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Non-Linear Radiative-Conductive Heat Transfer in a Heterogeneous Gray Plane-Parallel Participating Medium

Marco T.M.B. de Vilhena, Bardo E.J. Bodmann and Cynthia F. Segatto
Universidade Federal do Rio Grande do Sul
Brazil

1. Introduction

Radiative transfer considers problems that involve the physical phenomenon of energy transfer by radiation in media. These phenomena occur in a variety of realms (Ahmad & Deering, 1992; Tsai & Ozişik, 1989; Wilson & Sen, 1986; Yi et al., 1996) including optics (Liu et al., 2006), astrophysics (Pinte et al., 2009), atmospheric science (Thomas & Stamnes, 2002), remote sensing (Shabanov et al., 2007) and engineering applications like heat transport by radiation (Brewster, 1992) for instance or radiative transfer laser applications (Kim & Guo, 2004). Furthermore, applications to other media such as biological tissue, powders, paints among others may be found in the literature (see ref. (Yang & Kruse, 2004) and references therein). Although radiation in its basic form is understood as a photon flux that requires a stochastic approach taking into account local microscopic interactions of a photon ensemble with some target particles like atoms, molecules, or effective micro-particles such as impurities, this scenario may be conveniently modelled by a radiation field, i.e. a radiation intensity, in a continuous medium where a microscopic structure is hidden in effective model parameters, to be specified later. The propagation of radiation through a homogeneous or heterogeneous medium suffers changes by several isotropic or non-isotropic processes like absorption, emission and scattering, respectively, that enter the mathematical approach in form of a non-linear radiative transfer equation. The non-linearity of the equation originates from a local thermal description using the Stefan-Boltzmann law that is related to heat transport by radiation which in turn is related to the radiation intensity and renders the radiative transfer problem a radiative-conductive one (Ozisik, 1973; Pomraning, 2005). Here, local thermal description means, that the domain where a temperature is attributed to, is sufficiently large in order to allow for the definition of a temperature, i.e. a local radiative equilibrium.

The principal quantity of interest is the intensity I, that describes the radiation energy flow through an infinitesimal oriented area $d\hat{\Sigma} = \hat{n}d\Sigma$ with outward normal vector $\hat{n}$ into the solid angle $d\hat{\Omega} = \hat{\Omega}d\Omega$, where $\hat{\Omega}$ represents the direction of the flow considered, with angle θ of the normal vector and the flow direction $\hat{n} \cdot \hat{\Omega} = \cos\theta = \mu$. In the present case we focus on the non-linearity of the radiative-conductive transfer problem and therefore introduce the simplification of an integrated spectral intensity over all wavelengths or equivalently all frequencies that contribute to the radiation flow and further ignore possible effects due to polarization. Also possible effects that need in the formalism properties such as coherence

and diffraction are not taken into account. In general the Radiative-Conductive Transfer Equation is difficult to solve without introducing some approximations, like linearisation or a reduction to a diffusion like equation, that facilitate the construction of a solution but at the cost of predictive power in comparison to experimental findings, or more sophisticated approaches. The present approach is not different in the sense that approximations shall be introduced, nevertheless the non-linearity that represents the crucial ingredient in the problem is solved without resorting to linearisation or perturbation like procedures and to the best of our knowledge is the first approach of its kind. The solution of the modified or approximate problem can be given in closed analytical form, that permits to calculate numerical results in principle to any desired precision. Moreover, the influence of the non-linearity can be analysed in an analytical fashion directly from the formal solution. Solutions found in the literature are typically linearised and of numerical nature (see for instance (Asllanaj et al., 2001; 2002; Attia, 2000; Krishnapraka et al., 2001; Menguc & Viskanta, 1983; Muresan et al., 2004; Siewert & Thomas, 1991; Spuckler & Siegel, 1996) and references therein). To the best of our knowledge no analytical approach for heterogeneous media and considering the non-linearity exists so far, that are certainly closer to realistic scenarios in natural or technological sciences. A possible reason for considering a simplified problem (homogeneous and linearised) is that such a procedure turns the determination of a solution viable. It is worth mentioning that a general solution from an analytical approach for this type of problems exists only in the discrete ordinate approximation and for homogeneous media as reported in reference (Segatto et al., 2010).

Various of the initially mentioned applications allow to segment the medium in plane parallel sheets, where the radiation field is invariant under translation in directions parallel to that sheet. In other words the only spatial coordinate of interest is the one perpendicular to the sheet that indicates the penetration depth of the radiation in the medium. Frequently, it is justified to assume the medium to have an isotropic structure which reduces the angular degrees of freedom of the radiation intensity to the azimuthal angle θ or equivalently to its cosine μ. Further simplifications may be applied which are coherent with measurement procedures. One the one hand measurements are conducted in finite time intervals where the problem may be considered (quasi-)stationary, which implies that explicit time dependence may be neglected in the transfer equation. On the other hand, detectors have a finite dimension (extension) with a specific acceptance angle for measuring radiation and thus set some angular resolution for experimental data. Such an uncertainty justifies to segment the continuous angle into a set of discrete angles (or their cosines), which renders the original equation with angular degrees of freedom a set of equations known as the S_N approximation to be introduced in detail in section 3.

Our chapter is organised as follows: in the next section we motivate the radiative-conductive transfer problem. Sections 3 and 4 are dedicated to the hierarchical construction procedure of analytical solutions for the heterogeneous radiative-conductive transfer problem from its reduction to the homogeneous case, using two distinct philosophies. In section 4.3 we apply the method to specific cases and present results. Last, we close the chapter with some remarks and conclusions.

2. The radiative conductive transfer problem

In problems of radiative transfer in plane parallel media it is convenient to measure linear distances normal to the plane of stratification using the concept of optical thickness τ which is measured from the boundary inward and is related through the density ρ, an attenuation

coefficient κ and the geometrical projection on the direction perpendicular to that plane, say along the z-axis, so that $d\tau = -\kappa\rho dz$. Further the temperature is measured in multiples of a reference temperature $T(\tau) = \Theta(\tau)T_r$, typically taken at $\tau = 0$.

Based on the photon number balance and in the spirit of a Boltzmann type equation one arrives at the radiative transfer equation in a volume that shall be chosen in a way so that no boundaries that separate media with different physical properties cross the control volume. To this end, five photon number changing contributions shall be taken into account which may be condensed into the four terms that follow. The first term describes the net rate of streaming of photons through the bounding surface of an infinitesimal control volume, the second term combines absorption and out-scattering from μ to all possible directions μ' in the control volume. The third term contemplates in-scattering from all directions μ' into the direction μ, and last not least a black-body like emission term according to the temperature dependence of Stefan-Boltzmann's law for the control volume.

$$\frac{dI(\tau,\mu)}{d\tau} + \frac{1}{\mu}I(\tau,\mu) = \frac{\omega(\tau)}{2\mu}\int_{-1}^{1}\mathcal{P}(\mu')I(\tau,\mu')\,d\mu' + \frac{1-\omega(\tau)}{\mu}\Theta^4(\tau) \tag{1}$$

Here, ω is the single scattering albedo and $\mathcal{P}(\mu)$ signifies the differential scattering coefficient or also called the phase function, that accounts for the rate at which photons are scattered into an angle $d\mu'$ and with inclination μ with respect to the normal vector of the sheet. Note, that the phase function is normalized $\frac{1}{2}\int\mathcal{P}(\mu)\,d\mu = 1$.

