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Numerical Solution of Many-Body Wave Scattering Problem for Small Particles and Creating Materials with Desired Refraction Coefficient

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1. Introduction

Theory of wave scattering by small particles of arbitrary shapes was developed by A. G. Ramm in papers (Ramm, 2005; 2007;a;b; 2008;a; 2009; 2010;a;b) for acoustic and electromagnetic (EM) waves. He derived analytical formulas for the S-matrix for wave scattering by a small body of arbitrary shape, and developed an approach for creating materials with a desired spatial dispersion. One can create a desired refraction coefficient $n^2(x,\omega)$ with a desired x,ω-dependence, where ω is the wave frequency. In particular, one can create materials with negative refraction, i.e., material in which phase velocity is directed opposite to the group velocity. Such materials are of interest in applications, see, e.g., (Hansen, 2008; von Rhein et al., 2007). The theory, described in this Chapter, can be used in many practical problems. Some results on EM wave scattering problems one can find in (Tatseiba & Matsuoka, 2005), where random distribution of particles was considered. A number of numerical methods for light scattering are presented in (Barber & Hill, 1990). An asymptotically exact solution of the many body acoustic wave scattering problem was developed in (Ramm, 2007) under the assumptions $ka << 1$, $d = O(a^{1/3})$, $M = O(1/a)$, where a is the characteristic size of the particles, $k = 2\pi/\lambda$ is the wave number, d is the distance between neighboring particles, and M is the total number of the particles embedded in a bounded domain $D \subset \mathrm{R}^3$. It was not assumed in (Ramm, 2007) that the particles were distributed uniformly in the space, or that there was any periodic structure in their distribution. In this Chapter, a uniform distribution of particles in D for the computational modeling is assumed (see Figure 1). An impedance boundary condition on the boundary S_m of the m-th particle D_m was assumed, $1 \leq m \leq M$. In (Ramm, 2008a) the above assumptions were generalized as follows:

$$\zeta_m = \frac{h(x_m)}{a^\kappa}, \qquad d = O(a^{(2-\kappa)/3}), \qquad M = O(\frac{1}{a^{2-\kappa}}), \qquad \kappa \in (0,1), \tag{1}$$

where ζ_m is the boundary impedance, $h_m = h(x_m)$, $x_m \in D_m$, and $h(x) \in C(D)$ is an arbitrary continuous in $\overline{D}$ function, $\overline{D}$ is the closure of D, $\mathrm{Im}h \leq 0$. The initial field u_0 satisfies the Helmholtz equation in R^3 and the scattered field satisfies the radiation condition. We assume in this Chapter that $\kappa \in (0,1)$ and the small particle D_m is a ball of radius a centered at the point $x_m \in D$, $1 \leq m \leq M$.

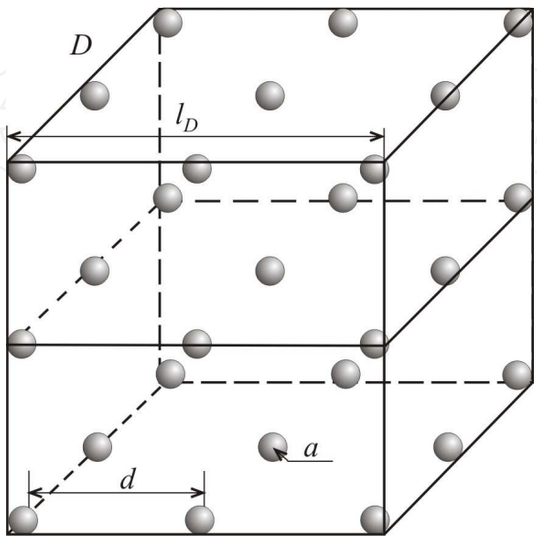

Fig. 1. Geometry of problem with $M = 27$ particles

2. Solution of the scattering problem

The scattering problem is

$$[\nabla^2 + k^2 n_0^2(x)]u_M = 0 \qquad \text{in} \qquad \mathrm{R}^3 \setminus \bigcup_{m=1}^{M} D_m, \tag{2}$$

$$\frac{\partial u_M}{\partial N} = \zeta_m u_M \qquad \text{on} \qquad S_m, 1 \leq m \leq M, \tag{3}$$

where

$$u_M = u_0 + v_M, \tag{4}$$

u_0 is a solution to problem (2), (3) with $M = 0$ (i.e., in the absence of the embedded particles) and with the incident field $e^{ik\alpha \cdot x}$. The scattered field v_M satisfies the radiation condition. The refraction coefficient $n_0^2(x)$ of the material in a bounded region D is assumed for simplicity a bounded function whose set of discontinuities has zero Lebesgue measure in R^3, and $\mathrm{Im}n_0^2(x) \geq 0$. We assume that $n_0^2(x) = 1$ in $D' := \mathrm{R}^3 \setminus D$. It was proved in (Ramm, 2008) that the unique solution to problem (2) - (4) exists, is unique, and is of the form

$$u_M(x) = u_0(x) + \sum_{m=1}^{M} \int_{S_m} G(x,y)\sigma_m(y)dy, \tag{5}$$

where $G(x,y)$ is Green's function of the Helmholtz equation (2) in the case when $M = 0$, i.e., when there are no embedded particles, and $\sigma_m(y)$ are some unknown functions. If these

functions are chosen so that the boundary conditions (3) are satisfied, then formula (5) gives the unique solution to problem (2) - (4). Let us define the "effective field" u_e, acting on the m-th particle:

$$u_e(x) := u_e(x,a) := u_e^{(m)}(x) := u_M(x) - \int\limits_{S_m} G(x,y)\sigma_m(y)dy, \tag{6}$$

where $|x - x_m| \sim a$. If $|x - x_m| >> a$, then $u_M(x) \sim u_e^{(m)}(x)$. The $\sim$ sign denotes the same order as $a \to 0$. The function $\sigma_m(y)$ solves an exact integral equation (see (Ramm, 2008)). This equation is solved in (Ramm, 2008) asymptotically as $a \to 0$, see formulas (12)-(15) in Section 3. Let $h(x) \in C(D)$, $\mathrm{Im}h \leq 0$, be arbitrary, $\Delta \subset D$ be any subdomain of D, and $\mathcal{N}(\Delta)$ be the number of the embedded particles in Δ. We assume that

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_\Delta N(x)dx[1 + o(1)], \ \ a \to 0, \tag{7}$$

where $N(x) \geqslant 0$ is a given continuous function in D. The following result was proved in (Ramm, 2008).

Theorem 1. There exists the limit $u(x)$ of $u_e(x)$ as $a \to 0$:

$$\lim_{a \to 0} ||u_e(x) - u(x)||_{C(D)} = 0, \tag{8}$$

and $u(x)$ solves the following equation:

$$u(x) = u_0(x) - 4\pi \int\limits_D G(x,y)h(y)N(y)u(y)dy. \tag{9}$$

This is the equation, derived in (Ramm, 2008) for the limiting effective field in the medium, created by embedding many small particles with the distribution law (7).

3. Approximate representation of the effective field

Let us derive an explicit formula for the effective field u_e. Rewrite the exact formula (5) as:

$$u_M(x) = u_0(x) + \sum_{m=1}^{M} G(x,x_m)Q_m + \sum_{m=1}^{M} \int\limits_{S_m} [G(x,y) - G(x,x_m)]\sigma_m(y)dy, \tag{10}$$

where

$$Q_m = \int\limits_{S_m} \sigma_m(y)dy. \tag{11}$$

Using some estimates of $G(x,y)$ (see (Ramm, 2007)) and the asymptotic formula for Q_m from (Ramm, 2008), one can rewrite the exact formula (10) as follows:

$$u_M(x) = u_0(x) + \sum_{m=1}^{M} G(x,x_m)Q_m + o(1) \ , \qquad a \to 0, \qquad |x - x_m| \geqslant a. \tag{12}$$

The numbers Q_m, $1 \leq m \leq M$, are given by the asymptotic formula

$$Q_m = -4\pi h(x_m)u_e(x_m)a^{2-\kappa}[1 + o(1)], \quad a \to 0, \tag{13}$$

and the asymptotic formula for σ_m is (see (Ramm, 2008)):

$$\sigma_m = -\frac{h(x_m)u_e(x_m)}{a^\kappa}[1 + o(1)], \quad a \to 0. \tag{14}$$

The asymptotic formula for $u_e(x)$ in the region $|x - x_j| \sim a$, $1 \leq j \leq M$, is (see (Ramm, 2008)):

$$u_e^{(j)}(x) = u_0(x) - 4\pi \sum_{m=1, m\neq j}^{M} G(x, x_m)h(x_m)u_e(x_m)a^{2-\kappa}[1 + o(1)]. \tag{15}$$

Equation (9) for the limiting effective field $u(x)$ is used for numerical calculations when the number M is large, e.g., $M = 10^b, b > 3$. The goal of our numerical experiments is to investigate the behavior of the solution to equation (9) and compare it with the asymptotic formula (15) in order to establish the limits of applicability of our asymptotic approach to many-body wave scattering problem for small particles.

