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Matlab Solutions of Chaotic Fractional Order Circuits

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1. Introduction

In the last two decades, integral and differential calculus of an arbitrary (or fractional) order has become a subject of great interest in different areas of physics, biology, economics and other sciences. It is accepted today as a new tool that extends the descriptive power of the conventional calculus, supporting mathematical models that, in many cases, describe more accurately the dynamic response of actual systems in various applications. While the theoretical and practical interest of these fractional order operators is nowadays well established, its applicability to science and engineering can be considered as an emerging topic. Among other, the need to numerically compute the fractional order derivatives and integrals arises frequently in many fields, especially in electronics, telecommunications, automatic control and digital signal processing.
The purpose of this chapter is to introduce the fractional calculus and its applications to solutions of fractional order circuits. Systematic methods and MATLAB-based computer routines are given. The next section provides a brief review of fractional calculus followed by useful approximations for these fractional operators (Section 2). In Section 3 we briefly present fundamental issues of the fractional order circuits in relation to nonlinear dynamical phenomena together with its fractional-order vector space representation - a generalization of the state space concept – and the modified Chua's circuit of fractional order as an example. Brief summary and conclusions follow in Section 4.

2. Fractional calculus

2.1 Fundamentals

The recent increased interest in the study of dynamic systems of non-integer orders (Baleanu et al., 2010, Caponeto et al., 2010, Das, 2008) stems from the premise that most of the processes associated with complex systems have non-local dynamics involving long-memory in time, and that fractional integral and fractional derivative operators share some of those characteristics. Many originally considered systems with lumped and/or distributed parameters can be more exactly described by fractional order systems (Dorčák et al. 2007; Kilbas et al., 2006; Monje et al., 2010; Nonnenmacher & Metzler, 2000; Sheu Long-Jye et al. 2007; Trzaska, 2010).
Extending derivatives and integrals from integer orders to non-integer orders has a firm and long standing theoretical foundation. For example, Leibniz mentioned this concept in a

letter to L′Hospital over three hundred years ago and the earliest more or less systematic studies have been made in the beginning and middle of the 19th century by Liouville, Riemann and Holmgren. In the literature, people often use the term "*fractional* order (FO) calculus", or "*fractional* order dynamic system" where "*fractional*" actually means "*non-integer*" (Oldham, 1974; Petras, 2006; Podlubny, 1999; Tavazoei et al., 2008; Zhao & Xue, 2008).

The definitions most commonly used in the literature are the Riemann-Liouville definition, the Grünwald-Letnikov definition, and the Caputo definition, as given below:

$$ {}_aD_t^{\alpha}[f(t)] = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d^n}{dt^n}\int_a^t \frac{f(\tau)}{(t-\tau)^{1-(n-\alpha)}}d\tau\right) \tag{1} $$

$$ {}_aD_t^{\alpha}[f(t)] = \lim_{h\to 0}\frac{1}{\Gamma(\alpha)h^{\alpha}}\sum_{k=0}^{\lceil (t-a)/h \rceil}\frac{\Gamma(\alpha+k)}{\Gamma(k+1)}f(t-kh) \tag{2} $$

$$ {}_aD_t^{\alpha}[f(t)] = \frac{1}{\Gamma(\alpha-n)}\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau \tag{3} $$

for $n-1 < a < n$ and where $\Gamma(\cdot)$ is the *Gamma* function, and $\lceil z \rceil$ means the integer part of z.

The "memory" effect of these operators is demonstrated by (1) and (2), where the convolution integral in (1) and the infinite series in (2) reveal the unlimited memory of these operators, ideal for modeling hereditary and memory properties in physical systems and materials. The initial conditions for the fractional order differential equations with the Caputo derivative (3) are of the same form as the initial conditions for the integer-order differential equations (Baleanu et al., 2010; Kilbas et al., 2006; Tavazoei & Haeri, 2008).

Two general properties of the fractional-order derivative are of major interest here. The first is the composition of fractional with integer-order derivative and the second is the property of linearity. Similar to integer-order differentiation, fractional-order differentiation fulfils the relations

$$ {}_aD_t^{\alpha}({}_aD_t^{\beta}f(t)) = {}_aD_t^{\alpha+\beta}f(t) - \sum_{k=0}^{n}[{}_aD_t^{\beta-k}]_{t=a}\frac{(t-a)^{-\alpha-k}}{\Gamma(1-\alpha-k)} \tag{4} $$

$$ {}_aD_t^{\alpha}[\eta f(t) + \lambda g(t)] = \eta[{}_aD_t^{\alpha}f(t)] + \lambda[{}_aD_t^{\alpha}g(t)] \tag{5} $$

In addition, fractional order systems may have other features that make them more suitable for the study of electronic circuits when most of the desired specifications are not readily achieved by traditional models (Petras, 2006; Trzaska, 2009).

2.2 Fractional order differential equations

A fractional order linear time invariant system can be represented in the form of a scalar differential equation of fractional order as follows:

$$ \begin{aligned} a_nD^{\alpha_n}y(t) + a_{n-1}D^{\alpha_{n-1}}y(t) + a_{n-2}D^{\alpha_{n-2}}y(t) + \cdots + a_0D^{\alpha_0}y(t) = b_mD^{\beta_m}u(t) + b_{m-1}D^{\beta_{m-1}}u(t) \\ + b_{m-2}D^{\beta_{m-1}}u(t) + \cdots + b_0D^{\beta_0}u(t) \end{aligned} \tag{6} $$

where $y(t)$ is the system response and $u(t)$ the excitation signal, D the derivative operator and $a_n > a_{n-1} > \ldots > a_0$ and $\beta_m > \beta_{m-1} > \ldots > \beta_0$ non-integer positive numbers. Coefficients a_k, $k = 0, 1, 2, \ldots, n$ and b_h, $h = 0, 1, 2, \ldots, m$ are constants.
An alternative representation of (6), more useful for the analysis of fractional-order systems, is given in the following state space form:

$$\frac{d^{\alpha}\mathbf{x}(t)}{dt^{\alpha}} + \boldsymbol{\varphi}(\mathbf{x},\alpha,\mathbf{A},\beta,t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \quad \varphi(\mathbf{x},\alpha,\mathbf{A},\beta,t) \text{ given for } t > \beta \quad (7)$$

where $\boldsymbol{x}(t) \in R^n$, $\boldsymbol{u}(t) \in R^m$, $\boldsymbol{y}(t) \in R^p$ are states, inputs, and outputs vectors of the system and $\boldsymbol{A} \in R^{n\times n}$, $\boldsymbol{B} \in R^{n\times m}$, $\boldsymbol{C} \in R^{p\times n}$, $D \in R^{p\times m}$ denote constant element matrices, and α is the fractional commensurate order (Das, 2008; Kilbas et al., 2006).
The fractional dynamic variables in the system of equations (7) are not states in the true sense of the 'state' space. Consequently, as the initialization function vector is generally required, the set of elements of the vector $\boldsymbol{x}(t)$, evaluated at any point in time, does not specify the entire 'state' of the system. Thus, for fractional-order systems, the ability to predict the future response of a system requires the set of fractional differential equations along with their initialization functions, that is, equation (7).

3. Fractional order circuits

Today, micro- and nanoelectronic products are wide-spread in all kinds of industries and in commonly used household devices and the needs of microelectronic industry often drive science and technology research. One of such needs, and the primary task in modeling of micro- and nanoelectronic devices, is the prediction of circuits' behavior in various situations. This means, on the one hand, the estimation of the effective (or overall) properties of a circuit from its structural composition, commonly referred to as homogenization and, on the other hand, allowing for localization, i.e. estimation of the local load state within the individual constituents as response to an overall applied (far field) load. In an effort to better understand dynamical properties of electronic circuits, their stability features and the impact of various parameters, the fractional order approach seems to be very promising (see for example Das, 2008; Dorčák et al., 2007; Petras, 2006; Santhiah et al., 2011; Trzaska, 2008; Marszałek & Trzaska, 2011): many real dynamical circuits are better characterized by using non-integer order models based on fractional order differential or integral calculations. However, the newly produced nano-devices need superior accuracies and possibilities of correlating their hierarchies to models of the fractional order α (Baleanu et al., 2010). While stepping from micro- to nanoelectronics one is faced with heterogeneous multi-order models which couple mathematical descriptions using integer order and fractional order differential equations.
This approach requires the theoretical understanding and numerical modeling of the fundamental fractional order phenomena that underlie integrated devices. As a first step understanding the possible dynamic behavior of linear fractional order circuits (FOCs) is fundamental as most properties and conclusions of integer order circuits (IOCs) cannot be simply extended to that of the FOCs (Caponetto, 2010). The models of the FOCs exhibit more degrees of freedom and they contain unlimited memory that make the circuit behave in more complicated manner.

