4

Exact Solutions Expressible in Hyperbolic and Jacobi Elliptic Functions of Some Important Equations of Ion-Acoustic Waves

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1. Introduction

Many phenomena in physics and other fields are often described by nonlinear partial differential equations (NLPDEs). The investigation of exact and numerical solutions, in particular, traveling wave solutions, for NLPDEs plays an important role in the study of nonlinear physical phenomena. These exact solutions when they exist can help one to well understand the mechanism of the complicated physical phenomena and dynamical processes modeled by these nonlinear evolution equations (NLEEs). The ion-acoustic solitary wave is one of the fundamental nonlinear wave phenomena appearing in fluid dynamics [1] and plasma physics [2, 3]. It has recently became more interesting to obtain exact analytical solutions to NLPDEs by using appropriate techniques and symbolical computer programs such as Maple or Mathematica. The capability and power of these software have increased dramatically over the past decade. Hence, direct search for exact solutions is now much more viable. Several important direct methods have been developed for obtaining traveling wave solutions to NLEEs such as the inverse scattering method [3], the tanh-function method [4], the extended tanh-function method [5] and the homogeneous balance method [6]. We assume that the exact solution is expressed by a simple expansion $u(x,t) = U(\xi) = \sum_{i=0}^{N} A_i F^i(\xi)$ where A_i are constants to be determined and the function $F(\xi)$ is defined by the solution of an auxiliary ordinary differential equation (ODE). The tanh-function method is the well known method as a direct selection of the function $F(\xi) = tanh(\xi)$. Recently, many exact solutions expressed by various Jacobi elliptic functions (JEFs) of many NLEEs have been obtained by Jacobi elliptic function expansion method [7-10], mapping method [11, 12], F-expansion method [13], extended F-expansion method [14], the generalized Jacobi elliptic function method [15] and other methods [16-20]. Various exact solutions were obtained by using these methods, including the solitary wave solutions, shock wave solutions and periodic wave solutions.
The main steps of the F-expansion method [13] are outlined as follows:
Step 1. Use the transformation $u(x,t) = u(\xi);\ \xi = k(x - \omega t) + \xi_0$, ξ_0 is an arbitrary constant, and reduce a given NLPDE, say in two independent variables,

$$F(u, u_t, u_x, u_{tt}, u_{xx}, ...) = 0, \tag{1.1}$$

to the (ODE)

$$G(u,u',u'',...) = 0, \qquad u' = \frac{du}{d\xi}. \tag{1.2}$$

In general, the left hand side of Eq. (1.1) is a polynomial in u and its various derivatives.
Step 2. The F-expansion method gives the solution of (1.1) in the form

$$u(x,t) = u(\xi) = \sum_{i=0}^{N} a_i F^i(\xi), \qquad a_N \neq 0, \tag{1.3}$$

where a_i $(i = 0,1,2,...,N)$ are constants to be determined and $F(\xi)$ satisfies the first order nonlinear ODE in the form

$$(F'(\xi))^2 = q_0 + q_2 F^2(\xi) + q_4 F^4(\xi), \tag{1.4}$$

where q_0, q_2 and q_4 are constants and N in Eq. (1.3) is a positive integer that can be determined by balancing the nonlinear term(s) and the highest order derivatives in Eq. (1.1).
Step 3. Substituting the F-expansion (1.3) into (1.2) and using (1.4); setting each coefficient of the polynomial to zero yields a system of algebraic equations involving a_0, a_1, ...a_N, k and ω.
Step 4. Solving these equations, probably with the aid of Mathematica or Maple, then a_0, a_1, ...a_N, k and ω can be expressed by q_0, q_2, q_4.
Step 5. Substituting these results into F-expansion (1.3), then a general form of traveling wave solution of the NLPDE (1.1) can be obtained. Many solutions of equation (1.4) have been reported in [13, 14]. Substituting the values of q_0, q_2, q_4 and the corresponding JEF solution $F(\xi)$ into the general form of solution, we may get several classes of exact solutions of equations (1.1) involving JEFs.
Also, we give a brief description of the mapping method to seek the traveling wave solutions of (1.1) in the form $u(x,t) = u(\eta)$, $\eta = kx - \omega t + \eta_0$, η_0 is an arbitrary constant. Thus, Eq. (1.1) reduces to Eq. (1.2), whose solution can be express in the form

$$u(\eta) = \sum_{i=0}^{n} A_i f^i(\eta), \tag{1.5}$$

where n is a balancing number, A_i are constants to be determined and $f(\eta)$ satisfies the nonlinear ODE

$$f'^2(\eta) = 2\,p\,f(\eta) + q\,f^2(\eta) + \frac{2}{3}\,r\,f^3(\eta). \tag{1.6}$$

Here p, q and r are constants. After substituting Eq. (1.5) into the ODE (1.2) and using Eq. (1.6), the constants A_i, k and ω may be determined. By using the solutions of auxiliary nonlinear equation (1.6), many JEF solutions of NLEEs have been obtained [19, 20].
The JEFs $\mathrm{sn}(\xi) = \mathrm{sn}(\xi,m)$, $\mathrm{cn}(\xi) = \mathrm{cn}(\xi,m)$ and $\mathrm{dn}(\xi) = \mathrm{dn}(\xi,m)$ are double periodic and have the following properties:

$$\mathrm{sn}^2(\xi) + \mathrm{cn}^2(\xi) = 1, \qquad \mathrm{dn}^2(\xi) + m^2\,\mathrm{sn}^2(\xi) = 1.$$

In the limit $m \longrightarrow 1$, the JEFs degenerate to the hyperbolic functions, i.e.,

$$\mathrm{sn}(\xi,1) \longrightarrow \tanh(\xi),\ \ \mathrm{cn}(\xi,1) \longrightarrow \mathrm{sech}(\xi),\ \ \mathrm{dn}(\xi,1) \longrightarrow \mathrm{sech}(\xi).$$

Detailed explanations about JEFs can be found in [21].
Some of the nonlinear models in fluids, plasma and dust plasma are described by canonical models and include the Korteweg-de Vries (KdV) and the modified KdV equations [22-25].