4. Reduction of the scattering problem to solving linear algebraic systems

The numerical calculation of the field u_e by formula (15) requires the knowledge of the numbers $u_m := u_e(x_m)$. These numbers are obtained by solving the following linear algebraic system (LAS):

$$u_j = u_{0j} - 4\pi \sum_{m=1, m\neq j}^{M} G(x_j, x_m)h(x_m)u_m a^{2-\kappa}, \quad j = 1, 2, ..., M, \tag{16}$$

where $u_j = u(x_j)$, $1 \leq j \leq M$. This LAS is convenient for numerical calculations, because its matrix is sometimes diagonally dominant. Moreover, it follows from the results in (Ramm, 2009), that for sufficiently small a this LAS is uniquely solvable. Let the union of small cubes Δ_p, centered at the points y_p, form a partition of D, and the diameter of Δ_p be $O(d^{1/2})$. For finitely many cubes Δ_p the union of these cubes may not give D. In this case we consider the smallest partition containing D and define $n_0^2(x) = 1$ in the small cubes that do not belong to D. To find the solution to the limiting equation (9), we use the collocation method from (Ramm, 2009), which yields the following LAS:

$$u_j = u_{0j} - 4\pi \sum_{p=1, m\neq j}^{P} G(x_j, x_p)h(y_p)N(y_p)u_p|\Delta_p|, \qquad p = 1, 2, ..., P, \tag{17}$$

where P is the number of small cubes Δ_p, y_p is the center of Δ_p, and $|\Delta_p|$ is volume of Δ_p. From the computational point of view solving LAS (17) is much easier than solving LAS (16) if $P << M$. We have two different LAS: one is (16), the other is (17). The first corresponds to formula (15). The second corresponds to a collocation method for solving equation (9). Solving these LAS, one can compare their solutions and evaluate the limits of applicability of

the asymptotic approach from (Ramm, 2008) to solving many-body wave scattering problem in the case of small particles.

5. EM wave scattering by many small particles

Let D is the domain that contains M particles of radius a, d is distance between them. Assume that $ka \ll 1$, where $k > 0$ is the wavenumber. The governing equations for scattering problem are:

$$\nabla \times E = i\omega\mu H, \quad \nabla \times H = -i\omega\varepsilon'(x)E \text{ in } \mathrm{R}^3, \tag{18}$$

where $\omega > 0$ is the frequency, $\mu = \mu_0 = \text{const}$ is the magnetic constant, $\varepsilon'(x) = \varepsilon_0 = \text{const} > 0$ in $D' = \mathrm{R}^3 \backslash D$, $\varepsilon'(x) = \varepsilon(x) + i\frac{\sigma(x)}{\omega}$; $\sigma(x) \geq 0$, $\varepsilon'(x) \neq 0 \ \forall \ x \in \mathrm{R}^3$, $\varepsilon'(x) \in C^2(\mathrm{R}^3)$ is a twice continuously differentiable function, $\sigma(x) = 0$ in D', $\sigma(x)$ is the conductivity. From (18) one gets

$$\nabla \times \nabla \times E = K^2(x)E, \quad H = \frac{\nabla \times E}{i\omega\mu}, \tag{19}$$

where $K^2(x) = \omega^2\varepsilon'(x)\mu$. We are looking for the solution of the equation

$$\nabla \times \nabla \times E = K^2(x)E \tag{20}$$

satisfying the radiation condition

$$E(x) = E_0(x) + v, \tag{21}$$

where $E_0(x)$ is the plane wave

$$E_0(x) = \mathcal{E}e^{ik\alpha \cdot x}, \quad k = \frac{\omega}{c}, \tag{22}$$

$c = \omega\sqrt{\varepsilon\mu}$ is the wave velocity in the homogeneous medium outside D, $\varepsilon = \text{const}$ is the dielectric parameter in the outside region D', $\alpha \in S^2$ is the incident direction of the plane wave, S^2 is unit sphere in R^3, $\mathcal{E} \cdot \alpha = 0$, $\mathcal{E}$ is a constant vector, and the scattered field v satisfies the radiation condition

$$\frac{\partial v}{\partial r} - ikv = o(\frac{1}{r}), \quad r = |x| \to \infty \tag{23}$$

uniformly in directions $\beta := x/r$. If E is found, then the pair $\{E, H\}$, where H is determined by second formula (19), solves our scattering problem. It was proved in (Ramm, 2008a), that scattering problem for system (18) is equivalent to solution of the integral equation:

$$E(x) = E_0(x) + \sum_{m=1}^{M} \int_{D_m} g(x,y)p(y)E(y)dy + \sum_{m=1}^{M} \nabla_x \int_{D_m} g(x,y)q(y) \cdot E(y)dy, \tag{24}$$

where M is the number of small bodies, $p(x) = K^2(x) - k^2$, $p(x) = 0$ in D', $q(y) = \frac{\nabla K^2(x)}{K^2(x)}$, $q(x) = 0$ in D', $g(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$. Equation (24) one can rewrite as

$$E(x) = E_0(x) + \sum_{m=1}^{M} [g(x,x_m)V_m + \nabla_x g(x,x_m)v_m] + \sum_{m=1}^{M} (J_m + K_m), \tag{25}$$

where

$$J_m := \int_{D_m} [g(x,y) - g(x,x_m)p(y)E(y)dy, \tag{26}$$

$$K_m := \nabla_x \int_{D_m} [g(x,y) - g(x,x_m)q(y)E(y)dy. \tag{27}$$

Neglecting J_m and K_m, let us derive a linear algebraic system for finding V_m and v_m. If V_m and v_m , $1 \leq m \leq M$, are found, then the EM wave scattering problem for M small bodies is solved by the formula

$$E(x) = E_0(x) + \sum_{m=1}^{M} [g(x,x_m)V_m + \nabla_x g(x,x_m)v_m] \tag{28}$$

with an error $O(\frac{a}{d} + ka)$ in the domain $\min_{1\leq m\leq M} |x - x_m| := d \gg a$. To derive a linear algebraic system for V_m and v_m multiply (25) by $p(x)$, integrate over D_j, and neglect the terms J_m and K_m to get

$$V_j = V_{0j} + \sum_{m=1}^{M} (a_{jm}V_m + B_{jm}v_m), 1 \leq j \leq M, \tag{29}$$

where

$$V_{0j} := \int_{D_j} p(x)E_0(x)dx, \quad V_j := \int_{D_j} p(x)E(x)dx, \tag{30}$$

$$a_{jm} := \int_{D_j} p(x)g(x,x_m)dx, \tag{31}$$

$$B_{jm} := \int_{D_j} p(x)\nabla_x g(x,x_m)dx. \tag{32}$$

Take the dot product of (25) with $q(x)$, integrate over D_j, and neglect the terms J_m and K_m to get

$$v_j = v_{0j} + \sum_{m=1}^{M} (C_{jm}V_m + d_{jm}v_m), 1 \leq j \leq M, \tag{33}$$

where

$$v_{0j} := \int_{D_j} q(x) \cdot E_0(x)dx, \quad v_j := \int_{D_j} q(x) \cdot E(x)dx, \tag{34}$$

$$C_{jm} := \int_{D_j} q(x)g(x,x_m)dx, \tag{35}$$

$$d_{jm} := \int_{D_j} q(x) \cdot \nabla_x g(x,x_m)dx. \tag{36}$$

Equations (29) and (33) form a linear algebraic system for finding V_m and v_m, $1 \leq m \leq M$. This linear algebraic system is uniquely solvable if $ka \ll 1$ and $a \ll d$. Elements B_{jm} and C_{jm}

are vectors, and a_{jm}, d_{jm} are scalars. Under the conditions

$$\max_{1\leq j\leq M}\sum_{m=1}^{M}\left(|a_{jm}|+|d_{jm}|+||B_{jm}||+||C_{jm}||\right)<1 \tag{37}$$

one can solve linear algebraic system (29), (33) by iterations. In (37), $||B_{jm}||$ and $||C_{jm}||$ are the lengths of corresponding vectors. Condition (37) holds if $a \ll 1$ and M is not growing too fast as $a \to 0$, not faster than $O(a^{-3})$. In the process of computational modeling, it is necessary to investigate the solution of system (29), (33) numerically and to check the condition (37) for given geometrical parameters of problem.

6. Evaluation of applicability of asymptotic approach for EM scattering

One can write the linear algebraic system corresponding to formula (24) as follows (Ramm, 2008a)

$$\begin{aligned} &E_j = E_{0j} + \sum_{j\neq p,p=1}^{P} g(x_j,x_p)p(x_p)E(x_p) + \nabla_x \sum_{j\neq p,p=1}^{P} g(x_j,x_p)\,q(x_p)\cdot E(x_p), \\ &j=1,2,...,P,\ x_j,x_p\in D\ , \end{aligned} \tag{38}$$

where $E_j = E(x_j)$. Having the solution to (38), the values of $E(x)$ in all R^3 one can calculate by

$$E(x) = E_0(x) + \sum_{p=1}^{P} g(x,x_p)p(x_p)E(x_p) + \nabla_x \sum_{p=1}^{P} g(x,x_p)\,q(x_p)\cdot E(x_p). \tag{39}$$

The values $E(x_p)$ in (39) correspond to set $\{E(x_p), p = 1,...,P\}$, which is determined in (38), where P is number of collocation points. In the process of numerical calculations the integration over regions D_m in formula (24) is replaced by calculation of a Riemannian sum, and the derivative ∇_x is replaced by a divided difference. This allows one to compare the numerical solutions to system (38) with asymptotical ones calculated by the formula (28).