Upon simplifying the phase function in plane geometry one may expand the angular dependence in Legendre Polynomials $P_n(\mu)$,

$$\mathcal{P}(\mu,\mu') = \sum_{\ell=0}^{\infty}\beta_n P_n(\mu' - \mu)\,,$$

with β_n the expansion coefficients that follow from orthogonality. Further one may employ the addition formula for Legendre polynomials using azimuthal symmetry (hence the zero integral)

$$P_\ell(\mu' - \mu) = P_\ell(\mu)P_\ell(\mu') + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\mu)P_n^m(\mu')\underbrace{\int_0^{2\pi}\cos(m(\phi - \phi'))\,d\phi'}_{=0}\,,$$

and write the integral on the right hand side of side of equation (1) as

$$\int_{-1}^{1}\mathcal{P}(\mu,\mu')I(\tau,\mu')\,d\mu' = \sum_{\ell=0}^{\infty}\beta_\ell\int_{-1}^{1}P_\ell(\mu)P_\ell(\mu')I(\tau,\mu')\,d\mu'\,,$$

where the summation index refers to the degree of anisotropy. For practical applications only a limited number of terms indexed with ℓ have to be taken into account in order to characterise qualitatively and quantitatively the anisotropic contributions to the problem. Also higher ℓ terms oscillate more significantly and thus suppress the integral's significance in the solution. The degree of anisotropy may be indicated truncating the sum by an upper limit L. The integro-differential equation (1) together with the afore mentioned manipulations may be cast into an approximation known as the S_N equation upon reducing the continuous angle cosine to a discrete set of N angles. This procedure opens a pathway to apply standard vector algebra techniques to obtain a solution from the equation system, discussed in detail in section 3.

In order to define boundary conditions we have to specify in more details the scenario in consideration. In the further we analyse non-linear radiative-conductive transfer in a grey plane-parallel participating medium with opaque walls, where specular (mirror like) as well as diffuse reflections occur besides thermal photon emission according to the Stefan-Boltzmann law (see (Elghazaly, 2009) and references therein). If one thinks the medium being subdivided into sheets of thickness $\Delta\tau$ with sufficiently small depth so that for each sheet a homogeneous medium applies, than for each face or interface the condition for the top boundary (at $\tau = \tau_i$) is

$$I(\tau,\mu) = \epsilon(\tau)\Theta^4(\tau) + \rho^s(\tau)I(\tau,-\mu) + 2\rho^d(\tau)\int_0^1 I(\tau,-\mu')\mu'\,d\mu' , \qquad (2)$$

with ρ^s and ρ^d the specular and diffuse reflections at the boundary, which are related to the emissivity ϵ by $\epsilon + \rho^s + \rho^d = 1$. For the limiting bottom boundary ($\tau = \tau_i + \Delta\tau$) μ and μ' change their sign in the argument of $I(\tau,\mu^{(\prime)})$ in equation (2). Suppose we have N_S sheets and $N_S + 1$ boundaries, one might think that for a first order differential equation (1) in τ the supply of $N_S + 1$ boundary conditions results in an ill-posed problem with no solutions at all. However, we still have to set up an equation that uniquely defines the non-linearity in terms of the radiation intensity.

The relation may be established in two steps, first recognizing that the dimensionless radiative flux is expressed in terms of the intensity by

$$q_r^* = 2\pi \int_{-1}^1 I(\tau,\mu)\mu\,d\mu , \qquad (3)$$

and the energy equation for the temperature that connects the radiative flux to a temperature gradient is

$$\frac{d^2}{d\tau^2}\Theta(\tau) = \frac{1}{4\pi N_c}\frac{d}{d\tau}q_r^*(\tau) = \frac{1}{4\pi N_c}\frac{d}{d\tau}\left(2\pi\int_{-1}^1 I(\tau,\mu)\mu\,d\mu\right) . \qquad (4)$$

Here N_c is the conduction-radiation parameter, defined as

$$N_c = \frac{k\beta_{ext}}{4\sigma\mathfrak{n}^2 T_r^3} , \qquad (5)$$

with k the thermal conductivity, β_{ext} the extinction coefficient, σ the Stefan-Boltzmann constant and $\mathfrak{n}$ the refractive index. Note that the radiative flux results from the integration of the intensity over angular variables, so that the thermal conductivity is considered here isotropic. Equation (4) is subject to prescribed temperatures at the top- and bottommost boundary

$$\Theta(0) = \Theta_T \qquad \text{and} \qquad \Theta(\tau_0) = \Theta_B . \qquad (6)$$

3. The S_N approximation for the heterogeneous problem

The set of equations (1) and (4), that are continuous in the angle cosine, may be simplified using an enumerable set of discrete angles following the collocation method, that defines the

radiative convective transfer problem in the S_N approximation

$$\frac{dI_n(\tau)}{d\tau} + \frac{1}{\mu_n}I_n(\tau) = \frac{\omega(\tau)}{2\mu_n}\sum_{\ell=0}^{L}\beta_\ell P_\ell(\mu_n)\sum_{k=1}^{N}w_k P_\ell(\mu_k)I_k(\tau) + \frac{1-\omega(\tau)}{\mu_n}\Theta^4(\tau)\,, \quad (7)$$

$$\frac{d\Theta(\tau)}{d\tau} - \left.\frac{d\Theta(\tau)}{d\tau}\right|_{\tau=0} = \frac{1}{2N_c}\sum_{k=1}^{N}w_k\left(I_k(\tau) - I_k(0)\right)\mu_k\,, \quad (8)$$

for $n = 1, \ldots, N$ and are subject to the following boundary conditions.

$$I_n(0) = \epsilon(0)\Theta^4(0) + \rho^s(0)I_{N-n+1}(0) + 2\rho^d(0)\sum_{k=1}^{\frac{N}{2}}w_k I_{N-k+1}(0)\mu_k$$

$$I_{N-n+1}(\tau_0) = \epsilon(\tau_0)\Theta^4(\tau_0) + \rho^s(\tau_0)I_n(\tau_0) + 2\rho^d(\tau_0)\sum_{k=1}^{\frac{N}{2}}w_k I_k(\tau_0)\mu_k \quad (9)$$

Note, that the integrals over the angular variables are replaced by a Gaussian quadrature scheme with weight factors w_k, where k refers to one of the discrete directions μ_k.

3.1 The S_N approach in matrix representation

For convenience we introduce a shorthand notation in matrix operator form, where the column vector

$$\mathbf{\Phi}(\tau) = (\mathbf{I}, \Theta(\tau))^T = (I_1(\tau), \cdots, I_N(\tau), \Theta(\tau))^T$$

combines the anisotropic intensities and the isotropic temperature function, the non-linear terms and boundary terms from integration (i.e. the temperature gradient and the conduction radiation intensity at $\tau = 0$) are absorbed in an inhomogeneity

$$\mathbf{\Psi} = \left(\frac{1-\omega(\tau)}{\mu_1}\Theta^4(\tau), \cdots, \frac{1-\omega(\tau)}{\mu_n}\Theta^4(\tau), \frac{d\Theta}{d\tau}(0) - \frac{1}{2N_c}\sum_{k=1}^{N}w_k I_k(0)\mu_k\right)^T$$

which allows to cast the equation system (7) and (8) in compact form

$$\frac{d}{d\tau}\mathbf{\Phi} - \mathcal{L}_M\mathbf{\Phi} = \mathbf{\Psi} \quad (10)$$

where $\mathcal{L}_M$ has the following elements.

$$(\mathcal{L}_M)_{nk} = \delta_{nk}(1 - \delta_{n,N+1})\frac{1}{\mu_n} + f_{nk} \qquad \text{for} \quad n, k = 1, \ldots, N+1 \quad (11)$$

Here, δ_{ij} is the Kronecker delta, θ_H the Heaviside functional

$$\delta_{ij} = \begin{cases} 1 \text{ for } i = j \\ 0 \text{ else} \end{cases}, \qquad \theta_H(x) = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ else} \end{cases}$$

and the factors f_{nk} are

$$f_{nk} = \theta_H(N - n + 1/2)\theta_H(N - k + 1/2)\frac{\omega(\tau)}{2\mu_n}\sum_{\ell=0}^{L}\beta_\ell P_\ell(\mu_n)w_k P_\ell(\mu_k)$$

$$+ (1 - \delta_{k,N+1})\delta_{n,N+1}\frac{\mu_k}{2N_c}\,. \quad (12)$$

Note, that the increment $1/2$ in the Heaviside functional was introduced merely to make the argument positive definite in the range of interest which otherwise could lead to conflicts with possible definitions for $\theta_H(x)$ at $x = 0$.

