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Parametric Robust Stability

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1. Introduction

Robust stability of LTI systems with parametric uncertainty is a very interesting topic to study, industrial world is contained in parametric uncertainty. In industrial reality, there is not a particular system to analyze, there is a family of systems to be analyzed because the values of physical parameters are not known, we know only the lower and upper bounds of each parameter involved in the process, this is known as *Parametric Uncertainty* (Ackermann et al., 1993; Barmish, 1994; Bhattacharyya et al., 1995). The set of parameters involved in a system makes a *Parametric Vector*, the set of all vectors that can exists such that each parameter is kept within its lower and upper bounds is called a *Parametric Uncertainty Box*.

The system we are studying is now composed of an infinite number of systems, each system corresponds to a parameter vector contained in the parametric uncertainty box. So as to test the stability of the LTI system with parametric uncertainty we have to prove that all the infinite number of systems are stable, this is called *Parametric Robust Stability*. The parametric robust stability problem is considerably more complicated than determine the stability of an LTI system with fixed parameters. The stability of a LTI system can be analyzed in different ways, this chapter will be analyzed by means of its characteristic polynomial, in the case of parametric uncertainty now exists a set with an infinite number of characteristic polynomials, this is known as a *Family of Polynomials*, and we have to test the stability of the whole family.

The parametric robust stability problem in LTI systems with parametric uncertainty is solved in this chapter by means of two tools, the first is a recent stability criterion for LTI systems (Elizondo, 2001B) and the second is the mathematical tool "Sign Decomposition" (Elizondo, 1999). The recent stability criterion maps the prametric robust stability problem to a robust positivity problem of multivariable polynomic functions, sign decomposition solves this problem in necessary and sufficient conditions.

By means of the recent stability criterion (Elizondo, 2001B) is possible to analyze the characteristic polynomial and determine the number of unstable roots on the right side in the complex plane. This criterion is similar to the Routh criterion although without using the traditional division of the Routh criterion. This small difference makes a big advantage when it is analized the robust stability in LTI systems with parametric uncertainty, the elements of the first column of the table (Elizondo, 2001B) they are multivariable polynomic functions and these must be positive for stability conditions. Robust positivity of a multivariable polynomial function is more easier to prove that in the case of quotients of this class of functions, therefore, the recent criterion (Elizondo, 2001B) is easier to use than Routh criterion. There are other

criterions whose its elements are multivariable polynomic functions, such as the Hurwitz criterion and Lienard-Chipart criterion (Gantmacher, 1990), but both use a huge amount of mathematical operations in comparison with the recently stablished stability criterion Elizondo et al. (2005). When industrial cases are analyzed, the difference of mathematical operations is paramount, if the recently stability criterion takes several hours to determine the robust stability, the other criterions take several days. For these reasons the recently stability criterion is used in this chapter instead of other criterions.

Sign Decomposition (Elizondo, 1999) also called by some authors as *Sign definite Decomposition* is a mathematical tool able to determine in necessary and sufficient conditions the robust positivity of multivariable polynomic functions by means of extreme points analysis. Sign Decomposition begun as incipient orthogonal ideas of the author in his PhD research. It was not easy to develop this tool as thus it happens in orthogonal works with respect to the contemporary research line, the orthogonal ideas normally are not well seen. This is a very difficult situation on any research work, there may be many opinions, but we must accept that the world keeps working by the aligned but it changes by the orthogonals.

In LTI systems with parametric uncertainty applications, the multivariable polynomic functions to be analyzed depend on bounded physical parameters and some bounds could be negative. So sign decomposition begins with a coordinates transformation from the physical parameters to a set of mathematical parameters such that all the vectors of the new parameters are contained in a positive convex cone; in other words, all the new parameters are non-negatives. In this way, the multivariable polynomic function is made by non-decreasing terms, some of them are preceded by a positive sign and some by a negative sign. Grouping all the positive terms and grouping all the negative terms, then factorizing the negative sign and defining a "positive part" and a "negative part" of the function we obtain two non-decreasing functions. Now the function can be expressed as the positive part minus the negative part. It is obvious that both parts are independent functions, so they can be taken as a basis in with a graphical representation using two axis, the axis of the negative part and the axis of the positive part. Now, suppose that we have a particular vector contained in the parametric uncertainty box , then evaluating the negative part and the positive part a point on the "negative part, positive part plane" is obtained, this point represents the function evaluated in the particular vector in . The forty five degree line crossing at the origin on the "negative part, positive part plane" represents the set of functions with zero value, a point above this line represents a function with positive value and a point below this line represents a function with negative value.

The decomposition of the function in its negative and positive parts may look very simple and non-transcendent but taking into account that the negative and positive parts are made by the addition of non-decreasing terms, then the negative and positive parts are nondecreasing functions in a vector space, this implies that the positive part and the negative part are bonded. So, geometrically, any point representing the function evaluated at any parameter vector is contained in a rectangle on the "negative part, positive part plane" and if the lowest right vertex is above the forty five degree line then the function is robust positive, obtaining in this way the basis of the "rectangle theorem". By means of this theorem upper and lower bounds of the multivariable polynomic function in the parametric uncertainty box are obtained. Sign decomposition contains a set of definitions, propositions, facts, lemmas, theorems and corollaries, sign decomposition can be applied to several disciplines; in the case of LTI systems with parametric uncertainty, this mathematical tool can be applied to robust controllability,

obsevability or stability analysis. In this chapter sign decomposition is applied to parametric robust stability.

In this chapter the following topics are studied: recent stability criterion, linear time invariant systems with parametric uncertainty, brief description of sign decomposition and finally a solution for the parametric robust stability problem. All demonstrations of the criterions, theorems, corollaries, lemmas, etc, will be omitted because they are results previously published.

2. A recent stability criterion for LTI systems

The study of stability of the LTI systems begun approximately one and a half century ago with three important criterions: *Hermite* in 1856 (Ackermann et al., 1993), 1854 (Bhattacharyya et al., 1995); *Routh* in 1875 (Ackermann et al., 1993), 1877 (Gantmacher, 1990) and *Hurwitz* in 1895 (Gantmacher, 1990). Routh, using Sturm's theorem and Cauchy Index theory of a real rational function, set up a theorem to determine the number k of roots of polynomial with real coefficients on the right half plane of the complex numbers.

Theorem 1. *(Routh) (Gantmacher, 1990) The number of roots of the real polynomial $p(s) = c_0 + c_1 s + c_2 s^2 + \cdots + c_n s^n$ in the right half of the complex plane is equal to the number of variations of sign in the first column of the Routh's table with coefficients: $a_{i,j} = (a_{i-1,1} a_{i-2,j+1} - a_{i-2,1} a_{i-1,j+1})/a_{i-1,1}$ $\forall i \geq 3$, $a_{i,j} = c_{n+1-i-2(j-1)}$ $\forall i \leq 2$*

There are several results related to the Routh criterion, for example (Fuller, 1977; Meinsma, 1995), but they are not appropriate to use in the parametric uncertainty case and they use more mathematical calculations than the Routh criterion.

In this chapter a recent criterion, an arrange similar to the Rouht table, it is presented. The stability in this recent criterion depends on the positivity of a *sign column*. The recent criterion has two advantages: 1) the numerical operations are reduced with respect to above mentioned criterions; 2) the coefficients are multivariable polynomic functions in the case of parametric uncertainty and robust positivity is easier to test than Routh criterion. The criterion is as described below.

Theorem 2. *(Elizondo, 2001B) Given a polynomial $p(s) = c_0 + c_1 s + c_2 s^2 + \cdots + c_{n-1} s^{n-1} + c_n s^n$ with real coefficients, the number of roots on the right half of the complex plane is equal to the number of variations of sign in the sign σ column on the follow arrange.*

$$
\begin{array}{cccc}
\sigma_1 & c_n & c_{n-2} & c_{n-4} \cdots \\
\sigma_2 & c_{n-1} & c_{n-3} & c_{n-5} \cdots \\
\sigma_3 & e_{3,1} & e_{3,2} & \cdots \\
\vdots & \vdots & \vdots &
\end{array}
$$

$$e_{i,j} = (e_{i-1,1} e_{i-2,j+1} - e_{i-2,1} e_{i-1,j+1}), \forall 3 \leq i \leq n+1$$
$$e_{i,j} = c_{n+1-i-2(j-1)} \ \forall i \leq 2$$
$$\sigma_i = Sign(e_{i,1}) \ \forall i \leq 2, \qquad \sigma_i = Sign(e_{i,1}) \prod_{j=1}^{(i+1-m)/2} Sign(e_{m+2(j-1),1}) \ \forall i \geq 3$$

The procedure for calculating the elements $(e_{i,j})$ is similar to the Routh table but without using the division. On the other hand, the calculation of an element σ_i is more easier than it looks mathematical expression. We can get the sign σ_i, multiplying the sign of the

element $(e_{i,1})$ by the sign of the immediate superior element $(e_{i-1,1})$ and then jumping in pairs. For example $\sigma_6 = Sign(e_{6,1})Sign(e_{5,1})Sign(e_{3,1})Sign(e_{1,1})$. Also $\sigma_1 = Sign(c_n)$ and $\sigma_2 = Sign(c_{n-2})$. So also it is not necessary to calculate the last element $(e_{n+1,1})$, only its sign is necessary to calculate. Each row of $(e_{i,j})$ elements is obtained by means of $(e_{i-1,j})$ and $(e_{i-2,j})$ elements previously calculated and in Hurwitz criterion a principal minor is not calculated from previous, then the Elizondo-González criterion is more advantageous than Hurwitz criterion as shown in table (1)

Remark 3. *a) Given the relation of the above criterion with the Routh criterion, the cases in that one element $e_{i,j}$ is equal to zero or all the elements of a row are zero, they are treated as so as it is done in the Routh criterion. b) The last element $e_{n+1,1}$ is not necessary to calculate, but it is necessary to obtain only its sign*

Mathematical operations in polynomials n degree

degree	Hurwitz		C. Elizondo	
n	$\times$	$+o-$	$\times$	$+o-$
3	4	1	2	1
4	9	2	5	2
5	66	18	9	4
6	193	45	14	6
7	780	145	20	9

Table 1. A comparison of stability criterions.

2.1 Examples

Example 1. Given the polynomial $p(s) = s^5 + 2s^4 + 1s^3 + 5s^2 + 2s + 2$ by means of criterion 2 determine the number of roots in the right half of the complex plane and compare the results with the Routh criterion.

Applying 2 criterion we obtain the left table. As an example of the procedure to obtain the elements $e_{i,j}$ and σ_i, we have: $e_{3,1} = 2 \times 1 - 1 \times 5$, $e_{3,2} = 2 \times 2 - 1 \times 2$, $\sigma_6 = Sign(+) \times Sign(-56) \times Sign(-3) \times Sign(1)$, $\sigma_5 = Sign(-56) \times Sign(-19) \times Sign(2)$.

