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An Optimal Distribution of Actuators in Active Beam Vibration – Some Aspects, Theoretical Considerations

Brański Adam
Rzeszow University of Technology
Poland

1. Introduction

The reduction of the effects of mechanical vibration fall into the of vibration isolation, design for vibration or vibration control (de Silva, 2000). The vibration control is subdivided into two group: passive control and active one. The core of the vibration control is to detect the level of vibration in a system and to counteract the effects of the vibration, so it needs two devices.
Hence, the passive devices do not require external power for their operation. Hence, passive control is relatively simple, reliable and economical. But it has limitations namely, the control force depends entirely on the natural dynamics and it may not be adjust on line. Furthermore, in a passive device, there is no supply of power from an external source. It leads to the incomplete control, particularly in complex and high-order systems.
The shortcomings of passive control can be overcome using an active one. In this case, the system response is directly sensed on line and on that basis, the specific control actions are applied to any locations of the system. But the active control needs external power, namely to apply control forces to vibrating system through actuators and to measure vibration response using sensors.
Two different types of actuators can be applied (Shimon et al., 2005). The first, inertial actuators, make up a piezoelectric material to vibrate large masses. Their vibrations are used to counteract the vibrations of the structure (Jiang et al., 2000). The advantages and disadvantages are enumerated in above reference.
The second type of actuators is a layer of smart or intelligent materials. The sensors also belong to these materials; together they are well–known as piezoelectric elements (Tylikowski & Przybyłowicz, 2004). It was shown that these elements can offer excellent potential for an active vibration reduction of the structure vibrating with low frequencies (Croker, 2007; Fuller at al, 1997; Hansen & Snyder, 1997; Kozień, 2006; Przybyłowicz, 2002; Wiciak, 2008). As a general, piezoelectric elements are glued to the host structure. It makes the advantage, namely their incorporating into the structure is that the actuating mechanism becomes part of the structure. Both sensors and actuators are relatively light, compared to the structure, and can be made in arbitrary shape. The disadvantage is that they once bonded and they cannnot be used again. In recent years the measure of the vibration with the sensors are replaced by touch less measures. For this reason, hereafter in research the sensors are omitted and only second type actuators will be considered. Nowadays actuators

are used to very original structures for example to the satellite boom (Moshrefi-Torbati et al., 2006) or to sun plate (Qiu et al., 2007).
To make the reduction more effective, many problems should be solved.

- dynamic effects (mass loading and stiffness) of the actuators on the structure vibration (Charette et al., 1998; Gosiewski & Koszewnik, 2007; Hernandes et al., 2000; Q. Wang & C. Wang, 2001)
- dynamic effects of the glue (between actuators and structure) on the structure vibration (Pietrzakowski, 2004; Sheu et al., 2008).
- actuators' geometric-technical features (Frecker, 2003; Hong et al., 2007; Wang, 2007),
- orientation of the actuators on the structure (Bruant et al., 2010; Ip & Tse, 2001; Qiu et al., 2007),
- appropriate actuators distribution on the structure (Bruant et al., 2010),
- others, but they play a minor part.

Reviewing the literature, it appears that the actuators distribution play a major part. Now, a question arises about an optimal distribution of actuators. In the recent year, a great number of papers has been published on this subject. It is obvious that there are a lot of optimization techniques; an excellent survey is given in (Bruant et al., 2010). Two main approaches are distinguished to this problem.
First of them is the coupling of the optimization of actuators/sensors locations and controller parameters. In this case the following criterions are taken into account for the optimization:

- quadratic cost function of the measure error and the control energy (Bruant et al., 2001),
- maximization of dissipation energy during the control (Yang, 2005),
- spatial H_2 norm of the closed-loop transfer matrix from the disturbance to the distributed controlled output (Liu et al., 2006),
- simultaneous simple H_∞ controller (Guney & Eskinat, 2007).

As can be seen, the optimization criterions are dependent on the choice of controllers. Therefore, the optimal location obtained using one controller may not be a suitable choice for another one.
At the latter approach, the optimal location is obtained independently of the controller definition. In this case, the following criterions are used:

- maximization controllability/observability criterion using the gramian matrices (Bruant & Proslier, 2005; Jha & Inman, 2003),
- modal controllability index based on singular value analysis of the control vector (Dhuri, & Seshu, 2006),
- maximization of the control forces transmitted by the actuators to the structure (Q. Wang & C. Wang, 2001),
- using the H_2 norm (Halim & Reza Moheimani, 2003; Qiu et al., 2007).

In the quoted references, it was not provided the actuators distribution in explicite; only the general rules (criterions) were formulated. However, this problem was partially solved; it was proved in (Brański & Szela, 2007; Brański & Szela, 2008; Szela, 2009; Brański & Szela 2010; Brański & Lipiński, 2011) that the most effective actuators distribution was on the structure sub-domains with the largest curvatures; such distribution was called quasi-optimal one. As the research object, a right-angled triangle plate with clamped-free-free boundary conditions was taken into account. The quasi-optimal distribution was deduced based on the heuristic reasons and the conclusions were confirmed only numerically. Furthermore, the problem was solved merely for the separate modes.

Basing on the quasi-optimal distribution of the actuators, the protection beam vibration is achieved (Brański et al., 2010; Brański & Lipiński, 2011). In this case always the separate modes were considered. The problem was solved based on heuristic reasons and was confirmed analytically. In the latest own research, the results presented in (Brański et al., 2010) were substantiated analytically (Brański & Lipiński, 2011).
In this chapter, the above attitude to the optimal actuators distribution is continued and extended. First at all, the optimal problem is formulated. For this purpose, the optimization criterion is defined. It is assumed that a measure of the vibration reduction is a reduction coefficient (Szela, 2009; Brański & Szela 2010) and here it becomes the objective function. This attitude is quite similar to the maximization of the control forces transmitted by the actuators to the structure (Q. Wang & C. Wang, 2001).
Dynamics effects of the glue and actuators are also considered. Furthermore, the solution of active vibrations reduction is derived for general solution, not only for separate modes. Since analytical solution was attained with separation of variables method, first of all the modes of the problem are derived. Next, the orthogonality condition of the modes is derived too.
The simple supported beam is chosen as the research object. The study of beams is very important in a variety of practical cases, noteworthy, the vibration analysis of structures like bridges, tall buildings, and so on. Loosing a bit on generality, it is considerably easier to realize the aim of the paper. It is assumed that the beam is excited with evenly spread and harmonic force. The material inner damping coefficients of all elements of the research system are taken into account. It seems that all main factors having the influence on the beam vibration were considered.
To solve the problem analytically, a few simplifications are made. Namely, the energy provided to the system is in the form of voltage applied to the surface of the actuators. Assuming that the charge is homogenously distributed, as a result of piezoelectric effect, the actuators interact with the beam with moments for couple of forces homogenously distributed along the actuators' edges. Next, these moments are replaced with the couple of forces and finally, they are counteracted the vibrations.
All problems were considered only theoretically; no calculations are run. It seems that presented considerations will be the base to many numerical simulations and experiments. To the author's knowledge, the theoretical description of the optimal actuators distribution on even simple structure like the beam, up to now have not been brought up.

2. Active beam vibration reduction with additional elements

In this problem, the additional elements make the concentrated masses and actuators and all constitute the mechanical set beam-actuators-masses. Adding actuators (and the glue at the same time) is the technical necessity but they introduce to the mechanical set the additional dynamics effects namely, local stiffness and concentrated masses. As far as concentrated masses are concerned, adding them is substantiated as follows. The proposed optimal distribution of the actuators needs asymmetrical beam vibrations and these ones may be ensured by at least one concentrated mass.

2.1 Uniform beam vibration with damping

There are four theories (models) for the transversely vibrating uniform beam (Han et al., 1999): Euler-Bernoulli, Rayleigh, shear and Timoshenko. The first of them, called the classical beam theory, is applied here. It is simple and provides reasonable results for formulated problem.