The boundary conditions are combined accordingly, except for the limiting temperatures (equation (6)) that are kept separately for simplicity because they would add only an additional diagonal block leading to a reducible representation and thus this does not bring any advantage.

$$\mathcal{B}_D \mathbf{I} - \mathcal{B}_M \mathbf{I} = \mathbf{\Gamma} \tag{13}$$

Equation (13) has a block form where one block represents forward angle contributions $\mu > 0$ and the other one backward terms $\mu < 0$ originating from the top and bottom boundary, respectively. Here, $\mathcal{B}_D$ is the $N \times N$ unit matrix, and

$$\mathcal{B}_M = \begin{pmatrix} 0 & \rho^s \mathcal{C}_{N/2} + 2\rho^d \mathcal{G}_{N/2}^- \\ \rho^s \mathcal{C}_{N/2} + 2\rho^d \mathcal{G}_{N/2}^+ & 0 \end{pmatrix} \tag{14}$$

with $\mathcal{C}_{N/2}$ an $N/2 \times N/2$ matrix which results from column reversion in the unit matrix, i.e. after mapping column position k to position $N/2 - k + 1$. The remaining matrices that control the diffuse forward and backward reflection ($\mathcal{G}_{N/2}^\pm$), respectively have the elements

$$\left(\mathcal{G}_{N/2}^+\right)_{nk} = \theta_H(N/2 - n + 1/2)\theta_H(k - N/2 - 1/2)\mu_{N-k+1} w_{N-k+1}$$

$$\left(\mathcal{G}_{N/2}^-\right)_{nk} = \theta_H(n - N/2 - n - 1/2)\theta_H(N/2 - k + 1/2)\mu_k w_k \,. \tag{15}$$

In these expressions the Heaviside functions restrict the non-zero elements to the off-diagonal blocks with row indices $n \in \{1, \ldots, N/2\}$ and column indices $k \in \{N/2 + 1, \ldots, N\}$ and with row indices $n \in \{N/2 + 1, \ldots, N\}$ and column indices $k \in \{1, \ldots, N/2\}$, respectively. The vector representation for the intensity is

$$\mathbf{I} = (\mathbf{I}_+, \mathbf{I}_-)^T \quad \text{with} \quad \mathbf{I}_+ = (I_1(\tau), \ldots, I_{N/2}(\tau)) \quad \text{and} \quad \mathbf{I}_- = (I_{N/2+1}(\tau), \ldots, I_N(\tau)) \,.$$

The inhomogeneity $\mathbf{\Gamma}$ has the same emission term in each component.

$$\Gamma_n = \epsilon(\tau)\Theta^4(\tau) \qquad \forall n$$

3.2 Constructing the solution by the decomposition method

The principal difficulty in constructing a solution for the radiative conductive transfer problem in the S_N approximation (10) subject to the boundary conditions (13) and (6) is due to the fact that the single scattering albedo $\omega(\tau)$, the emissivity $\epsilon(\tau)$ and the specular and diffuse reflection ($\rho^s(\tau)$ and $\rho^d(\tau)$) have an explicit dependence on the optical depth τ, that is the heterogeneity of the medium in consideration. It is worth mentioning that the proposed methodology is quite general in the sense that it can be applied to other approximations of equation (1) that make use of spectral methods, as for instance the spherical harmonic P_N-, the Chebychev Ch_N- and the Walsi W_N-approximation (Vilhena & Segatto, 1999; Vilhena et al., 1999), among others.

In the sequel we report on two approaches to solve the heterogeneous problem (equations. (10), (13), (6)). The principal idea of this techniques relies on the reduction of the Radiative Conductive transfer problem in heterogeneous media to a set of problems in domains of homogeneous media. In the first approach we consider the standard approximation of the
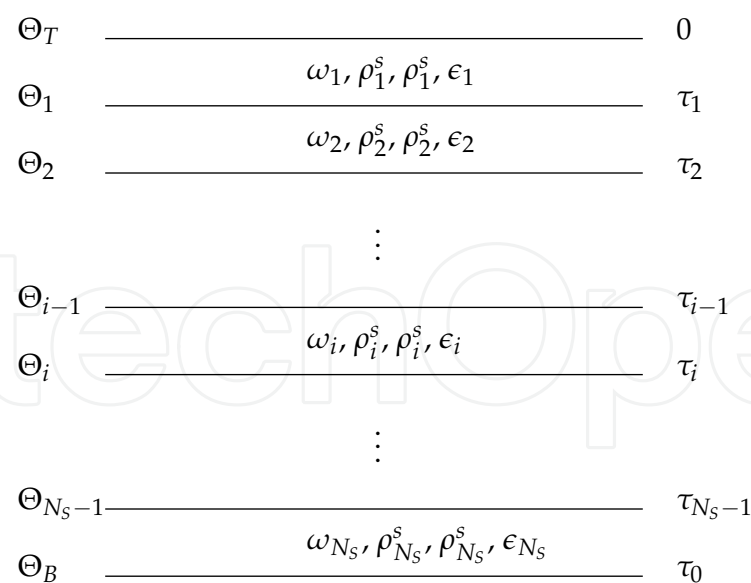
Fig. 1. Schematic illustration of a heterogeneous medium in form of a multi-layer slab.

heterogeneous medium in form of a multi-layer slab (see figure 1). For each of the layers the problem reduces to a homogeneous problem but with the same number of boundary conditions as the original problem. The procedure that determines the solution for each slab is presented in detail in section 4. In order to solve the unknown boundary values of the intensities and the temperatures at the interfaces between the slabs, matching these quantities using the bottom boundary values of the upper slab and the top boundary values of the lower slab eliminates these incognitos.