3.1 Basic properties of fractional order circuits

Generally, the descriptions of the components of electronic circuits need not to be limited to the separate characteristics of ideal resistors, capacitors or inductors: a component may have characteristics somewhere between the characteristics of standard components such as fractional order electrical impedance, that is between the characteristics of a resistor and a capacitor. For instance, a supercapacitor also known as ultracapacitor shows the constant phase behavior or capacitance dispersion, and thus simple RC circuit does not give an adequate description of the AC response of such elements. Its simplest model can be based on Curie's empirical law (Westerlund, 2002) which states that under DC voltage excitation U_0 applied at t =0 the current through a supercapacitor is

$$i(t) = \frac{U_0}{h_1 t^m} \tag{8}$$

where h_1 and m are constants.
However, for a general excitation voltage $u(t)$ we have the current

$$i(t) = C_f \frac{d^m u(t)}{dt^m} \tag{9}$$

where C_f is fractional capacitance of the supercapacitor depending on the kind of dielectric and rate of development of the electrode surfaces (Baleanu et al., 2010; Petras, 2006).
For a real coil with proximity and skin effects the relation between its voltage and current takes the form

$$u(t) = L_f \frac{d^n i(t)}{dt^n} \tag{10}$$

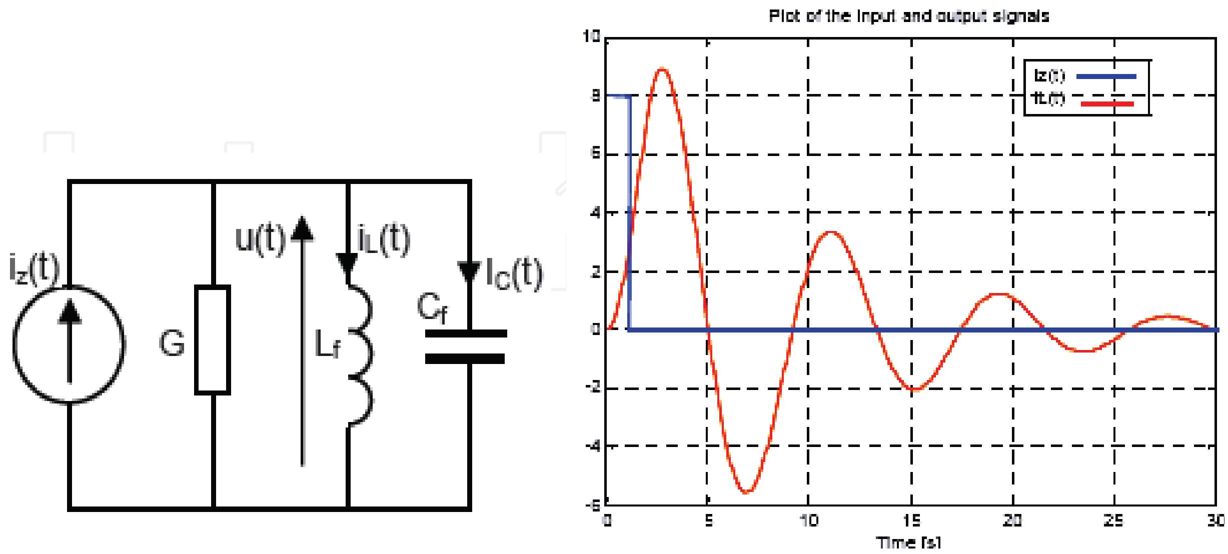
where L_f is the fractional inductance and $n \in (0, 1)$ depends on the coil form and its materials.
The importance of (9) and (10) lies mainly in the fact that they can easily be used to describe many other practical electronic and electrical devices because there is a large number of electric and magnetic phenomena where the fractional order models appear to be the most appropriate (M. Trzaska & Trzaska, 2007). It should be emphasized however that, although the real microelectronic objects are generally fractional (Dorčák et al., 2007), for many of them the fractionality is very low. To demonstrate these facts we shall consider some selected cases of FOCs.
Let us begin with a brief description of the application of fractional order calculus to analysis of a linear circuit shown in Fig. 1a which contains fractors L_f and C_f. The fractors represent a real coil and a supercapacitor, respectively, and are described by

$$u(t) = L_f \frac{d^{3/2} i_L(t)}{dt^{3/2}}, \quad i_C(t) = C_f \frac{d^{1/2} u(t)}{dt^{1/2}} \tag{11}$$

Denoting $x(t) = i_L(t)$ and then applying the Kirchhoff laws and taking into account (11) we obtain the following mathematical model of the fractional order

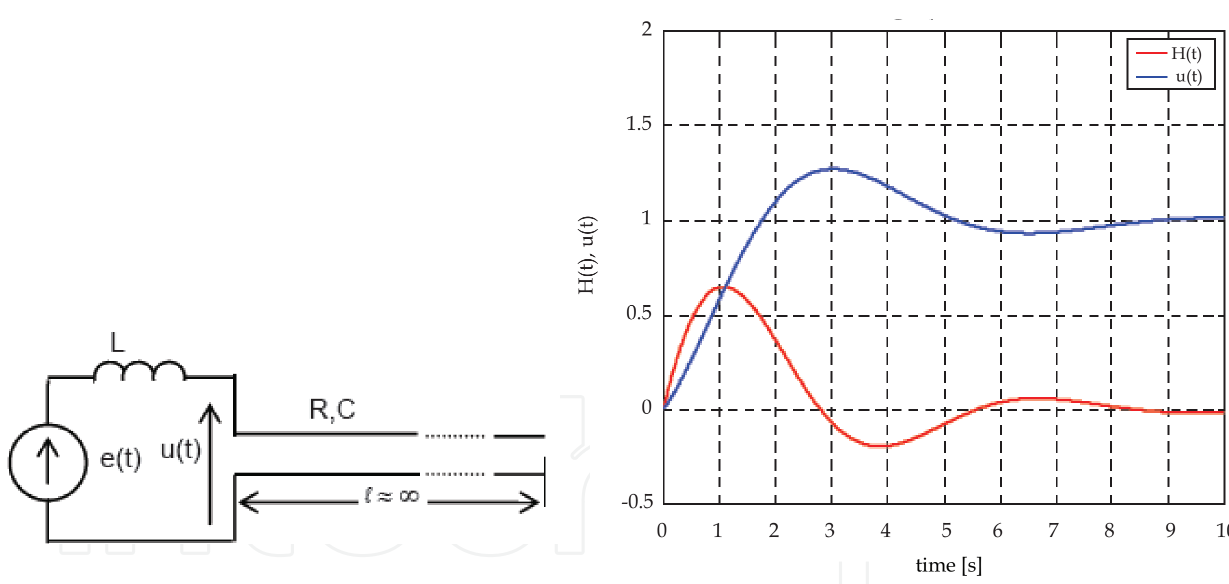
$$A_f \frac{d^2 x(t)}{dt^2} + B_f \frac{d^{3/2} x(t)}{dt^{3/2}} + x(t) = i_z(t) \tag{12}$$

where $A_f = C_f L_f$, $B_f = GL_f$ denote constant coefficients and $i_z(t)$ is a time varying right hand side term.



(a) (b)

Fig. 1. Linear circuit containing fractors L_f and C_f: a) structure, b) input and output signals



(a) (b)

Fig. 2. Voltage source connected to uniform RC line: a) scheme of the circuit, b) impulse $H(t)$ and unit-step $u(t)$ responses

To determine the solution of (12) we have used a special numerical procedure based on the Bagley-Torvik scheme (Trzaska, 2010). Taking into account $A_f = 1.25s^2$, $B_f = 0.5s^{3/2}$ and $i_z(t) = 8A$ for $0 < t < 1s$ with $i_z(t) = 0$ for $t > 1s$ and applying the mentioned numerical procedure implemented in MATLAB (Attia, 1999; Redfern & Campbell, 1998) we obtain the solution shown in Fig. 1b. It is easily seen that the circuit displays comprehensive dynamical behavior, such that the transient output signal varies very slowly in respect to the excitation.