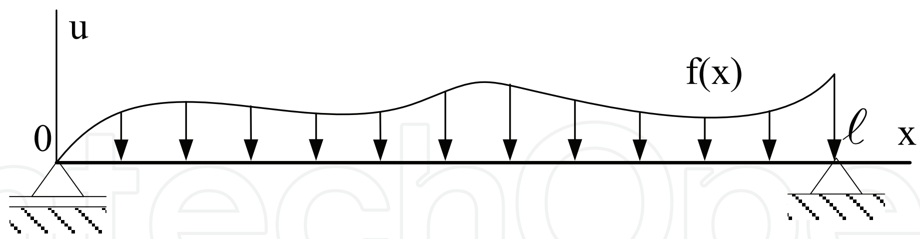

Fig. 1. The geometry of the simple supported beam

Let be the beam as depicted in Fig. 1. The Bernoulli-Euler equation governs transverse vibration (or bending or lateral vibration) of the beam has a following standard form (Kaliski, 1986; Pietrzakowski, 2004),

$$EJ\,D^4u + \mu EJ\,D^4(D_t u) + \rho S D_t^2 u = -f \tag{1}$$

where $u = u(x,t)$ – beam deflection at the point x and the time t, $f = f(x,t)$ – load force, $D^4(.) = \partial^4(.)/\partial x^4$, $D_t(.) = \partial(.)/\partial t$; hereafter the rest symbols are jointly explained.
To solve Eq. (1) explicitly, four boundary conditions, at the ends of the beam, are needed. In general, boundary conditions represent displacement, slope, moment and shear respectively. Here, it is assumed that the beam is simple supported, then both displacement and the bending moment equal zero

$$u(0,t) = 0, \qquad D^2u(0,t) = 0 \tag{2}$$

$$u(\ell,t) = 0, \qquad D^2u(\ell,t) = 0 \tag{3}$$

To solve over determined problem, one needs to know initial conditions. But here, the harmonic steady state plays a major part, so that the initial conditions are omitted.

2.2 Beam vibration with concentrated masses

To solve the intended problem, Eq. (1) must be rounded out. First of all, to obtain asymmetric modes and consistently asymmetric general vibration, a few concentrated masses are added to the beam (Low & Naguleswaran, 1998; Majkut, 2010; Naguleswaran, 1999). They are marked by $\{m_r\}$, and their distribution is described with set of coordinates $\{x_r\}$, see Fig. 2, hence

$$\textstyle\sum_r m_r\,\delta(x-x_r) = m_1\,\delta(x-x_1) + m_2\,\delta(x-x_2) + \ldots + m_r\,\delta(x-x_r) + \ldots \tag{4}$$

where $r = 1,2,\ldots,n_r$, $\delta(.)$ – Dirac's delta function.

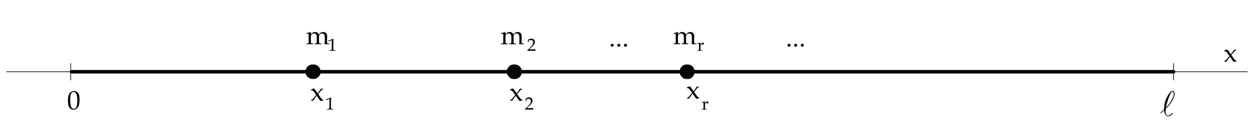

Fig. 2. Distribution of the concentrated masses

Furthermore, the dynamic effects of the actuators and glue on the beam vibration are introduced. The location and length of separate actuators, and the glue layers simultaneously, are denoted commonly with coordinates $\{x_s\}$ and $\{\ell_s\}$ respectively and they are arranged as depicted in Fig. 3.

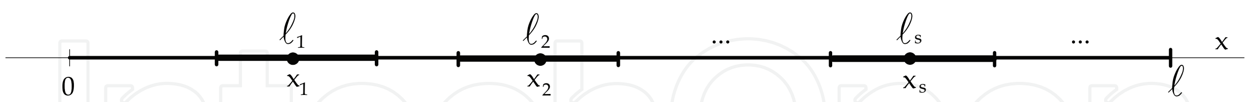

Fig. 3. Distribution of actuators and glue layers

For simplicity, let $P=\{E,J,h,\rho,S,\mu\}$ means the physical and geometrical parameters of the beam, actuators and glue, i.e. {Young's modulus, surface moment of inertia, thickness, mass density, surface of the rectangular cross-section, inner damping factor} respectively. Furthermore all parameters are supplemented with following index $\vartheta=\{b,a,g\}=\{[b]eam,[a]ctuator,[g]lue\}$, for example $S_\vartheta = bh_\vartheta$ means the surface of the rectangular cross-section, b – beam / glue layer width. Moments of inertia are calculated relatively of y-axis, see Fig. 4, where the neutral axis displacement d is neglected , hence $J_b=(bh_b^3)/12$, $J_g=(bh_g^3)/12+S_g(h_b/2+h_g/2)^2$, $J_a=(bh_a^3)/12+S_a(h_b/2+h_g+h_a/2)^2$.

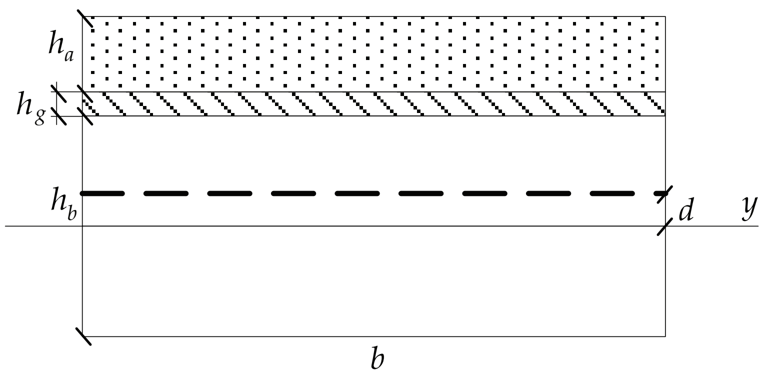

Fig. 4. Cross-sections of the set beam–actuator–glue

The parameters of the set beam-actuators-glue may be written as

$$P = P_b + \sum_s P_s H(x_{1s}-x_{2s}) = P_b + \sum_s P_s \langle H\rangle^0 \tag{5}$$

where $s=1,2,...,n_s$, $P_s=P_a+P_g$, $\langle H\rangle^0 = H(x_{1s}-x_{2s}) = H(x-x_{1s})-H(x-x_{2s})$, $H(x-x_{1s})$ – Heaviside step function in point x_{1s} and so on, $\{x_{1s},x_{2s}\}=\{x_s-\ell_s/2, x_s+\ell_s/2\}$.
For n_s aktuators (n_s glue layers) and n_r concentrated masses, Eq. (1) takes the form

$$\left(E_bJ_b+\sum_s E_sJ_s\langle H\rangle^0\right)D^4u + \left(\mu_b E_b J_b + \sum_s \mu_s E_s J_s \langle H\rangle^0\right)D^4(D_t u) + \left(\rho_b S_b + \sum_s \rho_s S_s \langle H\rangle^0\right)D_t^2 u + \\ + \sum_r m_r \delta(x-x_r) D_t^2 u = -f \tag{6}$$

The Eq. (6) may be written down quite similar like Eq. (1), namely

$$EJD^4u + \mu EJD^4(D_t u) + (\rho S + \alpha_r) D_t^2 u = -f \tag{7}$$

where hereafter

$$\mathrm{EJ} = \mathrm{E_b J_b} + \sum\nolimits_{s} \mathrm{E_s J_s} \langle \mathrm{H} \rangle^0, \qquad \mu \mathrm{EJ} = \mu_b \mathrm{E_b J_b} + \sum\nolimits_{s} \mu_s \mathrm{E_s J_s} \langle \mathrm{H} \rangle^0,$$
$$\rho \mathrm{S} = \rho_b \mathrm{S_b} + \sum\nolimits_{s} \rho_s \mathrm{S_s} \langle \mathrm{H} \rangle^0, \qquad \alpha_r = \sum\nolimits_{r} \mathrm{m_r}\, \delta(x - x_r) \tag{8}$$

On the ground of the EJ, $\rho\mathrm{S}$ and α_r form, Eq. (7) can not be understood in a classical manner. To solve it, some methods may be applied. One of them is presented in (Ercoli & Laura, 1987; Kasprzyk & Wiciak, 2007; Majkut, 2010); another attitude may be found in (C.N. Bapat & C. Bapat, 1987) and it is applied here.

Fig. 5. Geometry of the set beam–one actuator–one mass

At the latter attitude, the beam is divided into some uniform elements. The division may not be coincidental. To clearly explain this problem, for simplicity consider a set beam-one actuator (and glue)-one concentrated mass, Fig. 5. The division is imposed out of the change of physical properties namely, properties of the actuators (and glue) and concentrated masses. So, the beam is divided into $j = 1, 2, ..., n_j = 4$ elements. All elements may be considered separately and the solution to Eq. (7) can be expressed as

$$u(x,t) = \sum\nolimits_{j} u_j(x,t) \tag{9}$$

where $u_j(x,t)$ is the solution on j-element and it is fulfilled the following equation

$$E_j J_j D^4 u_j + \mu_j E_j J_j D^4 \left(D_t u_j \right) + \left(\rho_j S_j + \alpha_r \right) D_t^2 u_j = -f_j \tag{10}$$

To find $u_j(x,t)$ with the separation of variables method, the eigenvalues and eigenfunctions for each element are needed.