7. Determination of refraction coefficient for EM wave scattering

Formula (28) does not contain the parameters that characterize the properties of D, in particular, its refraction coefficient $n^2(x)$. In (Ramm, 2008a) a limiting equation, as $a \to 0$, for the effective field is derived:

$$E_e(x) = E_0(x) + \int_D g(x,y)C(y)E_e(y)dy, \tag{40}$$

and an explicit formula for refraction coefficient $n^2(x)$ is obtained. These results can be used in computational modeling. One has $E_e(x) := \lim_{a\to 0} E(x)$, and

$$C(x_m) = c_{1m}N(x_m). \tag{41}$$

Formula (41) defines uniquely a continuous function $C(x)$ since the points x_m are distributed everywhere dense in D as $a \to 0$. The function $C(x)$ can be created as we wish, since it is

determined by the numbers c_{1m} and by the function $N(x)$, which are at our disposal. Apply the operator $\nabla^2 + k^2$ to (40) and get

$$[\nabla^2 + \mathrm{K}^2(x)]E_e = 0,\ \mathrm{K}^2(x) := k^2 + C(x) := k^2 n^2(x). \tag{42}$$

Thus, the refraction coefficient $n^2(x)$ is defined by the formula

$$n^2(x) = 1 + k^{-2}C(x). \tag{43}$$

The functions $C(x)$ and $n^2(x)$ depend on the choice of $N(x)$ and c_{1m}. The function $N(x)$ in formula (7) and the numbers c_{1m} we can choose as we like. One can vary $N(x)$ and c_{1m} to reduce the discrepancy between the solution to equation (40) and the solution to equation (39). A computational procedure for doing this is described and tested for small number of particles in Section 9.

8. Numerical experiments for acoustic scattering

The numerical approach to solving the acoustic wave scattering problem for small particles was developed in (Andriychuk & Ramm, 2010). There some numerical results were given. These results demonstrated the applicability of the asymptotic approach to solving many-body wave scattering problem by the method described in Sections 3 and 4. From the practical point of view, the following numerical experiments are of interest and of importance:
a) For not very large M, say, M=2, 5, 10, 25, 50, one wants to find a and d, for which the asymptotic formula (12) (without the remainder $o(1)$) is no longer applicable;
b) One wants to find the relative accuracy of the solutions to the limiting equation (9) and to the LAS (17);
c) For large M, say, $M = 10^5$, $M = 10^6$, one wants to find the relative accuracy of the solutions to the limiting equation (9) and of the solutions to the LAS (16);
d) One wants to find the relative accuracy of the solutions to the LAS (16) and (17);
e) Using Ramm's method for creating materials with a desired refraction coefficient, one wants to find out for some given refraction coefficients $n^2(x)$ and $n_0^2(x)$, what the smallest M (or, equivalently, largest a) is for which the corresponding $n^2_{M(x)}$ differs from the desired $n^2(x)$ by not more than, say, 5% - 10%. Here $n^2_{M(x)}$ is the value of the refraction coefficient of the material obtained by embedding M small particles into D accoring to the recipe described below.
We take $k = 1$, $\kappa = 0.9$, and $N(x) = \text{const}$ for the numerical calculations. For $k = 1$, and a and d, used in the numerical experiments, one can have many small particles on the wavelength. Therefore, the multiple scattering effects are not negligible.

8.1 Applicability of asymptotic formulas for small number of particles

We consider the solution to LAS (17) with 20 collocation points along each coordinate axis as the benchmark solution. The total number P of the collocation points is $P = 8000$. The applicability of the asymptotic formulas is checked by solving LAS (16) for small number M of particles and determining the problem parameters for which the solutions to these LAS are close. A standard interpolation procedure is used in order to obtain the values of the solution to (17) at the points corresponding to the position of the particles. In this case the number P of

the collocation points exceeds the number M of particles. In Fig. 2, the relative errors of real (solid line) and imaginary (dashed line) parts, as well as the modulus (dot-dashed line) of the solution to (16) are shown for the case $M = 4$; the distance between particles is $d = a^{(2-\kappa)/3}C$, where C is an additional parameter of optimization (in our case $C = 5$, that yields the smallest error of deviation of etalon and asymptotic field components), $N(x) = 5$. The minimal relative error of the solution to (16) does not exceed 0.05% and is reached when $a \in (0.02, 0.03)$. The value of the function $N(x)$ influences (to a considerable degree) the quality of approximation. The relative error for $N(x) = 40$ with the same other parameters is shown in Fig. 3. The error is smallest at $a = 0.01$, and it grows when a increases. The minimal error that we were able to obtain for this case is about 0.01% . The dependence of the error on the distance d between

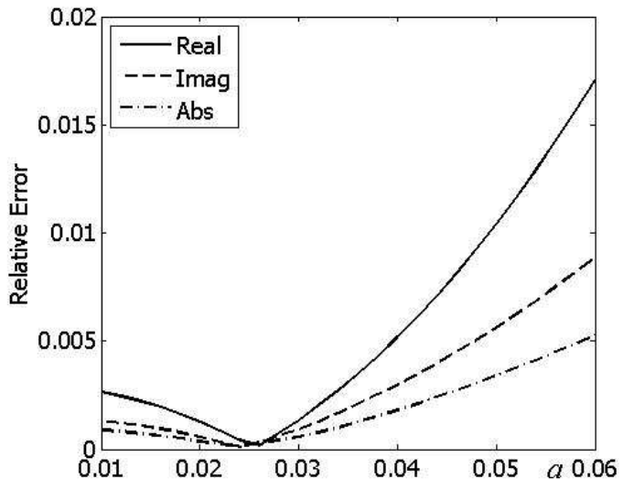

Fig. 2. Relative error of solution to (16) versus size a of particle, $N(x) = 5$

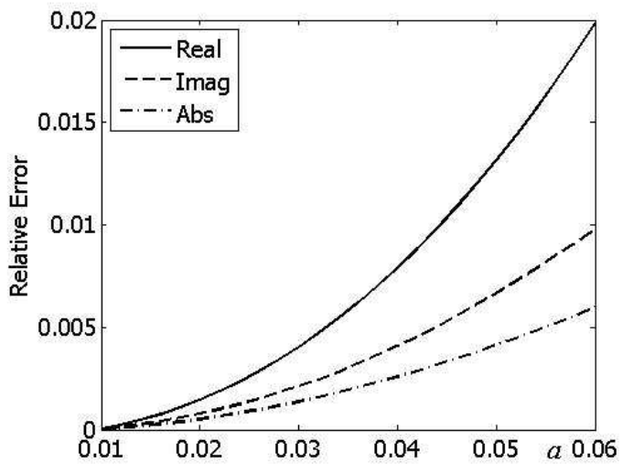

Fig. 3. Relative error of solution to (16) versus size a of particle, $N(x) = 40$

particles for a fixed a was investigated as well. In Fig. 4, the relative error versus parameter d is shown. The number of particles $M = 4$, the radius of particles $a = 0.01$. The minimal error

was obtained when $C = 14$. This error was 0.005% for the real part, 0.0025% for the imaginary part, and 0.002% for the modulus of the solution.
The error grows significantly when d deviates from the optimal value, i.e., the value of d for which the error of the calculated solution to LAS (16) is minimal. Similar results are obtained for the case $a = 0.02$ (see Fig. 5). For example, at $M = 2$ the optimal value of d is 0.038 for $a = 0.01$, and it is 0.053 for $a = 0.02$. The error is even more sensitive to changes of the distance d in this case. The minimal value of the error is obtained when $C = 8$. The error was 0.0078% for the real part, 0.0071% for the imaginary part, and 0.002% for the modulus of the solution.
The numerical results show that the accuracy of the approximation of the solutions to LAS

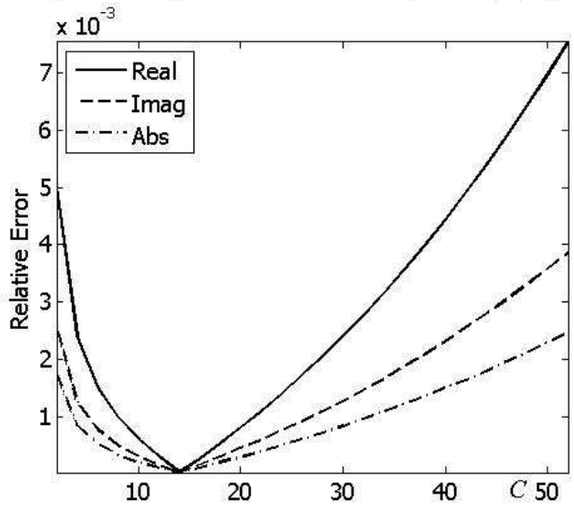

Fig. 4. Relative error of solution versus distance d between particles, $a = 0.01$

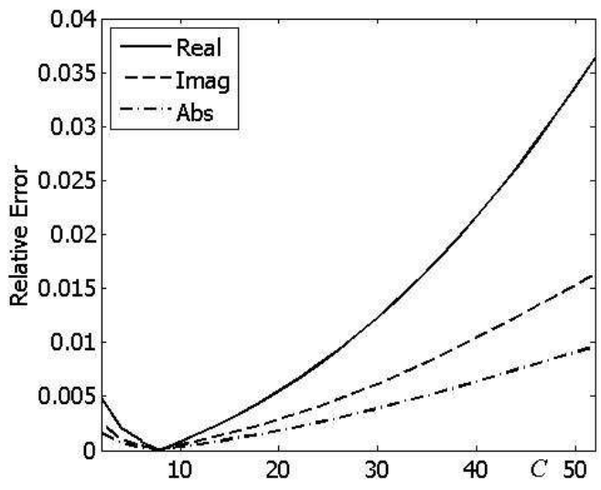

Fig. 5. Relative error of solution versus distance d between particles, $a = 0.02$

