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# Some Thermodynamic Problems in Continuum Mechanics

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## 1. Introduction

Classical thermodynamics discusses the thermodynamic system, its surroundings and their common boundary. It is concerned with the state of thermodynamic systems at equilibrium, using macroscopic, empirical properties directly measurable in the laboratory (Wang, 1955; Yunus, Michael and Boles, 2011). Classical thermodynamics model exchanges of energy, work and heat based on the laws of thermodynamics. The first law of thermodynamics is a principle of conservation of energy and defines a specific internal energy which is a state function of the system. The second law of thermodynamics is a principle to explain the irreversible phenomenon in nature. The entropy of an isolated non-equilibrium system will tend to increase over time, approaching a maximum value at equilibrium. Thermodynamic laws are generally valid and can be applied to systems about which only knows the balance of energy and matter transfer. The thermodynamic state of the system can be described by a number of state variables. In continuum mechanics state variables usually are pressure  $p$ , volume  $V$ , stress  $\sigma$ , strain  $\varepsilon$ , electric field strength  $E$ , electric displacement  $D$ , magnetic induction density  $B$ , magnetic field strength  $H$ , temperature  $T$ , entropy per volume  $s$ , chemical potential per volume  $\mu$  and concentration  $c$  respectively. Conjugated variable pairs are  $(p, V), (\sigma, \varepsilon), (E, D), (H, B), (T, S), (\mu, c)$ . There is a convenient and useful combination system in continuum mechanics: variables  $V, \varepsilon, E, H, T, \mu$  are used as independent variables and variables  $p, \sigma, D, B, S, c$  are used as dependent variables. In this chapter we only use these conjugated variable pairs, and it is easy to extend to other conjugated variable pairs. In the later discussion we only use the following thermodynamic state functions: the internal energy  $\mathcal{U}$  and the electro-magneto-chemical Gibbs free energy  $g_{e\mu}(\varepsilon, E, H, T, \mu)$  per volume in an electro-magneto-elastic material. They are taken as

$$\begin{aligned} d\mathcal{U}(\varepsilon, D, B, s, c) &= \sigma : d\varepsilon + E \cdot dD + H \cdot dB + Tds + \mu dc ; \quad \sigma : d\varepsilon = \sigma_{ij} d\varepsilon_{ij} \\ dg_{e\mu}(\varepsilon, E, H, T, \mu) &= d(\mathcal{U} - E \cdot D - H \cdot B - Ts - \mu c) = \sigma : d\varepsilon - D \cdot dE - B \cdot dH - sdT - cd\mu \end{aligned} \quad (1)$$

Other thermodynamic state functions and their applications can be seen in many literatures (Kuang, 2007, 2008a, 2008b, 2009a, 2009b, 2010, 2011a, 2011b). For the case without chemical potential  $g_{e\mu} = g_e$  is the electromagnetic Gibbs free energy. For the case without electromagnetic field  $g_{e\mu} = g_\mu$  is the Gibbs free energy with chemical potential. For the case without chemical potential and electromagnetic field  $g_{e\mu} = g$  is the Helmholtz free energy.

In this chapter two new problems in the continuum thermodynamics will be discussed. The first is that in traditional continuum thermodynamics including the non-equilibrium theory the dynamic effect of the temperature is not fully considered. When the temperature  $T$  is varied, the extra heat or entropy should be input from the environment. When  $c$  is varied, the extra chemical potential  $\mu$  is also needed. So the general inertial entropy theory (Kuang, 2009b, 2010) is introduced into the continuum thermodynamics. The temperature and diffusion waves etc. with finite phase velocity can easily be obtained from this theory. The second is that usually we consider the first law only as a conservation law of different kinds of energies, but we found that it is also containing a physical variational principle, which gives a true process for all possible process satisfying the natural constrained conditions (Kuang, 2007, 2008a, 2008b, 2009a, 2011a, 2011b). Introducing the physical variational principle the governing equations in continuum mechanics and the general Maxwell stress and other theories can naturally be obtained. When write down the energy expression, we get the physical variational principle immediately and do not need to seek the variational functional as that in the usual mathematical methods. The successes of applications of these theories in continuum mechanics are indirectly prove their rationality, but the experimental proof is needed in the further.