2.3 Eigenvalues and eigenfunctions problem

In this problem it is assumed that $f_j(x,t) = 0$ and $\mu_j = 0$, hence based on Eq. (10) one obtains

$$\mathrm{E_j J_j} D^4 u_j + \rho_j \mathrm{S_j} D_t^2 u_j = 0 \tag{11}$$

where $\mathrm{E_j J_j}$ and $\rho_j \mathrm{S_j}$ may be different on the separate elements, but here, as depicted in Fig. 5, is

$$\mathrm{E_1 J_1} = \mathrm{E_3 J_3} = \mathrm{E_4 J_4} = \mathrm{E_b J_b}, \qquad \mathrm{E_2 J_2} = \mathrm{E_b J_b} + \mathrm{E_a J_a} + \mathrm{E_g J_g} \tag{12}$$

$$\rho_1 \mathrm{S_1} = \rho_3 \mathrm{S_3} = \rho_4 \mathrm{S_4} = \rho_b \mathrm{S_b}, \qquad \rho_2 \mathrm{S_2} = \rho_b \mathrm{S_b} + \rho_a \mathrm{S_a} + \rho_g \mathrm{S_g} \tag{13}$$

The boundary conditions for the j–element consist of boundary conditions of the problem and coupling conditions between neighboring elements. The concentrated mass $\mathrm{m_r}$ is

considered in coupling conditions between third and forth elements and therefore it is omitted in Eq. (11).
Let the solution be represented by a product of spatial and temporal functions

$$u_j(x,t) = X_j(x)\,T(t) \tag{14}$$

Substituting (14) into (11) gives

$$E_jJ_j\,D^4X_jT + \rho_jS_j\,X_j\,D_t^2T = 0 \tag{15}$$

or

$$\frac{E_jJ_j}{\rho_jS_j}\frac{D^4X_j}{X_j} = -\frac{D_t^2T}{T} = \omega^2 \tag{16}$$

hence

$$D^4X_j - \lambda_j^4X_j = 0 \tag{17}$$

$$D_t^2T + \omega^2\,T = 0 \tag{18}$$

where the dispersion relationship is given by

$$\lambda_j^4 = \omega^2\,\frac{\rho_jS_j}{E_jJ_j} = \frac{\omega^2}{\gamma_j} \tag{19}$$

The Eq. (17) is very important and the solution to it is

$$X_j(x) = A_jK_1(\lambda_jx) + B_jK_2(\lambda_jx) + C_jK_3(\lambda_jx) + D_jK_4(\lambda_jx) \tag{20}$$

where Krylov functions are defined as, (Kaliski, 1986),

$$\begin{aligned} &K_1(z) = \left(\mathrm{ch}(z)+\cos(z)\right)/2, \qquad K_2(z) = \left(\mathrm{sh}(z)-\sin(z)\right)/2, \\ &K_3(z) = \left(\mathrm{ch}(z)-\cos(z)\right)/2, \qquad K_4(z) = \left(\mathrm{sh}(z)+\sin(z)\right)/2 \end{aligned} \tag{21}$$

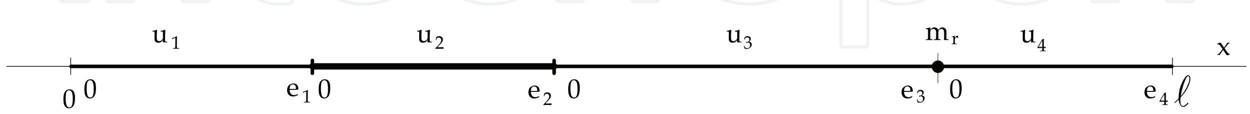

Fig. 6. Geometry of the set beam-one actuator-one mass in local coordinates

The boundary conditions in local coordinates, $x \in [0, e_j]$, to the separate j-element have the form, Fig. 6,

- boundary conditions at the left end of the 1st–element

$$X_1(0) = 0, \qquad D^2X_1(0) = 0 \tag{22}$$

- coupling conditions between 1st and 2nd–elements

$$\mathrm{X}_1(\lambda_1 \mathrm{e}_1) = \mathrm{X}_2(\lambda_2 0), \qquad \mathrm{DX}_1(\lambda_1 \mathrm{e}_1) = \mathrm{DX}_2(\lambda_2 0),$$
$$\mathrm{E}_1\mathrm{J}_1\,\mathrm{D}^2\mathrm{X}_1(\lambda_1 \mathrm{e}_1) = \mathrm{E}_2\mathrm{J}_2\,\mathrm{D}^2\mathrm{X}_2(\lambda_2 0), \qquad \mathrm{E}_1\mathrm{J}_1\,\mathrm{D}^3\mathrm{X}_1(\lambda_1 \mathrm{e}_1) = \mathrm{E}_2\mathrm{J}_2\,\mathrm{D}^3\mathrm{X}_2(\lambda_2 0) \tag{23}$$

- coupling conditions between 2nd and 3rd–elements

$$\mathrm{X}_2(\lambda_2 \mathrm{e}_2) = \mathrm{X}_3(\lambda_3 0), \qquad \mathrm{DX}_2(\lambda_2 \mathrm{e}_2) = \mathrm{DX}_3(\lambda_3 0),$$
$$\mathrm{E}_2\mathrm{J}_2\,\mathrm{D}^2\mathrm{X}_2(\lambda_2 \mathrm{e}_2) = \mathrm{E}_3\mathrm{J}_3\,\mathrm{D}^2\mathrm{X}_3(\lambda_3 0), \qquad \mathrm{E}_2\mathrm{J}_2\,\mathrm{D}^3\mathrm{X}_2(\lambda_2 \mathrm{e}_2) = \mathrm{E}_3\mathrm{J}_3\,\mathrm{D}^3\mathrm{X}_3(\lambda_3 0) \tag{24}$$

- coupling conditions between 3rd and 4th–elements

$$\mathrm{X}_3(\lambda_3 \mathrm{e}_3) = \mathrm{X}_4(\lambda_4 0), \qquad \mathrm{DX}_3(\lambda_3 \mathrm{e}_3) = \mathrm{DX}_4(\lambda_4 0), \qquad \mathrm{E}_3\mathrm{J}_3\,\mathrm{D}^2\mathrm{X}_3(\lambda_3 \mathrm{e}_3) = \mathrm{E}_4\mathrm{J}_4\,\mathrm{D}^2\mathrm{X}_4(\lambda_4 0)$$

and

$$\mathrm{E}_3\mathrm{J}_3\,\mathrm{D}^3\mathrm{X}_3(\lambda_3 \mathrm{e}_3) + \mathrm{m}_\mathrm{r}\omega^2\mathrm{X}_3(\lambda_3 \mathrm{e}_3) = \mathrm{E}_4\mathrm{J}_4\,\mathrm{D}^3\mathrm{X}_4(\lambda_4 0)$$

or

$$\mathrm{E}_3\mathrm{J}_3\,\mathrm{D}^3\mathrm{X}_3(\lambda_3 \mathrm{e}_3) = \mathrm{m}_\mathrm{r}\omega^2\mathrm{X}_4(\lambda_3 0) + \mathrm{E}_4\mathrm{J}_4\,\mathrm{D}^3\mathrm{X}_4(\lambda_4 0) \tag{25}$$

- boundary conditions at the right end of the 4th–element

$$X_4(\lambda_4 e_4) = 0, \qquad D^2 X_4(\lambda_4 e_4) = 0 \tag{26}$$

Since $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_4$ then, to calculate them, the Eq. (19) must be used. It is convenient to express $\{\lambda_2, \lambda_3, \lambda_4\}$ as a function λ_1, hence

$$\lambda_1^4\,\gamma_1 = \lambda_2^4\,\gamma_2 = \lambda_3^4\,\gamma_3 = \lambda_4^4\,\gamma_4 = \omega^2 \tag{27}$$

or

$$\lambda_2^4 = \lambda_1^4(\gamma_1/\gamma_2),\ \lambda_3^4 = \lambda_1^4(\gamma_1/\gamma_3),\ \lambda_4^4 = \lambda_1^4(\gamma_1/\gamma_4) \tag{28}$$