The evolution of small but finite-amplitude solitary waves, studied by means of the Korteweg-de Vries (KdV) equation, is of considerable interest in plasma dynamics. In the study of multidimensional version two type of nonlinear waves are well known, the so called Kadomtsev-Petviashvilli (KP) equation and Zakharov - Kuzentsov (ZK) equation. Employing the reductive perturbation technique on the system of equations for hydrodynamics and the dynamics of plasma waves to derive such equation.
We construct several classes of exact JEF solutions of some nonlinear evolution equations of plasma physics by using the mapping method and the F-expansion method. The rest of this chapter is organized as follows: in section 2, we present the JEF solutions to the KdV equation, combined KdV - modified KdV equation. In section 3, we apply the F-expansion method to the Schamel- KdV equation. Moreover, using the ansatz solution (1.5) and the solutions of nonlinear ODE (1.6), many exact solutions of Schamel equation, ZK equation and modified fifth order KdV equation are given in sections 4, 5, 6.

2. The KdV and modified KdV equations

The Korteweg de-Vries (KdV) equation

$$u_t + \alpha u u_x + u_{xxx} = 0,$$

models a variety of nonlinear phenomena, including ion acoustic waves in plasmas, dust acoustic solitary structures in magnetized dusty plasmas, and shallow water waves. On the other hand, the modified KdV equation (mKdV)

$$u_t + b u^2 u_x + u_{xxx} = 0,$$

models the dust-ion acoustic waves, electromagnetic waves in size-quantized films, ion acoustic solitons, traffic flow problems, and in other applications. The KdV equation and the modified KdV equation are completely integrable equations that have multiple-soliton solutions and possess infinite conservation quantities. The KdV equation is the earliest soliton equation that was firstly derived by Korteweg and de Vries to model the evolution of shallow water wave in 1895. In the study of the KdV equation, traveling wave solution leads to periodic solution which is called cnoidal wave solution [22, 23]. Exact solutions of KdV equation have been studied extensively since they were first found. Solitary wave solutions and periodic wave solutions were obtained for the KdV and modified KdV equations [3, 7, 22]. The JEF solutions to two kinds of KdV equations with variables coefficients have been constructed by using the method of the auxiliary equation [19]. The reductive perturbation method [24] has been employed to derive the KdV equation for small but finite amplitude electrostatic ion-acoustic waves [23, 25, 26]. The basic equations describing the system in dimensionless variables is studied by El-Labany [26] and the KdV equation for the first-order perturbed potential has been obtained using the reductive perturbation method.
We consider the combined KdV and mKdV equation [22, 27, 28]

$$u_t + \alpha u u_x + \beta u^2 u_x + \delta u_{xxx} = 0, \quad \beta \neq 0. \tag{2.1}$$

where α, β and δ are constants. Equation (2.1) is widely used in various fields such as quantum field theory, dust-acoustic waves, ion acoustic waves in plasmas with a negative ion, solid-state physics and fluid dynamics.
Let $u = u(\xi)$, equation (2.1) transformed to the reduced equation

$$-\omega u' + \alpha u u' + \beta u^2 u' + \delta k^2 u''' = 0. \tag{2.2}$$

Balancing u''' with u^2u' yields $N = 1$, so the F-expansion method gives

$$u(x,t) = a_0 + a_1 F(\xi). \tag{2.3}$$

Substituting (2.3) into (2.2) and equating the coefficients of like powers of $F(\xi)$ to zero, we obtain a set of algebraic equations. Solving these algebraic equations, we obtain the exact solutions of (2.1) as follows:
When $q_0 = 1,\ q_2 = -1 - m^2,\ q_4 = m^2$, solutions of Eq. (1.4) is $F(\xi) = \mathrm{sn}\xi$, we have

$$u = -\frac{\alpha}{2\beta} \pm k\sqrt{\frac{-6m^2\delta}{\beta}}\,\mathrm{sn}\Big(k(x + (\frac{\alpha^2 + 4\beta\delta k^2(m^2+1)}{4\beta})t) + \xi_0\Big), \tag{2.4}$$

If $q_0 = m^2 - 1,\ q_2 = 2 - m^2,\ q_4 = -1$, the solution of Eq (1.4) is $F(\xi) = \mathrm{dn}\xi$. Thus, we obtain the periodic wave solutions of Eq. (2.1)

$$u = -\frac{\alpha}{2\beta} \pm k\sqrt{\frac{6\delta}{\beta}}\,\mathrm{dn}(k(x + \frac{\alpha^2 - 4\beta\delta k^2(2-m^2)}{4\beta}\,t) + \xi_0), \tag{2.5}$$

Selecting the values of the q_0, q_2 and q_4 of equation (1.4) and the corresponding function F, we can construct various JEF solutions of (2.1). Other JEF solutions are omitted here for simplicity. If we put $\alpha = 0$ in (2.4), we get the periodic solution of the modified KdV equation which coincides with that given by Liu et al. [7]. Moreover, the solutions (2.5) to equation (2.1) given in [28] are recovered. With $m \longrightarrow 1$ in (2.4) , (2.5), the solitary wave solutions to (2.1) given in [7, 27, 28] are also recovered.
We notice that the solutions of the KdV equation cannot obtain from (2.4) and (2.5) as $\beta = 0$. In this case, the general form of cnoidal wave solutions of the KdV equation are given by

$$u(x,t) = -\frac{3\omega q_4}{\alpha\, q_2}\,F^2(\xi), \quad \xi = \sqrt{\frac{\omega}{4\delta q_2}}\,(x - \omega t) + \xi_0. \tag{2.6}$$

Thus we can obtain abundant cnoidal wave solutions of the KdV equation in terms of JEFs. Some periodic wave solutions of the KdV equation and modified KdV equation have been studied in [7,23, 28]. As $m \longrightarrow 1$, these solutions will degenerate into the corresponding solitary wave solutions.