Elizondo-González 2001

$\sigma_1 = +$	1	1	2
$\sigma_2 = +$	2	5	2
$\sigma_3 = -$	-3	2	
$\sigma_4 = +$	-19	-6	
$\sigma_5 = +$	-56		
$\sigma_6 = +$	$+$		

Routh

1	1	2
2	5	2
-1.5	1	
6.3333	2	
1.4737		
$+$		

Table 2. Example 1. Comparison of stability criterions.

The left arrangement shows two sign changes in σ column so the polynomial has two roots on the right half of the complex plane. By means of Routh criterion is obtained the right table, it shows too two sign changes in the first column which is the same previous result. An interesting observation (see table (2)) is that the left table presents a minus sign in the third row of the σ column and the right table presents a minus sign in the same third row but in the first column.

Example 2. Given the polynomial $p(s) = s^5 + 2s^4 + 2s^3 + 2s^2 + s + 3$ by means of criterion 2 determine the number of roots in the right half of the complex plane and compare the results with the Routh criterion.

Elizondo-González 2001				Routh		
$\sigma_1 = +$	1	2	1	1	2	1
$\sigma_2 = +$	2	2	3	2	2	3
$\sigma_3 = +$	2	-1		1	-0.5	
$\sigma_4 = +$	6	6		3	3	
$\sigma_5 = -$	-18			-1.5		
$\sigma_6 = +$	$-$			$+$		

Table 3. Example 2. Comparison of stability criterions.

It is easy to see by means of two criterions that the polynomial has two roots on the right half of the complex plane in accordance to the table (3).

Example 3. Given the polynomial $p(s) = s^5 + 1s^4 + 2s^3 + 2s^2 + 2s + 1$ by means of criterion 2 determine the number of roots in the right half of the complex plane.

When we try to make the table by means of Elizondo-González 2001 criterion or Routh criterion, it is truncated because $e_{3,1} = 0$

σ_1	1	2	2
σ_2	1	2	1
σ_3	0	1	

Table 4. Example 3. Presence of a zero in the first column of elements.

Since the element $e_{3,1}$ is equal zero (see table (4)) then this element is replaced by by an $\epsilon > 0$, thus obtaining the following arrangement.

σ_1	1	2	2
σ_2	1	2	1
σ_3	ϵ	1	
σ_4	$2\epsilon - 1$	ϵ	
σ_5	$2\epsilon - 1 - \epsilon^2$		
σ_6	$(2\epsilon - 1 - \epsilon^2)\epsilon$		

Table 5. Example 3. Solution of the problem of zero in the first column.

Applying the limit $\epsilon \to 0$ in table (5) is obtained the table (6).

$\sigma_1 = +$	1	2	2
$\sigma_2 = +$	1	2	1
$\sigma_3 = +$	ϵ	1	
$\sigma_4 = -$	-1	ϵ	
$\sigma_5 = +$	-1		
$\sigma_6 = +$	$-\epsilon$		

Table 6. Example 3. Final result to the solution of the problem of zero in the first column.

From the table (6) is easy to see that the polynomial has two roots on the right half of the complex plane.

Example 4. Given the polynomial $p(s) = s^5 + 1s^4 + 2s^3 + 2s^2 + 1s + 1$ by means of criterion 2 determine the number of roots in the right half of the complex plane. Applying this criterion we get as following.

σ_1	1	2	1
σ_2	1	2	1
σ_3	0	0	

Table 7. Example 4. A row equal zero.

The table (7) generated, it shows the third row equal zero. Then obtaining the derivative of the polynomial "corresponding" to the immediately superior row $p(s) = s^4 + 2s^2 + 1$ is obtained $p(s) = 4s^3 + 4s$. Now the coefficients of this polynomial replace the zeros of the third row and the procedure continues, obtaining in this way the follow arrangement.

$\sigma_1 = +$	1	2	1
$\sigma_2 = +$	1	2	1
$\sigma_3 = +$	4	4	
$\sigma_4 = +$	4	4	
$\sigma_5 = +$	ϵ		
$\sigma_6 = +$	4ϵ		

Table 8. Example 4. Solution to the problem of a row equal zero.

We can see in table (8) that there is no sign change in *sigma* column, then there are not roots in the right half complex plane.

3. Linear time invariant systems with parametric uncertainty

3.1 Parametric uncertainty

All physical systems are dependent on parameters q_i and in the physical world does not know the value of the parameters, only know the lower q_i^- and upper q_i^+ bounds of each parameter, so that $q_i^- \leq q_i \leq q_i^+$, this expression is also written as $q_i \in [q_i^-, q_i^+]$.

For example if we have several electrical resistances with color code of 1,000 ohm, if one measures one of them, the measurement can be: 938, 1,024, or a value close to 1,000 ohm but it is rather difficult that it is exactly 1,000 ohm. By means of tolerance code can be deduced that the resistance will be greater than 900 and less than 1,100 ohm. Another example is the mass of a commercial aircraft, it can fly with few passengers and little baggage or with with many passengers and much baggage, then the mass of the plane is not known until the last passenger to be registered, but not when the plane was designed, however the plane is designed to fly from a minimum mass to a maximum mass.

The set of ℓ parameters involved in a system makes a *Parametric Vector* $q = [q_1, q_2, \cdots, q_\ell]^T$, $q \in \Re^\ell$ and the set of all the possible parameter vectors that may exist makes a *Parametric Uncertainty Box* $Q = \{ q = [q_1, q_2, \cdots, q_\ell]^T | q_i \in [q_i^-, q_i^+] \ \forall i \}$. In the case of $q_i > 0 \ \forall i$ then $Q = \{ q = [q_1, q_2, \cdots, q_\ell]^T | q_i > 0, q_i \in [q_i^-, q_i^+] \ \forall i \}$ and Q is contained in a positive convex cone P, $Q \subset P \subset \Re^\ell$.

For the study of cases involving parametric uncertainty is necessary to define the minimum and maximum vertices of the parametric uncertainty box, so the minimum $v^{\min}$ and

maximum $v^{\max}$ Euclidean vertices of Q are defined as so as $\|v^{\min}\|_2 = \min\limits_{q \in Q} \|q\|_2$, $\|v^{\max}\|_2 = \max\limits_{q \in Q} \|q\|_2$.

3.2 Parametric robust stability in LTI systems

In the LTI systems with parametric uncertainty, the characteristic polynomial has coefficients dependent on physical parameters, $p(s, q) = c_0(q) + c_1(q)s + c_2(q)s^2 + \cdots c_n(q)s^n$; so Routh criterion is very difficult to use because it is necessary to test the robust positivity of rational functions dependent on physical parameters. By means of Hurwitz criterion is possible to solve the problem of parametric robust stability by means of robust positivity of principal minors of a matrix dependent on physical parameters, this procedure uses a lot of mathematical calculations. The robust positivity of rational function dependent on physical parameters can be considered as so as a very much difficult problem since only the robust positive test of multivariable polynomic function is very difficult problem (Ackermann et al., 1993) (page 93). So the parametric robust stability problem in LTI systems with parametric uncertainty in the general case is not an easy problem to solve, however in this chapter is presented a solution.

The characteristic polynomials are classified according to its coefficient of maximum complexity; from the simplest structure coefficient to the most complex are: *Interval, Affine, Multilinear and Polynomic*. For example, the coefficients: $c_i(q) = q_i$, $c_i(q) = 2q_1 + 3q_2 + 5q_3 + q_4$, $c_i(q) = 5q_1q_2 + 2q_2q_4 + 5q_3 + q_4$, $c_i(q) = 2q_1^3q_2 + 2q_2^2q_4^5 + q_3$, correspond to classification: Interval, Affine, Multilinear and Polynomic respectively. The number of polynomials $p(s, q)$ that can exist is infinite since the number of vectors that exist is infinite, the collection of all polynomials that exist is a *Family of Polynomials* $P(s, Q) = \{p(s, q)|q \in Q\}$.

The families of polynomials interval and afin are convex sets and these families have subsetting test. This concept, *subsetting test*, means that a family of polynomials is robustly stable if and only if all polynomials contained in the subsetting test are stable.

Kharitonov in (Kharitonov, 1978), by means of his theorem demonstrates that a family of interval polynomials is robust stable if and only if a set of four polynomials are stable. In (Bartlett et al., 1988) by means of their edge theorem, demonstrated that a family of afin polynomials is robustly stable if and only if all the polynomials corresponding to the edges of the parametric uncertainty box are stable. The multilinear an polynomic families are not convex set and they do not have subsetting test. So parametric robust stability of these families can not be resolved by tools based on convexity. In (Elizondo, 1999) was presented a solution for parametric robust stability of any kind of family: Interval, Affine, Multilinear or Polynomic. The solution is based on sign decomposition, and by means of this tool can also solve the problem of robust controllability or robust observability.

3.3 Robust stability mapped to robust positivity

The parametric robust stability problem of LTI systems can be mapped to a problem of robust positivity of polynomial functions for at least three ways.

The first two are: the Hurwitz and Lienard-Chipart criterions, the other is the recently stability criterion (2). By Hurwitz or Lienard-Chipart criterions can do the mapping but as explained these require making a lot of mathematical calculations. The criterion (2) requires much less mathematical calculations that the criterions mentioned as was shown in table (1), (Elizondo et al., 2005)

4. Brief description of sign decomposition

In different areas of sciences the fundamental problem can be mapped to a problem of robust positivity of multivariable polynomic functions. For example the no singularity of a matrix can be analyzed by mean of the robust positivity of its determinant, so it is very useful to have a mathematical tool that solves the problem of robust positivity of multivariable polynomic functions. Practically there are three tools for this purpose: Interval Arithmetic (Moor, 1966); Bernstein Polynomials (Zettler, et all 1998) and Sign Decomposition ((Elizondo, 1999)) whose complete version is developed in (Elizondo, 1999) and its partial versions are presented in (Elizondo, 2000; 2001A;B; 2002A;B), for simplicity only will be mentioned (Elizondo, 1999). Interval arithmetic is very difficult to use because it requires much more calculations than other methods. When robust positivity is analyzed in a very simple function, Bernstein polynomials have advantages over sign decomposition, but when the function is not simple, sign decomposition has advantages over Bernstein polynomials (Graziano et al., 2004). There are several works using sign decomposition instead of Bernstein polynomials, some of them are: (Bhattacharyya et al., 2009; Guerrero, 2006; Keel et al., 2008; 2009; Keel, 2011; Knap et al., 2010; 2011)

4.1 Definition of sign decomposition

The following is a brief description of the more relevant results of Sign Decomposition (Elizondo, 1999). By means of this tool it is possible to determine, in necessary and sufficient conditions, the robust positivity of a multivariable polynomic function depending on ℓ parameters, employing extreme points analysis.

