^3}\left[n_i\gamma\left(\frac{3}{2},a\right) - 2\gamma\left(\frac{5}{2},a\right)r,_i n_j r,_j\right] \tag{A-20}$$

$$\frac{\partial T_3^{\,*}}{\partial x_i} = \frac{-1}{8\pi^{3/2}}\left\{\gamma\left(\frac{1}{2},a\right) - \frac{1}{u}\gamma\left(\frac{3}{2},a\right)\right\}\frac{\partial r}{\partial x_i} \tag{A-21}$$

$$\frac{\partial^2 T_3^{\,*}}{\partial n\partial x_i} = \frac{1}{8\pi^{3/2}r}\{n_i[-\gamma(0.5,a) + \frac{1}{u}\gamma(1.5,a)] + r,_i r,_j n_j[\gamma(0.5,a) - \frac{3}{u}\gamma(1.5,a)]\} \tag{A-22}$$

$$\begin{aligned}\frac{dT_{3B}^*}{dx_i} = &-\frac{2a(kt)^{3/2}}{3\pi^{1/2}r}\frac{\partial r}{\partial x_i}[\frac{1}{r}\{2u_1^{\,3/2}\gamma(1.5\,,u_1) - 2\gamma\left(3\,,u_1\right)\;-6u_1\gamma(2\,,u_1)\;+6u_1^{\,1/2}\gamma\left(2.5\,,u_1\right) \\ &-\frac{1}{2}u_1^{\,2}\;-6u_1^{\,1/2}\gamma\left(1.5\,,u_1\right) + 6\gamma(2,u_1)\;+3u_1 + 3\exp(-u_1)\}\;-\frac{2}{r_1}\{3u_1^{\,1/2}\gamma(1.5\,,u_1) \\ &-6\gamma(2\,,u_1)\;+3u_1^{\,-3/2}\gamma\left(2.5\,,u_1\right)\;-1\;-3u_1^{\,-3/2}\gamma\left(1.5\,,u_1\right)\;+3u_1^{\,-1}\;-3u_1^{\,-1}\exp(-u_1)\}]\end{aligned} \tag{A-23}$$

The time integrals of Eqs. (A-18), (A-20) and (A-22) can be obtained as follows:

$$\int_{t_f}^{t_F}\frac{\partial}{\partial n}\frac{\partial T_1^*}{\partial x_i}d\tau = \frac{1}{2k\pi^{3/2}r^3}\left\{2r,_i n_j r,_j \Gamma\left(\frac{5}{2},a_f\right) - n_i\Gamma\left(\frac{3}{2},a_f\right)\right\} \tag{A-24}$$

$$\int_{t_f}^{t_F}\frac{\partial}{\partial n}\frac{\partial T_2^*}{\partial x_i}d\tau = \frac{1}{8\pi^{3/2}kr}\left\{(n_i - 3r,_i n_j r,_j)\frac{1}{a_f}\gamma\left(\frac{3}{2},a_f\right) + (n_i - r,_i n_j r,_j)[\Gamma\left(\frac{1}{2}\right) - \gamma\left(\frac{1}{2},a_f\right)]\right\} \tag{A-25}$$

$$\begin{aligned}\int_{t_f}^{t_F}\frac{\partial^2 T_3^*(p,q,t,\tau)}{\partial n\partial x_i}d\tau = \frac{r}{192\pi^{3/2}\kappa}&\left\{n_i\left[3a_f^{\,-2}\gamma\left(\frac{3}{2},a_f\right) - 6a_f^{\,-1}\gamma\left(\frac{1}{2},a_f\right) - 3\Gamma\left(-\frac{1}{2},a_f\right) + 16a_f^{-1/2}\right]\right. \\ &\left.+r,_i r,_j n_j\left[-9a_f^{\,-2}\gamma\left(\frac{3}{2},a_f\right) + 6a_f^{\,-1}\gamma\left(\frac{1}{2},a_f\right) - 3\Gamma\left(-\frac{1}{2},a_f\right)\right]\right\}\end{aligned} \tag{A-26}$$