## 2. Inertial entropy theory

### 2.1 Basic theory in linear thermoelastic material

In this section we discuss the linear thermoelastic material without chemical reaction, so in Eq. (1) the term  $-\mathbf{D} \cdot d\mathbf{E} - \mathbf{B} \cdot d\mathbf{H} - c d\mu$  is omitted. It is also noted that in this section the general Maxwell stress is not considered. The classical thermodynamics discusses the equilibrium system, but when extend it to continuum mechanics we need discuss a dynamic system which is slightly deviated from the equilibrium state. In previous literatures one point is not attentive that the variation of temperature should be supplied extra heat from the environment. Similar to the inertial force in continuum mechanics we modify the thermodynamic entropy equation by adding a term containing an inertial heat or the inertial entropy (Kuang, 2009b), i.e.

$$\begin{aligned} T\dot{s} + T\dot{s}^{(a)} &= \dot{r} - q_{i,i} = \dot{r} - (T\dot{\eta}_i)_{,i}, \quad \dot{s}^{(a)} = \rho_s \ddot{T}, \quad s^{(a)} = \rho_s \dot{T} = \rho_{s0} (C/T_0) \dot{T} \\ \dot{s} &= \dot{s}^{(r)} + \dot{s}^{(i)}; \quad \dot{s}^{(r)} + \dot{s}^{(a)} = \dot{r}/T - \dot{\eta}_{i,i}; \quad \dot{\eta} = \mathbf{q}/T \\ T\dot{s}^{(i)} &= T\dot{s} - T\dot{s}^{(r)} = T\dot{s} + T\dot{s}^{(a)} - \dot{r} + T\dot{\eta}_{i,i} = -\dot{\eta}_i T_{,i} \geq 0; \quad \dot{s}^{(i)} = -\dot{\eta}_i T_{,i}/T \end{aligned} \quad (2)$$

where  $s^{(a)}$  is called the reversible inertial entropy corresponding to the inertial heat;  $\rho_s$  is called the inertial entropy coefficient,  $\rho_{s0}$  is also a constant having the dimension of the time;  $s$  is the entropy saved in the system,  $\dot{s}^{(r)}$  and  $\dot{s}^{(i)}$  are the reversible and irreversible parts of the  $s$ ,  $T\dot{s}$  is the absorbed heat rate of the system from the environment,  $T\dot{s}^{(a)} = \rho_s T \ddot{T}$  is the inertial heat rate and  $\dot{s}^{(a)}$  is proportional to the acceleration of the temperature;  $r$  is the external heat source strength,  $\mathbf{q}$  is the heat flow vector per interface area supplied by the environment,  $\boldsymbol{\eta}$  is the entropy displacement vector,  $\dot{\boldsymbol{\eta}}$  is the entropy flow vector. Comparing Eq. (2) with the classical entropy equation it is found that in Eq. (2) we use  $T\dot{s} + T\dot{s}^{(a)}$  to instead of  $T\dot{s}$  in the classical theory. In Eq. (2)  $s$  is still a state function because  $s^{(a)}$  is

reversible. As in classical theory the dissipative energy  $\dot{h}'$  and its Legendre transformation or “the complement dissipative energy”  $h$  are respectively

$$\dot{h}' = d\dot{h}'/dt = T\dot{s}^{(i)} = -T_{,i}\dot{\eta}_i, \quad \dot{h} = -T_{,i}\dot{\eta}_i + (T_{,i}\eta_i)^{\cdot} = \eta_i\dot{T}_{,i} \quad (3)$$

Using the theory of the usual irreversible thermodynamics (Groet, 1952; Gyarmati, 1970; Jou, Casas-Vzquez, Lebon, 2001; Kuang, 2002) from Eq. (3) we get

$$\begin{aligned} \dot{\eta}_i &= \dot{\eta}_i(T_{,j}), \quad \text{or} \quad \dot{\eta}_i = -\lambda_{ij}T^{-1}T_{,j}, \quad T\dot{\eta}_i = q_i = -\lambda_{ij}T_{,j} \\ T_{,j} &= -\hat{\lambda}_{ij}T\dot{\eta}_i = -\hat{\lambda}_{ij}q_i, \quad \hat{\lambda}_{ij} = \lambda_{ij}^{-1} \end{aligned} \quad (4)$$

where  $\lambda$  is the usual heat conductive coefficient. Eq. (4) is just the Fourier's law.