	M value			
	$M = 2$	$M = 4$	$M = 6$	$M = 8$
$a = 0.01$	0.038	0.025	0.026	0.027
$a = 0.02$	0.053	0.023	0.027	0.054

Table 1. Optimal values of d for small M

	M value			
	$M = 10$	$M = 20$	$M = 30$	$M = 40$
$a = 0.01$	0.011	0.0105	0.007	0.006
$a = 0.02$	0.016	0.018	0.020	0.023

Table 2. Optimal values of d for medium M

(16) and (17) depends on a significantly, and it improves when a decreases. For example, the minimal error, obtained at $a = 0.04$, is equal to 0.018%. The optimal values of d are given in Tables 1, and 2 for small and not so small M respectively. The numerical results show that the distribution of particles in the medium does not influence significantly the optimal values of d. By optimal values of d we mean the values at which the error of the solution to LAS (16) is minimal when the values of the other parameters are fixed. For example, the optimal values of d for $M = 8$ at the two types of the distribution of particles: $(2 \times 2 \times 2)$ and $(4 \times 2 \times 1)$ differ by not more than 0.5% . The numerical results demonstrate that to decrease the relative error of solution to system (16), it is necessary to make a smaller if the value of d is fixed. One can see that the quality of approximation improves as $a \to 0$, but the condition $d >> a$ is not valid for small number M of particles: the values of the distance d is of the order $O(a)$.

8.2 Accuracy of the solution to the limiting equation

The numerical procedure for checking the accuracy of the solution to equation (9) uses the calculations with various values of the parameters k, a, l_D, and $h(x)$, where l_D is diameter of D. The absolute and relative errors were calculated by increasing the number of collocation points. The dependence of the accuracy on the parameter ρ, where $\rho = \sqrt[3]{P}$, P is the total number of small subdomains in D, is shown in Fig. 6 and Fig. 7 for $k = 1.0$, $l_D = 0.5$, $a = 0.01$ at the different values of $h(x)$. The solution corresponding to ρ =20 is considered as "exact" solution (the number P for this case is equal to 8000). The error of the solution to equation (9) is equal to 1.1% and 0.02% for real and imaginary part, respectively, at $\rho = 5$ (125 collocation points), it decreases to values of 0.7% and 0.05% if $\rho = 6$ (216 collocation points), and it decreases to values 0.29% and 0.02% if $\rho = 8$ (512 collocation points), $h(x) = k^2(1 - 3i)/(40\pi)$. The relative error smaller than 0.01% for the real part of solution is obtained at $\rho = 12$, this error tends to zero when ρ increases. This error depends on the function $h(x)$ as well, it diminishes when the imaginary part of $h(x)$ decreases. The error for the real and imaginary parts of the solution at $\rho = 19$ does not exceed 0.01%. The numerical calculations show that the error depends much on the value of k. In Fig. 8 and Fig. 9 the results are shown for $k = 2.0$ and $k = 0.6$ respectively ($h(x) = k^2(1 - 3i)/(40\pi)$). It is seen that the error is nearly 10 times larger at $k = 2.0$. The maximal error (at $\rho = 5$) for $k = 0.6$ is less than 30% of the error for $k = 1.0$. This error tends to zero even faster for smaller k.

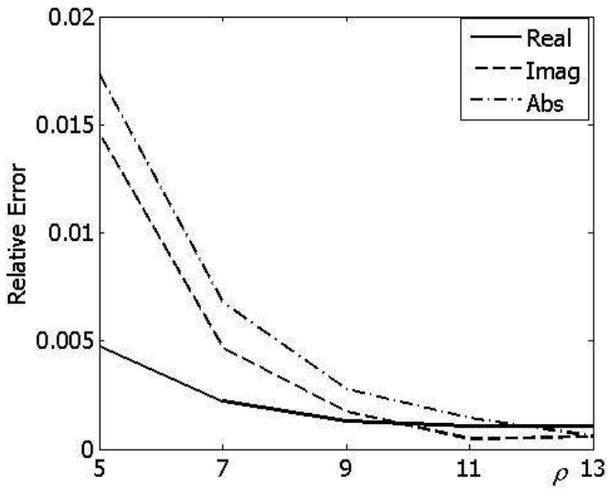

Fig. 6. Relative error versus the ρ parameter, $h(x) = k^2(1-7i)/(40\pi)$

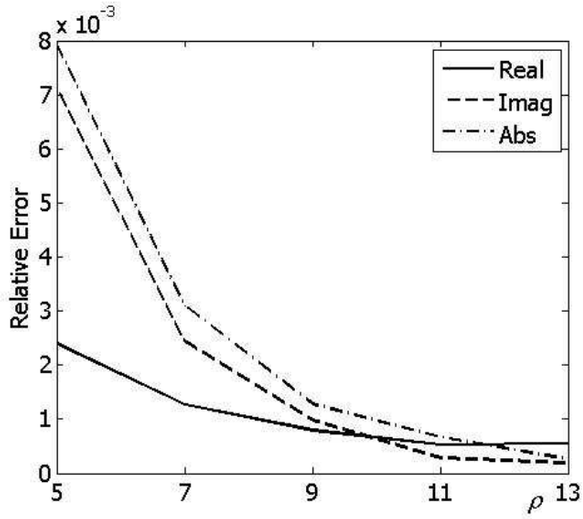

Fig. 7. Relative error versus the ρ parameter, $h(x) = k^2(1-3i)/(40\pi)$

8.3 Accuracy of the solution to the limiting equation (9) and to the asymptotic LAS (16)

As before, we consider as the "exact" solution to (9) the approximate solution to LAS (17) with $\rho = 20$. The maximal relative error for such ρ does not exceed 0.01% in the range of problem parameters we have considered ($k = 0.5 \div 1.0$, $l_D = 0.5 \div 1.0$, $N(x) \geq 4.0$). The numerical calculations are carried out for various sizes of the domain D and various function $N(x)$. The results for small values of M are presented in Table 3 for $k = 1$, $N(x) = 40$, and $l_D = 1.0$. The second line contains the values of a_{est}, the estimated value of a, calculated by formula (7), with the number $\mathcal{N}(\Delta_p)$ replacing the number M. In this case the radius of a particle is calculated as

$$a_{est} = (M/\int\limits_{\Delta_p} N(x)dx)^{1/(2-\kappa)}. \tag{44}$$

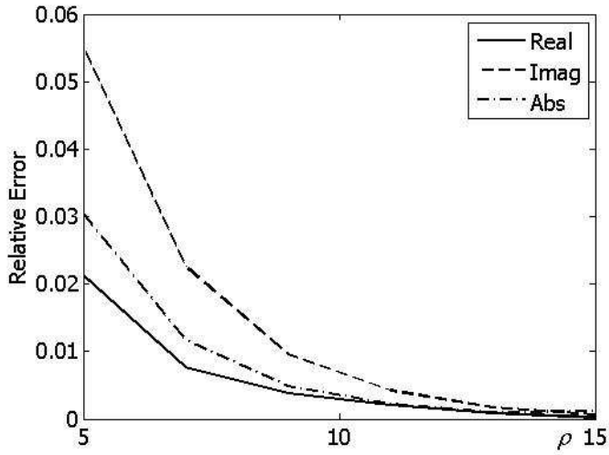

Fig. 8. Relative error versus the ρ parameter, $k = 2.0$

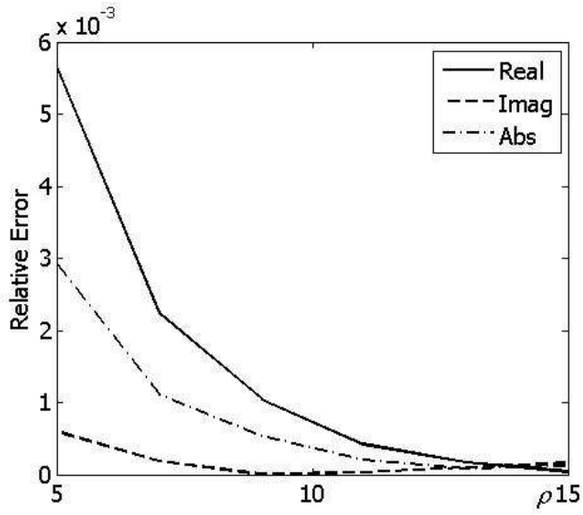

Fig. 9. Relative error versus the ρ parameter, $k = 0.6$

The values of a_{opt} in the third line correspond to optimal values of a which yield minimal relative error of the modulus of the solutions to equation (9) and LAS (16). The fourth line contains the values of the distance d between particles. The maximal value of the error is obtained when $\mu = 7$, $\mu = \sqrt[3]{M}$ and it decreases slowly when μ increases. The calculation results for large number of μ with the same set of input parameters are shown in Table 4. The minimal error of the solutions is obtained at $\mu = 60$ (total number of particles $M = 2.16 \cdot 10^5$. Tables 5 and 6 contain similar results for $N(x) = 4.0$, other parameters being the same. It is seen that the relative error of the solution decreases when number of particles M increases. This error can be decreased slightly (on 0.02%-0.01%) by small change of the values a and l_D as well. The relative error of the solution to LAS (16) tends to the relative error of the solution to LAS (17) when the parameter μ becomes greater than 80 ($M = 5.12 \cdot 10^5$). The relative error of the solution to LAS (17) is calculated by taking the norm of the difference of the solutions