Since mathematically exist the possibility that a parameter $\hat{q}_i$ has negative value, then this tool begins by a "coordinates transformation" from $\hat{q}_i$ to q_i such that the new parameters will be positive $q_i > 0$, then an uncertainty box $Q = \{\ q = [q_1, q_2, \cdots, q_\ell]^T \,|\, q_i > 0,\ q_i \in [q_i^-,\ q_i^+]\ \}$ is makes, in other words, Q is in a positive convex cone P, $Q \subset P \subset \Re^\ell$ with minimum $v^{\min}$ and maximum $v^{\max}$ Euclidean vertices. The transformation is very easy as shown in the equation (1)

$$q_i = q_i^- + \frac{\hat{q}_i - \hat{q}_i^-}{\hat{q}_i^+ - \hat{q}_i^-}(q_i^+ - q_i^-) \tag{1}$$

From here on we will assume that if necessary, the transformation was made and work with parameters $q_i > 0$. Under this consideration will continue with the rest of this topic.

Definition 4. *(Elizondo, 1999) Let $f : \Re^\ell \to \Re$ be a continuous function and let $Q \subset P \subset \Re^\ell$ be a box. It is said that $f(q)$ has **Sign Decomposition** in Q if there exist two bounded continuous nondecreasing and nonnegative functions $f_n(\cdot) \geq 0$, $f_p(\cdot) \geq 0$, such that $f(q) = f_p(q) - f_n(q)$ $\forall\ q \in Q$. In this way there are defined the **Positive Part** $f_p(q)$ and **Negative Part** $f_n(q)$ of the function.*

Negative Part is only a name since Negative Part and Positive Part are nonnegative.

4.2 (f_n, f_p) representation

Is obvious that for the general case, $f_n(\cdot)$ and $f_p(\cdot)$ are independent functions then they make a basis in $\Re^2$ with graphical representation in the $(f_n(\cdot), f_p(\cdot))$ plane in accordance with figure (1).

If we take a particular vector $q \in Q$ and evaluated the $f_n(q)$ and $f_p(q)$ parts, we obtain the coordinates $(f_n(q), f_p(q))$ of the function in the (f_n, f_p) plane. The 45° line is the set of points where the function is equal zero because $f_p(q) = f_n(q)$ so $f(q) = f_p(q) - f_n(q) = 0$. If a point is above the 45° line means that $f_p(q) > f_n(q)$ then $f(q) > 0$. If a point is below the 45° line means that $f_p(q) < f_n(q)$ then $f(q) < 0$.
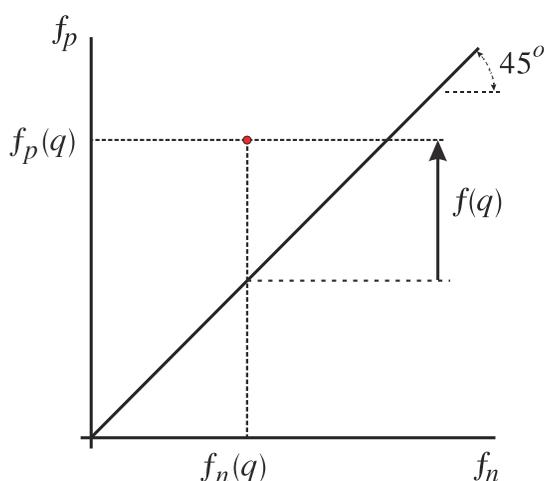

Fig. 1. (f_n, f_p) plane

It should be noted that independently of the number of parameters in which the function depends on, the function will always be represented in $\Re^2$ via $(f_n(q), f_p(q))$. For example, the function $f(q) = 4 - q_2 + q_1 q_3 + 8q_1^2 q_2 - 9q_1 q_2^2 q_3^3$ such that $q \in Q \subset P \subset \Re^3$, $Q = \{ q = [q_1, q_2, q_3]^T | q_i \in [0, 1] \}$. The function has sign decomposition because it is decomposed in two bounded continuous nondecreasing and nonnegative functions $f_p(q) = 4 + q_1 q_3 + 8q_1^2 q_2$, $f_n(q) = q_2 + 9q_1 q_2^2 q_3^3$ and $f(q) = f_p(q) - f_n(q)$. The figure (2) was obtained by plotting a hundred lines blue color, (one hundred fifty points per line) of variable q_3 holding (q_1, q_2) constant uniformly distributed in different positions. The process was repeated varying q_2 in green color and finally varying q_1 in red color. According to the position shown in the graph of the function with respect to the 45° line, it appears that the function is robustly positive. But it must be demonstrated mathematically.
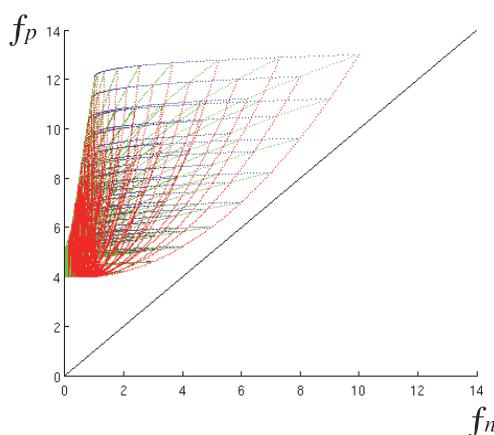
Fig. 2. Function in (f_n, f_p) plane

Some preliminary properties of the continuous functions $f(q)$, $g(q)$, $h(q)$ with sign decomposition in Q and for all $u(q)$ nondecreasing function in Q, are proved in (Elizondo, 1999) as so facts, lemmas and theorems. This properties are employed on the following theorems.
a) $(f_n(q) + u(q), f_p(q) + u(q))$ is a (f_n, f_p) representation of the function $f(q)$ $\forall q \in Q$; **b)** the representation $(f_n(q) + u(q), f_p(q) + u(q))$ of the function is reduced to its minimum expression: $(f_p(q), f_n(q))$; **c)** $f(q) + g(q)$; **d)** $f(q) - g(q)$ and **e)** $f(q)g(q)$ are functions with sign decomposition in Q; **f)** if $f(q) = g(q) + h(q)$, then the positive and negative parts of $f(q) - g(q)$ are reduced to their minimum expressions, as follows: $f(q) - g(q) = (f(q) - g(q))_p - (f(q) - g(q))_n$, $(f(q) - g(q))_n = f_n(q) - g_n(q)$, $(f(q) - g(q))_p = f_p(q) - g_p(q)$.

4.3 The rectangle theorem
Since negative part and positive part are bounded continuous nondecreasing functions, then the following inequalities (2) are fulfilled.

$$f_n(v^{min}) \le f_n(q) \le f_n(v^{max})$$
$$f_p(v^{min}) \le f_p(q) \le f_p(v^{max})$$
(2)

This means that a function $f(q)$ with sign decomposition, evaluated at any vector $q \in Q$, its negative part is contained in a segment and also the positive part is contained in another segment. So, on (f_n, f_p) plane the function is contained in a rectangle as expressed by the following theorem according to figure (3).

Theorem 5. *(Elizondo, 1999)* **Rectangle Theorem**. *Let $f : \Re^\ell \to \Re$ be a continuous function with sign decomposition in a box $Q \subset P \subset \Re^\ell$ with minimum and maximum Euclidean vertices v^{min}, v^{max}, then: **a)** $f(q)$ is lower and upper bounded by $f_p(v^{min}) - f_n(v^{max})$ and $f_p(v^{max}) - f_n(v^{min})$ respectively; **b)** The graphical representation of the function $f(q)$, $\forall q \in Q$ in (f_n, f_p) plane is contained in the rectangle with vertices $(f_n(v^{min}), f_p(v^{min}))$, $(f_n(v^{max}), f_p(v^{max}))$, $(f_n(v^{min}), f_p(v^{max}))$ and $(f_n(v^{max}), f_p(v^{min}))$; **c)** if the lower right vertex $(f_n(v^{max}), f_p(v^{min}))$ is over the 45° line then $f(q) > 0$ $\forall q \in Q$; **d)** if the upper left vertex $(f_n(v^{min}), f_p(v^{max}))$ is below the 45° line then $f(q) < 0$ $\forall q \in Q$. In accordance with figure (3).*

The above result seems to be very useful, we can say that the rectangle is the "house" where the multivariable function lives in $\Re^2$. We can know the robust positivity of a function analyzing only one point. It is important to note that this is only sufficient conditions, the lower right vertex can be below the 45° line and the function could be robustly positive or not be. But if the lower right vertex is above the 45° line then the function is robustly positive.
For example, the function $f(q) = 4 - q_2 + q_1q_3 + 8q_1^2q_2 - 9q_3^3q_1q_2^2$ such that $q \in Q \subset P \subset \Re^3$, $Q = \{ q = [q_1, q_2, q_3]^T \mid q_i \in [0, 1] \}$, has sign decomposition, its minimum and maximum Euclidean vertices are $v^{min} = [0, 0]^T$, $v^{max} = [1, 1]^T$, their positive and negative parts are: $f_p(q) = 4 + q_1q_3 + 8q_1^2q_2$, $f_n(q) = q_2 + 9q_3^3q_1q_2^2$. Then the lower bound is $f_p(v^{min}) - f_n(v^{max})$, $f_p(v^{min}) = 4 + (0)(0) + 8(0)(0) = 4$, $f_n(v^{max}) = 1 + 9(1)(1)(1) = 10$, the lower bound is $4 - 10 = -9$. The function could be robustly positive, but for now we do not know, It is necessary see more signs of decomposition items.

Remark 6. *Should be noted three important concepts:*
The graph of the function does not "fills" the whole rectangle, but it is contained in.