$$\begin{aligned}\int_{t_f}^{t_F}\frac{\partial T_{3B}^*}{\partial x_i}d\tau = &\frac{2a}{3\pi^{1/2}}\frac{\partial r}{\partial x_i}\frac{r^4}{32k}\{2\gamma\left(1.5,a_f\right)\frac{1}{a_f} + \frac{1}{5}\Gamma(0.5,a_f)\;-\frac{4}{5}\gamma\left(3,a_f\right)\frac{1}{a_f^{\,5/2}} \\ &-4\gamma\left(2,a_f\right)\frac{1}{a_f^{\,3/2}}\;+3\gamma\left(2.5,a_f\right)\frac{1}{a_f^{\,2}}\;-\frac{1}{a_f^{\,1/2}}\end{aligned}$$

$$-3\gamma\left(1.5,a_f\right)\frac{1}{{a_f}^2}-\frac{3}{5}\Gamma\left(-0.5,a_f\right)+\frac{12}{5}\gamma\left(2,a_f\right)\frac{1}{{a_f}^{5/2}}+\frac{2}{{a_f}^{3/2}}+3\ \Gamma(-2.5,a_f)\} \quad \text{(A-27)}$$

Appendix B (1D)

The functions for 1D unsteady heat conduction are

$$T_2^*\left(p,q,t,\tau\right)=\frac{r}{2\pi^{1/2}}\left[\gamma(a,0.5)+a^{-1/2}\exp(-a)\right] \quad \text{(B-1)}$$

$$\frac{\partial T_2^*\left(p,q,t,\tau\right)}{\partial n}=\frac{1}{2\pi^{1/2}}\gamma(0.5,a)\frac{\partial r}{\partial n} \quad \text{(B-2)}$$

$$T_3^*\left(p,q,t,\tau\right)=\frac{r^3}{12\pi^{1/2}}\{(a+1.5)a^{-1}\gamma(0.5,a)+a^{-1/2}\exp(-a)+2a^{-3/2}+a^{-3/2}\exp(-a)\} \quad \text{(B-3)}$$

$$\frac{\partial T_3^*\left(p,q,t,\tau\right)}{\partial n}=\frac{\partial r}{\partial n}\frac{r^2}{4\pi^{1/2}}[(1+a^{-1})\gamma(0.5,a)-a^{-1}\gamma(1.5,a)], \quad \text{(B-4)}$$

where $\gamma(,)$ is an incomplete gamma function of the first kind (Abramowitz, 1970). The time integral of Eqs. (49) and (B-1)-(B-4) can be obtained as follows:

$$\int_{t_f}^{t_F} T_1(p,q,t,\tau)d\tau=\frac{-r}{2\kappa\pi^{1/2}}[\Gamma(0.5,a_f)+\frac{1-\exp(-a_f)}{{a_f}^{0.5}}] \quad \text{(B-5)}$$

$$\int_{t_f}^{t_F}\frac{\partial T_1^*(p,q,t,\tau)}{\partial n}d\tau=\frac{\partial r}{\partial n}\frac{-1}{2\kappa\pi^{1/2}}\Gamma(0.5,a_f) \quad \text{(B-6)}$$

$$\int_{t_f}^{t_F} T_2^*(p,q,t,\tau)d\tau=\frac{r^3}{8\kappa\pi^{1/2}}\left\{\frac{1}{a_f}\gamma(0.5,a_f)-\frac{2}{3}\Gamma(0.5,a_f)+\frac{2}{3}\frac{(a_f^{1/2}+a_f^{3/2})}{a_f^2}\exp(-a_f)\right\} \quad \text{(B-7)}$$

$$\int_{t_f}^{t_F}\frac{\partial T_2^*(p,q,t,\tau)}{\partial n}d\tau=\frac{\partial r}{\partial n}\frac{r^2}{8\kappa\pi^{1/2}}\{-\frac{1}{a_f}\gamma(0.5,a_f)-2[\Gamma(0.5,a_f)-\frac{1}{a_f^{1/2}}\exp(-a_f)]\} \quad \text{(B-8)}$$