## 2.2 Temperature wave in linear thermoelastic material

The temperature wave from heat pulses at low temperature propagates with a finite velocity. So many generalized thermoelastic and thermopiezoelectric theories were proposed to allow a finite velocity for the propagation of a thermal wave. The main generalized theories are: Lord–Shulman theory (1967), Green–Lindsay theory (1972) and the inertial entropy theory (Kuang, 2009b).

In the Lord–Shulman theory the following Maxwell-Cattaneo heat conductive formula for an isotropic material was used to replace the Fourier's law, but the classical entropy equation is kept, i.e. they used

$$q_i + \tau_0\dot{q}_i = -\lambda T_{,i}, \quad T\dot{s} = \dot{r} - q_{i,i} \quad (5)$$

where  $\tau_0$  is a material parameter with the dimension of time. After linearization and neglecting many small terms they got the following temperature wave and motion equations for an isotropic material:

$$\begin{aligned} \lambda T_{,ii} &= C(\dot{T} + \tau_0\ddot{T}) + [2G(1+\nu)/(1-2\nu)]\alpha T_0(\dot{\varepsilon}_{jj} + \tau_0\ddot{\varepsilon}_{jj}) \\ [G/(1-2\nu)]u_{j,ij} &+ Gu_{i,jj} - [2G(1+\nu)/(1-2\nu)]\alpha T_{,i} = \rho\ddot{u}_i \end{aligned} \quad (6)$$

where  $C$  is the specific heat,  $\alpha$  is the thermal expansion coefficient,  $G$  and  $\nu$  are the shear modulus and Poisson's ratio respectively. From Eq. (5) we can get

$$T\dot{s} - \tau_0 T\ddot{s} = \lambda T_{,ii} + (\dot{r} + \tau_0\ddot{r})$$

From above equation it is difficult to consider that  $s$  is a state function.

The Green–Lindsay theory with two relaxation times was based on modifying the Clausius-Duhemin inequality and the energy equation; In their theory they used a new temperature function  $\phi(T, \dot{T})$  to replace the usual temperature  $T$ . They used

$$\begin{aligned} \int_V \dot{s} dV - \int_V (r/\phi) dV + \int_a (q_i/\phi) n_i da &\geq 0, \quad \phi = \phi(T, \dot{T}), \quad T = \phi(T, 0) \\ g &= \mathcal{U} - \phi s, \quad g = g(T, \dot{T}, \varepsilon_{ij}) \end{aligned} \quad (7)$$

After linearization and neglecting small terms, finally they get (here we take the form in small deformation for an isotropic material)

$$\begin{aligned}\lambda T_{,ii} &= C(\dot{T} + \tau_0 \ddot{T}) + \gamma T_0 \dot{\varepsilon}_{ij}, & \sigma_{ji,j} + \rho f_i &= \rho \ddot{u}_i \\ \sigma_{ij} &= [2G\nu/(1-2\nu)] \varepsilon_{kk} \delta_{ij} + 2G\varepsilon_{ij} - \gamma(\theta + \tau_1 \dot{\theta})\end{aligned}\quad (8)$$

where  $\tau_0$ ,  $\tau_1$  and  $\gamma$  are material constants.

Now we discuss the inertial entropy theory (Kuang, 2009b). The Helmholtz free energy  $g$  and the complement dissipative energy  $h$  assumed in the form

$$\begin{aligned}g(\varepsilon_{kl}, \vartheta) &= (1/2)C_{ijkl}\varepsilon_{ji}\varepsilon_{lk} - \alpha_{ij}\varepsilon_{ij}\vartheta - (1/2T_0)C\vartheta^2 \\ \delta h &= -\left[\int_0^t (\lambda_{ij}/T)\vartheta_{,i}d\tau\right]\delta\vartheta_{,j}, & \vartheta &= T - T_0 \\ C_{ijkl} &= C_{jikl} = C_{ijlk} = C_{klij}, & \alpha_{ij} &= \alpha_{ji}, \quad \lambda_{ij} = \lambda_{ji}\end{aligned}\quad (9a)$$

where  $T_0$  is the reference (or the environment) temperature,  $C_{ijkl}, \alpha_{ij}$  are material constants. In Eq. (9a) it is assumed that  $s = 0$  when  $T = T_0$  or  $\vartheta = 0$ . It is obvious that  $T_{,j} = \vartheta_{,j}, \dot{T} = \dot{\vartheta}$ . The constitutive (or state) and evolution equations are

$$\begin{aligned}\sigma_{ij} &= \partial g / \partial \varepsilon_{ij} = C_{ijkl}\varepsilon_{kl} - \alpha_{ij}\vartheta, & s &= -\partial g / \partial \vartheta = \alpha_{ij}\varepsilon_{ij} + C\vartheta / T_0 \\ \eta_i &= \partial h / \partial \vartheta_{,i} = -\int_0^t (\lambda_{ij}/T)\vartheta_{,j}d\tau, & T\dot{\eta}_i &= q_i = -\lambda_{ij}\vartheta_{,j}\end{aligned}\quad (10)$$