The graph of the function always "touches" the rectangle in lower left vertice and upper right vertice.
*The graph of the function is not necessarily convex.*
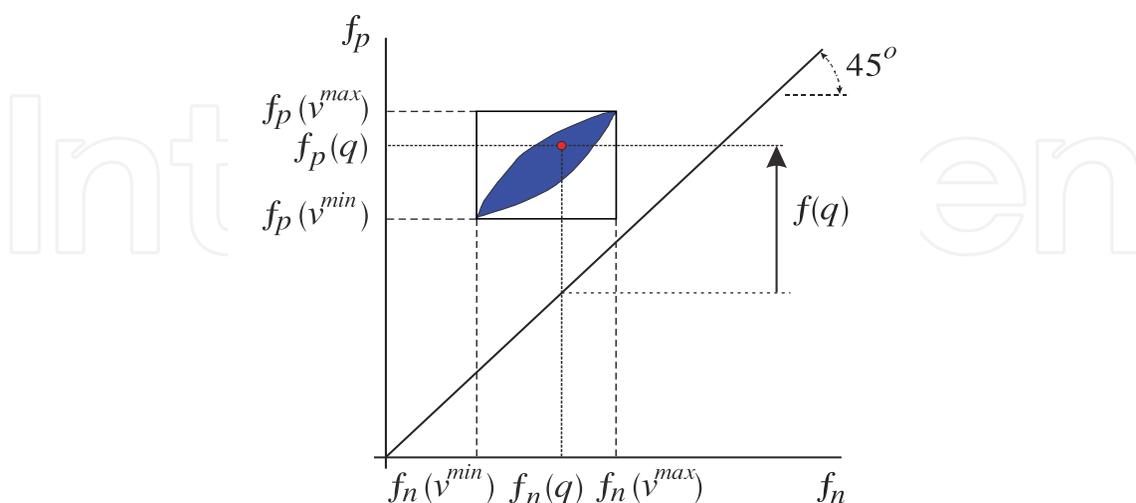
Fig. 3. Rectangle theorem

4.4 The polygon theorem

For the purpose of improving the results shown up to this point, the following proposition is necessary. In some cases it is necessary to analyze the function in a Γ box contained in Q, $\Gamma \subset Q$. The Γ box has Euclidean Vertices $\mu^{\min}$ and $\mu^{\max}$. So, a vector in Γ is expressed as so as $q = \mu^{\min} + \delta$, where δ is a vector in Γ, with origins in $\mu^{\min}$.

Proposition 7. *(Elizondo, 1999) Let $f : \Re^\ell \to \Re$ be a continuous function in $Q \subset P \subset \Re^\ell$, let $\Gamma^j \subset Q$ be a box with its vertices set $\{\mu^i\}$ with minimum and maximum Euclidean vertices $\mu^{\min}$, $\mu^{\max}$, let $\Delta = \{\delta \mid \delta_i \in [0, \delta_i^{\max}], \delta_i^{\max} = \mu_i^{\max} - \mu_i^{\min}\} \subset P \subset \Re^\ell$ be a box with its vertices set $\{\delta^i\}$ with minimum and maximum Euclidean vertices 0, $\delta^{\max} = \mu^{\max} - \mu^{\min}$, and let $q \in \Gamma^j$ a vector such that $q = \mu^{\min} + \delta$ where $\delta \in \Delta$. Then the function $f(q)$ is expressed by its: linear, nonlinear and independent parts, in its minimum expression for all $q \in \Gamma^j$.*

$f(q) = f^{\min} + f_L(\delta) + f_N(\delta) \mid \delta \in \Delta \forall q \in \Gamma^j$
$f^{\min} \triangleq Independent\ Part = f(\mu^{\min})$
$f_L(\delta) \triangleq Linear\ Part = \nabla f(q)|_{\mu^{\min}} \cdot \delta \quad \forall \delta \in \Delta$
$f_N(\delta) \triangleq Nonlinear\ Part = f(\mu^{\min} + \delta) - f^{\min} - f_L(\delta) \quad \forall \delta \in \Delta$
$\nabla f(q)|_{\mu^{\min}} \cdot \delta = \dfrac{\partial f(q)}{q_1}\bigg|_{\mu^{\min}} \delta_1 + \dfrac{\partial f(q)}{q_2}\bigg|_{\mu^{\min}} \delta_2 + \cdots + \dfrac{\partial f(q)}{q_\ell}\bigg|_{\mu^{\min}} \delta_\ell$

Must be noted that $f^{\min} = f(\mu^{\min})$. On other hand, it is clear that we can use the concepts of positive part and negative part in the above proposition, So, $f_p(q) - f_n(q) = f_p^{\min} - f_n^{\min} + f_{Lp}(\delta) - f_{Ln}(\delta) + f_{Np}(\delta) - f_{Nn}(\delta)$ obtaining the following equations (3) where the relation between δ and q can be appreciated in the figure (4).

$$\begin{aligned} f_p(q) &= f_p^{\min} + f_{Lp}(\delta) + f_{Np}(\delta) \\ f_n(q) &= f_n^{\min} + f_{Ln}(\delta) + f_{Nn}(\delta) \end{aligned} \tag{3}$$
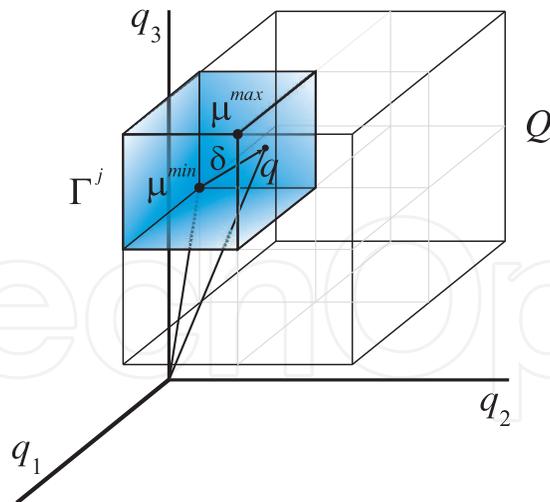
Fig. 4. Gamma box

Theorem 8. *Polygon Theorem (Elizondo, 1999). Let $f : \Re^\ell \to \Re$ be a continuous function with sign decomposition in Q, let q, δ, Γ^j and Δ in accordance with the proposition (7). Then, **a)** the lower and upper bounds of the function $f(q)$ are: Lower Bound $= f^{\min} + f_{L\min} - f_{Nn}(\delta^{\max})$ and Upper Bound $= f^{\min} + f_{L\max} + f_{Np}(\delta^{\max})$ $\forall q \in Q$, **b)** the bounds of incise "a", are contained in the interval defined by the bounds of the rectangle theorem 3. $f_p(\mu^{\min}) - f_n(\mu^{\max}) \leq$ Lower Bound $\leq$ Upper Bound $\leq f_p(\mu^{\max}) - f_n(\mu^{\min})$, **c)** The graphical representation of the function $f(q)$ $\forall q \in \Gamma$ in the (f_n, f_p) plane is contained in the polygon defined by the intersection of the rectangle of the rectangle theorem (5) and the space between the two 45° lines separated from the origin by the Lower Bound and Upper Bound in accordance with figure (5).*
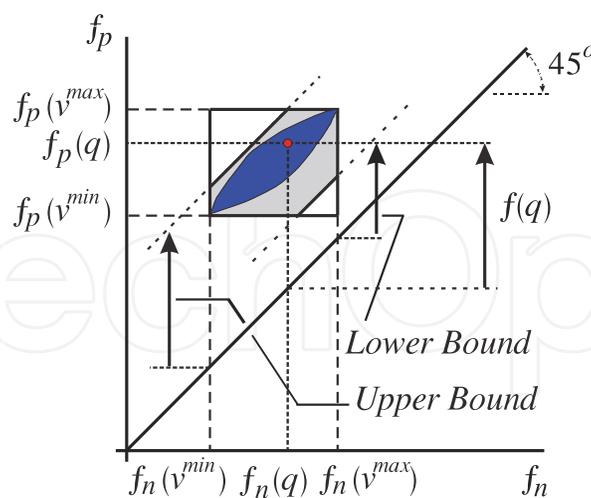
Fig. 5. Bounding of the function

The symbolic expression of the nonlinear part used in the above theorem is not necessary to obtain, because we will use only its numerical value. So, from the equations (3), the nonlinear parts are obtained as so as equations (4).

$$f_{Np}(\delta) = f_p(q) - f_p^{min} - f_{Lp}(\delta)$$
$$f_{Nn}(\delta) = f_n(q) - f_n^{min} - f_{Ln}(\delta)$$
$$f_{Lp}(\delta) = \nabla f_p(q)\big|_{\mu^{min}} \cdot \delta \qquad (4)$$
$$f_{Ln}(\delta) = \nabla f_n(q)\big|_{\mu^{min}} \cdot \delta$$

As an illustration of this theme, by means of rectangle theorem and polygon, we will analyze the lower bound of a function in a gamma box. Consider the function corresponding to the figure (2), $f(q) = 4 - q_2 + q_1 q_3 + 8q_1^2 q_2 - 9q_3^3 q_1 q_2^2$ such that $q \in Q \subset P \subset \Re^3$, $Q = \{ q = [q_1, q_2, q_3]^T | \ q_i \in [0, 1] \}$. Suppose that the function is analyzed into a gamma box $\Gamma \subset Q$, with Euclidean vertices $\mu^{min} = [0.2 \ 0.2 \ 0.2]^T$ and $\mu^{max} = [0.85 \ 0.85 \ 0.85]^T$. In accordance with the Rectangle Theorem (3) the lower bound is $f_p(v^{min}) - f_n(v^{max}) = -0.1403$. Applying the Polygon Theorem (8) the lower bound is $f^{min} + f_{L\,min} - f_{Nn}(\delta^{max})$, so it is necessary to obtain each of these expressions, the results are as follows: $f^{min} = f(\mu^{min}) = 3.9034$, $f_{L\,min} = -0.4457$, $f_{Nn}(\delta^{max}) = 3.3825$. The last value is obtained of equations (4), thus the lower bound is 0.0752. By means of the Rectangle Theorem is obtained $f(q) > -0.1403 \ \forall q \in \Gamma$, following the Polygon Theorem is obtained $f(q) > 0.0752 \ \forall q \in \Gamma$, so the function is robustly positive in the Γ box.

4.5 The box partition theorem

By means of Rectangle Theorem (3) and Polygon Theorem (8) are obtained sufficient conditions of robust positivity, so to obtain necessary and sufficient conditions is necessary to obtain new results.

When it is not possible to know whether the function is positive or not in $Q = [q_1^-, \ q_1^+] \times [q_2^-, q_2^+] \times \cdots \times [q_\ell^-, \ q_\ell^+]$. In this case it is possible to divide each variable $[q_i^-, \ q_i^+]$ in k parts, generating k new intervals: $[q_i^-, \ q_i^1], [q_i^1, \ q_i^2], \cdots, [q_i^j, \ q_i^{j+1}], \cdots [q_i^{k-1}, \ q_i^+]$, let $[\gamma_i^-, \ \gamma_i^+]$ be a k new interval, giving cause to the generation of k^ℓ new boxes $\Gamma^i = [\gamma_1^-, \ \gamma_1^+] \times [\gamma_2^-, \ \gamma_2^+] \times \cdots \times [\gamma_\ell^-, \ \gamma_\ell^+]$ with $\mu^{min}, \mu^{max} \in \Gamma^i$ minimum and maximum Euclidean vertices of Γ^i and $Q = \bigcup_i \Gamma^i$. Through these concepts, the following theorem is obtained.