3. The JEF solutions of Schamel- KdV equation

We consider the Schamel- KdV equation [29, 30]

$$u_t + (\alpha u^{1/2} + \beta u)u_x + \delta u_{xxx} = 0, \quad \beta \neq 0 \tag{3.1}$$

where α, β and δ are constants and u is the wave potential.
In order to find the periodic wave solution of (3.1), we use the transformations $u = v^2$, $v(x,t) = V(\xi);\ \xi = k(x - \omega t) + \xi_0$, then (2.7) becomes

$$-\omega V\,V' + (\alpha V^2 + \beta V^3)V' + \delta k^2[VV''' + 3V'V''] = 0. \tag{3.2}$$

The balancing procedure implies that $N = 1$. Therefore, the F-expansion method gives the solution

$$V(x,t) = V(\xi) = a_0 + a_1 F(\xi), \tag{3.3}$$

where a_0 and a_1 are constants to be determined and $F(\xi)$ is a solution of Eq. (1.4). Substituting Eq. (3.3) into Eq. (3.2) and equating the coefficients of the like powers of F to zero, yields a set of algebraic equations for a_0, a_1, k and ω:

$$\begin{aligned} &[\beta a_1^2 + 12\delta k^2 q_4]a_1 = 0, \\ &[\alpha a_1^2 + 3\beta a_0 a_1^2 + 6\delta k^2 a_0 q_4]a_1 = 0, \\ &[-\omega + 2\alpha a_0 + 3\beta a_0^2 + 4\delta k^2 q_2]a_1 = 0, \\ &[-\omega + \alpha a_0 + \beta a_0^2 + \delta k^2 q_2]a_0 = 0. \end{aligned} \tag{3.4}$$

Solving these algebraic equations, we gave a general form of traveling wave solutions of Eq. (3.1)

$$u = \frac{4\alpha^2}{25\beta^2}\left[1 \pm \sqrt{\frac{-2q_4}{q_2}}\,F(\xi)\right]^2. \tag{3.5}$$

Therefore, we obtained in [30] the JEF solutions of Eq. (3.1) as follows:
When $q_0 = 1,\ q_2 = -1 - m^2,\ q_4 = m^2$, solutions of Eq. (1.4) is $F(\xi) = \mathrm{sn}\xi$, we have

$$u_1 = \frac{4\alpha^2}{25\beta^2}\left[1 \pm \sqrt{\frac{2m^2}{m^2+1}}\mathrm{sn}\left(\frac{2\alpha}{5\sqrt{-6\delta\beta(m^2+1)}}(x + \frac{16\alpha^2}{75\beta}t) + \xi_0\right)\right]^2, \quad \beta\delta < 0, \tag{3.6}$$

If $q_0 = 1 - m^2,\ q_2 = 2m^2 - 1,\ q_4 = -m^2$, $F(\xi) = \mathrm{cn}\xi$, thus yields the exact solutions of Eq. (3.1)

$$u_2 = \frac{4\alpha^2}{25\beta^2}\left[1 \pm \sqrt{\frac{2m^2}{2m^2-1}}\,\mathrm{cn}\left(\frac{2\alpha}{5\sqrt{6\delta\beta(2m^2-1)}}(x + \frac{16\alpha^2}{75\beta}t) + \xi_0\right)\right]^2, \quad \beta\delta > 0, \tag{3.7}$$

If $q_0 = m^2 - 1,\ q_2 = 2 - m^2,\ q_4 = -1$, the solution of Eq (1.4) is $F(\xi) = \mathrm{dn}\xi$. So, we obtained the exact solutions of Eq. (3.1) in the form

$$u_3 = \frac{4\alpha^2}{25\beta^2}\left[1 \pm \sqrt{\frac{2}{2-m^2}}\,\mathrm{dn}\left(\frac{2\alpha}{5\sqrt{6\delta\beta(2-m^2)}}(x + \frac{16\alpha^2}{75\beta}t) + \xi_0\right)\right]^2, \quad \beta\delta > 0, \tag{3.8}$$

Many types of JEF solutions of Eq. (3.1) are given [30]. As $m \longrightarrow 1$, Eqs. (3.6)-(3.8) degenerate to

$$\begin{aligned} u_4 &= \frac{4\alpha^2}{25\beta^2}\left[1 \pm \tanh\left(\frac{\alpha}{5\sqrt{-3\delta\beta}}(x + \frac{16\alpha^2}{75\beta}t) + \xi_0\right)\right]^2, \quad \beta\delta < 0, \\ u_5 &= \frac{4\alpha^2}{25\beta^2}\left[1 \pm \sqrt{2}\,\mathrm{sech}\left(\frac{2\alpha}{5\sqrt{6\delta\beta}}(x + \frac{16\alpha^2}{75\beta}t) + \xi_0\right)\right]^2, \quad \beta\delta > 0, \end{aligned} \tag{3.9}$$

The solitary wave solutions (3.9) in terms of tanh are equivalent to the solutions given in [31]. The JEF solutions of (3.1) may be describe various features of waves and may be helpful in understanding the problems in ion acoustic waves.