Substituting Eq. (20) into boundary conditions (22) it appears that $\mathrm{A}_1 = 0$, $\mathrm{C}_1 = 0$. In the same way, the rest of conditions given by Eqs. (23) – (26) lead to the set of algebraic equations and it may be written in the matrix form

$$\mathbf{A}\,\mathbf{x} = \mathbf{0} \tag{29}$$

The matrix $\mathbf{A}$ is too large, to presented it in explicit form. Hence, its elements fall into blocks so that the matrix $\mathbf{A}$ can be written as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1' & \mathbf{A}_2'' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_1'' & \mathbf{B}_2''' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_1''' & \mathbf{C}_2'''' \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_1'''' \end{bmatrix} \tag{30}$$

In the current boundary problem, the separate blocks take the form

$$\begin{bmatrix} \mathbf{A}_1' & \mathbf{A}_2'' \\ \mathbf{0} & \mathbf{B}_1'' \end{bmatrix} = \begin{bmatrix} \mathrm{K}_2' & \mathrm{K}_4' & -1 & 0 & 0 & 0 & & \\ \lambda_1 \mathrm{K}_3' & \lambda_1 \mathrm{K}_1' & 0 & 0 & 0 & -\lambda_2 & & \\ \lambda_1^2 \mathrm{E}_1 \mathrm{J}_1 \mathrm{K}_4' & \lambda_1^2 \mathrm{E}_1 \mathrm{J}_1 \mathrm{K}_2' & 0 & 0 & -\lambda_2^2 \mathrm{E}_2 \mathrm{J}_2 & 0 & & \\ \lambda_1^3 \mathrm{E}_1 \mathrm{J}_1 \mathrm{K}_1' & \lambda_1^3 \mathrm{E}_1 \mathrm{J}_1 \mathrm{K}_3' & 0 & -\lambda_2^3 \mathrm{E}_2 \mathrm{J}_2 & 0 & 0 & & \\ & & \mathrm{K}_1'' & \mathrm{K}_2'' & \mathrm{K}_3'' & \mathrm{K}_4'' & & \\ & \mathbf{0} & \lambda_2 \mathrm{K}_2'' & \lambda_2 \mathrm{K}_3'' & \lambda_2 \mathrm{K}_4'' & \lambda_2 \mathrm{K}_1'' & & \\ & & \lambda_2^2 \mathrm{E}_2 \mathrm{J}_2 \mathrm{K}_3'' & \lambda_2^2 \mathrm{E}_2 \mathrm{J}_2 \mathrm{K}_4'' & \lambda_2^2 \mathrm{E}_2 \mathrm{J}_2 \mathrm{K}_1'' & \lambda_2^2 \mathrm{E}_2 \mathrm{J}_2 \mathrm{K}_2'' & & \\ & & \lambda_2^3 \mathrm{E}_2 \mathrm{J}_2 \mathrm{K}_4'' & \lambda_2^3 \mathrm{E}_2 \mathrm{J}_2 \mathrm{K}_1'' & \lambda_2^3 \mathrm{E}_2 \mathrm{J}_2 \mathrm{K}_2'' & \lambda_2^3 \mathrm{E}_2 \mathrm{J}_2 \mathrm{K}_3'' & & \end{bmatrix} \tag{31}$$

$$\begin{bmatrix} \mathbf{B}_1'' & \mathbf{B}_2''' \\ \mathbf{0} & \mathbf{C}_1''' \end{bmatrix} = \begin{bmatrix} & -1 & 0 & 0 & 0 \\ \mathbf{B}_1'' & 0 & 0 & 0 & -\lambda_3 \\ & 0 & 0 & -\lambda_3^2 \mathrm{E}_3 \mathrm{J}_3 & 0 \\ & 0 & -\lambda_3^3 \mathrm{E}_3 \mathrm{J}_3 & 0 & 0 \\ & \mathrm{K}_1''' & \mathrm{K}_2''' & \mathrm{K}_3''' & \mathrm{K}_4''' \\ \mathbf{0} & \lambda_3 \mathrm{K}_2''' & \lambda_3 \mathrm{K}_3''' & \lambda_3 \mathrm{K}_4''' & \lambda_3 \mathrm{K}_1''' \\ & \lambda_3^2 \mathrm{E}_3 \mathrm{J}_3 \mathrm{K}_3''' & \lambda_3^2 \mathrm{E}_3 \mathrm{J}_3 \mathrm{K}_4''' & \lambda_3^2 \mathrm{E}_3 \mathrm{J}_3 \mathrm{K}_1''' & \lambda_3^2 \mathrm{E}_3 \mathrm{J}_3 \mathrm{K}_2''' \\ & \lambda_3^3 \mathrm{E}_3 \mathrm{J}_3 \mathrm{K}_4''' & \lambda_3^3 \mathrm{E}_3 \mathrm{J}_3 \mathrm{K}_1''' & \lambda_3^3 \mathrm{E}_3 \mathrm{J}_3 \mathrm{K}_2''' & \lambda_3^3 \mathrm{E}_3 \mathrm{J}_3 \mathrm{K}_3''' \end{bmatrix} \tag{32}$$

$$\begin{bmatrix} \mathbf{C}_1''' & \mathbf{C}_2'''' \\ \mathbf{0} & \mathbf{D}_1'''' \end{bmatrix} = \begin{bmatrix} & -1 & 0 & 0 & 0 \\ \mathbf{C}_1''' & 0 & 0 & 0 & -\lambda_4 \\ & 0 & 0 & -\lambda_4^2 \mathrm{E}_4 \mathrm{J}_4 & 0 \\ & -\mathrm{m_r}\, \omega_\nu^2 & -\lambda_4^3 \mathrm{E}_4 \mathrm{J}_4 & 0 & 0 \\ \mathbf{0} & \mathrm{K}_1'''' & \mathrm{K}_2'''' & \mathrm{K}_3'''' & \mathrm{K}_4'''' \\ & \mathrm{K}_3'''' & \mathrm{K}_4'''' & \mathrm{K}_1'''' & \mathrm{K}_2'''' \end{bmatrix} \tag{33}$$

where the symbols in matrices are given by

$$\{\mathrm{K}_\upsilon\} = \{\mathrm{K}_1, \mathrm{K}_2, \mathrm{K}_3, \mathrm{K}_4\},$$

$$\mathrm{K}_\upsilon' = \mathrm{K}_\upsilon(\lambda_1 \mathrm{e}_1), \qquad \mathrm{K}_\upsilon'' = \mathrm{K}_\upsilon(\lambda_2 \mathrm{e}_2), \qquad \mathrm{K}_\upsilon''' = \mathrm{K}_\upsilon(\lambda_3 \mathrm{e}_3), \qquad \mathrm{K}_\upsilon'''' = \mathrm{K}_\upsilon(\lambda_4 \mathrm{e}_4) \tag{34}$$

The unknowns are collected in column matrix

$$\mathbf{x} = \left[\mathrm{B}_1, \mathrm{D}_1, \mathrm{A}_2, \mathrm{B}_2, \mathrm{C}_2, \mathrm{D}_2, \mathrm{A}_3, \mathrm{B}_3, \mathrm{C}_3, \mathrm{D}_3, \mathrm{A}_4, \mathrm{B}_4, \mathrm{C}_4, \mathrm{D}_4\right]^{\mathrm{T}} \tag{35}$$

To solve of the homogeneous matrix equation (29), one assumes that $\det \mathbf{A}(\lambda_1) = 0$ and it gives the set $\{\lambda_{1\nu}\}$, $\nu = 1,2,...,n$. Based on Eq. (28) one can calculate $\{\lambda_{2\nu}, \lambda_{3\nu}, \lambda_{4\nu}\}$ and finally, based on Eq. (19), the frequency $\{\omega_\nu\}$ of the system beam-actuator-mass.