In the second approach we introduce a new procedure to work the heterogeneity. To begin with, we take the averaged value for the albedo coefficient $\omega(\tau)$,

$$\bar{\omega} = \frac{1}{\tau_0} \int_0^{\tau_0} \omega(\tau) \, d\tau \tag{16}$$

and rewrite the problem as a homogeneous problem plus an inhomogeneous correction. Note that $\mathcal{L}_M$ as well as Ψ depend on the local albedo coefficient $\omega(\tau)$. Since the terms containing the coefficient are linear in ω permits to separate an average factor $\bar{\omega}$ and the difference $\omega(\tau) - \bar{\omega}$.

$$\frac{d}{d\tau}\Phi - \mathcal{L}_M(\bar{\omega})\Phi = \Psi(\bar{\omega}) + \mathcal{L}_M(\omega(\tau) - \bar{\omega})\Phi + \Psi(\omega(\tau) - \bar{\omega}) \tag{17}$$

Now, following the idea of the Decomposition method proposed originally by Adomian (Adomian, 1988), to solve non-linear problems without linearisation, we handle equation (17), constructing the following recursive system of equations. Here, $\Psi = \sum_{m=0}^{\infty} \Psi_m$ is a formal decomposition and the non-linearity is written in terms of the so-called Adomian polynomials $\Theta^4(\tau) = \sum_{m=0}^{\infty} \hat{A}_m(\tau)$. The first equation of the recursive system is the same as in a homogeneous slab, and the influence of the heterogeneity is governed by the source term. The homogeneous problem is explicitly solved in section 4 so that we concentrate here

on the inhomogeneity.

$$\frac{d}{d\tau}\mathbf{\Phi}_0 - \mathcal{L}_M(\bar{\omega})\mathbf{\Phi}_0 = \mathbf{\Psi}_0(\bar{\omega})$$

$$\frac{d}{d\tau}\mathbf{\Phi}_i - \mathcal{L}_M(\bar{\omega})\mathbf{\Phi}_i = \mathbf{\Psi}_i(\bar{\omega}) + \mathcal{L}_M(\omega(\tau) - \bar{\omega})\mathbf{\Phi}_{i-1} + \mathbf{\Psi}_{i-1}(\omega(\tau) - \bar{\omega}) \quad \text{for} \quad i \leq 1$$

$$\text{with} \quad \mathbf{\Psi}_{i-1}(\omega(\tau) - \bar{\omega}) = (\bar{\omega} - \omega(\tau))A_m(\tau)(\mu_1^{-1}, \ldots, \mu_1^{-N}, 0)^T \tag{18}$$

Note, that the $N + 1$-th component of $\mathbf{\Psi}_0(\bar{\omega})$ contains the inhomogeneous term of the temperature equation.

$$(\mathbf{\Psi}_0(\bar{\omega}))_{N+1} = \mathbf{\Psi}_{N+1} = \frac{d\Theta}{d\tau}(0) - \frac{1}{2N_c}\sum_{k=1}^{N} w_k I_k(0)\mu_k \tag{19}$$

The determination of the Adomian polynomials $A_m(\tau)$ in equation (18) in terms of the temperature is shown in section 4.

To complete our analysis considering the boundary conditions, the first equation of the recursive system satisfies the boundary condition, whereas the remaining equations satisfy homogeneous boundary conditions. By this procedure we guarantee that the solution $\mathbf{\Phi}$ determined from the recursive scheme and truncated at a convenient limit $\mathcal{M}$ satisfies the boundary conditions of the problem (13) and (6). Therefore we are now in a position to construct a solution with a prescribed accuracy by controlling the number of terms in the series solution given by equation (18). From the previous discussion it becomes apparent that it is possible by the proposed procedure to obtain a solution of the heterogeneous problem by a reduction to a set of homogeneous problems. To complete the construction of a solution for the heterogeneous problem in the next section we present the derivation of the solution of the S_N radiative-conductive transfer problem in a homogeneous slab.

4. The solution for the homogeneous radiative conductive heat transfer problem

In this section we consider the non-linear radiative-conductive transfer problem in a grey plane-parallel participating medium with combined specular and diffuse reflection (Siewert & Thomas, 1991) and its solution in an analytical form using a composite method by Laplace transform and the decomposition method (Adomian, 1988). Before the advent of the decomposition method analytical solutions were restricted to a few special problems like the Bernoulli and Ricatti equations, to mention only two. The basic idea of the decomposition technique understands the following steps: The non-linear problem is interpreted as an operator equation (as already introduced in section 3) and split into a sum of linear and non-linear terms. Next, one expands the solution (in the present discussion the intensity I) and the non-linear term (here the quartic dimensionless temperature term Θ^4), respectively, as a series $I(\tau) = \sum_{m=1}^{\infty} \mathbf{U}_m$ and $\Theta^4 = \sum_{m=1}^{\infty} \hat{A}_m$, where $\hat{A}_m$ are to be determined self consistently according to Adomian's procedure (Adomian, 1988). Upon insertion of these expansions in the split equation, one may construct a set of linear recursive problems that can be solved by classical methods for linear problems.

Although the method is designed for general non-linear problems, it is not straight forward to apply it to any given problem and to any desired precision. One specific equation system which we solve in the sequel considers the S_N problem equation (10) for non-linear radiative-conductive heat transfer in plane parallel geometry as introduced in ref. (Ozisik,

1973), the index N signifies here the number of the discrete directions of the angular space. More specifically, we circumvent limitations that arose in the discussion of the same problem in ref. (Vargas & Vilhena, 1999). Furthermore, differently than some iterative schemes found in the literature (Abulwafa, 1999; Ozisik, 1973; Siewert & Thomas, 1991), we construct an analytical solution sequence which in the limit of the truncation parameter $\mathcal{M} \to \infty$ converges to the exact solution of the equation that characterises the S_N problem.

For any arbitrary truncation and using Laplace transform (LT) the original S_N problem may be cast into LTS_N form, which allows for matrix orthogonalisation and thus opens the advantage of handling S_N problems with N as large as for instance ~ 1500; for further details see references. (Segatto et al., 1999) and (Goncalves et al., 2000). Once the $\hat{A}_m$ polynomials are known up to $\mathcal{M}$ the LTS_N provides the sum up to $\mathcal{M}$ of the expanded solution.

The hybrid LTS_N Adomian approach is not new, see references. (Vargas & Vilhena, 1999) and (Brancher et al., 1999), but in the present discussion we present a procedure based on the same reasoning but in a novel and optimised form. This progress is partially due to the more effective handling of the boundary conditions. In a previous attempt (Vargas et al., 2003) the boundary conditions entered in every step of recursion which posed limitations on the solutions so that it was only possible to resolve angles with $N = 30$ and truncate the expansion after the first term. As the following discussion will show, the present approach henceforth denoted the $D_{\mathcal{M}}LTS_N$ approach circumvents these shortcomings (here $D_{\mathcal{M}}LTS_N$ stands for Decomposition Laplace Transform S_N approach).

4.1 The LTS_N formalism

The dimensionless non-linear S_N radiative transfer equation in a grey plane-parallel homogeneous medium results from equation (7) upon substitution of the albedo coefficient by its average value.

$$\frac{d}{d\tau}I_n(\tau) + \frac{1}{\mu_n}I_n(\tau) = \frac{\bar{\omega}}{2\mu_n}\sum_{\ell=0}^{L}\beta_\ell P_\ell(\mu_n)\sum_{k=1}^{N}P_\ell(\mu_k)I_k(\tau) + \frac{1-\bar{\omega}}{\mu_n}\Theta^4(\tau) \qquad (20)$$

for $n = 1, \dots, N$ and subject to the boundary conditions with constant emissivity and reflectivity.

$$I_n(0) = \epsilon_1 \Theta_1^4 + \rho_1^s I_{N-n+1}(0) + 2\rho_1^d \sum_{k=1}^{N/2} w_k \mu_k I_{N-k+1}(0) , \qquad (21)$$

$$I_{N-n+1}(\tau_0) = \epsilon_2 \Theta_2^4 + \rho_2^s I_n(\tau_0) + 2\rho_2^d \sum_{k=1}^{N/2} w_k \mu_k I_k(\tau_0) . \qquad (22)$$

Note, that β_ℓ are the expansion coefficients explicitly given in case study 4.3.1.

The equation for the temperature (8) may be solved by integrating twice from the boundary $\tau = 0$ to any $\tau \in [0, \tau_0]$.

$$\Theta(\tau) = \Theta_1 + (\Theta_2 - \Theta_1)\frac{\tau}{\tau_0} - \frac{1}{4\pi N_c}\frac{\tau}{\tau_0}\int_0^{\tau_0}q_r^*(\tau')d\tau' + \frac{1}{4\pi N_c}\int_0^{\tau}q_r^*(\tau')d\tau' \qquad (23)$$

Recalling, that equation (8) relates the intensity to the temperature, equation (23) shows the connection between the temperature and the radiative flux that permits to cast the problem into a form that depends only on the directional intensity **I**.