Next, let us consider a fractional order circuit shown in Fig. 2a. It represents the connection of a voltage source *e(t)* exhibiting internal conventional inductance *L*=1H with a very long ($l \approx \infty$) uniform *RC* transmission line. The transmission line can be represented by a circuit with uniformly distributed constant parameters and the whole circuit considered at the line's input is described by the following fractional order model

$$L\sqrt{\frac{C}{R}}\frac{d^{3/2}u(t)}{dt^{3/2}} + u(t) = e(t) \tag{13}$$

where *u(t)* denotes the voltage at the line's input.
However, in order to effectively analyze such systems, it is necessary to develop approximations to the fractional operators using the standard integer order operators. In the work that follows, the approximations are effected in the Laplace variable $s = a + j\omega$. It should be pointed out that the resulting approximations provide sufficient accuracy for the time domain hardware implementations (Faleiros et al., 2006).
For the sake of simplicity of notations without any loss of generality we assume in what follows that $L\sqrt{CR^{-1}} = 1s^{3/2}$ and zero initial conditions are taken into consideration. Thus, applying Kirchhoff laws and relations describing circuit elements in the Laplace transform domain leads to the following fractional order model

$$U(s) = \frac{1}{s^{3/2}+1}E(s) \tag{14}$$

where $U(s)$ and $E(s)$ denote the Laplace transforms of $u(t)$ and $e(t)$, respectively. Now taking into account the inverse Laplace transforms of both sides of (14) yields

$$u(t) = \mathcal{L}^{-1}[U(s)] = \int_0^t H_{3/2}(-1,\tau)e(t-\tau)d\tau \tag{15}$$

where the function $H_{3/2}(-1,t)$ is determined by

$$H_{3/2}(-1,t) = \sqrt{t}\sum_{n=0}^{\infty}\frac{t^{3n/2}}{\Gamma(\frac{3}{2}(n+1))} \tag{16}$$

and represents the impulse response of the circuit. In the case of the source voltage $e(t) = E_0\eta(t)$ with $E_0 = const$ and $\eta(t)$ denoting Heaviside function the voltage at the line's input is determined as follows

$$u_\eta(t) = [\eta(t) - E_{3/2}(t^{3/2})]E_0 \tag{17}$$

where $E_{3/2}(t)$ denotes the Mittag-Leffler function for $a = 3/2$ and $b = 1$. Taking into account (16) and (17) we can plot the circuit response for the inputs in the form of Dirac impulse and unit-step functions ($E_0 = 1$) separately. The computed responses are presented in Fig. 2b. It has to be noted that in almost all cases the impulse responses of fractional order circuits are related to the Mittag-Leffler function (Podlubny, 1999), which is effectively the fractional order analog of the exponential function. With this knowledge, it has been possible to better clarify the time responses associated with fractional order circuits.

Interestingly, an alternative approach to the solution of the above circuit can be sought by applying *Symbolic Math Toolbox*. On the base of (14) we can determine the fractance (fractional transfer function) of the circuit, namely

$$\mathcal{F}(s) = \frac{1}{s^{3/2}+1} \tag{18}$$

Considering $s^{1/2}$ as basic complex variable we can expand right hand side of (18) in the following partial fractions

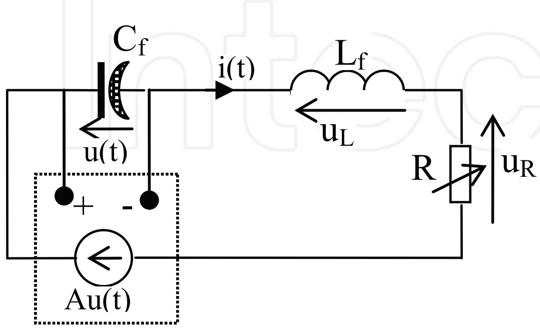
$$\mathcal{F}(s) = \frac{1}{3}\left(\frac{1}{s^{1/2}+1} - \frac{e^{j\pi/3}}{s^{1/2}-e^{j\pi/3}} - \frac{e^{-j\pi/3}}{s^{1/2}-e^{-j\pi/3}}\right) \tag{19}$$

Observe, that first term in (19) with pole at -1 on the complex plane with variable $s^{1/2}$ introduces an important damping of the impulse response and the remaining terms with poles at $e^{\pm j2\pi/3}$ are responsible on the damping oscillations and all terms taken in the whole give the circuit response which oscillates at an over damped rate. This is easily seen from the time-variations of H(t) presented in Fig. 2b. The diagram corresponds to the expression

$$\mathcal{L}^{-1}F(s) = H_{3/2}(-1,t) = \frac{1}{3}\Big[\frac{1}{\sqrt{\pi t}} - e^{\sqrt{t}} - \frac{e^{j2\pi/3}}{\sqrt{\pi t}} + e^{-j2\pi/3}e^{-j(4\pi/3)t}\operatorname{erfc}(e^{-j2\pi/3}\sqrt{t}) - \frac{e^{-j2\pi/3}}{\sqrt{\pi t}}$$
$$+e^{j2\pi/3}e^{j(4\pi/3)t}\operatorname{erfc}(e^{j2\pi/3}\sqrt{t})\Big] \tag{20}$$

which has been determined by applying the inverse Laplace commands from the *Symbolic Math Toolbox*. It is worth noting that the form of (20) is an alternative form of (16).

Now, to highlight numerous benefits from the application of suitable procedures contained in the MATLAB package we consider the circuit shown in Fig. 3a. It is composed of a real coil L_f and a supercapacitor C_f, both elements considered as fractors and a zero order voltage



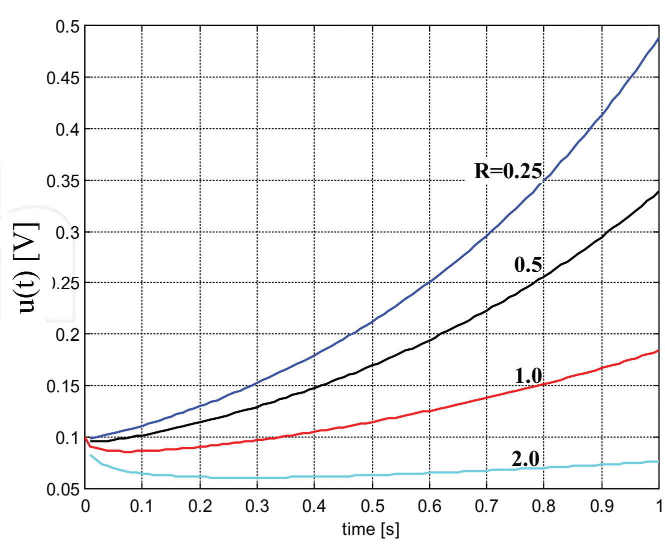


Fig. 3. Fractional order circuit with controlled source and resistor: a) circuit scheme, b) capacitor's voltage for various resistances

source controlled by the supercapacitor's voltage v_C (t) = Au(t). Applying circuit laws we obtain the following set of expression in terms of fractional order derivatives

$$L_f \frac{d^{\alpha_1} i(t)}{dt^{\alpha_1}} = u_L(t), \quad C_f \frac{d^{\alpha_2} u(t)}{dt^{\alpha_2}} = i(t), \quad Ri(t) = u_R(t), \quad Au(t) = v_C(t) \tag{21}$$

with the same symbol meanings as in Fig. 3a. Applying the voltage Kirchhoff law and taking into account (21) and rearranging the terms we obtain the following fractional order equation

$$L_f C_f \frac{d^{\alpha_1+\alpha_2} i(t)}{dt^{\alpha_1+\alpha_2}} + RC_f \frac{d^{\alpha_2} u(t)}{dt^{\alpha_2}} + u(t) = Au(t) \tag{22}$$

If we assume that $L_f C_f \equiv 1$, $A = 3$, $R = var$, and $C_f = const$ then equation (22) takes the form

$$\frac{d^{\alpha_1+\alpha_2} u(t)}{dt^{\alpha_1+\alpha_2}} + R_\tau \frac{d^{\alpha_2} u(t)}{dt^{\alpha_2}} - 2u(t) = 0 \tag{23}$$

where $R_\tau = RC_f$ denotes the time constant of the R, C_f series connection. In order to analyze effectively the above equation it is convenient to apply the standard procedures supplied by the MATLAB programs package and particularly those from the *Symbolic Math Toolbox*. In this regard, the Laplace transform of the initialized fractional-order differential equation (23) with $\alpha_1 = \alpha_2 = ½$ takes the form

$$sU(s) - u(0) + \sqrt{s} R_\tau U(s) - \frac{d^{-1/2}}{dt^{-1/2}} u(0) - 2U(s) = 0 \tag{24}$$

where $U(s) = L\,[u(t)]$ denotes the Laplace transform of the capacitor voltage $u(t)$. Solving (24) with respect to $U(s)$ yields

$$U(s) = \frac{B}{s + R_\tau \sqrt{s} - 2} \tag{25}$$

where $B = u(0) + d^{-1/2}/dt^{-1/2}\, u(0)$ is a constant depending on circuit initial conditions. Because (25) contains the fundamental 'fractional' poles it can be rewritten, using partial fractions, in a more simple form as follows

$$U(s) = \frac{B}{p_2 - p_1} \left(\frac{1}{\sqrt{s} - p_1} - \frac{1}{\sqrt{s} - p_2} \right) \tag{26}$$

where $p_1 = \frac{1}{2}(-a + \sqrt{a^2 + 8})$ and $p_2 = \frac{1}{2}(-a - \sqrt{a^2 + 8})$ denote the fractional poles of $U(s)$ with $a = R_\tau$. The above result indicates that the response of the initialized fractional order circuit can be determined by inverse transforming of each term of (26) separately. To accomplish these tasks, it is convenient rewrite (26) in an equivalent form as follows

$$U(s) = U_1(s) + U_2(s) \tag{27}$$

with

$$U_1(s) = B / ((p_2 - p_1)(\sqrt{s} - p_1)), \quad U_2(s) = -B / ((p_2 - p_1)(\sqrt{s} - p_2)) \tag{28}$$