Now, the unknowns put down in column matrix, Eq. (35), should be determined. Let the main matrix elements $\mathbf{A}$ be written as two suffix quantities $A_{\alpha\beta}$, where α and β label the rows and columns respectively. Let $M_{\alpha\beta}$ be the minor of the $A_{\alpha\beta}$ element. The general solution to Eq. (29) is

$$B_1 : D_1 : A_2 : ... = (-1)^{\alpha+1} M_{\alpha 1} : (-1)^{\alpha+2} M_{\alpha 2} : (-1)^{\alpha+3} M_{\alpha 3} : ... \tag{36}$$

Substituting $\{\lambda_{j\nu}\}$ and unknowns $\mathbf{x}$ to Eq. (20), the ν –eigenfunctions (ν –modes) assigned to the j –element are obtained. The solution to Eq. (10) is given by

$$X(x) = \sum_j X_j(x) = \sum_{j\nu} X_j(\lambda_{j\nu} x) = \sum_{j\nu} X_{j\nu}(x) = \sum_\nu X_\nu(x) \tag{37}$$

where $\sum_{j\nu}(...) = \sum_j \sum_\nu (...)$ and the separate modes are equal

$$X_{j\nu}(x) = A_j K_1(\lambda_{j\nu} x) + B_j K_2(\lambda_{j\nu} x) + C_j K_3(\lambda_{j\nu} x) + D_j K_4(\lambda_{j\nu} x) \tag{38}$$

2.4 Orthogonality of modes

Orthogonality condition of the uniform beam modes may be found in (Kaliski, 1986; de Silva, 2000). First of all, based on twice integration by parts, one has

$$\int_0^\ell X_\nu(x) D^4 X_\mu(x) dx = \left(X_\nu(x) D^3 X_\mu(x) - D X_\nu(x) D^2 X_\mu(x) \right)\Big|_0^\ell + \int_0^\ell D^2 X_\nu(x) D^2 X_\mu(x) dx \tag{39}$$

The separate modes $X_\mu(x)$, $X_\nu(x)$, fulfill the following modal equations

$$EJ D^4 X_\mu(x) = \omega_\mu^2 \rho S X_\mu(x) \tag{40}$$

$$EJ D^4 X_\nu(x) = \omega_\nu^2 \rho S X_\nu(x) \tag{41}$$

Multiplying above equations by $X_\nu(x)$ and $X_\mu(x)$ respectively, integrate both in range o integration $x \in [0, \ell]$, use Eq. (39), subtract the second result from the first one, one obtains (for simplicity an argument (x) is omitted)

$$\left(\omega_\nu^2 - \omega_\mu^2\right) \rho S \int_0^\ell X_\nu X_\mu dx = EJ \left[\left(X_\mu D^3 X_\nu - D X_\mu D^2 X_\nu \right) - \left(X_\nu D^3 X_\mu - D X_\nu D^2 X_\mu \right) \right]\Big|_0^\ell \tag{42}$$

For standard boundary conditions, the right-hand-side equals zero.
The procedure outlined above can be used to the problem presented in Fig. 6, but Eq. (39) must be applied to the separate j –element, namely

$$\begin{gathered} \int_0^{e_j} X_{j\nu}(x) D^4 X_{j\mu}(x) dx = \left(X_{j\nu}(x) D^3 X_{j\mu}(x) - D X_{j\nu}(x) D^2 X_{j\mu}(x) \right)\Big|_0^{e_j} + \\ + \int_0^{e_j} D^2 X_{j\nu}(x) D^2 X_{j\mu}(x) dx \end{gathered} \tag{43}$$

Considering both boundary conditions of the problem and coupling conditions between neighboring elements, Eqs. (22)–(26), instead of Eq. (42) one has

$$\left(\omega_\nu^2-\omega_\mu^2\right)\Big(\rho_1 S_1\int_0^{e_1} X_{1\nu}X_{1\mu}\,dx+\rho_2 S_2\int_0^{e_2} X_{2\nu}X_{2\mu}\,dx+\rho_3 S_3\int_0^{e_3} X_{3\nu}X_{3\mu}\,dx+\rho_4 S_4\int_0^{e_4} X_{4\nu}X_{4\mu}\,dx+$$
$$+\,m_r X_{4\nu}(0)X_{4\mu}(0)\Big)=E_4J_4\,\cdot$$
$$\cdot\Big[\left(X_{4\mu}(e_4)D^3X_{4\nu}(e_4)-DX_{4\mu}(e_4)D^2X_{4\nu}(e_4)\right)-\left(X_{4\nu}(e_4)D^3X_{4\mu}(e_4)-DX_{4\nu}(e_4)D^2X_{4\mu}(e_4)\right)\Big]\tag{44}$$

Because of Eq. (26), the right-hand-side is zero, hence

$$\left(\omega_\nu^2-\omega_\mu^2\right)\Big(\rho_1 S_1\int_0^{e_1} X_{1\nu}X_{1\mu}\,dx+\rho_2 S_2\int_0^{e_2} X_{2\nu}X_{2\mu}\,dx+\rho_3 S_3\int_0^{e_3} X_{3\nu}X_{3\mu}\,dx+\rho_4 S_4\int_0^{e_4} X_{4\nu}X_{4\mu}\,dx+$$
$$+\,m_r X_{4\nu}(0)X_{4\mu}(0)\Big)=0\tag{45}$$

Fig. 7. Geometry of set with n_j –elements

The orthogonality condition, Eq. (45), may be generalized in a simple way. Let the system beam-actuators-masses be divided into n_j –elements as depicted in Fig. 7. In this case one has

$$\left(\omega_\nu^2-\omega_\mu^2\right)\Big[\sum_j\Big(\rho_j S_j\int_j X_{j\nu}X_{j\mu}\,dx+m_j X_{j\nu}(0)X_{j\mu}(0)\Big)+m_{n_j+1}X_{n_j\nu}(e_{n_j})X_{n_j\mu}(e_{n_j})\Big]=0\tag{46}$$

Since the term $\omega_\nu^2-\omega_\mu^2$ is canceled for $\mu=\nu$, the general orthogonality condition is given by

$$\sum_j\Big(\rho_j S_j\int_j X_{j\nu}X_{j\mu}\,dx+m_j X_{j\nu}(0)X_{j\mu}(0)\Big)+m_{n_j+1}X_{n_j\nu}(e_{n_j})X_{n_j\mu}(e_{n_j})=\begin{cases}0, & \nu\neq\mu\\ \beta_\nu^2, & \nu=\mu\end{cases}\tag{47}$$

The Eq. (47) in particular case is used in deriving the solution to the forced vibration problem.

2.5 Forced vibrations with damping

A point departure for further consideration is Eq. (7); for j –element one has

$$E_jJ_jD^4u_j+\mu_jE_jJ_jD^4\left(D_tu_j\right)+\rho_jS_jD_t^2u_j=-f_j\tag{48}$$

The solution to Eq. (48) is forced vibrations with damping. Let be the load force in the form

$$f_j(x,t)=f_j(x)\exp(i\omega_f t)\tag{49}$$

where $i=(-1)^{1/2}$, ω_f – excited frequency.
Applying separation of variables method, the solution to Eq. (48) is assumed as

$$u_j(x,t)=X_{jf}(x)\exp(i\omega_f t)\tag{50}$$

Substituting Eqs. (49) and (50) to Eq. (48) one obtains

$$\gamma_j(1+i\mu_j\omega_f)D^4X_{jf}(x) - \omega_f^2 X_{jf}(x) = -\frac{1}{\rho_j S_j} f_j(x) \tag{51}$$

The solution of the above equation is given by

$$X_{jf}(x) = \sum_{\nu} C_{j\nu} X_{j\nu}(x) \tag{52}$$

where $C_{j\nu}$ – constants, $X_{j\nu}(x)$ – Eq. (38).
After some calculation, the constants $C_{j\nu}$ are expressed by

$$C_{j\nu} = \frac{1}{\rho_j S_j} \frac{1}{\alpha_{j\nu}^2} \frac{1}{\beta_\nu^2} I_{j\nu} = C_{j\nu}^* I_{j\nu} \tag{53}$$

where

$$C_{j\nu}^* = \frac{1}{\rho_j S_j} \frac{1}{\alpha_{j\nu}^2} \frac{1}{\beta_\nu^2},\ \frac{1}{\alpha_{j\nu}^2} = \frac{1}{(1+i\mu_j\omega_f)\omega_\nu^2 - \omega_f^2},\ I_{j\nu} = -\int_j f_j(x)\, X_{j\nu}(x)\, dx \tag{54}$$

In the end, the problem of the forced j –element beam vibration with damping, excited with the force $f_j(x)$ is solved; in the harmonic steady state it is given by

$$X_f(x) = \sum_j X_{jf}(x) = \sum_{j\nu} C_{j\nu} X_{j\nu}(x) \tag{55}$$

In current problem, two form of the forces have the practical meaning namely, the force with constant amplitude $f_{j0}(x) = f_0$ and the force acting at discrete point $f_{ja}(x_i)$. The former may be interpreted as the spread excitation forced, for example with plane acoustic wave, but the latter is the control force due to actuators, henceforth

$$f_j(x) = f_0 + f_{ja}(x_i) \tag{56}$$

2.6 Interaction between beam and actuators

It is assumed that the actuator is perfectly bonded to the beam surface. Exciting actuator, the interaction between actuator and the beam is appeared. The interaction process is explained in (Hansen & Snyder, 1997; Fuller at al, 1997) in detail and references cited therein. Assuming the spatially uniform actuator, it provides boundary induction solely in terms of the external line moment distributed along its edges (Burke & Hubbard, 1991; Sullivan et al., 1996). So, the bending moment in y –direction is given by the formula (Hansen & Snyder 1997), Fig. 8,

$$M_x = -M_0\left(< x - x_1 >^{-2} - < x - x_2 >^{-2}\right) \tag{57}$$

where $<.>^{-1} = \delta(.)$ and $<.>^{-2} = D\delta(.)$ – doublet function, M_0 – line moment amplitude

$$M_0 = C_a \frac{d_{31}}{h_a} V \tag{58}$$

where C_a – constant depending on geometry and mechanical properties of the actuator and plate, d_{31} – piezoelectric material strain constants, V – voltage in the direction of polarization.