In order to apply the decomposition method to the problem (20) and (23), we expand the non-linear source term into a series of Adomian polynomials Adomian (1988), which are determined in the next section.

$$\Theta^4(\tau) = \sum_{m=0}^{\infty} \hat{A}_m(\tau) \tag{24}$$

Upon inserting this *ansatz* in equation (20) yields a first order matrix differential equation:

$$\frac{d}{d\tau}\mathbf{I}(\tau) - \mathbf{A}\mathbf{I}(\tau) = \sum_{m=0}^{\infty} \hat{A}_m(\tau)\mathbf{M}. \tag{25}$$

Here $\mathbf{I}(\tau) = (\mathbf{I}_+(\tau), \mathbf{I}_-(\tau))^T$ is the intensity radiation vector, where the sub-vectors $\mathbf{I}_+(\tau)$ and $\mathbf{I}_-(\tau)$ are the intensity radiation for the positive ($0 < \mu < 1$) and negative ($-1 < \mu < 0$) directions, respectively, and of order $N/2$ each. Further, $\mathbf{M}$ is a vector of order N with entries:

$$\mathbf{M} = (1 - \omega)\left(\frac{1}{\mu_1}, \dots, \frac{1}{\mu_N}\right)^T \tag{26}$$

Finally, the components of matrix $\mathbf{A}$ have the form:

$$A_{ij} = -\frac{1}{\mu_i}\delta_{ij} + \frac{\omega}{2\mu_i}\sum_{\ell=0}^{L}\beta_\ell P_\ell(\mu_i)P_\ell(\mu_j), \tag{27}$$

where δ_{ij} is the Kronecker symbol. The radiation intensity can formally be written as a series:

$$\mathbf{I}(\tau) = \sum_{m=0}^{\infty}\mathbf{U}_m(\tau) \tag{28}$$

which upon substitution in equation (25) results in:

$$\sum_{m=0}^{\infty}\left(\frac{d}{d\tau}\mathbf{U}_m(\tau) - \mathbf{A}\mathbf{U}_m(\tau)\right) = \sum_{m=0}^{\infty}\hat{A}_m(\tau)\mathbf{M} \tag{29}$$

One possibility of solving the equation system (29) starts with the initialisation

$$\frac{d}{d\tau}\mathbf{U}_0(\tau) - \mathbf{A}\mathbf{U}_0(\tau) = 0 \tag{30}$$

$$\frac{d}{d\tau}\mathbf{U}_m(\tau) - \mathbf{A}\mathbf{U}_m(\tau) = \hat{A}_{m-1}(\tau)\mathbf{M}, \qquad m = 1,2,...,\infty \tag{31}$$

which is then solved by the Laplace transform procedure (i.e. the LTS_N method) for any arbitrary but finite $m \leq \mathcal{M}$. Here, $\mathcal{M}$ is a truncation of the series which has to be chosen such that the remaining dropped terms are only a small correction to the approximate solution. Details of the method may be found in references. (Segatto et al., 1999) and (Goncalves et al., 2000). In the further we make use of the results of the Laplace transformed equations (30) and (31) and write $\mathbf{U}_m$ in form of a Laplace inversion. The Adomian polynomials are given explicitly in equation (36).

So far the LTS_N solution to the first problem of the recursive system has the form:

$$\mathbf{U}_0(\tau) = \mathbf{X}\mathbf{E}(\mathbf{D}\tau)\mathbf{V}^{(0)} \tag{32}$$

where $\mathbf{D}$ and $\mathbf{X}$ are respectively the matrices of eigenvalues and eigenfunctions resulting from the spectral decomposition of the matrix $\mathbf{A}$. The components of the diagonal matrix $\mathbf{E}(\mathbf{D}\tau)$ are:

$$\mathbf{E}(\mathbf{D}\tau) = \begin{cases} e^{d_{ii}\tau} & \text{if} \quad d_{ii} < 0 \\ e^{d_{ii}(\tau-\tau_0)} & \text{if} \quad d_{ii} > 0 \end{cases} \tag{33}$$

Note, that the matrix expression $\mathbf{D}$, $\mathbf{E}$ and eigenvectors $\mathbf{X}$ are from the solution of the Laplace transformed problem equations (30) and (31). In equation (33) d_{ii} are entries of the eigenvalue matrix $\mathbf{D}$. Further the general solution for the remaining problems are given by

$$\mathbf{U}_m(\tau) = \mathbf{X}\mathbf{E}(\mathbf{D}\tau)\mathbf{V}^{(m)} + \mathbf{X}e^{\mathbf{D}\tau}\mathbf{X}^{-1} * \hat{A}_{m-1}(\tau)\mathbf{M} \tag{34}$$

for $m = 1, \ldots, \mathcal{M}$ and $(*)$ denotes the convolution operator. The constant vectors $\mathbf{V}^{(m)}$ are determined from the application of the inhomogeneous boundary conditions

$$\begin{aligned} \mathbf{U}_0(0) &= \mathbf{I}(0) \\ \mathbf{U}_0(\tau_0) &= \mathbf{I}(\tau_0) \end{aligned} \qquad \text{for} \qquad m = 0$$

and the homogeneous boundary conditions

$$\begin{aligned} \mathbf{U}_m(0) &= 0 \\ \mathbf{U}_m(\tau_0) &= 0 \end{aligned} \qquad \text{for} \qquad m = 1, \ldots, \mathcal{M}$$

on the left hand side of equation (34). The effectiveness of this recursive scheme is due to the fact that the boundary condition for the problem (20) is already absorbed in the first recursion instruction whereas the remaining problems satisfy homogeneous boundary conditions only. To complete the construction of the analytical solution of problem (20) by the decomposition method, we present in the next section, a convergent scheme to generate the Adomian polynomials $\hat{A}_m$ for $m \in \{1, \ldots, \mathcal{M}\}$, for any generic $\mathcal{M}$.

4.2 The determination of the $\hat{A}_m$ polynomials

The role of the Adomian polynomials is to constitute the non-linear term in equation (20), i.e. the dimensionless non-linear temperature term Θ^4. Using a finite functional expansion in $T_m(\tau)$ for the dimensionless temperature $\Theta(\tau) = \sum_{m=0}^{\mathcal{M}} T_m(\tau)$ implies

$$\Theta^4 = \sum_{m=0}^{\mathcal{M}} \hat{A}_m = T_0^4 + 4T_0^3 \sum_{i=1}^{\mathcal{M}} T_i + \frac{12T_0^2}{2!}\left(\sum_{i=1}^{\mathcal{M}} T_i\right)^2$$
$$+ \frac{24T_0}{3!}\left(\sum_{i=1}^{\mathcal{M}} T_i\right)^3 + \frac{24}{4!}\left(\sum_{i=1}^{\mathcal{M}} T_i\right)^4, \tag{35}$$

where one of the possible identifications of the $\hat{A}_m$ is to group together terms with T_i in the right hand side of the equation (35) in a way, such that the index i of T_i ranges from 0 to m. This can be seen explicitly in equation (36) where $\hat{A}_0$ depends on T_0 only, $\hat{A}_1$ on T_0, T_1, or generically, $\hat{A}_m = \hat{A}_m(T_0, \ldots, T_m)$. Note, that the significance of the T_m becomes clear further down in equation (39) and is used here merely as a term of a functional expansion. The