Using Eq. (10), Eq. (9a) can be rewritten as

$$g = (1/2)C_{ijkl}\varepsilon_{ji}\varepsilon_{lk} + g^{(T)}, \quad g^{(T)} = -(1/2)s\vartheta - (1/2)\alpha_{ij}\varepsilon_{ij}\vartheta \quad (9b)$$

where  $g^{(T)}$  is the energy containing the effect of the to temperature.

Substituting the entropy  $s$  and  $T\dot{\eta}_i$  in Eq. (10) and  $s^{(a)}$  in (2) into  $T\dot{s} + T\dot{s}^{(a)} = \dot{r} - (T\dot{\eta}_i)_{,i}$  in Eq. (2) we get

$$T(\alpha_{ij}\varepsilon_{ij} + C\vartheta / T_0)^{\cdot} + \rho_s T\ddot{\vartheta} = \dot{r} + (\lambda_{ij}\vartheta_{,j})_{,i} \quad (11)$$

When material coefficients are all constants from (11) we get

$$\rho_s T\ddot{\vartheta} + CT\dot{\vartheta} / T_0 - \lambda_{ij}\vartheta_{,ji} = \dot{r} - \alpha_{ij}T\dot{\varepsilon}_{ij} \quad (12a)$$

Eq. (12a) is a temperature wave equation with finite phase velocity. For an isotropic elastic material and the variation of the temperature is not large, from Eq. (12a) we get

$$\begin{aligned}(C / T_0)(\rho_{s0}\ddot{\vartheta} + \dot{\vartheta}) - (\lambda/T)\vartheta_{,ii} &= \dot{r}/T - \alpha\dot{\varepsilon}_{ii} \quad \text{or} \\ \lambda\vartheta_{,ii} &= C(\rho_{s0}\ddot{\vartheta} + \dot{\vartheta}) + \alpha T_0 \dot{\varepsilon}_{ii} - \dot{r}\end{aligned}\quad (12b)$$

Comparing the temperature wave equation Eq. (12b) with the Lord–Shulman theory (Eq. (6)) it is found that in Eq. (12b) a term  $\tau_0 \ddot{\varepsilon}_{ij}$  is lacked (in different notations), but with that in

the Green – Lindsay theory (Eq. (8)) is similar (in different notations). For the purely thermal conductive problem three theories are fully the same in mathematical form.

The momentum equation is

$$\sigma_{ij,j} + f_i = \rho \ddot{u}_i \quad (13)$$

where  $f$  is the body force per volume,  $\rho$  is the density. Substituting the stress  $\sigma$  in Eq. (10) into (13) we get

$$\left( C_{ijkl} \varepsilon_{kl} - \alpha_{ij} \vartheta \right)_{,j} + f_i = \rho \ddot{u}_i, \quad \text{or} \quad \rho \ddot{u}_i = C_{ijkl} u_{k,lj} - \alpha_{ij} \vartheta_{,j} + f_i \quad (14)$$

Comparing the elastic wave equation Eq. (14) with the Green – Lindsay theory (Eq. (8)) it is found that in Eq. (14) a term  $\gamma \tau_1 \dot{\vartheta}_{,i}$  is lacked (in different notations), but with the Lord – Shulman theory (Eq. (6)) is similar (in different notations).

### 2.3 Temperature wave in linear thermo - viscoelastic material

In the pyroelectric problem (without viscous effect) through numerical calculations Yuan and Kuang (2008, 2010) pointed out that the term containing the inertial entropy attenuates the temperature wave, but enhances the elastic wave. For a given material there is a definite value of  $\rho_{s0}^*$ , when  $\rho_{s0} > \rho_{s0}^*$  the amplitude of the elastic wave will be increased with time. For  $\text{BaTiO}_3$   $\rho_{s0}^*$  is about  $10^{-13} \text{ s}$ . In the Lord – Shulman theory critical value  $\tau_0^*$  is about  $10^{-8} \text{ s}$ . In order to substantially eliminate the increasing effect of the amplitude of the elastic wave the viscoelastic effect is considered as shown in this section.