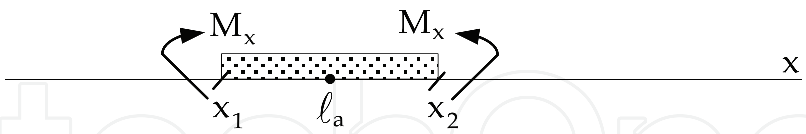

Fig. 8. External line moments of the actuator

The problem is to determine of the C_a, because it depends on the analysis method of the mutual interaction between beam-actuator (Hansen & Snyder 1997; Pietrzakowski, 2004). Let the static force coupling model is taken into account. If relatively thin actuator compared with beam thickness is assumed (so uniform normal stress distribution is accepted) and furthermore by ignoring the neutral axis displacement d, see Fig. 4, the constant C_a is come down to the form

$$C_a = \frac{E_b h_b E_a h_a (h_b + h_a)}{2(E_b h_b + E_a h_a)} \tag{59}$$

Since the beam vibration equation is the forces equation then to consider the action of actuator with the beam, moments M_x are replaced with two couples of forces, Fig. 9,

$$M_x = f_a \ell_a / 2 \tag{60}$$

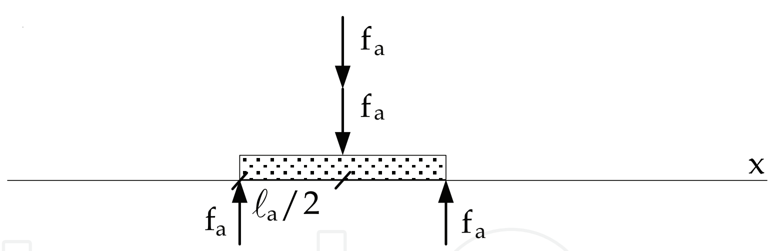

Fig. 9. External pair of forces of the actuator

Next, the separate forces are considered in Eq. (56).

2.7 Beam vibration reduction through actuators

For the problem presented in Fig. 5, the total load of the beam, described by Eq. (56), is given by

$$f_j(x) = -f_0 + \left(f_{js}\delta(x - x_{1s}) - 2f_{js}\delta(x_s) + f_{js}\delta(x + x_{2s}) \right) \tag{61}$$

where the symbol f_{ja} is replace by f_{js} in order to express, in the future, the interaction sum of actuators and the glue on the beam.
An expression in brackets is the sum of interacting forces actuator-beam. Hence, the integral $I_{j\nu}$, Eq. (54), for $f_j(x)$ expressed by above equation is given by

$$
\begin{aligned}
I_{j\nu} &= -\int_j f_j(x)\,X_{j\nu}(x)\,dx = -f_0\int_j X_{j\nu}(x)\,dx + f_{js}\int_j \left(\delta(x-x_{1s}) - 2\delta(x_s) + \delta(x+x_{2s})\right)X_{j\nu}\,dx = \\
&= -f_0\int_j X_{j\nu}(x)\,dx + f_{js}\left[X_{j\nu}(x_{1s}) - 2X_{j\nu}(x_s) + X_{j\nu}(x_{2s})\right] \qquad (62)
\end{aligned}
$$

The expression in square bracket constitutes the second-order central finite difference. Since the distance between nodes ℓ_s is constant, then the difference can be transformed into

$$\frac{1}{\ell_s^2}\left[X_{j\nu}(x_{1s}) - 2X_{j\nu}(x_s) + X_{j\nu}(x_{2s})\right] = D^2X_{j\nu}(x_s) \qquad (63)$$

where

$$D^2X_{j\nu}(x_s) = \pm\kappa_{j\nu}(x_s) \qquad (64)$$

The $\kappa_{j\nu}(x_s)$ is the curvature of the mode $X_{j\nu}(x)$ at the point $x = x_s$ (Brański & Szela, 2007; Brański & Szela, 2008). The sign of the $\kappa_{j\nu}(x_s)$ is contractual namely, if the bending of the beam is directed upwards, the sign is positive and vice versa. Substituting Eq. (64) into Eq. (62), one obtains

$$I_{j\nu} = -f_0\int_j X_{j\nu}(x)\,dx + f_{js}\,\ell_s^2\,\kappa_{j\nu}(x_s) = I_{j\nu 0} + I_{j\nu s} \qquad (65)$$

Next, substituting Eq. (65) into Eq. (52) through Eq. (53), the reduction vibration is obtained

$$X_{jf}(x) = \sum\nolimits_\nu C^*_{j\nu}\,I_{j\nu}\,X_{j\nu}(x) = \sum\nolimits_\nu C^*_{j\nu}\left(I_{j\nu 0} + I_{j\nu s}\right)X_{j\nu}(x) = \sum\nolimits_\nu A_{jf\nu}\,X_{j\nu}(x) \qquad (66)$$

Note, that the amplitude $A_{jf\nu}$ is the direct quantity which is liable to the reduction, in explicit form is

$$A_{jf\nu} = C^*_{j\nu}\,I_{j\nu} = C^*_{j\nu}\left(I_{j\nu 0} + I_{j\nu s}\right) \qquad (67)$$

At the same time, together with the vibration reduction amplitude $A_{jf\nu}$, the curvature is subjected to the reduction and based on Eq. (66) is

$$\kappa_{jf} = \pm D^2X_{jf} = \sum\nolimits_\nu \kappa_{jf\nu} = \pm\sum\nolimits_\nu A_{jf\nu}\,\kappa_{j\nu} \qquad (68)$$

Furthermore, the reduction of the $A_{jf\nu}$ leads to the reduction of the shear force $Q_{jf}(x)$ and bending moment $M_{jf}(x)$ (Brański et al., 2010; Kaliski, 1986; Kozień, 2006),

$$Q_{jf}(x) = \pm E_jJ_j\,D\kappa_{jf}(x) = \pm E_jJ_j\sum\nolimits_\nu D\kappa_{jf\nu}(x) = \pm E_jJ_j\sum\nolimits_\nu A_{jf\nu}\,D\kappa_{j\nu}(x) \qquad (69)$$

$$M_{jf}(x) = \pm E_jJ_j\,\kappa_{jf}(x) = \pm E_jJ_j\sum\nolimits_\nu \kappa_{jf\nu}(x) = \pm E_jJ_j\sum\nolimits_\nu A_{jf\nu}\,\kappa_{j\nu}(x) \qquad (70)$$

As can be seen, the vibration reduction undergo on the following amplitudes: of the beam vibration $X_{jf}(x)$ – Eq. (66), of the shear force $Q_{jf}(x)$ – Eq. (69) and of the bending moment $M_{jf}(x)$ – Eq. (70). Hereafter, the notion e.g. "shear force reduction" is used instead of "the reduction of the amplitude of the shear force", and so on.