Theorem 9. *Box Partition Theorem (Elizondo, 1999). Let $f : \Re^\ell \to \Re$ be a continuous function with sign decomposition in Q such that $Q \subset P \subset \Re^\ell$ is a box with minimum and maximum Euclidean vertices v^{min}, v^{max}. Then the function $f(q)$ is positive (negative) in Q if and only if a Γ boxes set exists, such that $Q = \bigcup_j \Gamma^j$ and Lower Bound $\geq c > 0$ for each Γ^j box (Upper Bound $\leq c < 0$ for each one Γ^j box).*

This theorem can be applied in two ways, one of them we call " **Analytical Partition**" and the other one "**Constant Partition**". In analytical partition, the box where the function has a negative lower bound is subdivided iteratively. In the case of the function is robustly positive is also obtained information about where the function is close to losing positivity. By means of constant partition is only obtained information on whether the function is robustly positive or not.

To illustrate both procedures, we analyze the robust positivity of the function (Elizondo, 1999) $f(q) = (4 + q_1 + 8q_1^2 q_2) - (q_2 + 9q_1 q_2^2)$, such that $Q = \{ q = [q_1, q_2]^T | \ q_i \in [0, 1] \ \forall i \}$. The robust positivity is analyzed by means of the rectangle theorem because it is more easier to

apply, although it must be said that the bounds of the polygon theorem are better than the rectangle theorem.

Analytical Partition (Elizondo, 1999). In the subfigure 1 of figure (6) shows that the function is robustly positive in boxes Γ^1 and Γ^3 but not in the boxes Γ^2 and Γ^4. So it is necessary apply iteratively the partition box to the boxes where the function is not robust positive, in this way is obtained the subfigure 2 of figure (6). Since there is a set of boxes such that $Q = \bigcup_j \Gamma^j \mid f(q) > 0 \ \forall \Gamma^j$, then the function is robustly positive in Q. The graphs were made to show the procedure in visual way, but for more than two dimensions, using software we can get the coordinates and dimensions of sub boxes where the function is close to losing positivity.

(a) Subfigure 1   (b) Subfigure 2
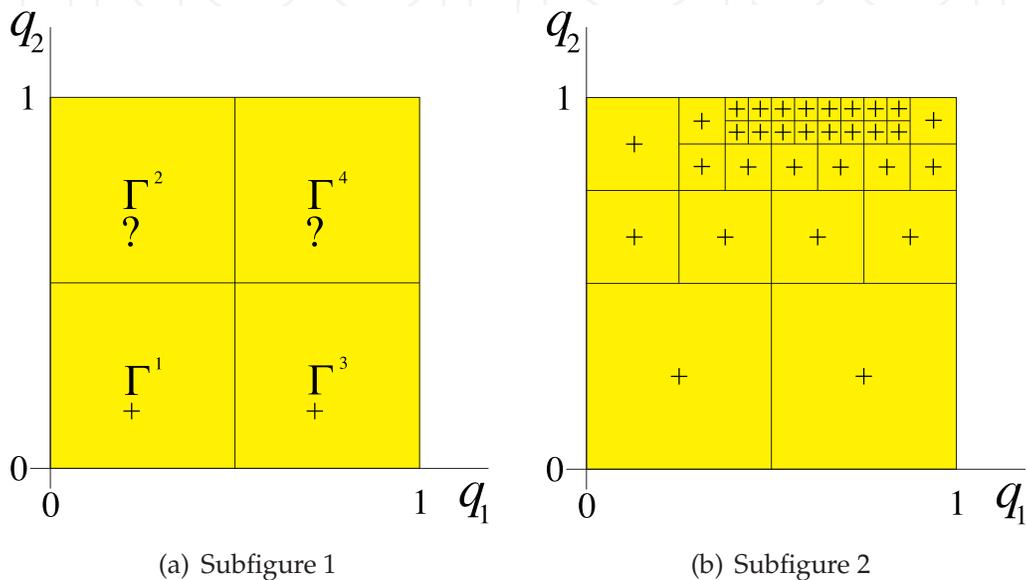
Fig. 6. Partition box

Constant Partition (Elizondo, 1999). In this procedure the domain of each one of the ℓ parameters is divide in k equal parts (not necessarily equal), in this way, it is generated a boxes set of k^ℓ sub boxes Γ^i such that $Q = \bigcup_j \Gamma^j$. The robust positivity of each Γ^i box can be analyzed by a computer program so that the computer give us the final result about the robust positivity of the function.

Another way is through a software which plot a $\times$ (blue) mark in the (f_n, f_p) plane in each $(f_n(\mu^{\min}), f_p(\mu^{\min}))$ and $(f_n(\mu^{\max}), f_p(\mu^{\max}))$ coordinates corresponding to the minimum and maximum vertices of each Γ^i box, and plot too a $+$ (red) mark corresponding to the lower bound of each Γ^i box, as can be appreciated in figure (7) that it was obtained with $k = 13$.

If a $\times$ (blue) mark is below the 45° line, means that there is at least one vector for which the function is negative and therefore the function is not robustly positive. If all the $\times$ (blue) marks are above the 45° line, and a $+$ (red) mark is below the 45° line means that it is necessary to increase the k number of partitions up to all the $+$ (red) and $\times$ (blue) marks are above the 45° line. If this is achieved then the function is robustly positive, as shown in figure (7).

In the figure (7) we can see that it is difficult to see that all $+$ (red) marks are above the 45° line, then with purpose to resolve this difficulty is proposed the following representation.
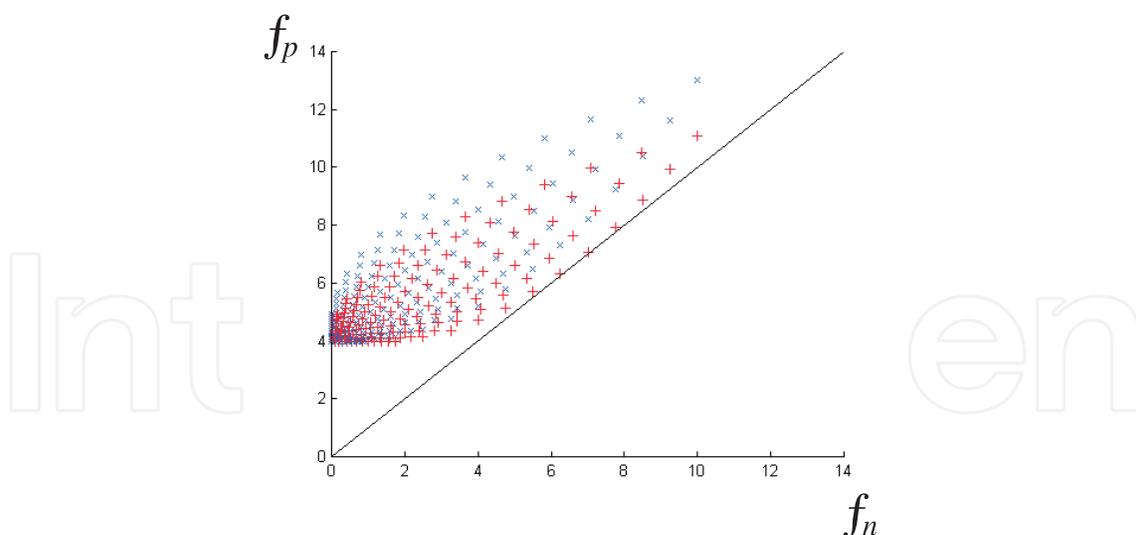
Fig. 7. Function in (f_n, f_p) plane

4.6 (α, β) Representation

In some cases as so as figure (7) it is not easy to determine in graphic way whether a point close to the 45° line is over this line or not. So in (Elizondo, 1999) the (α, β) representation was developed, $\alpha(q) = f_p(q) + f_n(q), \beta(q) = f_p(q) - f_n(q)$, it is similar to rotated 45° the axis with respect to (f_n, f_p) representation implying some graphical and algebraic advantages over the negative and positive representation.

Definition 10. *(Elizondo, 1999) Let $f_n(q)$ and $f_p(q)$ be the negative and positive parts of a continuous function $f(q)$ with sign decomposition in Q. Let T be the linear transformation described below such that T^{-1} exists, then it is called a representation of the function $f(q)$, in (α, β) coordinates, to the linear transformation $(\alpha(q), \beta(q)) = T(f_n(q), f_p(q))$ and the inverse transformation of an $(\alpha(q), \beta(q))$ representation is a $(f_n(q), f_p(q))$ representation of the function $f(q)$.*

$$T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad T^{-1} = \tfrac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \alpha(q) \\ \beta(q) \end{bmatrix} = T \begin{bmatrix} f_n(q) \\ f_p(q) \end{bmatrix} \qquad \begin{bmatrix} f_n(q) \\ f_p(q) \end{bmatrix} = T^{-1} \begin{bmatrix} \alpha(q) \\ \beta(q) \end{bmatrix}$$

$$\alpha(q) = f_p(q) + f_n(q) \qquad f_p(q) = \tfrac{1}{2}(\alpha(q) + \beta(q))$$

$$\beta(q) = f_p(q) - f_n(q) \qquad f_n(q) = \tfrac{1}{2}(\alpha(q) - \beta(q))$$

With the purpose to show the advantages of the (α, β) representation, by means of the rectangle theorem we analyze the same function in the previous subsection $f(q) = (4 + q_1 + 8q_1^2 q_2) - (q_2 + 9q_1 q_2^2)$ applying $k = 13$. We can see in the figure (8) beta axis scale is positive implying that all the bounds are positives and consequently the function is robustly positive.

The function $f(q) = 4 - q_2 + q_1 q_3 + 8q_1^2 q_2 - 9q_3^3 q_1 q_2^2$ corresponding to the figure (2) is shown in the figure (9) in (α, β) representation. We can see that beta axis scale is positive implying the function is robustly positive.