Using the irreversible thermodynamics (Groet, 1952; Kuang, 1999, 2002) we can assume

$$\begin{aligned} g &= (1/2) C_{ijkl} \varepsilon_{ji} \varepsilon_{lk} - \alpha_{ij} \varepsilon_{ij} \vartheta - (1/2 T_0) C \vartheta^2 \\ \sigma_{ij}^{(r)} &= \partial g / \partial \varepsilon_{ij} = C_{ijkl} \varepsilon_{kl} - \alpha_{ij} \vartheta, \quad s = -\partial g / \partial \vartheta = \alpha_{ij} \varepsilon_{ij} + C \vartheta / T_0 \\ \delta h &= \beta_{ijkl} \dot{\varepsilon}_{ji} \delta \dot{\varepsilon}_{lk} + \eta_j \delta \vartheta_{,j} = \beta_{ijkl} \dot{\varepsilon}_{ji} \delta \dot{\varepsilon}_{lk} - \left[ \int_0^t (\lambda_{ij} / T) \vartheta_{,i} d\tau \right] \delta \vartheta_{,j}, \\ \sigma_{ij}^{(i)} &= \partial h / \partial \dot{\varepsilon}_{ij} = \beta_{ijkl} \dot{\varepsilon}_{kl}, \quad \eta_i = \partial h / \partial \vartheta_{,i} = - \int_0^t (\lambda_{ij} / T) \vartheta_{,j} d\tau, \quad T \dot{\eta}_i = q_i = -\lambda_{ij} \vartheta_{,j} \\ \sigma_{ij} &= \sigma_{ij}^{(r)} + \sigma_{ij}^{(i)} = C_{ijkl} \varepsilon_{kl} + \beta_{ijkl} \dot{\varepsilon}_{kl} - \alpha_{ij} \vartheta \end{aligned} \quad (15)$$

where  $\sigma_{ij}^{(r)}$  and  $\sigma_{ij}^{(i)}$  are the reversible and irreversible parts of the stress  $\sigma_{ij}$ ,  $\dot{\varepsilon}_{ij} = d\varepsilon_{ij} / dt$ . Comparing Eqs. (9) and (10) with (15) it is found that only a term  $\beta_{ijkl} \dot{\varepsilon}_{ji} \delta \dot{\varepsilon}_{lk}$  is added to the rate of the complement dissipative energy in Eq. (15). Substituting the entropy  $s$  and  $T \dot{\eta}_i$  in Eq. (15) and  $s^{(a)}$  in (2) into  $T \dot{s} + T \dot{s}^{(a)} = \dot{r} - (T \dot{\eta}_i)_{,i}$  in Eq. (2) we still get the same equation (12). Substituting the stress  $\sigma$  in Eq. (15) into (13) we get

$$\left( C_{ijkl} \varepsilon_{kl} + \beta_{ijkl} \dot{\varepsilon}_{kl} - \alpha_{ij} \vartheta \right)_{,j} + f_i = \rho \ddot{u}_i, \quad \text{or} \quad \rho \ddot{u}_i = C_{ijkl} u_{k,lj} + \beta_{ijkl} \dot{u}_{k,lj} - \alpha_{ij} \vartheta_{,j} + f_i \quad (16)$$

In one dimensional problem for the isotropic material from Eq. (15) we have

$$\sigma = Y \varepsilon + \beta \dot{\varepsilon} - \alpha \vartheta, \quad s = \alpha \varepsilon + C \vartheta / T_0 \quad (17)$$

where  $Y$  is the elastic modulus,  $\beta$  is a viscose coefficient,  $\alpha$  is the temperature coefficient. When there is no body force and body heat source, Eqs. (12) and (16) are reduced to

$$\begin{aligned} C(\rho_{s0}\ddot{\vartheta} + \dot{\vartheta}) - \lambda\vartheta'' + \alpha T_0\dot{u}' &= 0 \\ \rho\ddot{u} - Y u'' - \beta\dot{u}'' + \alpha\vartheta' &= 0 \end{aligned} \quad (18)$$