The Eqs (66), (69) and (70) and may be written commonly

$$\Psi_{jf}(x) = \sum_{\nu} \Psi_{jf\nu}(x) = C_j \sum_{\nu} A_{jf\nu} \Phi_{j\nu}(x) \tag{71}$$

where

$$\Psi_{jf} = \{X_{jf}, Q_{jf}, M_{jf}\}, \quad \Phi_{j\nu} = \{X_{j\nu}, D\kappa_{j\nu}, \kappa_{j\nu}\}, \quad C_j = \{1, \pm E_j J_j, \pm E_j J_j\} \tag{72}$$

Omitting for simplicity the index "f", for entire system beam-actuators one has

$$\Psi(x) = \sum_j \Psi_j(x) = \sum_{j\nu} \Psi_{j\nu}(x) \tag{73}$$

or

$$\Psi(x) = \sum_{j\nu} C_j A_{j\nu} \Phi_{j\nu}(x) = CA\Phi(x) \tag{74}$$

where

$$A = C^{*}I = C^{*}\left(I_0 + \sum_s f_s \ell_s^2 \kappa(x_s)\right) = C^{*}\left(I_0 + I_{\Sigma}\right) \tag{75}$$

$$\Phi(x) = \{X(x), D\kappa(x), \kappa(x)\} \tag{76}$$

It is appeared from Eq. (75) that the active vibration reduction depends on the following parameters:

- ω_f – excited frequency, it is contained in C_{ν}^{*},
- x_s – distribution of the actuators on the beam,
- $\kappa(x_s)$ – value of the beam curvature at the point of the actuators distribution,
- f_s – interacting forces between beam-actuators or more generally – mechanical properties of the actuators,
- ℓ_s – actuators lengths or more generally – geometrical properties of the actuators,
- n_s – number of actuators.

As mentioned above, the optimal actuators distribution described with $\{x_s\}$ has an important meaning and finding of the $\{x_s\}$ is the aim of the chapter.

3. Optimal actuators distribution problem

Before the optimization problem will be formulated, any coefficients of the vibration reduction should be defined.

3.1 Reduction and effectiveness coefficients

Let be the difference between any quantities of the beam vibration

$$\Delta\Psi(x) = \Psi(x) - \Psi_R(x) \tag{77}$$

where $\Psi(x)$, $\Psi_R(x)$ – quantities calculated without and with actuators respectively; $\Psi(x)$, $\Psi_R(x)$ are given together by Eq. (74), where

$$A = C^{*} I_0 \tag{78}$$

and

$$A_R = C^*\left(I_0 + I_\Sigma\right) \tag{79}$$

The difference $\Delta\Psi(x)$ is interpreted as the quantity of the vibration reduction and it is the first measure of this reduction namely, the quantity reduction coefficient.
The second measure of the vibration reduction is defined as

$$R_\psi(x) = \frac{\Delta\Psi(x)}{\Psi(x)} = \frac{\Psi(x) - \Psi_R(x)}{\Psi(x)} \tag{80}$$

It is called as the reduction coefficient and it may be expressed in per cent. Note, that if the reduction coefficient equals one, the vibration reduction is total, $\Psi_R(x) = 0$.
An effectiveness of the vibration reduction is defined as a quotient of some vibration reduction measure by an amount of the energy W provided to the system in order to excite actuators. Hence, thirst measure of the vibration reduction may be defined by so called the effectiveness coefficient

$$E_\Psi(x) = R_\Psi(x)/W \tag{81}$$

The energy W provided to the system is translated into couples of forces, Fig. 9. Therefore, the energy W may be replaced by forces $f_R = \sum_s 4 f_s$, hence

$$E_\Psi(x) = R_\Psi(x)/f_R \tag{82}$$

The Eqs. (77) – (82) define the appropriate factors of the vibration reduction at the point x . In many cases, it is convenient to calculate mean values of these coefficients at whole beam domain or at the beam sub-domains. First of them is the mean quantity reduction coefficient and it is defined by the formula

$$\Delta\Psi_m = \frac{1}{n_i}\sum_i \left(\Psi(x_i) - \Psi_R(x_i)\right) \qquad i = 1,2,\ldots,n_i \tag{83}$$

Consequently, the mean reduction coefficient and the mean effectiveness coefficient are defined respectively

$$R_{\Psi m} = \Delta\Psi_m/\Psi_m , \qquad \Psi_m = \sum_i \Psi(x_i)\big/n_i \tag{84}$$

$$E_{\Psi m} = R_{\Psi m}/f_R \tag{85}$$

The coefficients defined above may constitute the base to formulate the optimization problem; hereafter the $R_\Psi(x)$ is chosen.

3.2 Formulation of the optimization problem

In this chapter, one formulates the following problem: find the optimal actuators distribution $\{x_s\}$ which maximize of the reduction coefficient $R_\Psi(x)$; hence $R_\Psi(x)$ is assumed as an objective function. In this case the maximal value of $R_\Psi(x)$ equals one and it means p-reduction; such instance is considered in (Brański et al., 2010; Brański & Lipiński, 2011) and it seems that it is possible only in for separate mode.

Let the energy provided to the actuators be constant and hence, the f_s is always constant. Now, for clarity of the disquisition, rewrite the effectiveness coefficient in explicit form

$$R_\Psi(x) = \frac{\Psi(x) - \Psi_R(x)}{\Psi(x)} = \frac{CA\Phi(x) - CA_R\Phi(x)}{CA\Phi(x)} \tag{86}$$

Working out on above assumption, the $R_\Psi(x)$ will be maximal, if $\Psi_R(x)$ is minimal. Hence, the optimal condition $R_\Psi(x) = R_{\Psi;max}(x)$ leads to the next condition $\Psi_R(x) \equiv \Psi_{R;min}(x)$. Note, that the $\Psi_R(x)$ depends on the reduction amplitude A_R. So, the $\Psi_R(x) \equiv \Psi_{R;min}(x)$, if the amplitude A_R is minimal and instead of the above condition, it leads to

$$A_R = A_{R;min} \tag{87}$$

3.3 Heuristic analysis of the optimization problem

Note, that the amplitude A_R comprises the factor $C^* \neq 0$, but it is constant and this is the factor $I_0 + I_\Sigma = I_R$ which is changed. In practice, instead of the condition (87), the following condition of the reduction must be fulfilled

$$I_R = I_0 + I_\Sigma = I_0 + \sum_s f_s \ell_s^2 \kappa(x_s) = I_{min} \tag{88}$$

For future considerations the sign of I_R is very important. The vibrations are reduced with actuators, if the I_Σ is positive, but must be fulfilled the following condition: $I_R = I_0 + I_\Sigma \geq 0$; $I_R = 0$ assures the total reduction. If this condition is not fulfilled, the actuators excite vibrations and thereby they are not accomplished owns role. Note, that the sign of I_0 is always negative, see Eq. (65). Then, the sign of I_Σ must be positive and it depends on the signs both forces f_s and curvatures $\kappa(x_s)$.

From physical point of view, the sign of $\kappa(x)$ is changed and as established above; it is positive, if the bending of the beam is directed upwards. Then, for many actuators one has

$$I_\Sigma = f_1 \ell_1^2 (+)\kappa(x_1) + f_2 \ell_2^2 (-)\kappa(x_2) + \ldots = \sum_s f_s \ell_s^2 (-1)^{s+1} \kappa(x_s) > 0 \tag{89}$$

To obtain the positive sign of I_Σ, the signs of f_s should alternates; this problem clearly expressed in the following way

$$I_\Sigma = (+)f_1 \ell_1^2 (+)\kappa(x_1) + (-)f_2 \ell_2^2 (-)\kappa(x_2) + \ldots = \sum_s (-1)^{s+1} f_s \ell_s^2 (-1)^{s+1} \kappa(x_s) > 0 \tag{90}$$

First at all, it is possible if the signs of $\kappa(x_s)$ and f_s are the same, namely positive or negative, and they take their extremes. To fulfill this requirement, the actuators should be specially distributed on the beam. An idea of description of the sign of $\kappa(x_s)$ is advance determined. The value of $\kappa(x_s)$ is determined by means of the distribution of the actuators; they are bended at $\{x_s\}$. Hence, the distribution has a great significance; this problem was solved in (Brański & Szela, 2007; Brański & Szela, 2008; Brański et al., 2010; Brański & Szela 2010; Szela, 2009). Interpreting Eq. (88) through Eq. (90) it is appear that the actuators ought to be bonded on the beam sub-domains in which the curvatures reach their extremum and consequently the highest and lowest values respectively, see Fig. 10. This is so called quasi-optimal actuators distribution and it is described with $x_Q \equiv x_s$ points, their number is $n_Q \equiv n_s$.