4. Schamel equation and modified KP equation

The equation describing ion-acoustic waves in a cold-ion plasma where electrons do not behave isothermally during their passage of the wave is

$$u_t + u^{1/2} u_x + \delta u_{xxx} = 0. \tag{4.1}$$

Schamel [29] derived this equation and a simple solitary wave solution having a sech^4 profile was obtained. Therefore the Schamel equation (4.1) containing a square root nonlinearity is very attractive model for the study of ion-acoustic waves in plasmas and dusty plasmas.
In order to find the periodic wave solution of (4.1), we use the transformations
$u = v^2,\ v(x,t) = V(\eta);\ \eta = kx - \omega t + \eta_0$,, then (4.1) becomes

$$-\omega V V' + k V^2 V' + \delta k^3 [V V''' + 3V' V''] = 0. \tag{4.2}$$

According to the mapping method, we assume that Eq. (4.2) has the following solution:

$$V(\eta) = A_0 + A_1 f(\eta), \tag{4.3}$$

where A_0 and A_1 are constants to be determined and $f(\eta)$ satisfies Eq. (1.6).
Substitution of Eq. (4.3) into Eq. (4.2) and selecting the values of p, q and r, we have the solutions of Eq. (4.1) which was given in [20] as follows:
Case 1. $p = 2,\ q = -4(1+m^2),\ r = 6m^2$. In this case, we have $f(\eta) = \text{sn}^2\eta$. Thus the periodic wave solutions of Eq. (4.1) are

$$\begin{aligned} u_1(x,t) &= 100\delta^2 k^4 \left[1 + m^2 \pm \sqrt{1 - m^2 + m^4} - 3m^2 \,\text{sn}^2 \eta\right]^2, \\ \eta &= kx \mp 16\,\delta k^3 \sqrt{1 - m^2 + m^4}\, t + \eta_0. \end{aligned} \tag{4.4}$$

Case 2. $p = \frac{-(1-m^2)^2}{2},\ q = 2(1+m^2),\ r = \frac{-3}{2}$. The solutions of Eq. (1.6) are $f(\eta) = (m\,\text{cn}\eta \pm \text{dn}\eta)^2$. Thus the exact solutions of Eq. (4.1) are

$$\begin{aligned} u_2(x,t) &= \tfrac{25\delta^2 k^4}{4} \left[-2(1+m^2) \pm \sqrt{1 + 14m^2 + m^4} + 3\,(m\,\text{cn}\eta \pm \text{dn}\eta)^2\right]^2, \\ \eta &= kx \mp 4\,\delta k^3 \sqrt{1 + 14m^2 + m^4}\, t + \eta_0. \end{aligned} \tag{4.5}$$

Case 3. $p = \frac{m^2}{2},\ q = 2(m^2 - 2),\ r = \frac{3m^2}{2}$. The solutions of Eq. (1.6) are
$f(\eta) = \left(\frac{m\,\text{sn}\eta}{1 \pm \text{dn}\,\eta}\right)^2$. So, we obtained the exact solutions of Eq. (4.1)in the form

$$\begin{aligned} u_3(x,t) &= \tfrac{25\delta^2 k^4}{4} \left[2(2 - m^2) \pm \sqrt{16 - 16m^2 + m^4} - 3m^4 \left(\frac{\text{sn}\,\eta}{1 \pm \text{dn}\,\eta}\right)^2\right]^2, \\ \eta &= kx \mp 4\,\delta k^3 \sqrt{16 - 16m^2 + m^4}\, t + \eta_0. \end{aligned} \tag{4.6}$$

There are several exact solutions for the Eq. (4.1) which are omitted here for simplicity. As $m \to 1$, these solutions reduce to the solitary wave solutions

$$\begin{aligned} u_4(x,t) &= 900\delta^2 k^4 \,\text{sech}^4(kx - 16\,\delta k^3 t + \eta_0), \\ u_5(x,t) &= 100\delta^2 k^4 \,[2 - 3\,\text{sech}^2(kx + 16\,\delta k^3 t + \eta_0)]^2. \end{aligned} \tag{4.7}$$

$$u_6(x,t) = \frac{225\delta^2 k^4}{4}\left[1 - \left(\frac{\tanh(kx-4\delta k^3 t+\eta_0)}{1+\operatorname{sech}(kx-4\delta k^3 t+\eta_0)}\right)^2\right]^2. \tag{4.8}$$

The KdV equation in two dimensions, known as Kadomtsev Petviashivili (KP) equation [32], was derived for ion-acoustic waves in a non magnetized plasma by Kako and Rowlands [33]. Therefore the modified KP equation containing a square root nonlinearity is very attractive model for the study of ion-acoustic waves in plasma and dusty plasma [34- 36]. Extensive work has been devoted to the study of nonlinear waves associated with the dust ion-acoustic waves, particularly the dust ion-acoustic solitary and shock waves in dusty plasmas in which dust particles are stationary and provide only the neutrality [37]. The KP equation is derived [38] for the propagation of nonlinear waves in warm dusty plasmas with variable dust charge, two-temperature ions and nonthermal electrons by using the reductive perturbation theory.
Consider the modified KP equation

$$(u_t + \alpha u^{1/2} u_x + \beta u_{xxx})_x + \delta u_{yy} = 0, \tag{4.9}$$

where α and β are constants. The modified KP equation (4.9) for ion-acoustic waves in a multi species plasma consisting of non-isothermal electrons have been derived by Chakraborty and Das [34]. We applied the mapping method with the ansatz solution (4.3) and the solutions of auxiliary equation (1.6) to find the solutions of equation (4.9) (see [39]).