μ	7	9	11	13	15
a_{est}	0.1418	0.0714	0.0413	0.0262	0.0177
a_{opt}	0.1061	0.0612	0.0382	0.0261	0.0172
d	0.1333	0.1105	0.0924	0.0790	0.0688
Rel.error	2.53%	0.46%	0.45%	1.12%	0.81%

Table 3. Optimal parameters of D for small μ, $N(x) = 40.0$

μ	20	30	40	50	60
a_{est}	0.0081	0.0027	0.0012	6.65×10^{-4}	4.04×10^{-4}
a_{opt}	0.0077	0.0025	0.0011	6.6×10^{-4}	4.04×10^{-4}
d	0.0526	0.0345	0.0256	0.0204	0.0169
Rel.error	0.59%	0.35%	0.36%	0.27%	0.19%

Table 4. Optimal parameters of D for big μ, $N(x) = 40.0$

μ	7	9	11	13	15
a_{est}	0.0175	0.0088	0.0051	0.0032	0.0022
a_{opt}	0.0179	0.0090	0.0052	0.0033	0.0022
d	0.1607	0.1228	0.0990	0.0828	0.0711
Rel.error	1.48%	1.14%	1.06%	1.05%	0.91%

Table 5. Optimal parameters of D for small μ, $N(x) = 4.0$

μ	20	30	40	50	60
a_{est}	9.97×10^{-4}	3.30×10^{-4}	1.51×10^{-4}	8.20×10^{-5}	4.98×10^{-5}
a_{opt}	1.02×10^{-3}	3.32×10^{-4}	1.50×10^{-4}	8.21×10^{-5}	4.99×10^{-5}
d	0.0542	0.0361	0.0265	0.0209	0.0172
Rel.error	0.21%	0.12%	0.11%	0.07%	0.03%

Table 6. Optimal parameters of D for big μ, $N(x) = 4.0$

to (17) with P and $2P$ points, and dividing it by the norm of the solution to (17) calculated for $2P$ points. The relative error of the solution to LAS (16) is calculated by taking the norm of the difference between the solution to (16), calculated by an interpolation formula at the points y_p from (17), and the solution of (17), and dividing the norm of this difference by the norm of the solution to (17).

8.4 Investigation of the relative difference between the solution to (16) and (17)

A comparison of the solutions to LAS (16) and (17) is done for various values of a, and various values of the number ρ and μ. The relative error of the solution decreases when ρ grows and μ remains the same. For example, when ρ increases by 50% , the relative error decreases by 12% (for $\rho = 8$ and $\rho = 12$, $\mu = 15$). The differences between the real parts, imaginary parts, and moduli of the solutions to LAS (16) and (17) are shown in Fig. 10 and Fig. 11 for $\rho = 7$, $\mu = 15$. The real part of this difference does not exceed 4% when $a = 0.01$, it is less than 3.5% at $a = 0.008$, less than 2% at $a = 0.005$; $d = 8a$, $N(x) = 20$. This difference is less than 0.08% when $\rho = 11$, $a = 0.001$, $N = 30$, and $d = 15a$ (μ remains the same). Numerical calculations for wider range of the distance d demonstrate that there is an optimal value of d, starting

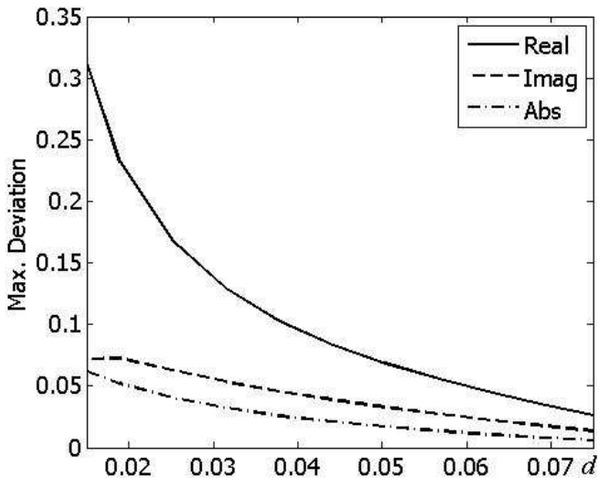

Fig. 10. Deviation of component field versus the distance d between particles, $N(x) = 10$

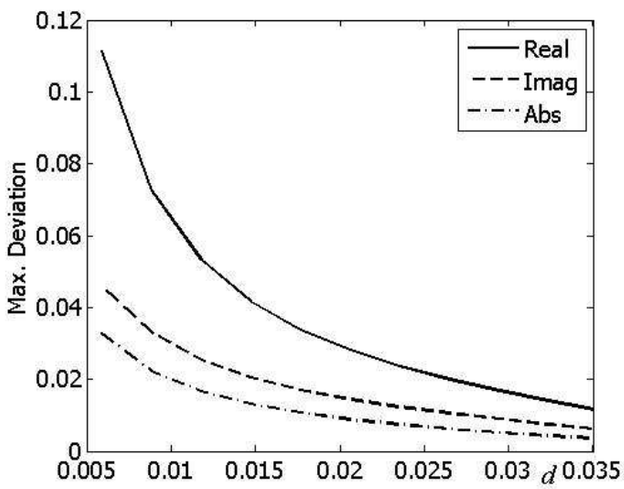

Fig. 11. Deviation of component field versus the distance d between particles, $N(x) = 30$

from which the deviation of solutions increases again. These optimal values of d are shown in Table 7 for various $N(x)$. The calculations show that the optimal distance between particles increases when the number of particles grows. For small number of particles (see Table 1 and Table 2) the optimal distance is the value of the order a. For the number of particles $M = 15^3$, i.e. $\mu = 15$, this distance is about $10a$.

The values of maximal and minimal errors of the solutions for the optimal values of distance d are shown in Table 8.

One can conclude from the numerical results that optimal values of d decrease slowly when the function $N(x)$ increases. This decreasing is more pronounced for smaller a. The relative error of the solution to (16) also smaller for smaller a.

	$N(x)$ value			
	$N(x) = 10$	$N(x) = 20$	$N(x) = 30$	$N(x) = 40$	$N(x) = 50$
$a = 0.005$	0.07065	0.04724	0.04716	0.04709	0.04122
$a = 0.001$	0.08835	0.07578	0.06331	0.06317	0.05056

Table 7. Optimal values of d for various $N(x)$

	$N(x)$ value			
	$N(x) = 10$	$N(x) = 20$	$N(x) = 30$	$N(x) = 40$	$N(x) = 50$
$a = 0.005$	0.77/0.12	5.25/0.56	0.52/0.1	0.97/0.12	0.32/0.05
$a = 0.001$	2.47/0,26	1.7/0.3	0.5/0.1	2.7/0,37	1.5/0.2

Table 8. Relative error of solution in % (max/min) for optimal d

8.5 Evaluation of difference between the desired and obtained refraction coefficients

The recipe for creating the media with a desired refraction coefficient $n^2(x)$ was proposed in (Ramm, 2008a). It is important from the computational point of view to see how the refraction coefficient $n_M^2(x)$, created by this procedure, differs from the one, obtained theoretically. First, we describe the recipe from (Ramm, 2010a) for creating the desired refraction coefficient $n^2(x)$. By $n_0^2(x)$ we denote the refraction coefficient of the given material.

The recipe consists of three steps.

Step 1. Given $n_0^2(x)$ and $n^2(x)$, calculate

$$\bar{p}(x) = k^2[n_0^2(x) - n^2(x)] = \bar{p}_1(x) + i\bar{p}_2(x). \tag{45}$$

Step 1 is trivial from the computational and theoretical viewpoints.

Using the relation

$$\bar{p}(x) = 4\pi h(x)N(x) \tag{46}$$

from (Ramm, 2008a) and equation (45), one gets the equation for finding $h(x) = h_1(x) + ih_2(x)$, namely:

$$4\pi[h_1(x) + ih_2(x)]N(x) = \bar{p}_1(x) + i\bar{p}_2(x). \tag{47}$$

Therefore,

$$N(x)h_1(x) = \frac{\bar{p}_1(x)}{4\pi}, \quad N(x)h_2(x) = \frac{\bar{p}_2(x)}{4\pi}. \tag{48}$$

Step 2. Given $\bar{p}_1(x)$ and $\bar{p}_2(x)$, find $\{h_1(x), h_2(x), N(x)\}$.

The system (48) of two equations for the three unknown functions $h_1(x)$, $h_2(x) \leq 0$, and $N(x) \geq 0$, has infinitely many solutions $\{h_1(x), h_2(x), N(x)\}$. If, for example, one takes $N(x)$ to be an arbitrary positive constant, then h_1 and h_2 are uniquely determined by (48). The condition $\mathrm{Im}n^2(x) > 0$ implies $\mathrm{Im}\bar{p} = \bar{p}_2 < 0$, which agrees with the condition $h_2 < 0$ if $N(x) \geq 0$. One takes $N(x) = h_1(x) = h_2(x) = 0$ at the points at which $\bar{p}_1(x) = \bar{p}_2(x) = 0$. One can choose, for example, N to be a positive constant:

$$N(x) = N = \text{const}, \tag{49}$$

$$h_1(x) = \frac{\bar{p}_1(x)}{4\pi N}, \quad h_2(x) = \frac{\bar{p}_2(x)}{4\pi N}. \tag{50}$$

Calculation of the values $N(x)$, $h_1(x)$, $h_2(x)$ by formulas (49)-(50) completes *Step 2* our procedure.