To obtain the inverse Laplace transform of (28) we have used quite simple procedure of symbolic computing, namely

```
syms a s t B;
p1=-a/2+0.5*sqrt(a^2+8);
p2=-a/2-0.5*sqrt(a^2+8);
U1=B/(p2-p1)/(sqrt(s)-p1);
U2=-B/(p2-p1)/(sqrt(s)-p2);
u1=ilaplace(U1),
u2=ilaplace(U2)
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The symbolically computed expressions take the form

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u1=-B/(a^2+8)^(1/2)*(1/(pi*t)^(1/2)-(-/2*(a^2+8)^(1/2)+1/2*a)*exp((-1/2*(a^2+8)^(1/2)
+1/2*a)^2*t)*erfc((2*(a^2+8)^(1/2)+1/2*a)*t^(1/2)));

u2=B/(a^2+8)^(1/2)*(1/(pi*t)^(1/2)-(1/2*(a^2+8)^(1/2)
+1/2*a)*exp((1/2*(a^2+8)^(1/2)
+1/2*a)^2*t)*erfc((1/2*(a^2+8)^(1/2)+1/2*a)*t^(1/2)))          (29)
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Thus, following (25) – (29) we can represent the desired response as

$$u(t) = u_1 + u_2 \tag{30}$$

The above expressions are useful for determining the influence of the resistance's control on the circuit response. Plots of the u-function versus time for B = -0.1 and various values of R_τ are given in Fig. 3b. Looking at the values of the poles p_1 and p_2 we conclude that they correspond to a saddle node equilibrium point which involves instability of the circuit response for the whole range of possible values of the resistance R_τ (Das, 2008).

The above results can be used in many possible applications of FOCs and help assessing whether the latter are capable of addressing the industry's problems. They can serve as an overall introduction to fractional order methods and approaches as well as hands on exercises using state-of-the art software capabilities.

3.2 Chaotic fractional order circuits

Although chaotic phenomena have been studied extensively for a few dozen of years, the chaos theory still remains a fascinating area for exploration and there always seems to be some new aspects that can be revealed. Chaotic behaviors have been observed in different areas of science and engineering and various mathematical definitions of chaos have been proposed, resulting in a lack of a commonly accepted definition for it (Awrejcewicz & Lamarque, 2003; Chua, 1994; Ogorzałek, 1997). Thus, instead of selecting a definition for 'chaos', it is much easier to list properties of chaotic systems: it is accepted that the chaotic behavior is a recurrent, bounded, nonperiodic, long-time evolution of a system leading to a strange attractor in phase space. Also, chaotic systems present an extreme sensitivity to initial conditions i.e. small differences in the initial states can lead to extraordinary differences in the system states (Marszalek & Trzaska, 2009; Tadeusiewicz & Halgas, 2005). Chaos can be a desirable feature in many applications. For example in electronics and telecommunications chaos could be induced to spread modal energy at resonance or to achieve optimal spatial emission of electromagnetic waves.

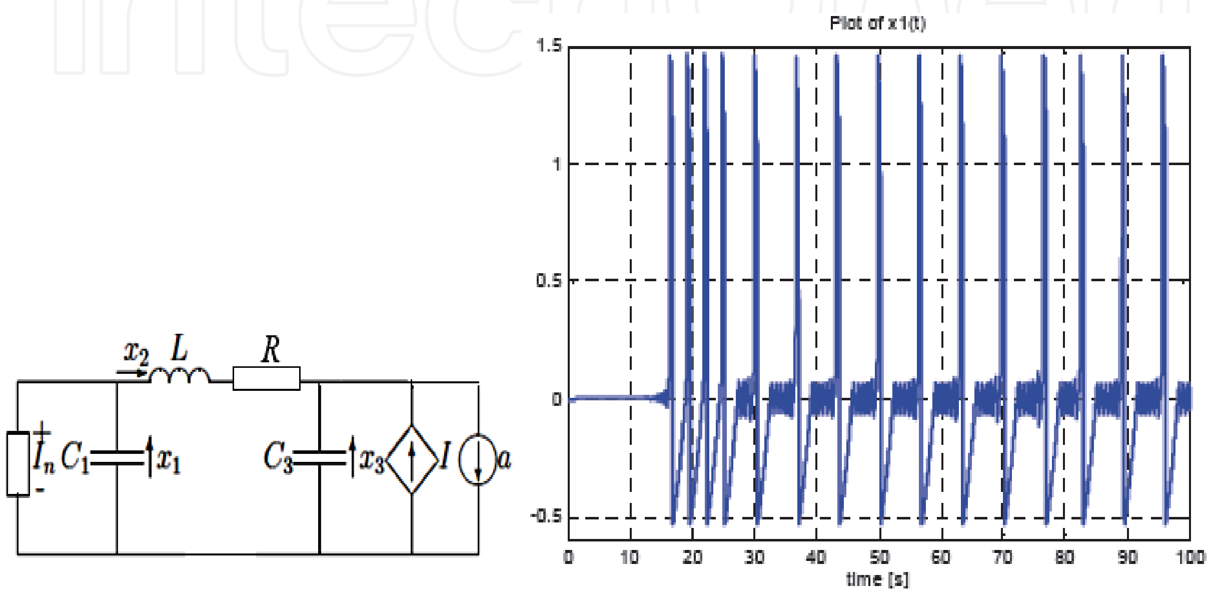
In this subsection the effects of fractional dynamics in chaotic circuits are studied. In particular, Chua's circuit is modified to include fractional order elements. It is worth mentioning that fractional order Chua's circuit has proven to be an excellent paradigm for generation of a multitude of different dynamical phenomena and can thus obviate the need to consider many different models to simulate those phenomena. One of the main reasons behind Chua's circuits' popularity is their flexibility and generality for representing virtually many practical structures, including those undergoing dynamic changes of topology. Varying the total circuit order incrementally demonstrates that systems of "order" less than three can exhibit chaos as well as other nonlinear behavior. This effectively forces a clarification of the definition of order which can no longer be considered only by the total number of differentiations or by the highest power of the Laplace variable *s*.

3.2.1 Analysis of oscillations in the modified Chua's circuit

Let us consider the fractional order circuit shown in Fig. 4a which represents a modified Chua's circuit (Marszalek & Trzaska, 2010). Standard Chua's circuit is well known and has been extensively studied in (Brown et al, 2001; Chua, 1994; Trzaska, 2005). The particular form to be considered here was presented by Trzaska (2008) and used further in Marszalek and Trzaska (2009). This circuit is different from the usual Chua's circuit in that the piecewise-linear nonlinearity is replaced by an appropriate cubic nonlinearity which leads to a very similar behavior. In a general case, this nonlinear oscillator comprises a nonlinear resistor with a cubic characteristic $I_n(V_1)$, three fractors represented by an inductor and two supercapacitors, a current controlled current source I and a biasing constant current source a. Its behavior depends on all six constants (parameters) involved. It can exhibit a wide spectrum of dynamical behaviors such as the relaxation, multi-mode oscillation, bifurcation and chaos. The mathematical description of dynamical components is based on expressions (16) and (17) and the studied circuit can be described as follows

$$\begin{aligned} C_1\frac{d^{q_1}x_1}{dt^{q_1}} &= -x_2+\alpha x_1^2+\beta x_1^3, \\ L\frac{d^{q_2}x_2}{dt^{q_2}} &= x_1-Rx_2-x_3, \\ C_3\frac{d^{q_3}x_3}{dt^{q_3}} &= a-bx_2 \end{aligned} \tag{31}$$

where q_1, q_3 and q_2 denote fractional orders of the supercapacitors C_1, C_3 and of the real coil L, respectively. Constant resistance is denoted by R. It is worth mentioning that even in the case of integer orders $q_1 = q_2 = q_3 = 1$ the above circuit can exhibit a number of exceptional behaviors depending on circuit linear element parameters, the form of the nonlinear resistor characteristic as well as on initial conditions. Of particular interest here are the mixed-mode oscillations (MMOs) which consist of a series of small-amplitude oscillations (also called the subthreshold oscillations, or STOs) and large-amplitude oscillations, or relaxations, occurring in various patterns. For instance, assuming C_1 = 0.01F, L = 1H, C_3 = 1F and a = 0.0005A, b = 0.0035, α =1.5 and β = -1 with $q_1 = q_2 = q_3 = 1$ we get the MMOs illustrated in Fig. 4b with respect to the state variable $x_1(t)$. The solution of the circuit equations and the illustration figure have been obtained by applying standard MATLAB's procedure ODE45 and plot commands, respectively. A series of numerical computations when various parameters of the circuits bifurcate have been performed and the one showed is typical for MMOs. The importance of MMOs lies mainly in the fact that during bifurcations the circuits' dynamics undergoes complex transitions between various stable and chaotic modes, including the mixed mode oscillations, period doubling bifurcations and chaotic responses. In most cases the solutions pertain to canard phenomenon. The most accepted and popular approach to explain the MMOs phenomenon in $\boldsymbol{R}^3$ is that they result from a combination of canard solutions around a fold singularity and relaxation spikes coupled together by a special ***global return mechanism*** (Trzaska & Marszalek, 2011).



(a) (b)

Fig. 4. Modified Chua's circuit: a) scheme with fractors C_1, L, C_3 and $I = (1 + b)x_2$, $I_n = \alpha x_1^2 + \beta x_1^3$, a = const > 0, b = const > 0, $\alpha < 0$ and $\beta > 0$, b) MMOs in an integer order case

Using the same element parameters values as the ones assumed above and changing only the form of the nonlinear characteristic to $0.88 \le \alpha \le 1$ with step size 0.0005 we obtain the solutions exhibiting bifurcations between MMOs and chaotic oscillations. In this chapter, we use two no-chaos criteria for a fractional order circuit represented by state variable equation $d^q x(t)/dt^q = f(x(t))$. Those criteria are: (*i*) x(t) approaches a fixed point; (*ii*) x(t) is bounded. It is clear that if a circuit contravenes both of the above conditions, it has necessary, albeit not sufficient, condition to demonstrate chaotic behavior (Marszalek & Trzaska, 2010).
The code presented below computes bifurcation diagram shown in Fig. 5a. The same code can be used to compute diagrams for other parameters provided that appropriate changes are made in the lines with parameters in bold-face.

Main code:

```
Clear
M = [1 0  0
  0 1  0
  0 0  1];
x0=[-0.5; 0.2; 0.5];
options = odeset('Mass',M,'RelTol',1e-12,'AbsTol',[1e-14 1e-14 1e-14], 'Vectorized','on');
```

```
global t x y z dt alpha
dt=0.01;
for alpha=0.8800:0.0005:1.6000
alpha
clear n
clear m
[t,x]=ode15s(@equations,0:dt:500,x0);
n=length(x(:,1));
m=floor(n/2);
y=diff(x(m,n,1))/dt;
z=diff(y)/dt;
k=1;
clear aa
for i=m:n
t0=t(i); %Comp. local max. pts for m<t<n
option=optimset('display','off');
zer=fsolve(@differ,t0,option);
if interp1(t(m:n-2),z,zer)>0
aa(k)=interp1(t(m:n),x(m:n,1),zer);
k=k+1;
end
end
kmax=k-1;
h=plot(alpha.*ones(1,kmax),aa,'r.');
hold on
set(h,'MarkerSize',0.1);
end
```

Function equations.m:

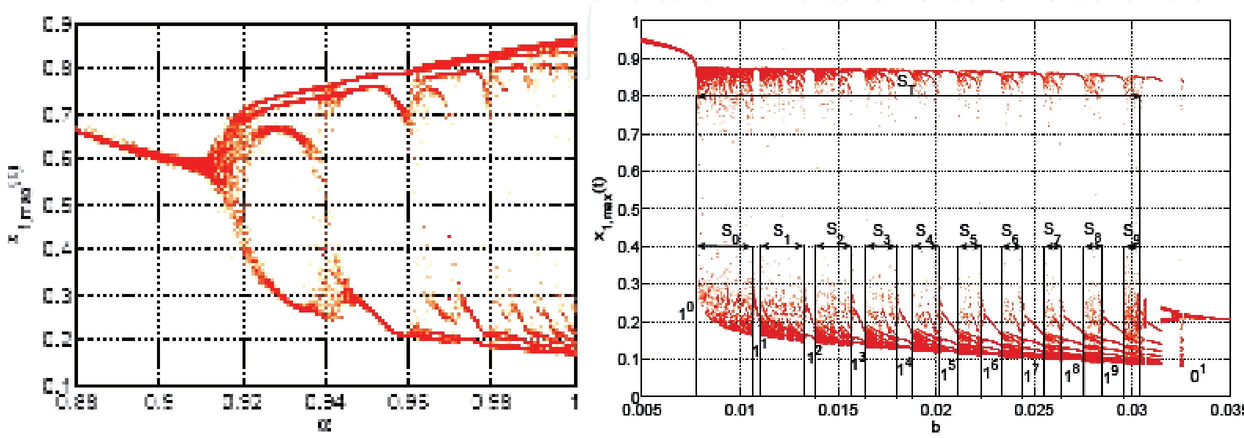
```
function xdot=equations(t,x)
global alpha
a=0.0005; b=0.01; eps=0.01; beta=-1; R=0.3;
xdot(1) = (-x(2) + alpha*x(1)^2 + beta*x(1)^3)/eps;
xdot(2) = x(1) - x(3) - R*x(2);
xdot(3) = a - b*x(2);
xdot = xdot';
end
```

Function differ.m:

```
function f=differ(a)
global t x dt
s=diff(x(m:n,1))/dt;
f=interp1(t(m:n-1),s,aa);
end
```

Fig. 5 shows the solutions obtained with local maximum values of $x_1(t)$ as a function of α. In all the integrations the initial conditions were zero for all three variables in (31). Note that the number of local maximum points at the level $0.2 - 0.3$ (on the lower fold of the surface $x_2 = \alpha x^2_1 + \beta x^3_1$) increases if α increases. The number of upper local maximum points (at the

0.7−0.9 level) increases, too. There are, however, sudden drops in the numbers of both lower and upper maximum points. This happens, for example, around α = 0.945 in Fig. 5a where we observe pure periodic MMOs of type 2-1. Thus, it is clear from Fig. 5a that by selecting α = 0.94 one may expect a chaotic response of the circuit. Similar diagram appears in the case of changing parameter b with α =1 and holding other parameters the same as above. The result is shown in Fig. 5b for $0.005 \le b \le 0.035$. The intervals of b leading to chaotic states are marked by S_0, S_1, …, S_9 and the total range of b equals S_T.



(a) (b)

Fig. 5. Local maximum values of $x_1(t)$: a) as a function of a , b) as function of b

The intervals S_i can be used to determine the fractal dimensions of the circuits, since a detailed analysis of the large–small (L_s) amplitude patterns clearly indicates that they are directly linked to the Farey sequence of the pairs of co-prime integers. Starting from the bifurcation diagram shown in Fig. 5b and using the values of S_k and S_T one can solve the following equation to compute the fractal dimension, denoted by D, namely

```
syms D
solve('(21/167)^D+(17/167)^D+(15/167)^D+(12/167)^D+(10/167)^D+(9/167)^D ...
      +2*(7/167)^D+ (8/167)^D+(6/167)^D-1')
```

In results we have