The original idea to develop the representation (α, β) (Elizondo, 1999) was to solve a visual geometric problem, but this representation has interesting algebraic properties on continuous functions $f(q), g(q), h(q)$ with sign decomposition in Q and for all $u(q)$ nondecreasing function in Q, (Elizondo, 1999) as the following:
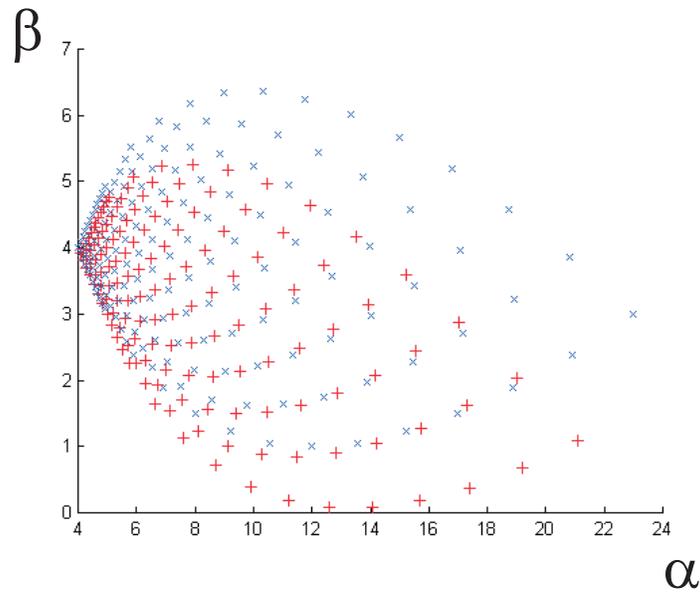
Fig. 8. Function in (α, β) representation
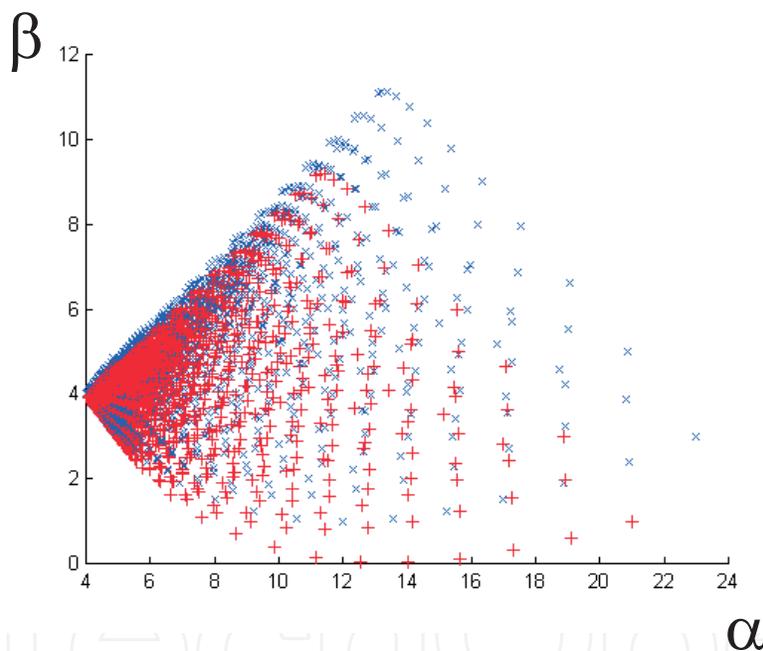
Fig. 9. Function in (α, β) representation

a) $\alpha(q)$ is a non-decreasing and non-negative function in Q; **b)** $\alpha(q) \geq \beta(q)$; **c)** $\beta(q) = f(q)$ $\forall f(q)$, $\forall q \in Q$; **d)** the $(\alpha(q) + u(q), \beta(q) + u(q))$ is a α, β representation of $f(q)$; **e)** the $(\alpha(q) + u(q), \beta(q))$ representation is reduced to its minimum expression $(\alpha(q), \beta(q))$; **f)** *Addition* $f(q) + g(q) : \alpha(q) = \alpha_f(q) + \alpha_g(q), \beta(q) = \beta_f(q) + \beta_g(q)$; **g)** *Subtraction* $f(q) - g(q) : \alpha(q) = \alpha_f(q) + \alpha_g(q), \beta(q) = \beta_f(q) - \beta_g(q)$; **h)** *Product* $f(q)g(q), \alpha(q) = \alpha_f(q)\alpha_g(q),$ $\beta(q) = \beta_f(q)\beta_g(q)$; **i)** the (α, β) representation of $-g(q)$ is as follows: $(\alpha_g(q), -\beta_g(q))$; **j)** if $f(q) = g(q) + h(q)$ then the alpha an beta parts of $f(q) - g(q)$ are reduced to its minimum expression as follows $\alpha(q) = \alpha_f(q) - \alpha_g(q), \beta(q) = \beta_f(q) - \beta_g(q)$.

Computationally the (α, β) representation is better than (f_n, f_p) because if the computer does not generate the negative scale in the β axis it is implying that all "marks" are positives. This

is an useful and interesting property, but above all properties there are three outstanding properties, it would be very useful if they were fulfilled in complex numbers, they are as follows:

$$
\begin{array}{llll}
Addition & f(q) + g(q) & \alpha(q) = \alpha_f(q) + \alpha_g(q) & \beta(q) = \beta_f(q) + \beta_g(q) \\
Subtraction & f(q) - g(q) & \alpha(q) = \alpha_f(q) + \alpha_g(q) & \beta(q) = \beta_f(q) - \beta_g(q) \\
Product & f(q)g(q) & \alpha(q) = \alpha_f(q)\alpha_g(q) & \beta(q) = \beta_f(q)\beta_g(q)
\end{array} \quad (5)
$$

Most be noted that the alpha component of subtraction is correct with $\alpha(q) = \alpha_f(q) + \alpha_g(q)$, it is an "addition" of alphas. It is also important to highlight the simplicity with which made the addition, subtraction and product in alpha beta representation.

4.7 Sign decomposition of the determinant

Sign decomposition of the determinant was developed in (Elizondo, 1999) and it was presented an application in (Elizondo, 2001A; 2002B), by simplicity only will mention (Elizondo, 1999). In parametric robust stability is not very useful the sign decomposition of the determinant, but it is a part of sign decomposition. We can analyze robust stability by means of the Hurwitz criterion means the robust positivity of determinants, but it is so much easier by means of criterion (2), see table (1). Taking account that the reader could work in other areas where the nonsingularity of a matrix dependent in parameters is important, then sign decomposition of the determinant is included in this chapter.

4.7.1 The (α, β) representation of the determinant

In order to achieve the procedure to determine the robust positivity in necessary and sufficient conditions of a determinant with real coefficients depending on ℓ parameters q_i, the following fact is presented. By means of the (α, β) properties (5) is obtained the following fact, in the development of the determinant appears the alpha part and beta part, as shown in the following fact.

Fact 1. *(Elizondo, 1999) Let $M(q)$ be a (2×2) matrix with elements $m_{i,j}(q) \in \Re$ with representation $(\alpha_{i,j}(q), \beta_{i,j}(q))$. Then the (α, β) representation of the determinant of the matrix $M(q)$ is:*

$(\det(M(q)))_\alpha = (\alpha_{1,1}(q)\alpha_{2,2}(q) + \alpha_{2,1}(q)\alpha_{1,2}(q))$
$(\det(M(q)))_\beta = (\beta_{1,1}(q)\beta_{2,2}(q) - \beta_{2,1}(q)\beta_{1,2}(q))$.

Definition 11. *(Elizondo, 1999) Let $M(q) = \left[m_{i,j}(q)\right]$ be a matrix with elements $m_{i,j}(q) \in \Re$ with $(\alpha_{i,j}(q), \beta_{i,j}(q))$ representation. Then the matrix $M_\alpha(q) = \left[\alpha_{i,j}(q)\right]$ will be called the alpha part of the matrix $M(q)$, and the determinant $\det_\alpha(M(q)) = |M(q)|_\alpha = |M_\alpha(q)|_\alpha$ will be called the alpha part of the determinant $|M(q)|$, which is similar to the usual determinant changing all the subtractions by additions including the sign rule of Cramer. In a similar way, the matrix $M_\beta(q) = \left[\beta_{i,j}(q)\right]$ will be called the beta part of the matrix $M(q)$, and the determinant $\det_\beta(M(q)) = |M(q)|_\beta = \left|M_\beta(q)\right|$ will be called the beta part of the determinant $|M(q)|$.*

Most be noted that: **a)** $\beta_{i,j}(q) = m_{i,j}(q)$, then, $M_\beta(q) = M(q)$ and $\det_\beta(M(q)) = \det(M(q))$, **b)** In accordance with the above fact, for a (2×2) matrix, the (α, β) representation of the determinant of the matrix $M(q)$ is $\left(\det_\alpha(M(q)), \det_\beta(M(q))\right)$. In the following lemma a generalization of the last expression for a $(n \times n)$ matrix is established.

Lemma 12. *(Elizondo, 1999) Let $M(q)$ be a $(n \times n)$ matrix with elements $m_{i,j}(q) \in \Re$ with representation $(\alpha_{i,j}(q), \beta_{i,j}(q))$. Then the (α, β) representation of the determinant of the matrix $M(q)$ is $\left(\det_\alpha(M(q)), \det_\beta(M(q)) \right)$. In accordance with definition (11)*

4.7.2 Linear, nonlinear and independent parts of the determinant

When the positivity of the determinant of a matrix with elements $m_{i,j}(q)$ is analyzed via sign decomposition, it is normally necessary to use the box partition and polygon theorems. Then, the independent, linear and nonlinear parts of the determinant need to be obtained. These are obtained in the following theorem.

Theorem 13. *(Elizondo, 1999) (Sign Decomposition of the Determinant Theorem) Let $q \in \Gamma \subseteq Q \mid q = \mu^{\min} + \delta$ be according to the proposition (7). Let $M(q) \in \Re^{n \times n}$ be a matrix with elements $m_{i,j}(q)$ with sign decomposition in Q with representation $(\alpha_{i,j}^{\min} + \alpha_{i,j,L}(\delta) + \alpha_{i,j,N}(\delta), \beta_{i,j}^{\min} + \beta_{i,j,L}(\delta) + \beta_{i,j,N}(\delta))$, then the (α, β) representation of the determinant of the matrix $M(q)$ is as follows:*

$$\alpha(q) = \alpha^{\min} + \alpha_L(\delta) + \alpha_N(\delta),$$
$$\beta(q) = \beta^{\min} + \beta_L(\delta) + \beta_N(\delta)$$
$$\alpha^{\min} = det_\alpha \left(\left[\alpha_{i,j}^{\min} \right] \right), \quad \beta^{\min} = det \left(\left[\beta_{i,j}^{\min} \right] \right)$$
$$\alpha_L(q) = \sum_{k=1}^{k=n} det_\alpha \left(\Phi(k) \left[\alpha_{i,j}^{\min} \right] + [I - \Phi(k)] \left[\alpha_{i,j,L}(\delta) \right] \right)$$
$$\beta_L(q) = \sum_{k=1}^{k=n} det \left(\Phi(k) \left[\beta_{i,j}^{\min} \right] + [I - \Phi(k)] \left[\beta_{i,j,L}(\delta) \right] \right)$$

$$\Phi(k) = \left[\varphi_{i,j}(k) \right] \mid$$
$$\varphi_{1,1}(k) = |sign(1 - k)|$$
$$\varphi_{2,2}(k) = |sign(2 - k)|$$
$$\vdots$$
$$\varphi_{n,n}(k) = |sign(n - k)|$$
$$\varphi_{i,j}(k) = 0 \ \forall i \neq j$$
$$\alpha_N(\delta) = \alpha(q) - \alpha^{\min} - \alpha_L(\delta), \quad \beta_N(\delta) = \beta(q) - \beta^{\min} - \beta_L(\delta)$$

4.7.3 Example

(Elizondo, 1999; 2001A). The Frazer and Duncan Theorem is presented in (Ackermann et al., 1993) in the boundary crossing version as follows. Let $P(s, Q) = \{p(s, q) \mid q \in Q \subset P \subset \Re^\ell\}$ be a family of polynomials of invariant degree with parametric uncertainty and real continuous coefficients, then the family $P(s, Q)$ is robust stable if and only if: 1) a stable polynomial $p(s, \hat{q}) \in P(s, Q)$ exists, 2) $\det(H(q)) \neq 0$ for all $q \in Q$.