resulting scheme for the Adomain polynomials reads then, which for later use we indicate in factorized form:

$$
\begin{aligned}
\hat{A}_0 &= T_0^4 = T_0 T_0 T_0^2 \\
\hat{A}_1 &= 4T_0^3 T_1 + 6T_0^2 T_1^2 + 4T_0 T_1^3 + T_1^4 = T_1(2T_0 + T_1)(2T_0^2 + 2T_0 T_1 + T_1^2) \\
\hat{A}_2 &= 4T_0^3 T_2 + 12T_0^2 T_1 T_2 + 12T_0 T_1^2 T_2 + 4T_1^3 T_2 + 6T_0^2 T_2^2 + 12T_0 T_1 T_2^2 \\
&\qquad + 6T_1^2 T_2^2 + 4T_0 T_2^3 + 4T_1 T_2^3 + T_2^4 \\
&= T_2(2T_0 + 2T_1 + T_2)(2T_0^2 + 4T_0 T_1 + 2T_1^2 + 2T_0 T_2 + 2T_1 T_2 + T_2^2) \\
&\ \vdots
\end{aligned}
\tag{36}
$$

In shorthand notation the recursive scheme for the Adomian polynomials may be written as

$$
\hat{A}_m = T_m S_m R_m
\tag{37}
$$

where S_m and R_m are determined by the formulas for $m = 1, \dots, \mathcal{M}$.

$$
S_m = S_{m-1} + T_m + T_{m-1} \qquad \text{and} \qquad R_m = R_{m-1} + S_{m-1} T_{m-1} + S_m T_m
\tag{38}
$$

The recursive procedure according to equation (36) starts with $S_0 = T_0$ and $R_0 = T_0^2$. From equation (23), we construct then the recursive formulation for the temperature.

$$
T_0(\tau) = \Theta_1 + (\Theta_2 - \Theta_1)\frac{\tau}{\tau_0}
\tag{39}
$$

$$
T_{m+1}(\tau) = -\frac{1}{2N_c}\frac{\tau}{\tau_0}\left\langle \mathbf{W}, \int_0^{\tau_0} \mathbf{U}_m(\tau')d\tau' \right\rangle + \frac{1}{2N_c}\left\langle \mathbf{W}, \int_0^{\tau} \mathbf{U}_m(\tau')d\tau' \right\rangle
$$

Here $m = 0, \dots, \mathcal{M}$ and the column vector $\mathbf{W} = (w_1\mu_1, \dots, w_N\mu_N)^T$ contains as components the discrete directions μ_i and the Gaussian quadrature weights w_i. The bracket signifies the vector inner product. Note, that equation (39) establishes the Adomian polynomials in terms of the temperature at the boundaries and the expansion terms of the intensity, which in principle could be determined until infinity.

4.3 Numerical results

In this section we present three cases that show the robustness and quantitative coincidence of the $D_{\mathcal{M}}LTS_N$ approach with solutions of the S_N radiative-conductive problem in a slab in the literature. As results we evaluate the normalised temperature, conductive, radiative and total heat fluxes.

$$
Q_r(\tau) = \frac{1}{4\pi N_c}q_r^*(\tau) \quad Q_c(\tau) = -\frac{d}{d\tau}\Theta(\tau) \quad \text{and} \quad Q(\tau) = Q_r(\tau) + Q_c(\tau)
$$

4.3.1 Case 1

In this case we determine the numerical values for $\mathcal{M}$ and N in order to get results with a considerable accuracy. The numerical values of the parameters used in cases 1 to 2 are given in table 1. The coefficient β_ℓ is defined considering a binomial scattering law which also permits a comparison with the results of (Siewert & Thomas, 1991).

$$
\beta_\ell = \left(\frac{2\ell+1}{2\ell-1}\right)\left(\frac{L+1-\ell}{L+1+\ell}\right)\beta_{\ell-1} \qquad 0 \leq \ell \leq L \qquad \text{and} \qquad \beta_0 = 1
$$

ϵ_1	ϵ_2	ρ_1^s	ρ_2^s	ρ_1^d	ρ_2^d	Θ_1	Θ_2	ω	τ_0	N_c	L
0.6	0.4	0.1	0.2	0.3	0.4	1.0	0.5	0.95	1.0	0.05	299

Table 1. Parameters of case 1.

$\mathcal{M}$	$\Theta(\tau)$	$Q_c(\tau)$	$Q_r(\tau)$	$Q(\tau)$
0	0.8177177955027018	0.5016457699309158	1.5158278540320405	2.0174736239629563
1	0.7698084721160902	0.4588373285860757	1.5859093685802235	2.0447466971662993
5	0.7775834780564881	0.4652872714487863	1.5788014097534921	2.0440886812022785
10	0.7775905224102305	0.4652925878442414	1.5787962534790447	2.0440888413232861
20	0.7775905213060152	0.4652925870115464	1.5787962542859926	2.044088412975391
50	0.7775905213060152	0.4652925870115464	1.5787962542859926	2.044088412975391
100	0.7775905213060152	0.4652925870115464	1.5787962542859926	2.044088412975391
200	0.7775905213060152	0.4652925870115464	1.5787962542859926	2.044088412975391

Table 2. The $D_{\mathcal{M}}LTS_{300}$ results for $\mathcal{M}$ ranging from 0 to 200, assuming $\tau/\tau_0 = 0.5$.

The numerical results for Θ, $Q_r(\tau)$, $Q_c(\tau)$ and $Q(\tau)$ are shown in table 2,3 and 4. The stability and convergence of the method was tested for $\tau/\tau_0 = 0.5$, varying $\mathcal{M}$ from 0 to 200, and using for N the values $300, 350$ and 400, respectively. The displayed precision with 16 digits was adopted to show the smooth convergence with increasing $\mathcal{M}$ in the three cases for N.

$\mathcal{M}$	$\Theta(\tau)$	$Q_c(\tau)$	$Q_r(\tau)$	$Q(\tau)$
0	0.8177176602853717	0.5016457476711904	1.5158274669152312	2.0174732145864214
1	0.7698083890454525	0.4588373328783077	1.5859091448625833	2.0447464777408908
5	0.7775833829280683	0.4652872682514292	1.5788011774222819	2.0440884456737112
10	0.7775904272568637	0.4652925846297308	1.5787960211505467	2.0440886057802774
20	0.7775904261526551	0.4652925837970406	1.5787960219574921	2.0440886057545327
50	0.7775904261526551	0.4652925837970406	1.5787960219574921	2.0440886057545327
100	0.7775904261526551	0.4652925837970406	1.5787960219574921	2.0440886057545327
150	0.7775904261526551	0.4652925837970406	1.5787960219574921	2.0440886057545327
200	0.7775904261526551	0.4652925837970406	1.5787960219574921	2.0440886057545327

Table 3. The $D_{\mathcal{M}}LTS_{350}$ results for $\mathcal{M}$ ranging from 0 to 200, assuming $\tau/\tau_0 = 0.5$.