```
ans =
.84815781317251170768166648997506
```

On the basis of similar series of computations with respect to the remaining state variables $x_2(t)$ and $x_3(t)$ we can conclude that all three variables yield the counting box fractal dimension, denoted by D_c, of the circuits at $D_c = 2.5446$. As a consequence, we shall conjecture that the dynamics of the circuits presented in Fig. 4a is in fact of fractal type. Note that it is well known that chaos cannot occur in conventional continuous systems of total integer order less than three.

As mentioned above, determination of proper parameter range for which a dynamical circuit exhibits chaotic behavior is not always simple and sometimes needs a large amount of numerical simulations. Although more than three decades have passed from the birth of chaos theory, our knowledge about conditions for chaos existence in dynamical systems is still incomplete. Nowadays, it is commonly accepted that the chaotic behavior of nonlinear systems exhibits highly organized shapes leading to a strange attractor in phase space. Moreover, even if a system is characterized by the geometrical form of the corresponding attractor with a defined shape, the modified Chua's circuit can generate both an unusual complexity and an astonishing unpredictability. Noteworthy, up to date no analytical solution has been found for studies of chaotic systems.

3.2.2 Numerical solution of the fractional-order Chua's circuit

In what follows the effects of fractional dynamics in chaotic modified Chua circuits with the structure shown in Fig. 4a are studied. Taking into account the circuit state variable equation (31) we can approximate its solution by applying the discretization procedure and numerical calculation of fractional-order derivation using an explicit relation derived from the Grunwald–Letnikov definition (2). Using the following relation for the explicit numerical approximation of q-*th* derivative at the points kh, (k = 1, 2, . . .) (Kilbas et al., 2006):

$$_{(k-Q_m)}D^q_{kh}x(t) = \lim_{h\to 0} h^{-q} \sum_{n=0}^{k} (-1)^n \binom{q}{n} x((k-n)h) \tag{32}$$

where Q_m is the ''memory length'', h is the time step size of the calculation and $(-1)^n \binom{q}{n}$ are binomial coefficients $b_n^{(q)}$, (n = 0, 1, 2, …, k) which are calculated as follows

$$b_0^{(q)} = 1, \quad b_n^{(q)} = (1 - \frac{1+q}{n}) b_{n-1}^{(q)} \tag{33}$$

For discretized evaluation purposes it is convenient to simplify notation and introduce the following substitutions

$$x(i) = x_1(ih); \quad y(i) = x_2(ih); \quad z(i) = x_3(ih); \quad i = 0, 1, 2, ..., \tag{34}$$

Initial conditions are taken in the form

$$x(1) = x_1(0); \; y(1) = x_2(0); \; z(1) = x_3(0); \tag{35}$$

Introducing the solution approximation in the polynomial form we obtain the following discrete mathematical model of the circuit for state variables x(i), y(i) and z(i), namely

$$\begin{aligned} &x(i) = (alpha * (-y(i) + f1(a0, a1, x(i-1))) * h \wedge q1 - back(x, b1, i), \\ &y(i) = (beta * (x(i) - R * y(i-1) - z(i-1)) * h \wedge q2 - back(y, b2, i), \\ &z(i) = (gamma * (a - b * y(i)) * h \wedge q3 - back(z, b3, i) \end{aligned} \tag{36}$$

where

$$alpha = C_1^{-1}; \quad beta = L^{-1}; \quad gamma = C_3^{-1}; \quad i = 2, 3, ..., n. \tag{37}$$

and for nonlinear characteristic f1(x)= αx^2 +βx^3 we have

$$a0 = \alpha; \; a1 = \beta; \tag{38}$$

To compute discrete state variable (36) and to plot their diagrams the code presented below can be used. The same code can be used for other parameters provided that appropriate changes are made in the lines with parameters in bold-face (Petras, 2010)

Main code:

```
Clc
clear
% Numerical Solution of the Fractional-Order Modified Chua's Circuit
% with cubic nonlinearity defined in function fm()
function [h, xdisc]=MOChua(parameters, orders, tfin, x0)
%
% time step
h=0.001; tfin=200;
% number of discrete points
n=round(tfin/h);
%orders of derivatives
q1=0.9; q2=0.9; q3=0.9;
%parameters
alpha=100.0; beta=1.0;gamma=1.0;R=0.1;a=0.0005; b=0.001;a0=1.5; a1=-1;
%initial conditions
x0=0;y0=0;z0=0;
% binomial coefficients
bp1=1; bp2=1; bp3=1;
for j=1:n
  b1(j)=(1-(1+q1)/j)*bp1;
  b2(j)=(1-(1+q2)/j)*bp2;
  b3(j)=(1-(1+q3)/j)*bp3;
  bp1=b1(j); bp2=b2(j); bp3=b3(j);
end
% initial conditions setting
x(1)=x0; y(1)=y0; z(1)=z0;
% discretized solution
for i=2:n
  x(i)=(alpha*(-y(i-1)+f1(a0,a1,x(i-1))))*h^q1 - back(x, b1, i);
  y(i)=(beta*(x(i)-R*y(i-1)-z(i-1)))*h^q2 - back(y, b2, i);
  z(i)=(gamma*(a-b*y(i)))*h^q3 - back(z, b3, i);
  f(i)=f1(a0,a1,x(i));
end
%plots
t=h:h:tfin;
plot(t,x),grid,pause
plot(x,y),grid,pause
plot(t,x,t,y,t,z),grid,pause
plot3(x,y,z),grid,pause
plot(x,f),grid
%
```

The codes *f*1(a0,a1,x) for the characteristic of a nonlinear resistor and back(x,b,i) are as follows

```
function [fn]=f1(a0, a1, un);
a0=1.5;a1=-1.;
fn=a0*un.^2 + a1*un.^3;
%
function [vo] =back(r, b, k)
%
temp = 0;
for j=1:k-1
  temp = temp + b(j)*r(k-j);
end
vo = temp;
%
```

Simulations were then performed using various q_k, k =1, 2, 3, from the value interval 0.8 ÷ 1.1. In all cases the Lyapunov exponents were computed and one of them at least exhibited positive value which indicates that the circuit is behaving chaotically. Moreover, the numerical simulations also indicated that the lower limit of the fractional derivative q of all state vector components leading to generating chaos takes values from the interval (0.8 ÷0.9). Therefore, using q = 0.9 as fractional state vector derivative yields the lowest value at 2.7 for circuit order generating chaos in the modified Chua circuit. The simulation results obtained for these circuits are presented in Fig. 6 for alpha =100.0, beta = gamma = 1.0, a = 0.0005, b = 0.005, a_0 = 1.5, a_1 = -1.

3.2.3 Fractional-order Chua's circuit with piecewise-linear resistor

Let us explore now the fractional-order modified Chua's circuit of Fig. 4a but with the nonlinear resistor exhibiting piecewise-linear current-voltage characteristic. In order to appreciate the richness of the dynamics of the present variant circuit we involve the resistor's piecewise-linear characteristic composed of three linear segments leading to the form quite similar to that shown in Fig. 6d. It can be described by the formula

$$I_n = \mathrm{abs}(u_1) + H_1 \cdot (u_2 - u_1) \tag{39}$$

where u_1 and u_2 determine successive segments of the piecewise-linear characteristic depending on the voltage at the nonlinear resistor terminals. These segments can be represented as follows

$$u_1 = a_1 x_1, \quad u_2 = a_2 x_1 + a_0, \tag{40}$$

where constant coefficients a_1 and a_2 determine the slopes of particular characteristic segments, and a_0 is the free term. To get a continuous piecewise-linear characteristic the segments are exactly matched as a result of the concatenation given by the shifted Heaviside function

$$H_1 = \frac{1}{2}\left(1 + \frac{\mathrm{abs}(x_1 - x_c)}{x_1 - x_c}\right) \tag{41}$$

with x_c fixed at the point of coordinate x_1 corresponding to characteristic folding of neighboring segments.

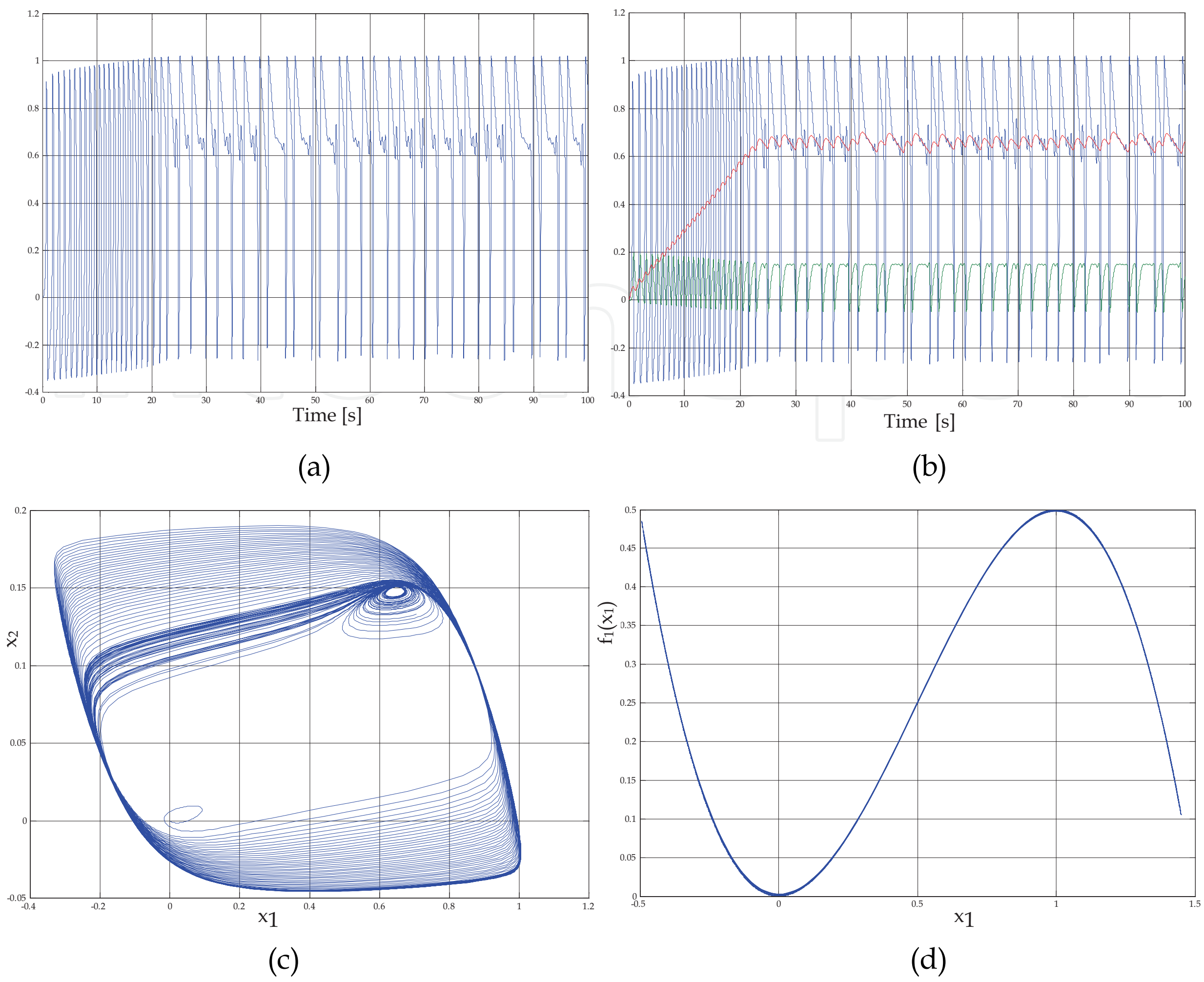


Fig. 6. Simulation results for a modified Chua's circuit of fractional orders $q_1 = q_2 = q_3 = 0.9$: a) state variable $x_1(t)$, b) all state variables: $x_1(t)$ - in blue, $x_2(t)$ - in green, $x_3(t)$ – in red, c) $x_2(t)$ versus $x_1(t)$, d) nonlinear resistor's characteristic $f_1(x_1)= 1.5x_1^2 - x_1^3$

To perform simulations of the fractional-order modified Chua's circuit with nonlinear resistor characteristic determined by (39) we can use the above presented MATLAB code after changing only the code corresponding to the nonlinear term by the following one