5. The ZK equation and modified ZK equation

The equation

$$u_t + \beta u^2 u_x + u_{xxx} + u_{yyx} = 0, \tag{5.1}$$

is the modified ZK in (2+1) dimensions which is a model for acoustic plasma waves [40, 41]. The ZK equation was first derived for describing weakly nonlinear ion- acoustic waves in a strongly magnetized lossless plasma in two dimension [41]. The ZK equation and modified ZK equation possess traveling wave structures [28, 42]. Peng [42] studied the exact solutions of ZK equation by using extended mapping method. Various types of solutions of Schamel-KdV equation and modified ZK equation arising in plasma and dust plasma are presented in [43].
We apply the F-expansion method to the modified ZK equation. Thus, Eq. (5.1) has a solution in the form

$$u(\xi) = a_0 + a_1 F(\xi), \quad \xi = k(x + ly - \omega t) + \xi_0.$$

Substituting this equation into Eq. (5.1), we obtain the following classes of exact solutions of the modified ZK equation:

$$\begin{aligned} u &= m\sqrt{\tfrac{6\omega}{(m^2+1)\beta}}\,\operatorname{sn}\Big(\sqrt{\tfrac{-\omega}{(m^2+1)(1+l^2)}}\,(x+ly-\omega t+\xi_0)\Big),\\ u &= m\sqrt{\tfrac{6\omega}{(2-m^2)\beta}}\,\operatorname{dn}\Big(\sqrt{\tfrac{\omega}{(2-m^2)(1+l^2)}}\,(x+ly-\omega t+\xi_0)\Big). \end{aligned} \tag{5.2}$$

In the following we apply the mapping method to the ZK equation

$$u_t + \alpha u\, u_x + u_{xxx} + u_{yyx} = 0. \tag{5.3}$$

In this case, we have $n = 1$. Thus Eq. (5.3) has a solution in the form

$$u(\eta) = A_0 + A_1 f(\eta), \quad \eta = kx + ly - \omega t + \eta_0.$$

Substituting this equation into Eq. (5.3) to determine A_0, A_1, k, ω and using the solutions of auxiliary equation (1.6), we obtained the following classes of exact solutions of the ZK equation [39]:

$$\begin{aligned} u_1(x,y,t) &= \tfrac{\omega}{k\,\alpha} + \tfrac{4(1+m^2)(l^2+k^2)}{\alpha} - \tfrac{12m^2(l^2+k^2)}{\alpha}\,\mathrm{sn}^2(\,kx+ly-\omega\,t+\eta_0), \\ u_2(x,y,t) &= \tfrac{\omega}{k\,\alpha} + \tfrac{4(m^2-2)(l^2+k^2)}{\alpha} + \tfrac{12(l^2+k^2)}{\alpha}\,\mathrm{dn}^2(\,kx+ly-\omega\,t+\eta_0), \end{aligned} \tag{5.4}$$

$$u_3 = \tfrac{\omega}{k\,\alpha} - \tfrac{2(1+m^2)(l^2+k^2)}{\alpha} + \tfrac{3(l^2+k^2)}{\alpha}\,[m\,\mathrm{cn}(\,\eta) \pm \mathrm{dn}(\,\eta)]^2, \tag{5.5}$$

$$u_4(x,y,t) = \tfrac{\omega}{k\,\alpha} - \tfrac{2(1+m^2)(l^2+k^2)}{\alpha} - \tfrac{3(1-m^2)(l^2+k^2)}{\alpha}\left(\frac{\mathrm{cn}(\,kx+ly-\omega\,t+\eta_0)}{1\pm\mathrm{sn}(\,kx+ly-\omega\,t+\eta_0)}\right)^2. \tag{5.6}$$

When $m \longrightarrow 1$, some of these solutions degenerate as solitary wave solutions of ZK equation. The solutions (5.3) are coincide with the solutions given in [44].

Recently, some properties of the quantum ion-acoustic waves were also investigated in dense quantum plasmas by studying the quantum hydrodynamical equations in different conditions, which includes the quantum Zakharov Kuznetsov equation, the extended quantum Zakharov Kuznetsov equation, and the quantum Zakharov system [45]. The three-dimensional extended quantum Zakharov Kuznetsov (QZK) equation [46] was investigated in dense quantum plasmas which arises from the dimensionless hydrodynamics equations describing the nonlinear propagation of the quantum ion-acoustic waves. The three-dimensional extended QZK equation was given in [46]

$$\Phi_t + (A\Phi + B\,\Phi^2)\Phi_x + C\,\Phi_{zzz} + D\,(\Phi_{xxz} + \Phi_{yyz}) = 0, \tag{5.7}$$

where A, B, C and D are constants. This equation has the following JEF solutions (see [45, 46]):

$$\begin{aligned} \Phi_1 &= -\tfrac{A}{2B} + m\,k\sqrt{\tfrac{-6E}{B}}\,\mathrm{sn}(\,k(x+ly+\gamma z-\omega\,t+\eta_0)), \quad \omega = -\tfrac{4BEk^2(1+m^2)+A^2}{4\,B}, \quad BE<0, \\ \Phi_2 &= -\tfrac{A}{2B} + mk\sqrt{\tfrac{6E}{B}}\,\mathrm{cn}(\,k(x+ly+\gamma z-\omega\,t+\eta_0)), \quad \omega = -\tfrac{4BEk^2(1-2m^2)+A^2}{4\,B}, \quad BE>0, \end{aligned} \tag{5.8}$$

with $E = C\gamma^2 + D(1+l^2)$. Moreover, many types of analytical solutions of the extended QZK equation are constructed in terms of some powerful ansatze, which include doubly periodic wave solutions, solitary wave solutions, kink-shaped wave solutions, rational wave solutions and singular solutions [46].

6. The modified fifth order KdV equation

Higher order KdV equations have many applications in different fields of mathematical physics. For example the fifth-order KdV equations can be derived in fluid dynamics and in magneto-acoustic waves in plasma and its exact solutions was given in [47-51]. The higher-order KdV equation can be derived for magnetized plasmas by using the reductive perturbation technique. Traveling wave solutions of Kawahara equation and modified Kawahara equation have been studied [9, 48, 49]. Moreover, the solitary wave solutions of nonlinear equations with arbitrary odd-order derivatives were studied by many authors [47, 51].