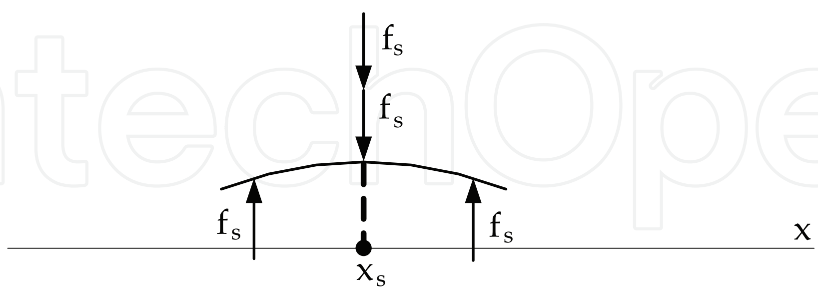

Fig. 10. Optimal distribution of the actuators

As far as signs and values of $\{f_s\}$ are concerned, it was assumed that the added energy exciting actuators is constant. So, the values $\{f_s\}$ of the separate actuators are known and always are constant, while the sign f_s springs from the physical interpretation of the interaction beam-actuator. As can be seen in Fig. 10, the forces $2f_s$ are placed at the point of local extreme, namely at the x_s, with the opposite direction to the bending of the beam $X(x)$. At the same time, the forces f_s on the actuator edges are in the direction of the beam bending and let assume that this sign of f_s is positive. Another way, the vectors f_s and the beam bending $X(x)$ are in the same direction. In such sign convection, both f_s and $\kappa(x_s)$ in Eq. (90) have the same signs and all terms are positive. Furthermore, the actuators distribution described with $x_Q \equiv x_s$ ensures the maximum of the reduction coefficient.
The heuristic analysis described above was substantiated numerically for the separate beam and triangular modes and the details may be found in own papers.

3.4 Analytical analysis of the optimization problem

The aim of this section is to work out of the analytical method, which will describe such distribution of the actuators in order to assure the maximum of the reduction coefficient. It is expected that the analytical method will confirm the quasi-optimal distribution which has been found above with heuristic method. Therefore the assumptions are the same like in heuristic method, namely n_s, f_s and ℓ_s are settled.
Let the distribution of actuators be marked with the set of unknown coordinates $\{x_s\}$ for the moment; that are exactly these coordinates $\{x_s\} \equiv \{x_Q\}$ of which are looked for. One starts from Eq. (88), hence

$$I_R = I_0 + I_\Sigma = I_0 + f_1 \ell_1^2 \kappa(x_1) + f_1 \ell_2^2 \kappa(x_2) + \ldots \tag{91}$$

Since the $\kappa(x)$ is the function which changes the sign, it is appropriate to search the points x_s which assure the extreme $I_R(x)$, not minimum only. The function $I_R(x)$ can have the extreme only at points x_s, at which $DI_R(x)$ is equal to zero or $f_1 \ell_1^2 D\kappa(x_1) + f_1 \ell_2^2 D\kappa(x_2) + \ldots = 0$ does not exist (Fichtenholtz, 1999). Because I_0 is constant than a necessary condition for existing extreme value is

$$\mathrm{DI_R}(\mathrm{x_s}) = 0 \tag{92}$$

where $\mathrm{DI_R}(\mathrm{x_s}) = \mathrm{DI_R}(\mathrm{x} = \mathrm{x_s})$ and hence

$$\mathrm{f_1}\,\ell_1^2\,\mathrm{D}\kappa(\mathrm{x_1}) + \mathrm{f_1}\,\ell_2^2\,\mathrm{D}\kappa(\mathrm{x_2}) + \ldots = 0 \tag{93}$$

Because of $\mathrm{f_s}\,\ell_s^2 \neq 0$ then instead of Eq. (93) one has

$$\mathrm{D}\kappa(\mathrm{x_s}) = 0 \tag{94}$$

From the condition (94), a set of stationary points $\{\mathrm{x_s}\}$ is obtained. The sufficient condition for existing extreme demands, in order to the function be determined on either side of the point $\mathrm{x_s}$ and $\mathrm{D}\kappa(x)$ must change sign at this point (turning point); it is sufficient condition formulated in the first form. This condition is expressed in the other form

$$\mathrm{D}^2\kappa(\mathrm{x_s}) \neq 0 \tag{95}$$

If this condition is not fulfilling, then this point should be omitted.
One still needs to consider the biggest and the lowest values of $\kappa(\mathrm{x})$; they are in hypothetical points $\{\mathrm{x_{max}}, \mathrm{x_{min}}\}$. In order to find them, the values of $\kappa(\mathrm{x})$ at the stationary points $\{\mathrm{x_s}\}$ are calculated and they are compared to the values calculated at the end points of the appropriate interval. In the future consideration, the $\mathrm{n_s}$ points among stationary $\{\mathrm{x_s}\}$ and $\{\mathrm{x_{max}}, \mathrm{x_{min}}\}$ ones, at which $\kappa(\mathrm{x})$ takes in turn its absolute values, are taken into account. The problem of the signs of the $\kappa(\mathrm{x_s})$ and $\mathrm{f_s}$ is quite the same as in heuristic analysis.
Analytical analysis was applied for p-reduction and for the separate beam modes (Brański & Lipiński, 2011). As pointed out there, the analytical solution to the optimal actuators distribution problem confirms the results obtained with heuristic solution.

4. Conclusion

Deriving the shape of $\kappa(\mathrm{x})$, the influence both masses and stiffness of the actuators and glue on the shape of $\mathrm{X}(\mathrm{x})$, and consistently on the shape of $\kappa(\mathrm{x})$, were omitted; if not, an adaptation method must be applied. But after determining shape of $\kappa(\mathrm{x})$, all these parameters were considered.
As can be note, the actuators optimal distribution is attained assuming that the added energy to excite actuators is constant. It is translated into constant $\mathrm{f_s}$. Having the optimal distribution, the reduction coefficient may be improved by adding more energy or in order words, by increasing $\mathrm{f_s}$. This way, presented optimal method corresponds to that one presented in (Q. Wang & C. Wang, 2001), namely "maximization of the control forces transmitted by the actuators to the structure".
Based on theoretical considerations, and numerical ones presented in own papers, the following conclusion may be formulated.

1. The optimization problem of the actuators distribution assuring the maximal active vibration reduction of the beam, measured with reduction coefficient, may be solved both heuristically and analytically. In analyzed problem, it turned out that both methods give the same results.
2. The following algorithm of analytical method may be worked out:

- to search of stationary points $\{x_s\}$ of the beam curvature,
- to search of $\{x_{max}, x_{min}\}$ points of the beam curvature,
- n_Q points among stationary $\{x_s\}$ and $\{x_{max}, x_{min}\}$ ones, at which $\kappa(x)$ takes in turn its maximum absolute values, are selected, they are denoted by $\{x_Q\}$,
- to bond the actuators at the $\{x_Q\}$ points,
- to determine the value of the reduction coefficient,
- to increase the value f_s, through the energy increase which excites actuators, until the reduction coefficient will attain its maximum.

It seems that proposed optimization method is very simple and may be useful in many technical problems of active vibration reduction. This work is a starting point for many computer simulations and experiments.

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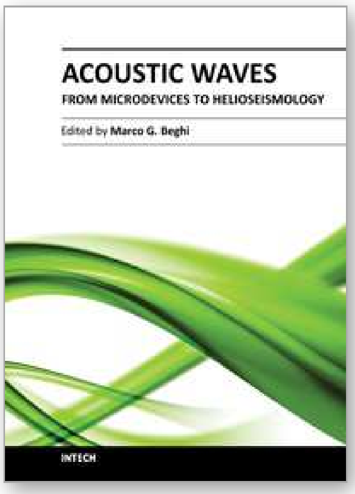

Acoustic Waves - From Microdevices to Helioseismology

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The concept of acoustic wave is a pervasive one, which emerges in any type of medium, from solids to plasmas, at length and time scales ranging from sub-micrometric layers in microdevices to seismic waves in the Sun's interior. This book presents several aspects of the active research ongoing in this field. Theoretical efforts are leading to a deeper understanding of phenomena, also in complicated environments like the solar surface boundary. Acoustic waves are a flexible probe to investigate the properties of very different systems, from thin inorganic layers to ripening cheese to biological systems. Acoustic waves are also a tool to manipulate matter, from the delicate evaporation of biomolecules to be analysed, to the phase transitions induced by intense shock waves. And a whole class of widespread microdevices, including filters and sensors, is based on the behaviour of acoustic waves propagating in thin layers. The search for better performances is driving to new materials for these devices, and to more refined tools for their analysis.

InTech Europe
University Campus STeP Ri
Slavka Krautzeka 83/A
51000 Rijeka, Croatia
Phone: +385 (51) 770 447
Fax: +385 (51) 686 166
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中国上海市延安西路65号上海国际贵都大饭店办公楼405单元
Phone: +86-21-62489820
Fax: +86-21-62489821