(Ackermann et al., 1993) Given the family of invariant degree polynomials with parametric uncertainty described by: $p(s, q) = c_0 + c_1 s + c_2 s^2 + c_3 s^3 + c_4 s^4$, with real continuous coefficients: $c_0(q) = 3, c_1(q) = 2, c_2(q) = 0.25 + 2q_1 + 2q_2, c_3(q) = 0.5(q_1 + q_2), c_4(q) = q_1 q_2$, such that $q_i \in [1, 5]$. Determine the robust stability of the family by means of the Frazer and Duncan theorem applying in graphical way the sign decomposition of the determinant theorem (13).

The Hurwitz matrix $H(q)$ is obtained, it is proved that the polynomial $p(s, \hat{q})$ is stable for $\hat{q} = [1\ 1]^T$ and that the determinant of the Hurwitz matrix $H(\hat{q})$ is positive. Having the first condition of the Frazer and Duncan theorem satisfied, and proving that the determinant is robust positive in Q, the second condition of the Frazer and Duncan theorem will be satisfied too.

$$H(q) = \begin{bmatrix} c_3(q) & c_1(q) & 0 & 0 \\ c_4(q) & c_2(q) & c_0(q) & 0 \\ 0 & c_3(q) & c_1(q) & 0 \\ 0 & c_4(q) & c_2(q) & c_0(q) \end{bmatrix}$$

The robust positivity of the determinant problem is solved by means of: the box partition theorem 9, the polygon theorem 8 in (α, β) representation and the sign decomposition of the determinant theorem (13). Taking the partition in 9 equal parts in each one of the two variables q_i and applying sign decomposition in constant partition way, the function values in minimum and maximum vertices "×" and lower bound "+" are plotted for each Γ^i box, as it appears in the figure (10). All lower bound marks "+" are above the alpha axis, then all of bounds are positive, therefore the determinant of the Hurwitz matrix $H(q)$ is robust positive implying that the polynomials family is robust stable.
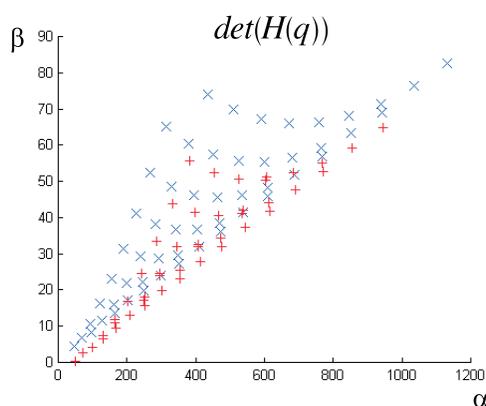
Fig. 10. Positivity of the determinant

5. A solution for the parametric robust stability problem

5.1 Problem identification

In control area, the robust stability of LTI systems with parametric uncertainty problem has been studied in different interesting ways. The problem can be divided in two parts. One of them is that it is not possible to be obtained roots of a polynomial by analytical means for the general case. The second is that we have now a family of polynomials to study instead of a single polynomial.

Since to obtain roots of polynomials for the general case is a difficult problem. Then the extraction of roots of polynomials went mapped firstly to a "position" of roots problem in the complex plane, Routh never tried to extract the roots, his work begun studying the position of the roots. This problem was subjected to a second mapping, it was transferred to mathematical problems of smaller level for example to a positivity problem, as it is the case of: Routh, Hurwitz, Lienard-Chipart and Elizondo-González 2001 criterions.

The objective in this chapter is to study the stability of a family of polynomials with invariant degree (the reader can see poles and zeros cancellation cases) and real continuous coefficients dependent on parameters with uncertainty. The essence of the problem is that we have now a set of roots in the the complexes plane, and for stability condition all of them must be in the left half of the complex plane for asymptotic stability. How to obtain that the set of roots remains in the left side of the complex plane?

A well known solution is: a) the family $P(s, Q)$ has at least one element $p(s, q^*)$ stable and b) $\mid H(q)\mid \neq 0 \ \forall q \in Q$. The explanation is because the determinant of a Hurwitz matrix is zero when the polynomial has roots in the imaginary axis, so if there is a $q^* \in Q$ vector such that $p(s, q^*)$ is stable then its roots are at the left half of the complex plane. On other hand, if a vector q slides into Q starting from q^* implies that the coefficients $c_i(q)$ will change in continuous way and the roots of $p(s, q^*)$ will slides too on the complex plane. But if $\mid H(q)\mid \neq 0 \ \forall q \in Q$, it means that does not exist a vector q for which $p(s, q)$ has roots in the imaginary axis, implying that the displacement of the roots never cross the imaginary axis. This solution is very difficult to use because to test the robust positivity of a determinant in the general case is a very difficult problem (Ackermann et al., 1993)(page 93).

Another solution was through the subsetting test, the idea worked well in convex families as interval (Kharitonov, 1978) and affine (Bartlett et al., 1988), but it was not in nonconvex families as multilinear and polynomic.

Then it can be concluded that the solution for robust stability of LTI systems with parametric uncertainty problem for the general case: interval, affine, multilinear, polinomic, cannot be sustained in convexity properties nor subsetting test.

5.2 A proposed solution

In (Elizondo, 1999) it was developed a solution for the general case of robust stability of LTI systems with parametric uncertainty without concerning the convexity of the families, the solution consists of two parts.

A part of the solution was the development of a stability criterion, operating with multivariable polynomic functions in parametric uncertainty case, simpler than Hurwitz and Lienard-Chipart criterions (Elizondo et al., 2005). The mentioned criterion is similar to criterion (Elizondo, 2001B) but without the σ column, therefore it does not determine the number of unstable roots, it only determines whether the polynomial is stable or not. The amount of mathematical operations required in this criterion is equal to the one of (Elizondo, 2001B) but they are much less that the required ones in Hurwitz and Lienard-Chipart criterions (Elizondo et al., 2005).

The other part of the solution was the development of a mathematical tool capable of solving robust positivity problems of multivariable polynomic functions in necessary and sufficient conditions by means of extreme point analysis.The mathematical tool developed in (Elizondo, 1999) was Sign Decomposition.

Then, the solution proposed for robust stability in LTI systems with parametric uncertainty in the general case is supported in two results: the stability criterion for LTI systems (Elizondo, 2001B) and sign decomposition (Elizondo, 1999). Given a polynomial $p(s, q) = c_n(q)s^n + c_{n-1}(q)s^{n-1} + \cdots + c_0(q)$ with real coefficients, where $q \in Q \subset P$, $Q = \{[q_1 \ q_2 \ \cdots q_\ell]^T | q_i \in [0, 1] \ \forall i\}$. The procedure easier to use is by means of the partition box theorem (9) in the modality "Constant Partition", its application could be of the following way.

a) Take the equations of the coefficients $c_i(q)$ and decompose them into positive and negative parts $c_{ip}(q)$ and $c_{in}(q)$. In symbolic way.

b) By means of the positive and negative parts, to obtain the components in alpha and beta representation. $\alpha_i = c_{ip}(q) + c_{in}(q)$, $\beta_i = c_{ip}(q) - c_{in}(q)$.

c) To make a table in accordance to the criterion (2).

d) By means of the rectangle theorem (5) or polygon theorem (8), to analyze the robust positivity in Q of the coefficients $c_n(q)$ and $c_{n-1}(q)$. In case of negative bound in a coefficient, include its graph in the following software.

e) To make a software to develop the table in accordance to the partition box theorem and to graph the wished $e_{i,1}$ element.

Remark 14. *The sigma column in the criterion (2) is not necessary calculate for robust stability*

5.3 Example

Given a LTI system with parametric uncertainty $Q = \{[q_1 \; q_2 \; q_3]^T | q_i \in [0, 1] \; \forall i\}$, its characteristic polynomial of invariant degree is $p(s, q) = c_4(q)s^4 + c_3(q)s^3 + +c_2(q)s^2 + +c_1(q)s + c_0(q)$. To analyze the robust stability of the system.

a) Positive and negative parts $c_{pi}(q)$ and $c_{ni}(q)$.

$$c_0(q) = 2 + q_1 q_2 q_3^3 - q_2 q_3$$
$$c_1(q) = 5 + q_1 q_2^3 - q_2 q_3$$
$$c_2(q) = 10 + 4q_1 q_3 - q_1 q_2^2 - q_2^3$$
$$c_3(q) = 5 + q_2^2 - q_1 q_2^2$$
$$c_4(q) = 3 + q_1 q_2^3 - q_2 q_3$$

$$c_{0p}(q) = 2 + q_1 q_2 q_3^3 \qquad\qquad c_{0n}(q) = q_2 q_3$$
$$c_{1p}(q) = 5 + q_1 q_2^3 \qquad\qquad c_{1n}(q) = q_2 q_3$$
$$c_{2p}(q) = 10 + 4q_1 q_3 \qquad\qquad c_{2n}(q) = q_1 q_2^2 + q_2^3$$
$$c_{3p}(q) = 5 + q_2^2 \qquad\qquad c_{3n}(q) = q_1 q_2^2$$
$$c_{4p}(q) = 3 + q_1 q_2^3 \qquad\qquad c_{4n}(q) = q_2 q_3$$

b) The alpha and beta representation of the coefficients is as follows.

$$\alpha_i = c_{pi}(q) + c_{ni}(q), \qquad\qquad \beta_i = c_{pi}(q) - c_{ni}(q)$$
$$\alpha_0 = c_{p0}(q) + c_{n0}(q) \qquad\qquad \beta_0 = c_{p0}(q) - c_{n0}(q)$$
$$\alpha_1 = c_{p1}(q) + c_{n1}(q) \qquad\qquad \beta_1 = c_{p1}(q) - c_{n1}(q)$$
$$\alpha_2 = c_{p2}(q) + c_{n2}(q) \qquad\qquad \beta_2 = c_{p2}(q) - c_{n2}(q)$$
$$\alpha_3 = c_{p3}(q) + c_{n3}(q) \qquad\qquad \beta_3 = c_{p3}(q) - c_{n3}(q)$$
$$\alpha_4 = c_{p4}(q) + c_{n4}(q) \qquad\qquad \beta_4 = c_{p4}(q) - c_{n4}(q)$$

c) To make a table in accordance to the criterion (2).