Consider the modified fifth order KdV equation

$$u_t + \beta u^2 u_x + c_3 u_{xxx} + c_5 u_{xxxxx} = 0, \tag{6.1}$$

where β, c_3 and c_5 are constants. Here, we review the exact traveling wave solutions of equation (6.1) using exact solutions of the auxiliary equation (1.5) and applied the mapping method. Thus, Eq. (6.1) has the solutions in the form

$$u(\eta) = A_0 + A_1 f(\eta), \quad \eta = kx - \omega t + \eta_0, \tag{6.2}$$

Substituting equation (6.2) into (6.1) and equating the coefficients of like powers of f to zero, yields a system of algebraic equations for A_0, A_1, k and ω and then solve it. Therefore, the solutions of the modified fifth order KdV equation (6.1) was given in [39] as follows:

$$\begin{aligned} u_1 &= \pm\frac{(10k^2c_5(1+m^2)+c_3)}{\sqrt{-10\beta c_5}} \mp k^2\sqrt{\frac{-90c_5}{4\beta}}\Big(m\,\mathrm{cn}\,\eta \pm \mathrm{dn}\,\eta\Big)^2, \\ \eta &= k\left[x + \frac{(15c_5^2k^4(m^4+14m^2+1)+c_3^2)}{10c_5}\,t\right] + \eta_0. \end{aligned} \tag{6.3}$$

If we choose $A_0 = 0$, equation (6.3) takes the form

$$\begin{aligned} u_2 &= \pm\frac{3c_3}{2(1+m^2)\sqrt{-10\beta c_5}}\Big(m\,\mathrm{cn}\,\eta \pm \mathrm{dn}\,\eta\Big)^2, \\ \eta &= \pm\sqrt{\frac{-c_3}{10c_5(1+m^2)}}\left[x + \frac{(23m^4+82m^2+23)\,c_3^2}{200c_5(1+m^2)^2}\,t\right] + \eta_0. \end{aligned} \tag{6.4}$$

Moreover, we have obtained the exact solutions

$$\begin{aligned} u_3 &= \pm\frac{3m^2c_3}{2(m^2-2)\sqrt{-10\beta c_5}}\left(\frac{m\,\mathrm{sn}\,\eta}{1\pm\mathrm{dn}\,\eta}\right)^2, \\ \eta &= \pm\sqrt{\frac{c_3}{10c_5(2-m^2)}}\left[x + \frac{(23m^4-128m^2+128)\,c_3^2}{200c_5(m^2-2)^2}\,t\right] + \eta_0, \end{aligned} \tag{6.5}$$

$$\begin{aligned} u_4 &= \pm\frac{3c_3}{2(1-2m^2)\sqrt{-10\beta c_5}}\left(\frac{\mathrm{sn}\,\eta}{1\pm\mathrm{cn}\,\eta}\right)^2, \\ \eta &= \pm\sqrt{\frac{-c_3}{10c_5(1-2m^2)}}\left[x + \frac{(128m^4-128m^2+23)\,c_3^2}{200c_5(1-2m^2)^2}\,t\right] + \eta_0. \end{aligned} \tag{6.6}$$

There are several other JEFs of Eq. (6.1) which are omitted here for simplicity. When $m \longrightarrow 1$, then (6.4)-(6.6) become the solitary wave solutions

$$u_5 = \pm\frac{3c_3}{\sqrt{-10\beta c_5}}\,\mathrm{sech}^2\Big(\frac{1}{2}\sqrt{\frac{-c_3}{5c_5}}\,(x + \frac{4\,c_3^2}{25c_5}\,t) + \eta_0\Big), \tag{6.7}$$

$$u_6 = \mp\frac{3c_3}{2\sqrt{-10\,\beta\,c_5}}\left(\frac{\tanh\eta}{1\pm\mathrm{sech}\,\eta}\right)^2, \quad \eta = \pm\sqrt{\frac{c_3}{10c_5}}\left[x + \frac{23\,c_3^2}{200c_5}\,t\right] + \eta_0. \tag{6.8}$$

We notice that Eq. (6.7) is the solution given by Example 2 in Ref. [47].
Finally, we can construct various types of exact and explicit solutions of the generalized ZK equation

$$u_t + (\alpha + \beta\,u^p)\,u^p u_x + u_{xxx} + \delta u_{yyx} = 0, \tag{6.9}$$

by using suitable method and using an appropriate transformation. Also, we can study the exact solution of the generalized KdV equation ($\delta = 0$) which studied by many authors [22, 23, 31]. The generalized ZK equation was first derived for describing weakly nonlinear ion-acoustic waves in strongly magnetized lossless plasma in two dimensions and governs the behavior of weakly nonlinear ion-acoustic waves in plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field. Eq. (6.9) includes considerable interesting equations, such as KdV equation, mKdV equation, ZK equation and mZK equation. Exact traveling wave solutions for the generalized ZK equation with higher-order nonlinear terms have obtained in [52-54]. Moreover, we can use the symbolic computations and apply the mapping method with the ansatz solution (1.5) to find the several classes of traveling wave solutions of the fifth order KdV equation

$$u_t + c_1 u\, u_x + c_2\, u_{xxx} + \delta u_{xxxxx} = 0.$$

This equation appears in the theory of shallow water waves with surface tension and the theory of magneto-acoustic waves in plasmas [9]. Wazwaz [55] studied soliton solutions of fifth-order KdV equation. We can use a suitable method to construct the exact solutions of some special types of nonlinear evolution equations aries in plasma physics such as Liouville, sine-Gordon and sinh-Poisson equations.

7. References

[1] G. Whitham, Linear and Nonlinear Waves, New York, Wiley (1974).

[2] R. Davidson, Methods in Nonlinear Plasma Theory, New York, Academic Press (1972).

[3] M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform, *Cambridge, Cambridge University Press* (1991).

[4] W. Malfliet, Solitary wave solutions of nonlinear wave equations, *Am. J. Phys.* 60 (1992) 650-654;
W. Malfliet, The tanh method: a tool for solving certain classes of nonlinear evolution and wave equations, *J. Comput. Appl. Math.* 164-156 (2004) 529-541.