```
function [fn]=f2(a0, a1, a2, u0, un);
a0=1.0; a1=1.5; a2=-0.75; u0=1.0;
J1=abs(un-u0)./(un-u0);H1=0.5*(1+J1);
u1=a1*un;u2=a2*un+a0;
fn=abs(u1)+H1.*(u2-u1);
```

where appropriate changes can be made in parameters in bold-face, similarly to the main code. The resulting characteristic for a range of parameters assumed as above are shown in Fig. 7a.
A slight modification of arguments of the code above enables to generate other forms of piecewise-linear characteristics of the nonlinear resistor. For instance, the case of 5-segment characteristic obtained by such modified code is presented in Fig. 7b.
It is worth noticing that the piecewise-linear models are an attractive alternative to continuous nonlinear ones because they are both efficient in memory use and economical in computation time, despite the fact that the derivation of a model usually requires two steps:

first, the piecewise-linear approximation of nonlinear elements' characteristics, and second, their algebraic representations. Applying such an approach to studies of nonlinear circuits leads to piecewise-linear differential equations yielding a pervasively better understanding of various problems in applications. For instance, the piecewise-linear models for operational amplifiers (op amps) and operational transconductance amplifiers (OTA's) are both simple and frequently used. Another way of approximating dynamic circuits with semiconductor diodes can be obtained by replacing the real characteristic by piecewise-linear functions between diode current and voltage.

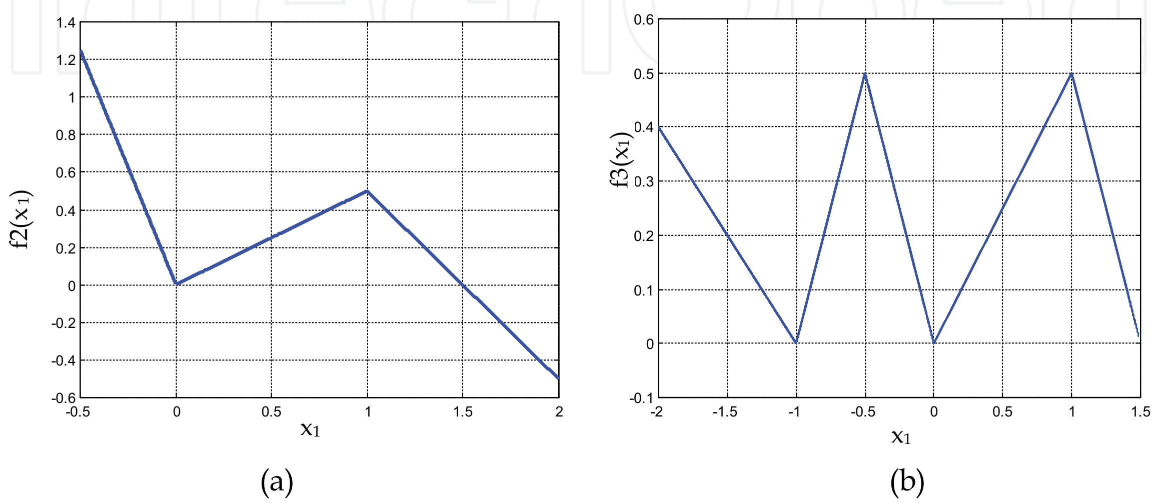
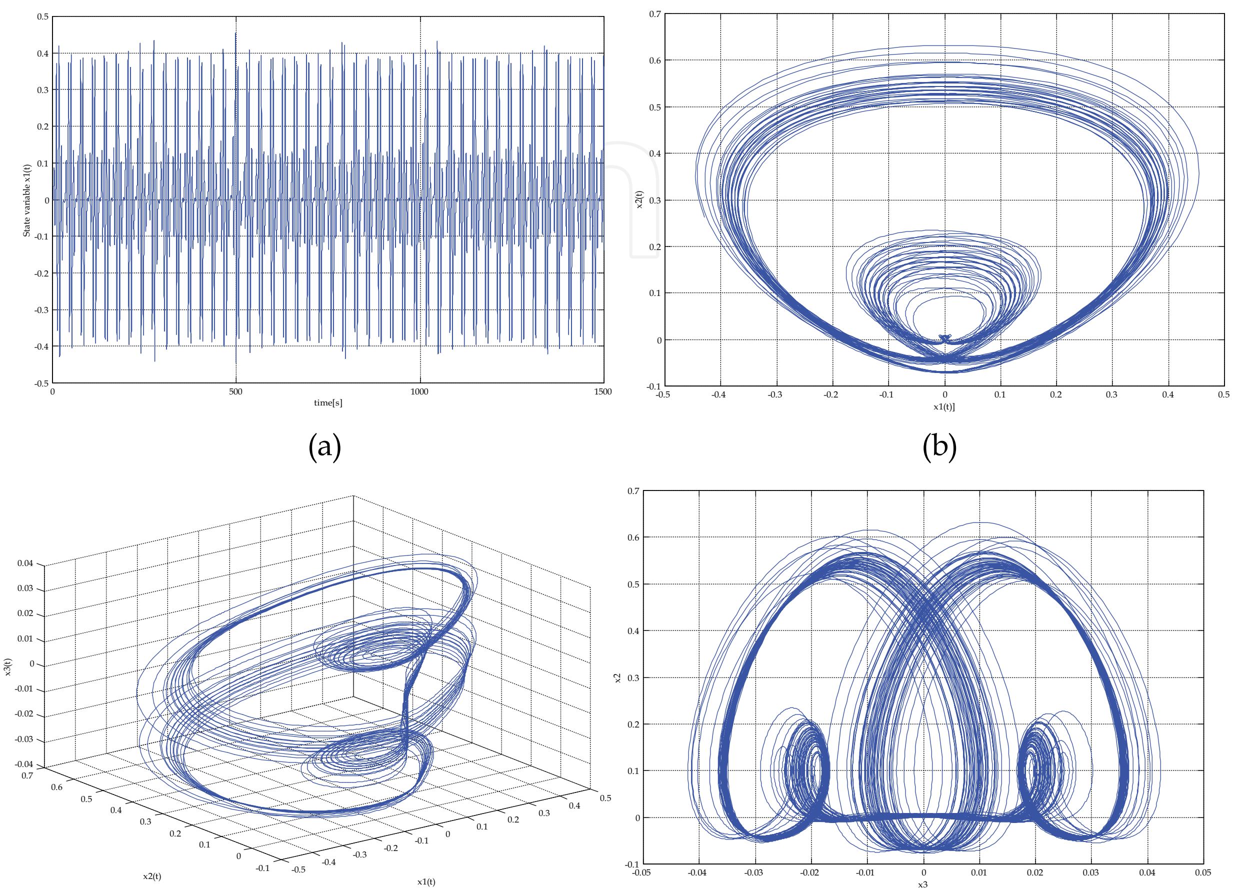


Fig. 7. Piecewise-linear characteristics of the nonlinear resistor: a) 3- segments, b) 5-segments

As it is well known nonlinear devices are already present in a great variety of applications in both power-electronic engineering and signal processing, and in many electrical networks controlling elements like thyristors. To reduce the simulation time of the transient behavior of such circuits and for analysis purposes (e.g., stability or chaos (Tang & Wang, 2005; Trzaska, 2007)), these circuit components are often modeled using piecewise-linear characteristics. Piecewise-linear circuits exhibit most of the phenomena of fractional order nonlinear circuits being yet weakly penetrating. Computer simulations of the suggested discrete maps with memory prove that the nonlinear dynamical circuits, which are described by the equations with fractional derivatives, exhibit a new type of chaotic behavior. This type of behavior demonstrates a fractional generalization of attractors. The subsequent numerical simulation results demonstrate this claim.

3.2.4 Chaotic oscillations in the circuit with piecewise-linear resistor

For the modified Chua's circuit of different fractional orders $\mathbf{q} = [q_1, q_2, q_3]'$ we have chosen function f(x) and the circuit parameters R, L, C_1, C_3, a, b, and h to satisfy the conditions of numerical stability, convergence and accuracy, so that the model (36) with appropriately adapted piecewise-linear characteristic (code for f2(a0, a1, a2, u0, un)) generates regular oscillations or demonstrates chaotic behaviors. For one of the chaotic circuits given in Fig. 4a with alpha = 1.5, beta = 0.85, gamma=1.0, R=0.0, a=0.005, b= 0.05, and piecewise-linear characteristic shown in Fig. 7a we have numerically investigated the sensitivity of the circuit states to the fractional orders q_1, q_2 and q_3. In all computations performed we have fixed initial conditions to be zero. The obtained numerical simulations demonstrated that the non-regular oscillations specified by chaotic attractors can be generated for the components of the vector fractional derivative **q** taking values from the set (0.8 ÷1.0). The chaotic dynamics was identified in two fractional order modified Chua's circuits with 3-segments and 5-segments piecewise-linearity.



(a) (b)

(c) (d)

Fig. 8. Results of simulations for modified Chua circuits with three segments of piecewise-linear characteristic: a) state variable x1(t), b) phase trajectory x2(t) versus x1(t), c) 3D phase trajectory, d) phase trajectory x2(t) versus x3(t)

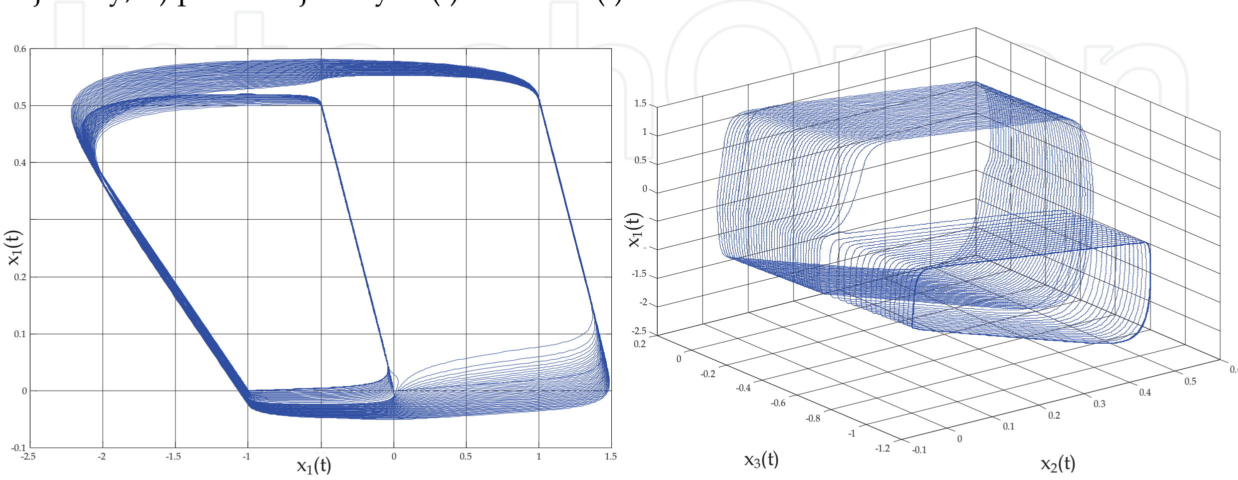


Fig. 9. Chaotic attractors for modified Chua circuits with quintuple segments of piecewise-linear characteristic: a) phase trajectory x2(t) versus x1(t), b) 3D phase trajectory