Step 2. is easy from computational and theoretical viewpoints.

Step 3. This step is clear from the theoretical point of view, but it requires solving two basic technological problems. First, one has to embed many (M) small particles into D at the approximately prescribed positions according to formula (7). Secondly, the small particles have to be prepared so that they have prescribed boundary impedances $\zeta_m = h(x_m)a^{-\kappa}$, see formula (1).

Consider a partition of D into union of small cubes Δ_p, which have no common interior points, and which are centered at the points $y^{(p)}$, and embed in each cube Δ_p the number

$$\mathcal{N}(\Delta_p) = \left[\frac{1}{a^{2-\kappa}} \int\limits_{\Delta_p} N(x)dx \right] \tag{51}$$

of small balls D_m of radius a, centered at the points x_m, where $[b]$ stands for the integer nearest to $b > 0$, $\kappa \in (0,1)$. Let us put these balls at the distance $O(a^{\frac{2-\kappa}{3}})$, and prepare the boundary impedance of these balls equal to $\frac{h(x_m)}{a^\kappa}$, where $h(x)$ is the function, calculated in Step 2 of our recipe. It is proved in (Ramm, 2008a) that the resulting material, obtained by embedding small particles into D by the above recipe, will have the desired refraction coefficient $n^2(x)$ with an error that tends to zero as $a \to 0$.

Let us emphasize again that Step 3 of our procedure requires solving the following technological problems:

(i) How does one prepare small balls of radius a with the prescribed boundary impedance? In particular, it is of practical interest to prepare small balls with large boundary impedance of the order $O(a^{-\kappa})$, which has a prescribed frequency dependence.

(ii) How does one embed these small balls in a given domain D, filled with the known material, according to the requirements formulated in Step 3 ?

The numerical results, presented in this Section, allow one to understand better the role of various parameters, such as a, M, d, ζ, in an implementation of our recipe. We give the numerical results for $N(x) = \text{const}$. For simplicity, we assume that the domain D is a union of small cubes (subdomains) Δ_p ($D = \bigcup\limits_{p=1}^{P} \Delta_p$). This assumption is not a restriction in practical applications. Let the functions $n_0^2(x)$ and $n^2(x)$ be given. One can calculate the values h_1 and h_2 in (50) and determine the number $\mathcal{N}(\Delta_p)$ of the particles embedded into D. The value of the boundary impedance $\frac{h(x_m)}{a^\kappa}$ is easy to calculate. Formula (51) gives the total number of the embedded particles. We consider a simple distribution of small particles. Let us embed the particles at the nodes of a uniform grid at the distances $d = O(a^{\frac{2-\kappa}{3}})$. The numerical calculations are carried out for the case $D = \bigcup\limits_{p=1}^{P} \Delta_p, P = 8000$, D is cube with side $l_D = 0.5$, the particles are embedded uniformly in D. For this P the relative error in the solution to LAS (16) and (17) does not exceed 0.1%. Let the domain D be placed in the free space, namely $n_0^2(x) = 1$, and the desired refraction coefficient be $n^2(x) = 2 + 0.01i$. One can calculate the value of $\mathcal{N}(\Delta_p)$ by formula (51). On the other hand, one can choose the number μ, such that

$M = \mu^3$ is closest to $\mathcal{N}(\Delta_p)$. The functions $\tilde{n}_1^2(x)$ and $\tilde{n}_2^2(x)$, calculated by the formula

$$\tilde{n}_1^2(x) = -\frac{4\pi h_1(x)N}{k^2} + n_0^2, \quad \tilde{n}_2^2(x) = -\frac{4\pi h_2(x)N}{k^2}, \tag{52}$$

differ from the desired coefficients $n_1^2(x)$ and $n_2^2(x)$. In (52), $N = \frac{Ma^{2-\kappa}}{V_D}$, V_D is volume of D, $\kappa < 1$ is chosen very close to 1, $\kappa = 0.99$. To obtain minimal discrepancy between $n_j^2(x)$ and $\tilde{n}_j^2(x)$, $j = 1, 2$, we choose two numbers μ_1 and μ_2 such that $M_1 < \mathcal{N}(\Delta_p) < M_2$, where $M_1 = \mu_1^3$ and $M_2 = \mu_2^3$. Hence, having the number $\mathcal{N}(\Delta_p)$ for a fixed a, one can estimate the numbers M_1 and M_2, and calculate the approximate values of $n_1^2(x)$ and $n_2^2(x)$ by formula (52). In Fig. 12, the minimal relative error of the calculated value $\tilde{n}^2(x)$ depending on the radius a of particle is shown for the case $N(x) = 5$ (the solid line corresponds to the real part of the error, and the dashed line corresponds to the imaginary part of the error in the Figs. 12-14). These results show that the error depends significantly on the relation between the

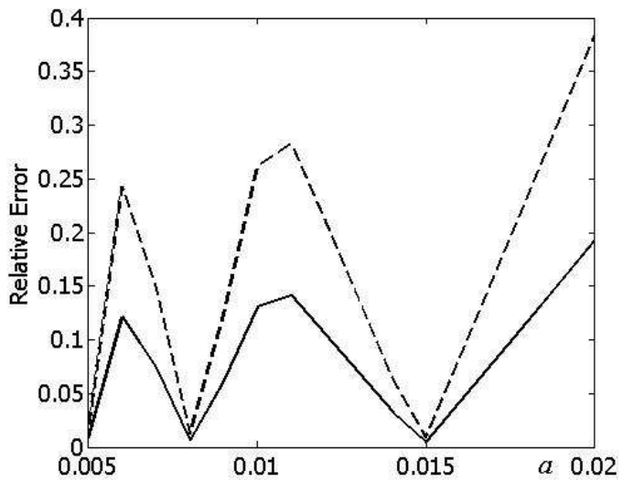

Fig. 12. Minimal relative error for calculated refraction coefficient $\tilde{n}^2(x)$, $N(x) = 5$

numbers M_1, M_2, and $\mathcal{N}(\Delta_p)$. The error is smallest when one of the values M_1 and M_2 is sufficiently close to $\mathcal{N}(\Delta_p)$. The error has quasiperiodic nature with growing amplitude as a increases (this is clear from the behavior of the function $\mathcal{N}(\Delta_p)$ and values M_1 and M_2). The average error on a period increases as a grows. Similar results are shown in Fig. 13 and Fig. 14 for $N = 20$ and $N = 50$ respectively. The minimal error is attained when $a = 0.015$, and this error is 0.51%. The error is 0.53% when $a = 0.008$, and it is equal to 0.27% when $a = 0.006$ for $N(x) = 20, 50$ respectively. Uniform (equidistant) embedding small particles into D is simple from the practical point of view. The results in Figs. 12-14 allow one to estimate the number M of particles needed for obtaining the refraction coefficient close to a desired one in a given domain D. The results for $l_D = 0.5$ are shown in Fig. 15. The value $\mu = \sqrt[3]{M}$ is marked on the y axes here. Solid, dashed, and dot-dashed line correspond to $N(x) = 5, 20, 50$, respectively. One can see from Fig. 15 that the number of particles decreases if radius a increases. The value $d = O(a^{(2-\kappa)/3})$ gives the distance d between the embedded particles. For example, for $N(x) = 5$, $a = 0.01$ d is of the order 0.1359, the calculated d is equal to 0.12 and to 0.16 for

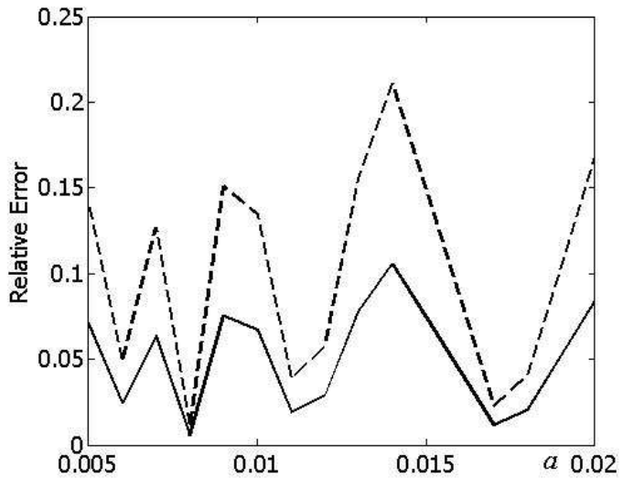

Fig. 13. Minimal relative error for calculated refraction coefficient $\tilde{n}^2(x), N(x) = 20$