where  $\dot{\varphi} = \partial\varphi/\partial t$ ,  $\varphi' = \partial\varphi/\partial x$  for any function  $\varphi$ . For a plane wave propagating along direction  $x$  we assume

$$u = U \exp[i(kx - \omega t)], \quad \vartheta = \Theta \exp[i(kx - \omega t)] \quad (19)$$

where  $U, \Theta$  are the amplitudes of  $u$  and  $\vartheta$  respectively,  $k$  is the wave number and  $\omega$  is the circular frequency. Substituting Eq. (19) into (18) we obtain

$$\begin{aligned} [(Y - i\beta\omega)k^2 - \rho\omega^2]U + i\alpha k\Theta &= 0 \\ \alpha T_0 k\omega U + [\lambda k^2 - C(\rho_{s0}\omega^2 + i\omega)]\Theta &= 0 \end{aligned} \quad (20)$$

In order to have nontrivial solutions for  $U, \Theta$ , the coefficient determinant of Eq. (20) should be vanished:

$$\begin{vmatrix} (Y - i\beta\omega)k^2 - \rho\omega^2 & i\alpha k \\ \alpha T_0 k\omega & \lambda k^2 - C(\rho_{s0}\omega^2 + i\omega) \end{vmatrix} = \begin{vmatrix} ak^2 - \rho\omega^2 & i\alpha k \\ \alpha T_0 k\omega & \lambda k^2 - Cb \end{vmatrix} = 0 \quad (21a)$$

where

$$\begin{aligned} a = Y - i\beta\omega &= r_Y e^{i\theta_Y}, \quad r_Y = \sqrt{aa} = \sqrt{Y^2 + \beta^2\omega^2}, \quad \sin\theta_Y = -\beta\omega/r_Y \\ b = \rho_{s0}\omega^2 + i\omega &= r_T e^{i\theta_T}, \quad r_T = \sqrt{bb} = \omega\sqrt{\rho_{s0}^2\omega^2 + 1}, \quad \sin\theta_T = \omega/r_T \end{aligned} \quad (21b)$$

From Eq. (21) we get

$$\begin{aligned} \lambda ak^4 - (Cab + \lambda\rho\omega^2 + i\alpha^2 T_0\omega)k^2 + \rho\omega^2 Cb &= 0 \\ k &= \left\{ \frac{1}{2\lambda r_Y} \left[ (Cr_Y r_T e^{i\theta_T} + \lambda\rho\omega^2 e^{-i\theta_Y} + i\alpha^2 T_0\omega e^{-i\theta_Y}) \right. \right. \\ &\quad \left. \left. \pm \sqrt{(Cr_Y r_T e^{i\theta_T} - \lambda\rho\omega^2 e^{-i\theta_Y} + i\alpha^2 T_0\omega e^{-i\theta_Y})^2 + 4i\alpha^2 T_0\omega r_Y r_T e^{i(\theta_T - \theta_Y)}} \right] \right\}^{1/2} \end{aligned} \quad (22)$$

where the symbol “+” is applied to the wave number  $k_T$  of the temperature wave and the symbol “−” is applied to the wave number of the viscoelastic wave  $k_Y$ . If the temperature wave does not couple with the elastic wave, then  $\alpha$  is equal to zero. In this case we have

$$\begin{aligned} k &= \sqrt{(2\lambda r_Y)^{-1} \left\{ (Cr_T r_Y e^{i\theta_T} + \lambda\rho\omega^2 e^{-i\theta_Y}) \pm (Cr_T r_Y e^{i\theta_T} - \lambda\rho\omega^2 e^{-i\theta_Y}) \right\}} \\ k_Y &= \omega\sqrt{\rho/r_Y} e^{-i\theta_Y/2}, \quad k_T = \sqrt{Cr_T/\lambda} e^{i\theta_T/2} \end{aligned} \quad (23)$$



Because  $-\theta_Y > 0$  due to  $\beta > 0$  and  $\theta_T > 0$  due to  $\rho_{s0} > 0$ , a pure viscoelastic wave or a pure temperature waves is attenuated. The pure elastic wave does not attenuate due to  $\beta = 0$ .