[5] E.G. Fan, Extended tanh-function method and its applications to nonlinear equations, *Phys. Lett.* A 277 (2000) 212-218;
E.G. Fan and Y.C. Hong, Generalized tanh method to special types of nonlinear equations, *Z. Naturforsch.* A 57 (2002) 692-700.

[6] M. Wang, Exact solutions for a compound KdV - Burgers equation, *Phys. Lett.* A 213 (1996) 279-287.

[7] S. K. Liu, Z. T. Fu, S. D. Liu, and Q. Zhao, Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, *Phys. Lett. A* 289 (2001) 69-74;

[8] Z.T.Fu, S. K. Liu, S. D. Liu, and Q. Zhao, New Jacobi elliptic function expansion method and new periodic solutions of nonlinear wave equations, *Phys. Lett. A* 290 (2001) 72-76.

[9] E.J. Parkes, B.R. Duffy and P.C. Abbott, The Jacobi elliptic-function method for finding periodic wave solutions to nonlinear evolution equations, *Phys. Lett.* A 295 (2002) 280-286.

[10] H. T. Chen and H. Q. Zhang, Improved Jacobin elliptic method and its applications, *Chaos, Solitons and Fractals* 15 (2003) 585-591.

[11] Y. Peng, Exact periodic wave solutions to a new Hamiltonian amplitude equation, *J. Phys. Soc. Japan* 72 (2003) 1356-1359;
Y. Peng, New exact solutions to a new Hamiltonian amplitude equation II, *J. Phys. Soc. Japan* 73 (2004) 1156-1158.

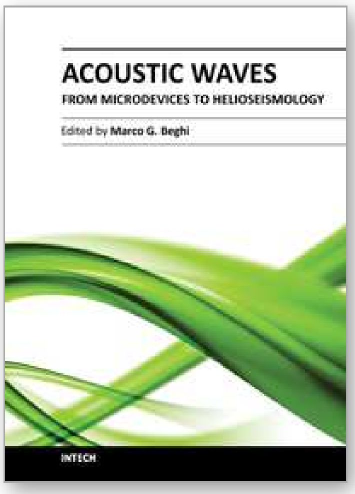
[12] Y. Peng, Exact periodic wave solutions to the Melnikov equation, *Z. Naturforsch A* 60 (2005) 321-327.

[13] Y. B. Zhao, M. L. Wang and Y. M. Wang, Periodic wave solutions to a coupled KdV equations with variable coefficients, *Phys. Lett. A* 308 (2003) 31-36.

[14] J. Liu and K. Yang, The extended F-expansion method and exact solutions of nonlinear PDEs, *Chaos, Solitons and Fractals* 22 (2004) 111-121.

[15] H. T. Chen and H. Q. Zhang, New double periodic and multiple soliton solutions of the generalized (2+1)-dimensional Boussinesq equation, *Chaos, Solitons and Fractals* 20 (2004) 765-769.
[16] S. A. Elwakil, S. K. El-labany, M. A. Zahran and R. Sabry, Modified extended tanh-function method for solving nonlinear partial differential equations, *Phys. Lett. A.* 299 (2002) 179-188.
[17] M. A. Abdou and S. Zhang, New periodic wave solutions via extended mapping method, *Commun. Nonlinear Sci. Numer. Simul.* 14 (2009) 2-11.
[18] D. Baldwin, Ü. Göktas, W. Hereman et al., Symbolic computions of exact solutions expressible in hyperbolic and elliptic functions for nonlinear PDEs, *J. Symb. Comput.* 37 (2004) 669-705.
[19] Taogetusang and Sirendaoerji, The Jacobi elliptic function-like exact solutions to two kinds of KdV equations with variable coefficients and KdV equation with forcible term, *Chinese Phys.* 15 (2006) 2809-2818.
[20] A. H. Khater, M. M. Hassan, E. V. Krishnan and Y.Z. Peng, Applications of elliptic functions to ion-acoustic plasma waves, *Eur. Phys. J. D* 50 (2008) 177-184.
[21] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, *Dover, New York* (1965).
[22] P. G. Drazin and R. S. Johnson, Solitons,: An Introduction, *Cambridge University press, Cambridge* (1989).
[23] A. Jeffrey and T. Kakutani, Weak nonlinear dispersive waves: A discussion centered around the Korteweg de Vries equation, *SIAM Rev.* 14 (1972) 582-643.
[24] H. Washimi and T. Taniuti, Propagation of ion acoustic solitary waves of small amplitude, J Phys Rev Lett. 17 (1966) 996 -998.
[25] E.K. El-Shewy, H.G. Abdelwahed and H.M. Abd-El-Hamid, Computational solutions for the KortewegŰdeVries equation in warm Plasma, Computat. Methods in Sci. Technology 16 (2010) 13-18.
[26] S. K. El-Labany, Contribution of higher-order nonlinearity to nonlinear ion acoustic waves in a weakly relativistic warm plasma. Part1. isothermal case, J. Plasma Phys. 50 (1993) 495 .
[27] J. F. Zhang, New solitary wave solution of the combined KdV and m KdV equation, *Int. J. Theoret. Phys.* 37 (1998) 1541-1546.
[28] A. H. Khater and M. M. Hassan, Travelling and periodic wave solutions of some nonlinear wave equations, *Z. Naturforsch.* 59 a (2004) 389-396.
[29] H. Schamel, A modified Korteweg de Vries equation for ion acoustic waves due to resonant electrons, *J. Plasma Phys.* 9 (1973) 377-387.
[30] A. H. Khater, M. M. Hassan and R. S. Temsah, Exact solutions with Jacobi elliptic functions of two nonlinear models for ion-acoustic plasma waves, *J. Phys. Soc. Japan* 74 (2005) 1431-1435.
[31] M. M. Hassan, Exact solitary wave solutions for a generalized KdV - Burgers equation, *Chaos, Solitons and Fractals* 19 (2004) 1201-1206.
[32] B. B. Kadomtsev and V. I. Petviashvili, On the stability of solitary in weakly dispersive media, *Soviet Phys. Dokl.* 15 (1970) 539-541.
[33] M. Kako and G. Rowlands, Two-dimensional stability of ion acoustic solitons, *Plasma Phys.* 18 (1976) 165-170.
[34] D. Chakraborty and K. P. Das, Stability of ion acoustic solitons in a multispecies plasma consisting of non-isothermal electrons, *J. Plasma Phys.* 60 (1998) 151-158.
[35] A. H. Khater, A. A. Abdallah, O. H. El-Kalaaway and D. K. Callebaut, Backlund transformations, a simple transformation and exact solutions for dust-acoustic solitary