$$\begin{aligned}\int_{t_f}^{t_F} T_3^*(p,q,t,\tau)d\tau=&\frac{r^5}{2880\kappa\pi^{1/2}}\{15(\frac{3}{a_f^2}+\frac{4}{a_f})\gamma(0.5,a_f)-12\Gamma(0.5,a_f)\\&+12\gamma(0.5,a_f)+6(\frac{2}{a_f^{1/2}}+\frac{9}{a_f^{3/2}}+\frac{4}{a_f^{5/2}})\exp(-a_f)+\frac{48}{{a_f}^{5/2}}\}\end{aligned} \quad \text{(B-9)}$$

$$\int_{t_f}^{t_F}\frac{\partial T_3^*(p,q,t,\tau)}{\partial n}d\tau=\frac{\partial r}{\partial n}\frac{r^4}{16\kappa\pi^{1/2}}\{(\frac{1}{a_f}-\frac{1}{2{a_f}^2})\gamma(0.5,a_f)+\frac{1}{3}\Gamma(0.5,a_f)-\frac{\exp(-a_f)}{3{a_f}^{1/2}}+\frac{2\exp(-a_f)}{3{a_f}^{3/2}}\}, \quad \text{(B-10)}$$

where

$$a_f = \frac{r^2}{4\kappa(t_F - t_f)}. \tag{B-11}$$

Appendix C (Linear time interpolation)

The time integrals of Eq. (62) using linear time interpolation in the two-dimensional case can be obtained as follows:

$$\int_{t_{f-1}}^{t_f}(t_f-\tau)T_1^* d\tau = \frac{1}{4\pi\kappa}\left[\frac{r^2}{4\kappa}\left\{\frac{\exp(a_{f-1})}{a_{f-1}}-\frac{\exp(a_f)}{a_f}\right\}-\left\{(t_F-t_f)+\frac{r^2}{4\kappa}\right\}\left\{E_1(a_{f-1})-E_1(a_f)\right\}\right] \tag{C-1}$$

$$\int_{t_{f-1}}^{t_f}(\tau-t_{f-1})T_1^* d\tau = \frac{1}{4\pi\kappa}\left[\frac{r^2}{4\kappa}\left\{\frac{\exp(a_{f-1})}{a_{f-1}}-\frac{\exp(a_f)}{a_f}\right\}-\left\{(t_F-t_{f-1})+\frac{r^2}{4\kappa}\right\}\left\{E_1(a_{f-1})-E_1(a_f)\right\}\right] \tag{C-2}$$

$$\int_{t_{f-1}}^{t_f}(t_f-\tau)\frac{\partial T_1^*}{\partial n}d\tau = \frac{1}{2\pi\kappa}\frac{\partial r}{\partial n}\left\{(t_F-t_f)\frac{1}{R^2}\left[\exp(-a_{f-1})-\exp(-a_f)\right]-\frac{1}{4\kappa}\left[E_1(a_{f-1})-E_1(a_f)\right]\right\} \tag{C-3}$$

$$\int_{t_{f-1}}^{t_f}(\tau-t_{f-1})\frac{\partial T_1^*}{\partial n}d\tau$$
$$= \frac{1}{2\pi\kappa}\frac{\partial r}{\partial n}\left\{(t_F-t_{f-1})\frac{1}{R^2}\left[\exp(-a_{f-1})-\exp(-a_f)\right]-\frac{1}{4\kappa}\left[E_1(a_{f-1})-E_1(a_f)\right]\right\} \tag{C-4}$$

$$\int_{t_{f-1}}^{t_f}(t_f-\tau)T_2^*(p,q,t,\tau)d\tau = \frac{r^2}{256\kappa\pi}\left\langle 16(t_F-t_f)\left[E_1(a_f)+\frac{E_1(a_f)+\ln(a_f)+C+1-\exp(-a_f)}{a_f}\right]\right.$$
$$-\frac{r^2}{\kappa}\left\{E_1(a_f)-\frac{1-\exp(-a_f)}{a_f}-\frac{2E_1(a_f)+2\ln(a_f)+2C+1-\exp(-a_f)+a_f}{a_f^2}\right\}$$
$$\left.+\frac{r^2}{\kappa}\left\{E_1(a_{f-1})-\frac{1-\exp(-a_{f-1})}{a_{f-1}}-\frac{2E_1(a_{f-1})+2\ln(a_{f-1})+2C+1-\exp(-a_{f-1})+a_{f-1}}{a_{f-1}^2}\right\}\right\rangle \tag{C-5}$$