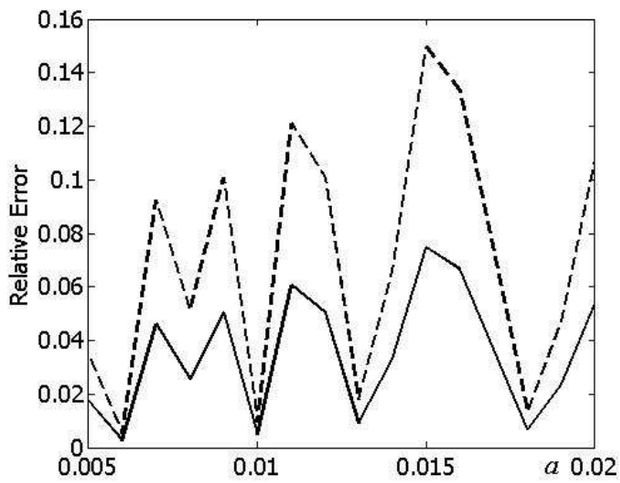

Fig. 14. Minimal relative error for calculated refraction coefficient $\tilde{n}^2(x), N(x) = 50$

$\mu = 5$ and $\mu = 4$, respectively. The calculations show that the difference between the both values of d is proportional to the relative error for the refraction coefficients. By the formula $d = O(a^{(2-\kappa)/3})$, the value of d does not depend on the diameter l_D of D. This value can be used as an additional optimization parameter in the procedure of the choice between two neighboring μ in Tables 9, 10. On the other hand, one can estimate the number of the particles embedded into D using formula (51). Given $\mathcal{N}(\Delta_p)$, one can calculate the corresponding number M of particles if the particles distribution is uniform. The distance between particles is also easy to calculate if l_D is given. The optimal values of $\mu, \mu = \sqrt[3]{M}$ are shown in the Tables 9 and 10 for $l_D = 0.5$ and $l_D = 1.0$ respectively.

The numerical calculations show that the relative error of $\tilde{n}^2(x)$ for respective μ can be decreased when the estimation of d is taken into account. Namely, one should choose μ from Tables 9 or 10 that gives value of d close to $(a^{(2-\kappa)/3})$.

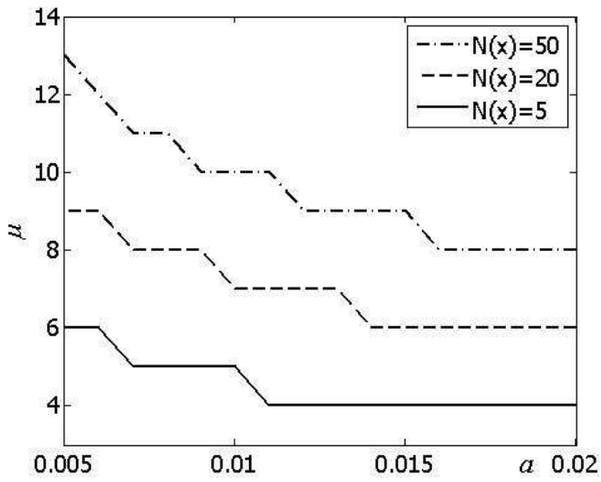

Fig. 15. Optimal value of μ versus the radius a for various $N(x)$

a	$\mathcal{N}(\Delta_p)$	Optimal μ
0.02	96.12	$4 \leq \mu \leq 5$
0.01	204.05	$4 \leq \mu \leq 5$
0.008	245.62	$6 \leq \mu \leq 7$
0.005	416.17	$7 \leq \mu \leq 8$
0.001	2442.1	$13 \leq \mu \leq 14$

Table 9. Optimal values of μ for $l_D = 0.5$

a	$\mathcal{N}(\Delta_p)$	Optimal μ
0.02	809.25	$9 \leq \mu \leq 10$
0.01	1569.1	$11 \leq \mu \leq 12$
0.008	1995.3	$12 \leq \mu \leq 13$
0.005	3363.3	$15 \leq \mu \leq 16$
0.001	19753	$27 \leq \mu \leq 28$

Table 10. Optimal values of μ for $l_D = 1.0$

9. Numerical results for EM wave scattering

Computing the solution by limiting formula (28) requires much PC time because one computes $3 - D$ integrals by formulas (30)-(32) and (34)-(36). Therefore, the numerical results, presented here, are restricted to the case of not too large number of particles ($M \leq 1000$). The modeling results demonstrate a good agreement with the theoretical predictions, and demonstrate the possibility to create a medium with a desired refraction coefficient in a way similar to the one in the case of acoustic wave scattering.

9.1 Comparison of "exact" and asymptotic solution

Let $\alpha = e_3$, where e_3 is unit vector along z axis, then the condition yields $E \cdot \alpha = 0$, that vector E is placed in the xOy plane, i. e. it has two components E_x and E_y only. In the case

M	$a = 0.1$	$a = 0.2$	$a = 0.3$	$a = 0.4$
8	0.351	0.798	0.925	1.457
27	0.327	0.825	0.956	1.596
64	0.315	0.867	1.215	1.691
125	0.306	0.935	1.454	1.894

Table 11. Minimal values of d guaranteeing the convergence of iterative process (29), (33)

if domain D is placed symmetrically to axis z and $\alpha = e_3$ one can consider the component E_x or E_y because of symmetry (this restriction is valid if $x-$ and $y-$components in E_0 are the same). The applicability of asymptotic approach was checked by comparison of solution by the limiting formula (28) and solution determined by the formula (39). The first solution implies the knowledge of vectors V_j and numbers v_j which are received from the solutions to LAS (29), (33). The second solution requires the values $\{E(y_p), p = 1, ..., P\}$, which are received as solution to LAS (38) by the collocation method (Ramm, 2009). We consider the solution to LAS (38) with 15 collocation points along each coordinate axis as a benchmark or "exact" solution. The total number P of the collocation points is $P = 3375$ and relative error of solution does not exceed 0.5% in the range of considered values a, d, and M. The LAS (29), (33) is solved by iterations and condition (37) superimposes considerable restriction on the relation d to a. The analytical estimation gives $d \sim 15a$ and greater. It means that dimensions of D at big number of M are very large that can not satisfy the engineering requirements. Therefore, the knowledge of minimal values d at which the iterative process for solution to system (29), (33) is still converged has a practical importance. In Table 11, the minimal values of d for several a at fixed number of particles M are shown.
One can see that allowable distance d is order $d \sim 4a$ that is less three times than theoretical estimation.
The investigation of the amplitude field deviation for the both solutions depending on the radius a of particle was performed for points in the middle and far zone at $M = 125$, $k = 0.1$, and $d = 1$. In Fig. 16, the results are presented for the far zone of D ($d_f = 15$, where d_f is distance from center of D to far zone). The thick curves correspond to the case of the same amplitude distribution of $x-$ and $y-$components of the field $E_0(x)$, and thin curves correspond to the case of various $x-$ and $y-$components. In the middle zone the solutions differ in the limits of 20% and greater at the small values of a, this difference grows if a increases. The results for the far zone are in good correspondence with theoretical condition, i. e., the asymptotical solution tends to "exact" one as $a \to 0$. The maximum deviation of field components is observed at $a = 0.05$ and it is equal to 5%, and it is equal to 25% if a grows to 0.5. The relative error can be decreased in the considerable extent if the value of d to icrease. In the above example the relation d/a is equal to 2 only, and it is complicate situation for our asymptotical approach.

9.2 Creating the desired refraction coefficient

In Section 7, the formula for refraction coefficient $n^2(x)$ for domain D with $\mathcal{N}(\Delta)$ embedded particles of radius a was derived. If $n^2(x)$ is presribed, one can easy to determine the parameters of D that can provide the desired value of refraction coefficient. Similarly to

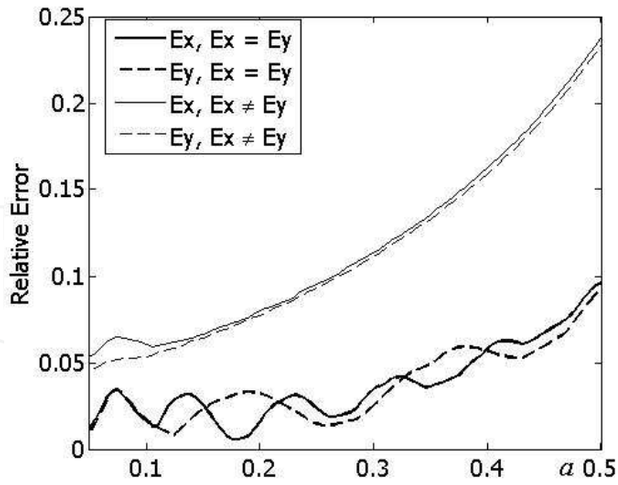

Fig. 16. Relative error of solution to limiting equation (28) for differing $E_0(x)$

M	$N(x)$	c_{1m}	γ_m	$\max\|p(x)\|$	Relative error
8	0.7407	0.0675	2.0250	64.4578	0.0005
27	0.5400	0.0912	2.7360	87.0896	0.0009
64	0.4665	0.1072	3.2154	102.1494	0.0023