waves in dusty plasma consisting of cold dust particles and two-temperature isothermal ions, *Phys. Plasmas* 6 (1999) 4542-4547.
[36] A. H. Khater and M. M. Hassan, Exact Jacobi elliptic function solutions for some special types of nonlinear evolution equations, *Il Nuovo Cimento* 121 B (2006) 613- 622.
[37] A. A. Mamun and P.K. Shukla, Cylindrical and spherical dust ion-acoustic solitary waves, *Phys. Plasmas 9* (2002) 1468.
[38] H.R. Pakzad, Soliton energy of the Kadomtsev Petviashvili equation in warm dusty plasma with variable dust charge, two-temperature ions, and nonthermal electrons, *Astrophys Space Sci.* 326 (2010) 69 -75.
[39] A. H. Khater, M. M. Hassan and D. K. Callebaut, Travelling wave solutions to some important equations of mathematical physics, *Reports on Math. Phys.* 66 (2010) 1-19.
[40] K. P. Das and F. Verheest, Ion acoustic solitons in magnetized multi-component plasmas including negative ions, *J. Plasma. Phys.* 41 (1989) 139-155.
[41] V. E. Zakharov and E. A. Kuznetsov, On three-dimensional solitons, *Sov. Phys. JETP* 39 (1974) 285-286.
[42] Y.Z. Peng, Exact travelling wave solutions of the Zakharov Kuznetsov equation, *Appl. Math. Comput.* 199 (2008) 397- 405.
[43] M. M. Hassan, New exact solutions of two nonlinear physical models, *Commun. Theor. Phys.* 53 (2010) 596-604.
[44] M. M. Hassan, Exact solutions for some models of nonlinear evolution equations, *J. Egypt Math. Soc.* 12 (2004) 31- 43.
[45] R. Sabry, W.M. Moslem, F. Haas, S. Ali, P.K. Shukla, Nonlinear structures: Explosive, soliton and shock in a quantum electron- positron-ion magentoplasma, Phys. Plasmas 15 (2008) 122308.
[46] Z. Yan, Periodic, Solitary and rational wave solutions of the 3D extended quantum Zakharov Kuznetsov equation in dense quantum plasmas, *Phys. Let.* A 373 (2009) 2432-2437.
[47] E.J. Parkes, Z. Zhu, B.R. Duffy and H. C. Hang, Sech-polynomail traveling solitary-wave solutions of odd-order generalized KdV equations, *Phys. Lett.* A 248 (1998) 219-224.
[48] Z.J. Yang, Exact solitary wave solutions to a class of generalized odd-order KdV equations, *Int. J. Theor. Phys.* 34 (1995) 641-647.
[49] D. Zhang, Doubly periodic solution of modified Kawahara equation, *Chaos, Solitons and Fractals* 25 (2005) 1155-1160.
[50] A. H. Khater, M. M. Hassan and R. S. Temsah, Cnoidal wave solutions for a class of fifth-order KdV equations, *Math. Comput. Simulation* 70 (2005) 221-226.
[51] J. Sarma, Solitary wave solution of higher-order Korteweg de Vries equation, *Chaos, Solitons and Fractals* 39 (2009) 277 - 281.
[52] L.- H. Zhang, Travelling wave solutions for the generalized Zakharov-Kuznetsov equation with higher-order nonlinear terms, *Appl. Math. Comput.* 208 (2009) 144-155.
[53] C. Deng, New exact solutions to the Zakharov -Kuznetsov equation and its generalized form, *Commun. Nonlinear Sci. Numer. Simulat.* 15 (2010) 857 - 868.
[54] SUN Yu-Huai, MA Zhi-Min and LI Yan, Explicit solutions for generalized (2+1)-dimensional nonlinear Zakharov- Kuznetsov equation, *Commun. Theor. Phys.* 54 (2010) 397-400.
[55] A.-M., Wazwaz, Soliton solutions for the fifth-order KdV equation and the Kawahara equation with time-dependent coefficients, *Phys. Scr.* 82 (2010) 035009.



Acoustic Waves - From Microdevices to Helioseismology

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The concept of acoustic wave is a pervasive one, which emerges in any type of medium, from solids to plasmas, at length and time scales ranging from sub-micrometric layers in microdevices to seismic waves in the Sun's interior. This book presents several aspects of the active research ongoing in this field. Theoretical efforts are leading to a deeper understanding of phenomena, also in complicated environments like the solar surface boundary. Acoustic waves are a flexible probe to investigate the properties of very different systems, from thin inorganic layers to ripening cheese to biological systems. Acoustic waves are also a tool to manipulate matter, from the delicate evaporation of biomolecules to be analysed, to the phase transitions induced by intense shock waves. And a whole class of widespread microdevices, including filters and sensors, is based on the behaviour of acoustic waves propagating in thin layers. The search for better performances is driving to new materials for these devices, and to more refined tools for their analysis.

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