For the general case in Eq. (22) a coupling term  $i\alpha^2 T_0 \omega k^2$  is appeared. It is known that

$$\left| \operatorname{Im} \left( C r_Y r_T e^{i\theta_T} + \lambda \rho \omega^2 e^{-i\theta_Y} + i\alpha^2 T_0 \omega e^{-i\theta_Y} \right) \right| > \left| \operatorname{Im} \left( C r_Y r_T e^{i\theta_T} - \lambda \rho \omega^2 e^{-i\theta_Y} + i\alpha^2 T_0 \omega e^{-i\theta_Y} \right) \right|$$

It means that  $\operatorname{Im} k_T > 0$  or the temperature wave is always an attenuated wave. If

$$\begin{aligned} & \operatorname{Im} \left[ \left( C r_Y r_T e^{i\theta_T} - \lambda \rho \omega^2 e^{-i\theta_Y} + i\alpha^2 T_0 \omega e^{-i\theta_Y} \right)^2 + 4i\alpha^2 T_0 \omega r_Y r_T e^{i(\theta_T - \theta_Y)} \right] \\ & < \operatorname{Im} \left( C r_Y r_T e^{i\theta_T} + \lambda \rho \omega^2 e^{-i\theta_Y} + i\alpha^2 T_0 \omega e^{-i\theta_Y} \right)^2 \end{aligned} \quad (24)$$

we get  $\operatorname{Im} k_Y > 0$  or in this case the elastic wave is an attenuated wave, otherwise is enhanced.

Introducing the viscoelastic effect in the elastic wave as shown in this section can substantially eliminate the increasing effect of the amplitude of the elastic wave with time.

## 2.4 Temperature wave in thermo-electromagneto-elastic material

In this section we discuss the linear thermo-electromagneto-elastic material without chemical reaction and viscous effect, so the electromagnetic Gibbs free energy  $g_e$  in Eq. (1) should keep the temperature variable. The electromagnetic Gibbs free energy  $g_e$  and the complement dissipative energy  $h_e$  in this case are assumed respectively in the following form

$$\begin{aligned} g_e(\varepsilon_{kl}, E_k, H_k, \vartheta) &= (1/2) C_{ijkl} \varepsilon_{ji} \varepsilon_{lk} - e_{kij}^e E_k \varepsilon_{ij} - (1/2) \varepsilon_{ij} E_i E_j - \tau_i^e E_i \vartheta \\ &\quad - e_{kij}^m H_k \varepsilon_{ij} - (1/2) \mu_{ij} H_i H_j - \tau_i^m H_i \vartheta - \alpha_{ij} \varepsilon_{ij} \vartheta - (1/2 T_0) C \vartheta^2 \\ \delta h_e &= - \left( \int_0^t (\lambda_{ij}/T) \vartheta_{,i} d\tau \right) \delta \vartheta_{,j} (= \eta_j \delta \vartheta_{,j}), \quad \vartheta = T - T_0 \\ C_{ijkl} &= C_{jikl} = C_{ijlk} = C_{kl ij}, \quad e_{kij}^e = e_{kji}^e, \quad \varepsilon_{kl} = \varepsilon_{lk}, \quad e_{kij}^m = e_{kji}^m, \quad \mu_{kl} = \mu_{lk}, \quad \alpha_{ij} = \alpha_{ji} \end{aligned} \quad (25)$$

where  $e_{kij}^e, \varepsilon_{kl}, \tau_i^e, e_{kij}^m, \mu_{kl}, \tau_i^m$  are material constants. The constitutive equations are

$$\begin{aligned} \sigma_{ij} &= C_{ijkl} \varepsilon_{kl} - e_{kij}^e E_k - e_{kij}^m H_k - \alpha_{ij} \vartheta, \quad D_i = \varepsilon_{ij} E_j + e_{ijk}^e \varepsilon_{jk} + \tau_i^e \vartheta \\ B_i &= \mu_{ij} H_j + e_{ijk}^m \varepsilon_{jk} + \tau_i^m \vartheta, \quad s = \alpha_{ij} \varepsilon_{ij} + \tau_i^e E_i + \tau_i^m H_i + C \vartheta / T_0 \\ \eta_i &= \partial h_v / \partial \vartheta_{,i} = - \int_0^t (\lambda_{ij}/T) \vartheta_{,j} d\tau, \quad T \dot{\eta}_i = q_i = - \lambda_{ij} \vartheta_{,j} \end{aligned} \quad (26)$$