σ_1	(α_4, β_4)		(α_2, β_2)	(α_0, β_0)
σ_2	(α_3, β_3)		(α_1, β_1)	
σ_3	$\alpha_{3,1} = c_{\alpha3}c_{\alpha2} + c_{\alpha4}c_{\alpha1}, \quad \beta_{3,1} = c_{\beta3}c_{\beta2} - c_{\beta4}c_{\beta1}$	$\alpha_{3,2} = c_{\alpha3}c_{\alpha0}, \quad \beta_{3,2} = c_{\beta3}c_{\beta0}$		
σ_4	$\alpha_{4,1} = \alpha_{3,1}c_{\alpha1} + c_{\alpha3}\alpha_{3,2}, \quad \beta_{4,1} = \beta_{3,1}c_{\beta1} - c_{\beta3}\beta_{3,2}$			
σ_5	Check robust positivity of $\beta_{4,1}$ and $\beta_{3,2}$			

d) The lower bound of $c_4(q)$ and $c_3(q)$ are as follows.

For $c_4(q)$ is $LB \; c_4 = c_{4p}([0\,0\,0]^T) - c_{4n}([1\,1\,1]^T) = 3 + (0)(0)^3 - (1)(1) = 2$.

For $c_3(q)$ is $LB \; c_3 = c_{3p}([0\,0\,0]^T) - c_{3n}([1\,1\,1]^T) = 5 + (0)^2 - (1)(1)^2 = 4$.

Then $c_4(q)$ and $c_3(q)$ are robustly positives in Q

e) By means of software applying 8 partitions the graphs $e_{3,1}$, $e_{3,2}$, $e_{4,1}$ were obtained as following.
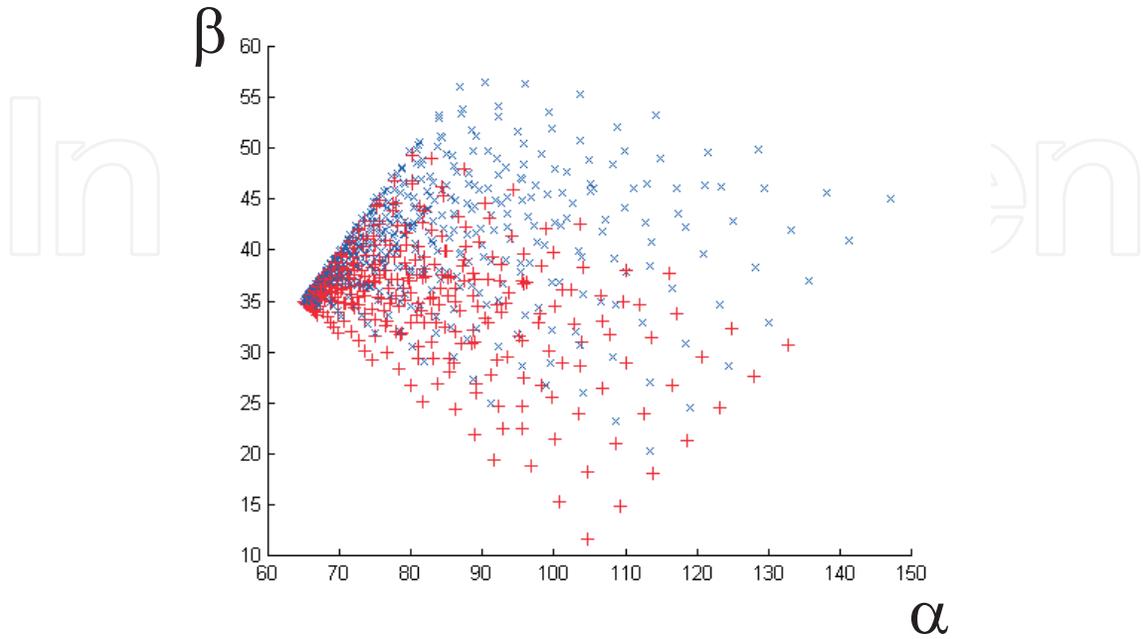

Fig. 11. Element e_{31} in (α, β) representation
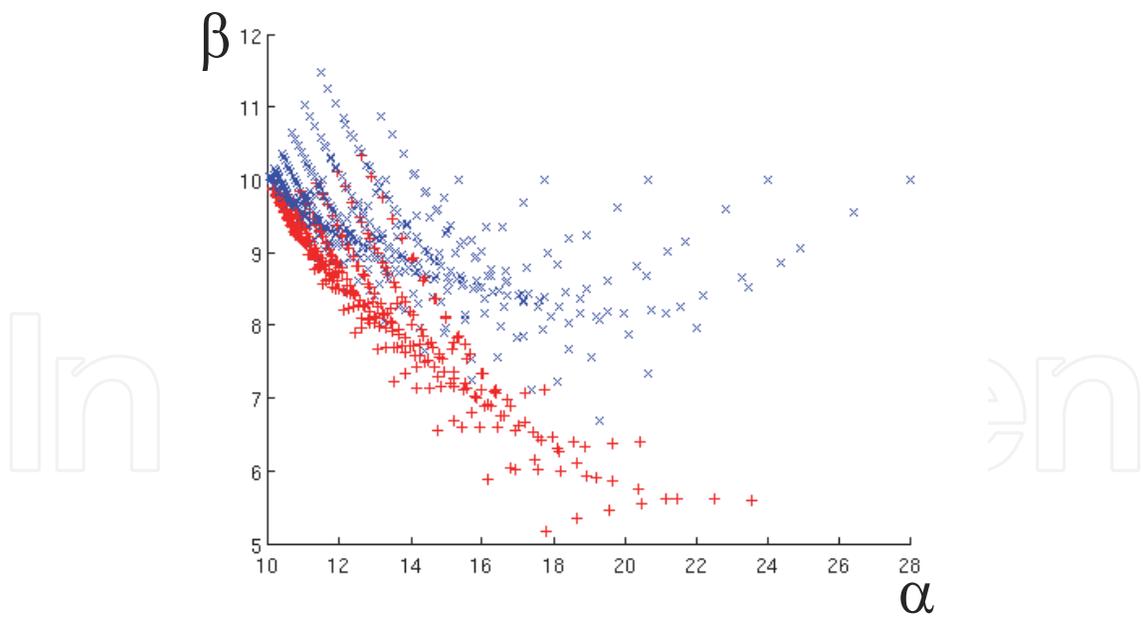
Fig. 12. Element e_{32} in (α, β) representation
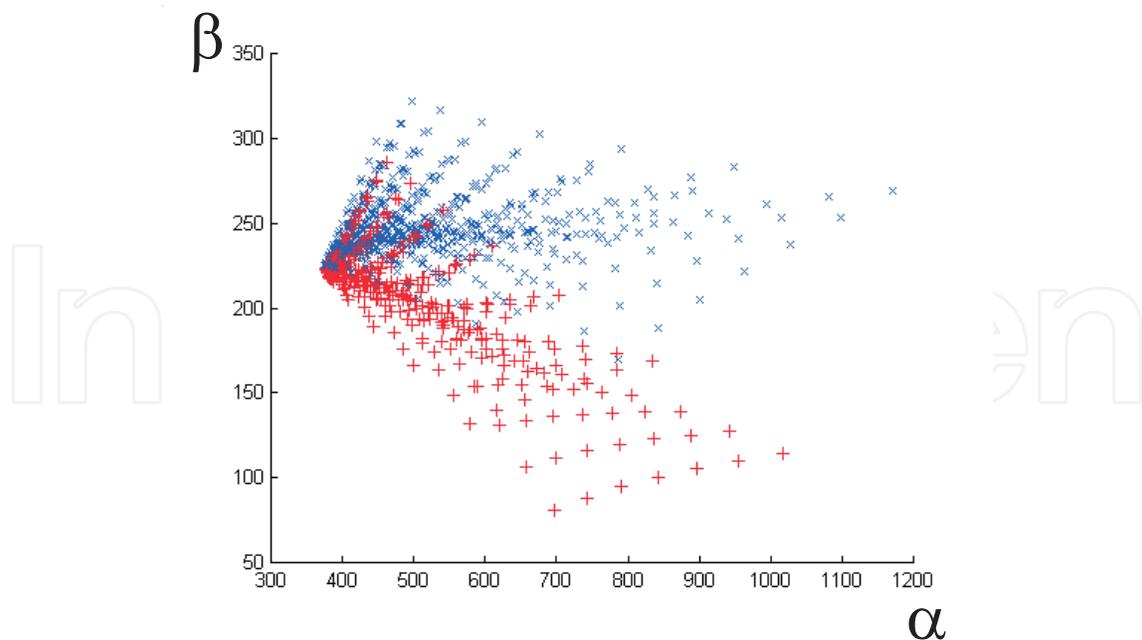
Fig. 13. Element e_{41} in $(\alpha,\ \beta)$ representation

Since $c_4(q), c_3(q), e_{31}(q), e_{32}(q), e_{41}(q)$ are robustly positive, then the system is robustly stable.

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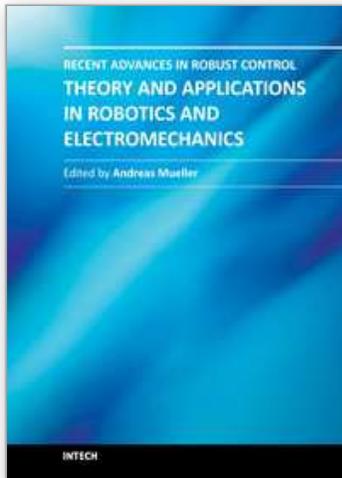
Robust control has been a topic of active research in the last three decades culminating in H_2/H_\infty and \mu design methods followed by research on parametric robustness, initially motivated by Kharitonov's theorem, the extension to non-linear time delay systems, and other more recent methods. The two volumes of Recent Advances in Robust Control give a selective overview of recent theoretical developments and present selected application examples. The volumes comprise 39 contributions covering various theoretical aspects as well as different application areas. The first volume covers selected problems in the theory of robust control and its application to robotic and electromechanical systems. The second volume is dedicated to special topics in robust control and problem specific solutions. Recent Advances in Robust Control will be a valuable reference for those interested in the recent theoretical advances and for researchers working in the broad field of robotics and mechatronics.

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