$$\int_{t_{f-1}}^{t_f}(\tau-t_{f-1})T_2^*(p,q,t,\tau)d\tau = \frac{r^2}{256\kappa\pi}\left\langle 16(t_F-t_{f-1})\left[E_1(a_f)+\frac{E_1(a_f)+\ln(a_f)+C+1-\exp(-a_f)}{a_f}\right]\right.$$
$$-\frac{r^2}{\kappa}\left\{E_1(a_f)-\frac{1-\exp(-a_f)}{a_f}-\frac{2E_1(a_f)+2\ln(a_f)+2C+1-\exp(-a_f)+a_f}{a_f^2}\right\}$$

$$+\frac{r^2}{\kappa}\left\{E_1(a_{f-1})-\frac{1-\exp(-a_{f-1})}{a_{f-1}}-\frac{2E_1(a_{f-1})+2\ln(a_{f-1})+2C+1-\exp(-a_{f-1})+a_{f-1}}{a_{f-1}^2}\right\}\Bigg\rangle \quad \text{(C-6)}$$

$$\int_{t_{f-1}}^{t_f}(t_f-\tau)\frac{\partial T_2^*}{\partial n}d\tau=\frac{r}{8\kappa\pi}\frac{\partial r}{\partial n}\Bigg\langle-(t-t_f)\left[E_1(a_{f-1})+\frac{1-\exp(-a_{f-1})}{a_{f-1}}\right]+(t-t_f)\left[E_1(a_f)+\frac{1-\exp(-a_f)}{a_f}\right]$$
$$+\frac{2\kappa(t-t_{f-1})^2}{r^2}\{1-\exp(-a_{f-1})+a_{f-1}\exp(-a_{f-1})-a_{f-1}^2\exp(-a_{f-1})\}$$
$$-\frac{2\kappa(t-t_f)^2}{r^2}\{1-\exp(-a_f)+a_f\exp(-a_f)-a_f^2\exp(-a_f)\}\ \Bigg\rangle \quad \text{(C-7)}$$

$$\int_{t_{f-1}}^{t_f}(t_f-\tau)T_3^*(p,q,t,\tau)d\tau=\frac{r^4}{9216\kappa\pi}\{36(t_F-t_f)\Bigg\langle\Bigg[E_1\left(a_f\right)+\frac{1}{a_f}\left\{4E_1(a_f)+4\ln(a_f)+4C-\exp(-a_f)+1\right\}$$
$$+\frac{2E_1(a_f)+2\ln(a_f)+2C-3\exp(-a_f)+3-5a_f}{{a_f}^2}\Bigg]$$
$$-\Bigg[E_1(a_{f-1})+\frac{1}{a_{f-1}}\left\{4E_1(a_{f-1})+4\ln(a_{f-1})+4C+1-\exp\left(-a_{f-1}\right)\right\}$$
$$+\frac{2E_1(a_f)+2\ln(a_f)+2C-3\exp(-a_f)+3-5a_f}{{a_{f-1}}^2}\Bigg]\Bigg\rangle$$
$$+\frac{r^2}{\kappa}\Bigg[\Bigg\{-E_1(a_{f-1})-\frac{1-\exp(-a_{f-1})}{a_f}+\frac{18E_1(a_{f-1})+18\ln(a_{f-1})+18C+a_{f-1}+1-\exp(-a_{f-1})}{a_{f-1}^2}$$
$$+\frac{12E_1(a_{f-1})+12\ln(a_{f-1})+12C+16-16\exp(-a_{f-1})-28a_{f-1}+11{a_{f-1}}^2}{a_{f-1}^3}\Bigg\}$$
$$-\Bigg\{-E_1(a_f)-\frac{9-\exp(-a_f)}{a_f}+\frac{18E_1(a_f)+18\ln(a_f)+18C+9a_f-\exp(-a_f)-27}{a_f^2}$$
$$+\frac{12E_1(a_f)+12\ln(a_f)+12C+16-16\exp(-a_f)}{a_f^3}\Bigg\}\Bigg]\} \quad \text{(C-8)}$$