Table 12. Optimal parameters of D for $n^2(x) = 1.2$

the case of acoustic wave scattering, we formulate constructive recipe to create the media with desired refraction coefficient. Let us denote the refraction coefficient of medium without embedded particles $n_0^2(x) = 1$. We develop a method to create a desired refraction coefficient $n^2(x)$. To do this, we impose some mild restrictions on the function $N(x)$ and $p(x)$. Let the domain D be a cube with M embedded particles. If one assumes that $N(x) = \text{const}$ in D, then $N(x) = Ma^2/(d+2a)^3$. Having the prescribed $n^2(x)$ and known $N(x)$, one can find c_{1m} from the relation $C(x) = c_{1m}N(x)$, and number γ_m by the formula $\gamma_m = 30c_{1m}$ (see (Ramm, 2008a)). In order to derive the limiting equation of the form (40), the function $p(x)$ is chosen as follows:

$$p(r) = p(r,a) = \begin{cases} \frac{\gamma_m}{4\pi a^\kappa}(1-t)^2, & 0 \le t \le 1, \\ 0, & t > 1; \ t := \frac{r}{a}, \ \kappa = \text{const} > 0. \end{cases} \tag{53}$$

The values of various parameters, calculated by above procedure, are presented in Tables 12 and 13. The relative error of the asymptotic solution is presented in the last columns in these Tables. This error is minimal at the value of $\max p(x)$ presented in the neighboring column.
In order to obtain greater $n^2(x)$ it is necessary to increase $p(x)$ remaining the same of rest parameters.
The dependence of $n^2(x)$ on a for the various d is shown in Fig. 17 at $M = 125$ and Fig. 18 at $M = 1000$. At the small values of a the scattering from D is negligible, therefore $n^2(x) \to n_0^2(x)$ as $a \to 0$. If a grows, then $n^2(x)$ decreases and differs considerably from $n_0^2(x)$.
The relative error of the solution to limiting equation (40) is shown in Fig. 19. The error gets smaller as $a \to 0$. The numerical results show that the relative error for various d gets larger if a approaches $d/3$.

M	$N(x)$	c_{1m}	γ_m	$\max\|p(x)\|$	Relative error
8	0.7407	0.1350	4.0500	128.9155	0.0008
27	0.5400	0.1825	5.4750	174.2747	0.0012
64	0.4665	0.2144	6.4309	204.7019	0.0033

Table 13. Optimal parameters of D for $n^2(x) = 1.4$

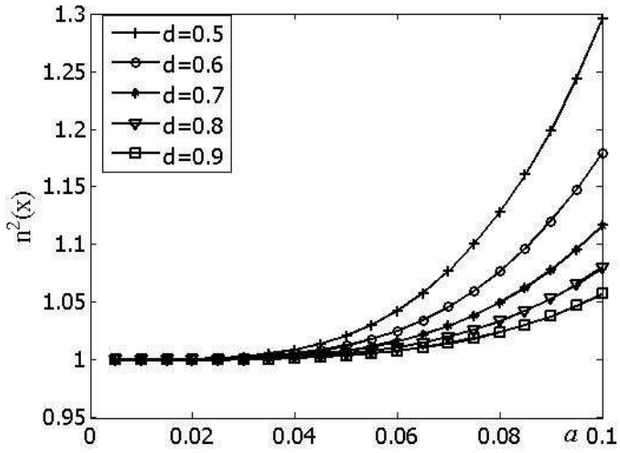

Fig. 17. The refraction coefficient $n^2(x)$ at $M = 125$

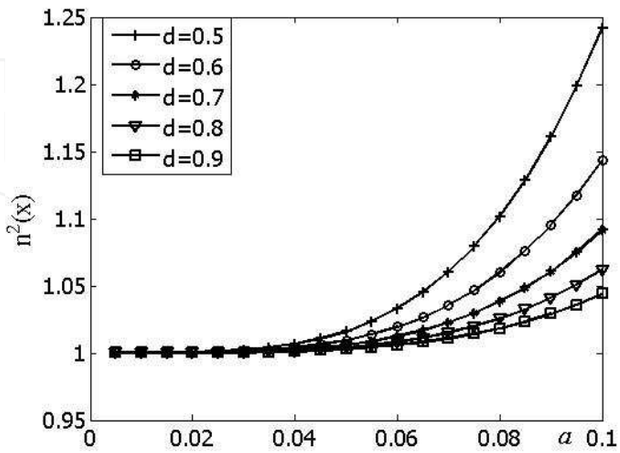

Fig. 18. The refraction coefficient $n^2(x)$ at $M = 1000$

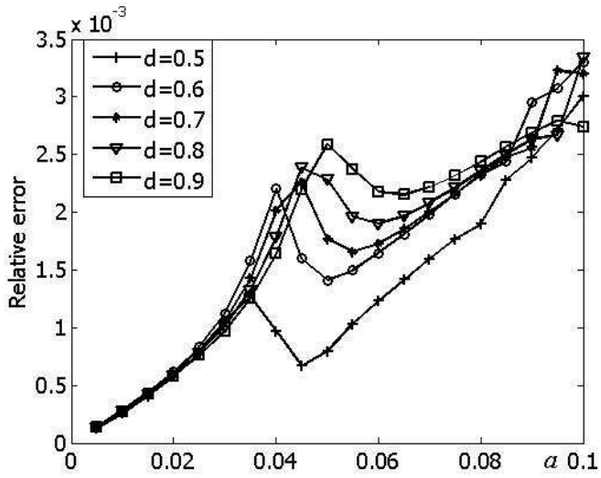

Fig. 19. The relative error of solution to limiting equation (40) for various d

10. Conclusions

The numerical results based on the asymptotical approach to solving the scattering problem in a material with many small particles embedded in it help to understand better the dependence of the effective field in the material on the basic parameters of the problem, namely, on $a, M, d, \zeta_m, N(x)$, and $h(x)$, and to give a constructive way for creating materials with a desired refraction coefficient $n^2(x)$, see (Ramm, 2009a), (Ramm, 2010), (Ramm, 2010a).

For acoustic wave scattering, it is shown that, for small number M of particles there is an optimal value of a, for which the relative error to asymptotic solution is minimal. When $a \to 0$ and M is small ($M < 100$) the matrix of (16) is diagonally dominant and the error goes to 0. This is confirmed by the numerical results as well. The relative error can be decreased by changing function $N(x)$ or by decreasing a, d being fixed, but the condition $d >> a$ is not necessary if M is small.

The accuracy of the solution to the limiting equation (9) depends on the values of k, a, and on the function $h(x)$. The accuracy of the solution improves as the number P increases.

The relative error of the solution to asymptotic LAS (16) depends essentially on the function $N(x)$ which is at our disposal. In our numerical experiments $N(x) = \text{const}$. The accuracy of the solution is improved if $N(x)$ decreases, while parameters M, a, and d are fixed. The error of the solution decreases if M grows, while d is fixed and satisfying condition $d \gg a$.

The relative difference between the solutions to LAS (16) and (17) can be improved by changing the distance d between the particles, a being fixed. The optimal values of d change slowly in the considered range of function $N(x)$. The relative error is smaller for smaller a.

A constructive procedure, described in Section 8, for prescribing the function $N(x)$, calculating the numbers μ, and determining the radius a, allows one to obtain the refraction coefficient approximating better the desired one.

These results help to apply the proposed technique for creating materials with a desired refraction coefficient using the recipe, formulated in this paper. Development of methods for embedding many small particles into a given domain D according to our recipe, and for preparing small balls with the desired large impedances $\zeta = \frac{h(x)}{a^\kappa}$, especially if one wants

to have function $h(x,\omega)$ with a desired frequency dependence, are two basic technological problems that should be solved for an immediate practical implementation of our recipe.
For EM wave scattering it is shown that, for convergence of iterative procedure (29), (33) condition (37) is not necessary, but only sufficient: in many examples we had convergence, but condition was voilated. Altough theoretically we assumed $d > 10a$, our numerical results show that the proposed method gives good results even for $d = 3a$ in many cases.
The relative error between the "exact" solution corresponding to equation (39) and limiting solution (28) depends essentially on the ratio d/a. For example, for fixed M and a, ($M = 125, a = 0.05$) this difference changed from 2.3% to 0.7% if d/a decreases twice.
As in the case of acoustic wave scattering, a simple constructive procedure for calculation of desired refraction coefficient $n^2(x)$ is given. The numerical experiments show that in order to change the initial value $n_0^2(x)$ one increases radius a while the number M is fixed and not too large, or increases M and decreases a if M is very large. The second way is more attractive, because it is in correspondence with our theoretical background.
The extension of the developed numerical procedures for very large M, $M \geq O(10^5)$, and their applications to solving real-life engineering problems is under consideration now.

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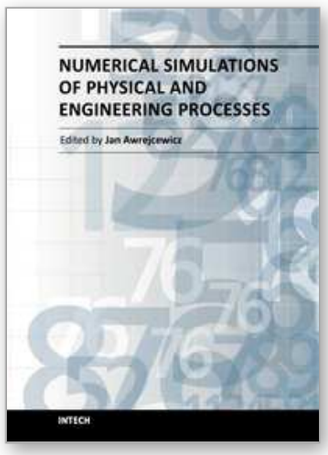

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Numerical Simulations of Physical and Engineering Process is an edited book divided into two parts. Part I devoted to Physical Processes contains 14 chapters, whereas Part II titled Engineering Processes has 13 contributions. The book handles the recent research devoted to numerical simulations of physical and engineering systems. It can be treated as a bridge linking various numerical approaches of two closely inter-related branches of science, i.e. physics and engineering. Since the numerical simulations play a key role in both theoretical and application oriented research, professional reference books are highly needed by pure research scientists, applied mathematicians, engineers as well post-graduate students. In other words, it is expected that the book will serve as an effective tool in training the mentioned groups of researchers and beyond.

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