Examples of numerical simulation results confirming the existence of chaos in circuits with components described by piecewise-linear characteristics are presented in Figs. 8 and 9. It should be noted that the qualitative features of the studied circuits are very well predicted using sufficiently small step h of integrations in such a way that the quantitative results are reasonably close to the ones obtained for continuous relations representing cubic and quintuple nonlinearities.
As mentioned in the previous paragraph, determination of proper parameter range for which a dynamical system exhibits chaotic behavior is not always simple and sometimes needs a large amount of numerical simulations. Unfortunately, outside of numerical simulation, there is currently no other method to distinguish between regular and chaotic oscillations.

4. Conclusion

Most properties and conclusions of integer order circuits cannot be simply extended to that of the fractional order circuits. The models of the fractional order circuits contain unlimited memory and they exhibit more degrees of freedom. Due to the lack of appropriate mathematical tools, chaos analysis in fractional order circuits is more complicated than that in integer order systems. However, it is possible to design a circuit with moderate characteristics, which behavior is not properly represented using conventional methods. In that case the fractional calculus provides a framework for more efficient circuit modeling and control. For instance, the microstructures containing such components as supercapacitors with nano- and microcrystalline surface deposited electrodes can be modeled more successfully by fractional order equations than by traditional models. However, many challenges in the field of fractional order nonlinear circuits remain, notably the ones related to regularity and chaoticity of oscillations. In the case of the modified Chua's circuit the degree of chaoticity can be determined by a measure D_c expressing the counting box fractal dimension of chaotic flows. In the context of the emerging field of computational nanoscience, and in particular in the area of algorithms devoted to numerically explore the electronic circuits, the reported examples reinforce the suitability of the modified Chua's circuits for emulating fractional emergent phenomena.
In this Chapter different circuit dynamics have been described. The wide range of oscillation forms and their combination reflect the complexity of fractional-order nonlinear circuits. Numerical investigations of the behavior of the modified Chua's circuit for different forms of the nonlinear resistor characteristic can be realized with appropriate procedures from the standard MATLAB program package and from the *Symbolic Math Tools* box. It should be pointed out that the use of discretized form of the Grünvald-Letnikov fractional derivative provides a very important tool in the study of the dynamics of fractional order nonlinear circuits. Additionally, examples of program codes by which fractional order nonlinear circuits can be effectively simulated have been provided for studies of circuits with continuous nonlinear and/or piecewise-linear output-input characteristics of respective elements. This approach seems very promising for predicting chaos in fractional order systems studied in domains such as electrical science, diffusion process, electrochemistry, control science, viscoelasticity, material science, etc.

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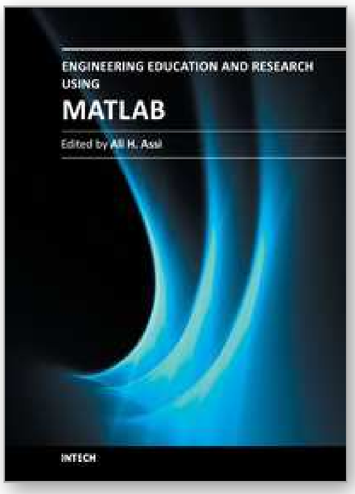

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MATLAB is a software package used primarily in the field of engineering for signal processing, numerical data analysis, modeling, programming, simulation, and computer graphic visualization. In the last few years, it has become widely accepted as an efficient tool, and, therefore, its use has significantly increased in scientific communities and academic institutions. This book consists of 20 chapters presenting research works using MATLAB tools. Chapters include techniques for programming and developing Graphical User Interfaces (GUIs), dynamic systems, electric machines, signal and image processing, power electronics, mixed signal circuits, genetic programming, digital watermarking, control systems, time-series regression modeling, and artificial neural networks.