$\mathcal{M}$	$\Theta(\tau)$	$Q_c(\tau)$	$Q_r(\tau)$	$Q(\tau)$
0	0.8177175726732399	0.5016457333417628	1.5158272165190925	2.0174729498608555
1	0.7698083350919457	0.4588373357092782	1.5859090000727045	2.0447463357819826
5	0.7775833211907642	0.4652872662455577	1.5788010270489736	2.0440882932945312
10	0.7775903655034493	0.4652925826127813	1.5787958707789687	2.0440884533917498
20	0.7775903643992450	0.4652925817800941	1.5787958715859125	2.0440884533660064
50	0.7775903643992450	0.4652925817800941	1.5787958715859125	2.0440884533660064
150	0.7775903643992450	0.4652925817800941	1.5787958715859125	2.0440884533660064
200	0.7775903643992450	0.4652925817800941	1.5787958715859125	2.0440884533660064

Table 4. The $D_{\mathcal{M}}LTS_{400}$ results for $\mathcal{M}$ ranging from 0 to 200, assuming $\tau/\tau_0 = 0.5$.
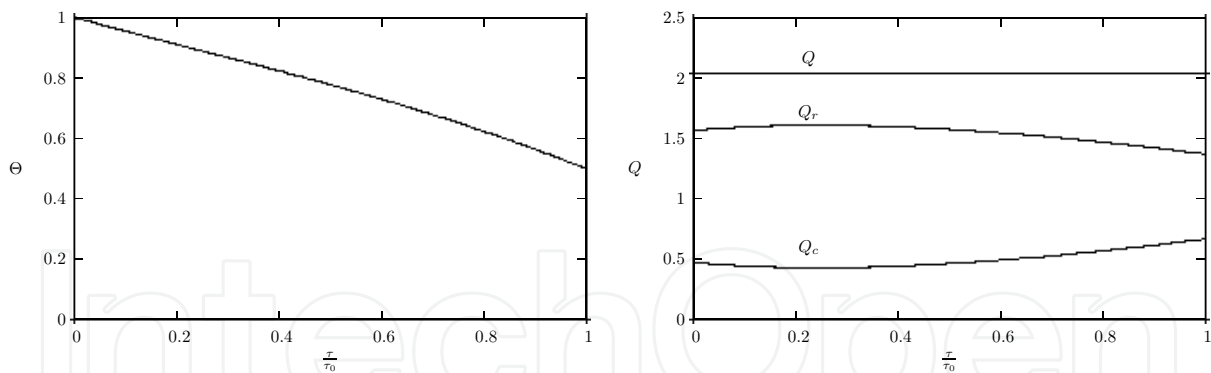
Fig. 2. Numerical results for $D_{10}LTS_{350}$ that exactly reproduce the results of reference (Siewert & Thomas, 1991) within the adopted precision. The temperature profile Θ (left), the conductive Q_c, radiative Q_r and total heat flux Q (right) against the relative optical depth $\frac{\tau}{\tau_0}$.

Comparing the corresponding lines in tables 2 to 4 for different N one observes that an analytical expression with $\mathcal{M} = 10$ and $N = 300$ is already close to the solution with $\mathcal{M}$ as large as 200 or ideally in the limit $\mathcal{M} \to \infty$. However, in the subsequent problems we set $\mathcal{M} = 10$ and $N = 350$ in order to reproduce within the adopted precision the numerical results of ref. (Siewert & Thomas, 1991) where the P_N method was applied to the same problem.

4.3.2 Case 2
The numerical evaluation in case 1 may now be refined varying the optical depth which was maintained fixed previously. To this end we determine $\Theta(\tau)$, $Q_c(\tau)$, $Q_r(\tau)$ and $Q(\tau)$ for τ/τ_0 ranging from 0 to 1. The numerical results are shown in figure 2 which coincide with the findings in reference (Siewert & Thomas, 1991) beyond a six digit precision. In their work Siewert and Thomas left open the question of convergence of their applied method, which by virtue of numerical coincidence with the present approach may be positively answered. Although not presented here with mathematical rigour, convergence of the decomposition method is formally guaranteed (see references (Adomian, 1988; Cherruault, 1989; Pazos & Vilhena, 1999a;b)) by the manifest exact solution in the limit $\mathcal{M} \to \infty$.

4.3.3 Case 3
A third comparison is elaborated making contact to a work by (Abulwafa, 1999), considering a conductive radiative problem in a slab assuming isotropy ($L = 0$) and with thickness τ_0, which also serves as a unit length. The parameter set is with either $\omega = 0.9$ or $\omega = 0.5$, with $\epsilon_1 = \epsilon_2 = \Theta_1 = 1$ and $\rho_1^d = \rho_2^d = \rho_1^s = \rho_2^s = 0$. In this article the author uses a variational technique to solve the radiative problem, while an iterative method is implemented to include the non-linearity effect of the temperature distribution of the medium from the conductive energy equation. Figure 3 shows the numerical findings of $D_{10}LTS_{350}$ in comparison to results from (Abulwafa, 1999) for two conduction-radiation parameters $N_c = 0.5$ and $N_c = 0.1$, respectively.

The comparison of the $D_{\mathcal{M}}LTS_N$ results with the ones of reference (Abulwafa, 1999) shows a fairly good agreement between the methods. It seems that a decrease in N_c opens slightly the difference between the two solutions, whereas increase in ω closes the difference between the solutions. Moreover, the larger N_c the closer one gets to a linear temperature profile. The difference is probably due to the fact that the approach in reference (Abulwafa, 1999) makes
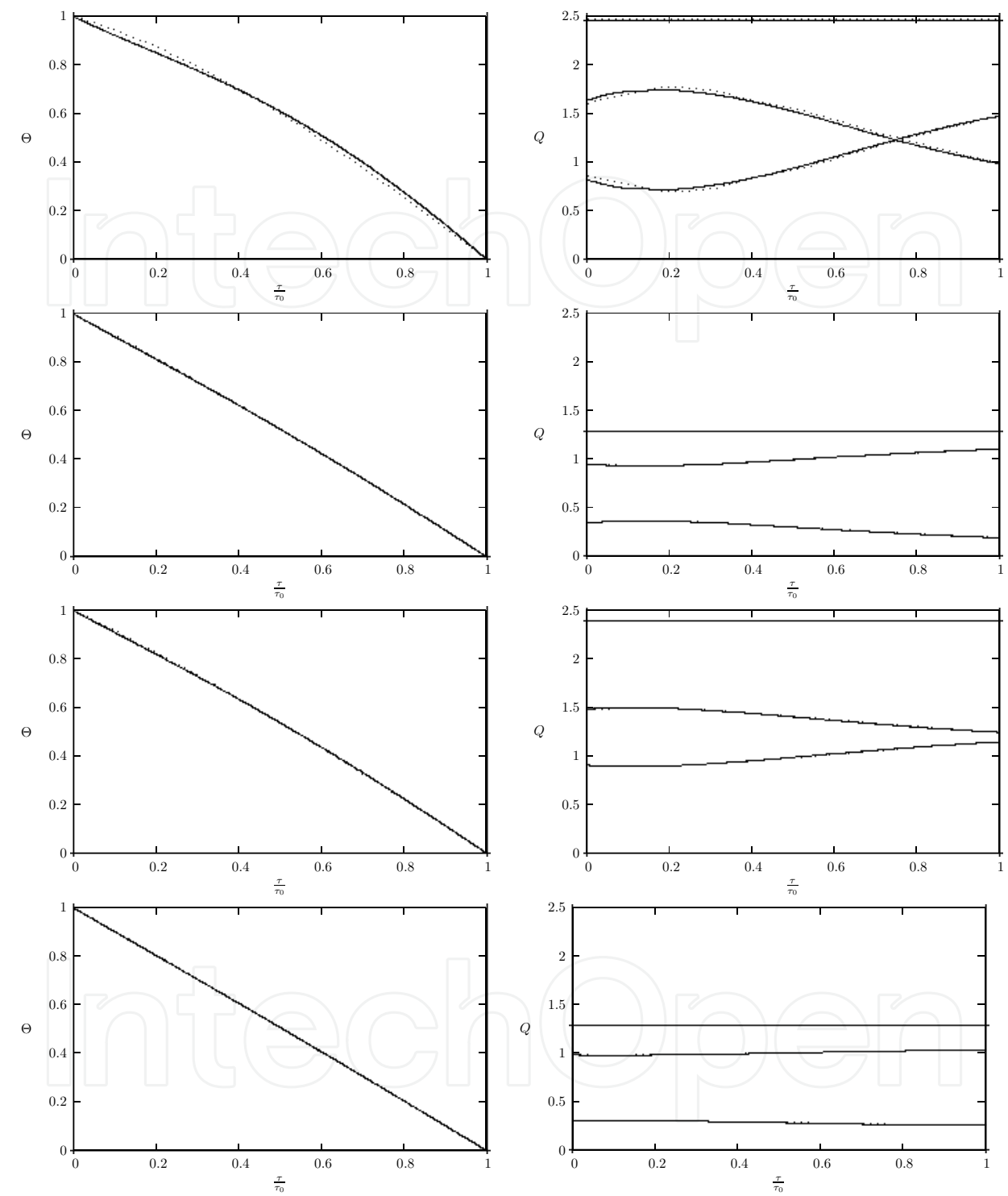
Fig. 3. Numerical comparisons of the $D_{\mathcal{M}}LTS_N$ (solid line) and Abulwafa's results (dotted line) for the parameter combinations $(\omega, N_c) = \{(0.5, 0.1), (0.5, 0.5), (0.9, 0.1), (0.9, 0.5)\}$ from top to bottom. The temperature profile Θ (left) and the conductive Q_c, radiative Q_r and total heat flux Q (right) against the relative optical depth $\frac{\tau}{\tau_0}$. The total heat fluxes are the constant curves, the conductive (radiative) heat fluxes show predominantly convex (concave) behaviour in the considered range (see also figure 2).