$$\int_{t_{f-1}}^{t_f}(t_f-\tau)\frac{\partial T_3^*}{\partial n}d\tau=\frac{r^5}{1536\kappa^2\pi}\frac{\partial r}{\partial n}\left\{\left[-E_1(a_{f-1})+\frac{\exp(-a_{f-1})}{a_{f-1}}+6\frac{E_1(a_{f-1})+\ln(a_{f-1})+C}{a_{f-1}^2}\right.\right.$$

$$\left.+\frac{1-\exp(-a_{f-1})}{a_{f-1}^2}+4\frac{1-\exp(-a_{f-1})-a_{f-1}}{a_{f-1}^3}\right]$$

$$-\frac{24\kappa(t_F-t_f)}{r^2}\left[E_1(a_{f-1})+2\frac{E_1(a_{f-1})+\ln(a_{f-1})+C}{a_{f-1}}+\frac{1-\exp(-a_{f-1})}{a_{f-1}}+\frac{1-\exp(-a_{f-1})-a_{f-1}}{a_{f-1}^2}\right]$$

$$-\left[-E_1(a_f)+\frac{\exp(-a_f)}{a_f}+6\frac{E_1(a_f)+\ln(a_f)+C}{a_f^2}+\frac{1-\exp(-a_f)}{a_f^2}+4\frac{1-\exp(-a_f)-a_f}{a_f^3}\right]$$

$$\left.+\frac{24\kappa(t_F-t_f)}{r^2}\left[E_1(a_f)+2\frac{E_1(a_f)+\ln(a_f)+C}{a_f}+\frac{1-\exp(-a_f)}{a_f}+\frac{1-\exp(-a_f)-a_f}{a_f^2}\right]\right\}. \quad \text{(C-9)}$$

4. References

Abramowitz, M. and Stegun, A. Eds., (1970), *Handbook of Mathematical Functions*, pp. 255-263, Dover, New York.

Brebbia, C. A., Tells, J. C. F. and Wrobel, L. C., (1984), *Boundary Element Techniques-Theory and Applications in Engineering*, pp. 47-107, Springer-Verlag, Berlin.

Carslaw, H. S. and Jaeger, J. C., (1938), *Some Problems in the Mathematical Theory of the Conduction of Heat*, Phil. Mag., Vol. 26, pp. 473-495.

Dyn, N., (1987), Interpolation of Scattered Data by Radial Functions, in *Topics in Multivariate Approximation*, Eds. C. K. Chui, L. L. Schumaker and F. I. Utreras, pp. 47-61, Academic Press, London.

Micchelli, C. A., (1986), Interpolation of Scattered Data, *Constructive Approximation*, Vol. 2, pp. 12-22.

Nowak, A. J., (1989), The Multiple Reciprocity Method of Solving Transient Heat Conduction Problems, *Advances in Boundary Elements*, Vol. 2, pp. 81-93, Eds. C. A. Brebbia and J. J. Connors, Computational Mechanics Publication, Southampton, Springer-Verlag, Berlin.

Nowak, A. J. and Neves, A. C., (1994), *The Multiple Reciprocity Boundary Element Method*, Computational Mechanics Publication, Southampton, Boston.

Ochiai, Y. and Sekiya, T., (1995a), Steady Thermal Stress Analysis by Improved Multiple-Reciprocity Boundary Element Method, *Journal of Thermal Stresses*, Vol. 18, No. 6, pp. 603-620.

Ochiai, Y., (1995b), Axisymmetric Heat Conduction Analysis by Improved Multiple-Reciprocity Boundary Element Method, *Heat Transfer-Japanese Research*, Vol. 23, No. 6, pp. 498-512.

Ochiai, Y. and Sekiya, T., (1995c), Generation of Free-Form Surface in CAD for Dies, *Advances in Engineering Software*, Vol. 22, pp. 113-118.

Ochiai, Y., (1996a), Generation Method of Distributed Data for FEM Analysis, *JSME International Journal*, Vol. 39, No. 1, pp. 93-98.

Ochiai, Y. and Sekiya, T., (1996b), Steady Heat Conduction Analysis by Improved Multiple-Reciprocity Boundary Element Method, *Engineering Analysis with Boundary Elements*, Vol. 18, pp. 111-117.