use of a trial function. In general in such type of approaches convergence depends crucially on how close the trial function is to the true solution. The comparison of cases 2 and 3 and the quantitative agreement with two different approaches shows the quality of the present method, especially because of the fact that it reproduces the exact analytical solution in the limit $\mathcal{M} \to \infty$ and thus allows to implement computationally a genuine convergence criterion. Some further information concerning the computational issue, the three cases were calculated on a Notebook computer with 64-bit Athlon 3200+ processor (1Ghz, 512kb Cache) and 1GB RAM. All calculations terminated with less than a minute execution time (some examples returned the result within seconds) and used typically between 10 and 20 iterations.

5. Conclusion

In the present work we discussed and compared an analytical approach to the non-linear S_N radiative-conductive transfer problem in plane-parallel geometry and a heterogeneous medium using a composite method by the Laplace transform and the Adomian decomposition (Adomian, 1988). We showed by two options how the heterogeneous problem may be cast into a set of homogeneous problems, so that the general solution may be obtained by a hierarchical algorithm. The Laplace technique opens pathways to resort to classical methods for linear problems, whereas the decomposition procedure allows to disentangle the non-linear contribution of the problem, that permits to solve the equations by a recursion scheme. It is worth mentioning two limiting cases, i.e. with single scattering albedo either $\omega = 0$ or $\omega = 1$. The latter case turns Adomian obsolete, because the non-linear term vanishes, whereas $\omega = 0$ diagonalizes the equation system and thus turns Laplace obsolete, since the solution may be obtained directly by integration.

The decomposition method as originally introduced is designed for general non-linear problems, but several ways are possible to construct a solution (Cardona et al., 2009; Segatto et al., 2008). The present study may be considered a guideline on how to distribute the influence of the boundary conditions and the non-linearity in order to solve the given problem. The boundary condition is absorbed in the part of the solution that belongs to the inversion of the differential operator without the non-linear contribution and the non-linear part simplifies to a problem for homogeneous boundary conditions only. Since existence and uniqueness of the solution for radiative-conductive transfer problems was discussed in references (Kelley, 1996; Thompson et al., 2004; 2008) the only critical issue of the recursive scheme is convergence. According to (Adomian, 1988; Cherruault, 1989; Pazos & Vilhena, 1999a;b) the resulting scheme is manifest exact and converges in the limit $\mathcal{M} \to \infty$ to the exact analytical solution. A genuine control of errors opens thus the possibility of model validation in comparison to experimental findings. In numerical approaches it is not straight forward to distinguish between model and numerical uncertainties, especially when non-linearities are present in the problem. Moreover, in general error analysis in numerical procedures is based on a heuristic basis, whereas the present approach permits a mathematical proof of convergence. This may not be that crucial for homogeneous problems, or those that permit linear approximations, but in the case of a heterogeneous problem in form of a multi-slab medium the question of convergence certainly plays a major role, especially because of the matching of solutions at the slab interfaces.

We are completely aware of the fact that the present procedure is limited by the convergence relation between the optical depth and the convergence radius of the Adomian approximation.

Nevertheless, we consider our work as an essential step for implementation of problems considering heterogeneous media. Furthermore, for thick media were the total optical depth lies outside the convergence radius a multi-slab treatment shall be used. In section 3 we showed explicitly how dependencies of the Albedo, the emissivity and the reflectivity on the optical depth are handled reducing partially the problem to a homogeneous one and including corrections in form of source terms. As an additional task due to this particular procedure matching of the partial solutions at the interfaces of the slabs has to be performed in addition, as a consequence of the local character of the physical parameter in each slab. So far, our findings show that for the S_N problem the series truncated with $\mathcal{M} = 10$ yields already a fairly good solution in form of an analytical expression. One advantage in comparison to numerical approaches lies in the fact that the dependence of the solution on the physical parameter may be analytically explored from the resulting expressions.

In section 4.3 we solved a selection of cases that may constitute a partial problem in a more complex medium and showed systematically, how a reliable solution may be obtained following the construction steps of 4.1 and 4.2. The application given in case 1 indicates the limits for $\mathcal{M}$ and N, and showed in case 2 that for a sequence of optical depths the same numerical results appear as given in ref. (Siewert & Thomas, 1991). Since convergence in the present approach is guaranteed one may elaborate a genuine convergence criterion depending on a desired precision. As a third test we compared our results for an isotropic problem to ref. (Abulwafa, 1999), where also agreement between the findings was verified. Since the proposed method reproduces the exact analytical solution in the limit $\mathcal{M} \to \infty$, approximate analytical expressions with finite $\mathcal{M}$ gain the character of benchmark results, which are of special interest in applications considering heterogeneous media.

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Convection and Conduction Heat Transfer

Edited by Dr. Amimul Ahsan

The convection and conduction heat transfer, thermal conductivity, and phase transformations are significant issues in a design of wide range of industrial processes and devices. This book includes 18 advanced and revised contributions, and it covers mainly (1) heat convection, (2) heat conduction, and (3) heat transfer analysis. The first section introduces mixed convection studies on inclined channels, double diffusive coupling, and on lid driven trapezoidal cavity, forced natural convection through a roof, convection on non-isothermal jet oscillations, unsteady pulsed flow, and hydromagnetic flow with thermal radiation. The second section covers heat conduction in capillary porous bodies and in structures made of functionally graded materials, integral transforms for heat conduction problems, non-linear radiative-conductive heat transfer, thermal conductivity of gas diffusion layers and multi-component natural systems, thermal behavior of the ink, primer and paint, heating in biothermal systems, and RBF finite difference approach in heat conduction. The third section includes heat transfer analysis of reinforced concrete beam, modeling of heat transfer and phase transformations, boundary conditions-surface heat flux and temperature, simulation of phase change materials, and finite element methods of factorial design. The advanced idea and information described here will be fruitful for the readers to find a sustainable solution in an industrialized society.

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