Ochiai, Y. and Kobayashi, T., (1999), Initial Stress Formulation for Elastoplastic Analysis by Improved Multiple-Reciprocity Boundary Element Method, *Engineering Analysis with Boundary Elements*, Vol. 23, pp. 167-173.

Ochiai, Y. and Yasutomi, Z., (2000), Improved Method Generating a Free-Form Surface Using Integral Equations, Computer Aided Geometric Design, Vol. 17, No. 3, pp. 233-245.

Ochiai, Y., (2001), Two-Dimensional Unsteady Heat Conduction Analysis with Heat Generation by Triple-Reciprocity BEM, *International Journal of Numerical Methods in Engineering,* Vol. 51, No. 2, pp. 143-157.

Ochiai, Y., (2003a), Multidimensional Numerical Integration for Meshless BEM, *Engineering Analysis with Boundary Elements*, Vol. 27, No. 3, pp. 241-249.

Ochiai, Y., (2003b), The Multiple-Reciprocity Method for Elastic Problems with Arbitrary Body Force, *Transformation of Domain Effects to the Boundary,* Y. F. Rashed Ed., Chapter 5, ISBN 1-85312-896-1, WIT Press.

Ochiai, Y. and Sladek, V., (2005), Numerical Treatment of Domain Integrals without Internal Cells in Three-Dimensional BIEM Formulations, *CMES(Computer Modeling in Engineering & Sciences)* Vol. 6, No. 6, pp. 525-536.

Ochiai, Y., Sladek, V. and Sladek, S., (2006), Transient Heat Conduction Analysis by Triple-Reciprocity Boundary Element Method, *Engineering Analysis with Boundary Elements*, Vol. 30, No. 3, pp. 194-204.

Ochiai, Y and Takeda, S., (2009a), Meshless Convection-Diffusion Analysis by Triple-Reciprocity Boundary Element Method, *Engineering Analysis with Boundary Elements*, Vol.33, No.2, pp.168-175.

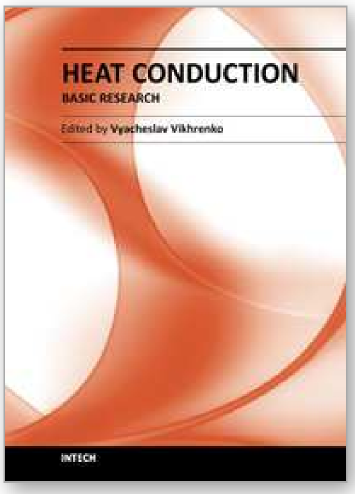
Ochiai, Y and Kitayama, Y., (2009b),Three-dimensional Unsteady Heat Conduction Analysis by Triple-Reciprocity Boundary Element Method, *Engineering Analysis with Boundary Elements*, Vol. 33, No. 6, pp. 789-795.

Partridge, P. W., Brebbia, C. A. and Wrobel, L. C., (1992), *The Dual Reciprocity Boundary Element Method*, Computational Mechanics Publications, pp. 223-253.

Sladek, J. and Sladek, V., (2003), Local Boundary Integral Equation Method for Heat Conduction Problem in an Anisotropic Medium, *Proceedings of ICCES2003*, Chap. 5.

Tanaka, M., Matsumoto, T. and Takakuwa, S., (2003), Dual Reciprocity BEM Based on Time-Stepping Scheme for the Solution of Transient Heat Conduction Problems, *Boundary Elements XXV*, WIT Press, pp. 299-308.

Wrobel, L. C., (2002), *The Boundary Element Method*, Vol. 1, John Wiley & Sons, West Sussex, pp. 97-117.



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The content of this book covers several up-to-date approaches in the heat conduction theory such as inverse heat conduction problems, non-linear and non-classic heat conduction equations, coupled thermal and electromagnetic or mechanical effects and numerical methods for solving heat conduction equations as well. The book is comprised of 14 chapters divided into four sections. In the first section inverse heat conduction problems are discuss. The first two chapters of the second section are devoted to construction of analytical solutions of nonlinear heat conduction problems. In the last two chapters of this section wavelike solutions are attained.The third section is devoted to combined effects of heat conduction and electromagnetic interactions in plasmas or in pyroelectric material elastic deformations and hydrodynamics. Two chapters in the last section are dedicated to numerical methods for solving heat conduction problems.

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