Similar to derivations in sections 2.2 and 2.3 it is easy to get the governing equations:

$$T \left( \alpha_{ij} \varepsilon_{ij} + \tau_i^e E_i + \tau_i^m H_i + C \vartheta / T_0 \right) + \rho_s T \ddot{T} = \dot{r} + \left( \lambda_{ij} \vartheta_{,j} \right)_{,i} \quad (27)$$



$$\begin{aligned} \left( C_{ijkl} \varepsilon_{kl} - e_{kij}^e E_k - e_{kij}^m H_k - \alpha_{ij} \vartheta \right)_{,j} + f_i &= \rho \ddot{u}_i, \\ \left( \epsilon_{ij} E_j + e_{kij}^e \varepsilon_{kl} + \tau_i^e \vartheta \right)_{,i} &= \rho_e, \quad \left( \mu_{ij} H_j + e_{kij}^m \varepsilon_{kl} + \tau_i^m \vartheta \right)_{,i} = 0 \end{aligned} \quad (28)$$

where  $\rho_e$  is the density of the electric charge. The boundary conditions are omitted here.

## 2.5 Thermal diffusion wave in linear thermoelastic material

The Gibbs equation of the classical thermodynamics with the thermal diffusion is:

$$\begin{aligned} T\dot{s} &= \dot{r} - q_{i,i} + \mu d_{i,i}, \quad T\dot{s} + \mu\dot{c} = \dot{r} - q_{i,i} = \dot{r} - (T\dot{\eta}_i)_{,i}, \quad d_{i,i} = -\dot{c} \\ \mathcal{L} &= \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + T\dot{s} + \mu\dot{c}, \quad \dot{g}_\mu = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - s\dot{T} - c\dot{\mu} \end{aligned} \quad (29)$$

where  $\mu$  is the chemical potential,  $\mathbf{d}$  is the flow vector of the diffusing mass,  $c$  is the concentration. In discussion of the thermal diffusion problem we can also use the free energy  $\dot{g}_c = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - s\dot{T} + \mu\dot{c}$  (Kuang, 2010), but here it is omitted. Using relations

$$T^{-1} q_{i,i} = (T^{-1} q_i)_{,i} + T^{-2} q_i T_{,i}, \quad T^{-1} \mu d_{i,i} = (T^{-1} \mu d_i)_{,i} - d_i (T^{-1} \mu)_{,i}$$

From Eq. (29) (Kuang, 2010) we get:

$$\begin{aligned} \dot{s} &= \dot{s}^{(r)} + \dot{s}^{(i)}; \quad T\dot{s}^{(r)} = \dot{r} - T(q_i/T - \mu d_i/T)_{,i} \\ T\dot{s}^{(i)} &= T\dot{s} - T\dot{s}^{(r)} = -T_{,i} \dot{\eta}_i - \mu'_{,i} \dot{\xi}_i \geq 0, \quad \mu'_{,i} = (\mu/T)_{,i}, \quad \dot{\xi}_i = T d_i \end{aligned} \quad (30)$$

where  $T\dot{s}^{(i)}$  is the irreversible heat rate. According to the linear irreversible thermodynamics the irreversible forces are proportional to the irreversible flow (Kuang, 2010; Gyarmati, 1970; De Groet, 1952), we can write the evolution equations in the following form

$$T\dot{\eta}_i = -\lambda_{ij}(T) T_{,i} - L_{ij}(T) T \mu'_{,i}, \quad T^{-1} \dot{\xi}_i = -D_{ij}(T) T \mu'_{,i} - L_{ij}(T) T_{,i} \quad (31a)$$

where  $D_{ij}$  is the diffusing coefficients and  $L_{ij}$  is the coupling coefficients. The linear irreversible thermodynamics can only give the general form of the evolution equation, the concrete exact formula should be given by experimental results. Considering the experimental facts and the simplicity of the requirement for the variational formula, when the variation of  $T$  is not too large, Eq. (31a) can also be approximated by

$$\begin{aligned} T\dot{s}^{(i)} &\approx -T_{,i} \dot{\eta}_i - \mu_{,i} \dot{\xi}_i \geq 0; \quad \dot{\xi}_i = d_i \\ T\dot{\eta}_i &= -\lambda_{ij}(T) T_{,i} - L_{ij}(T) \mu_{,i}, \quad \dot{\xi}_i = -D_{ij}(T) \mu_{,i} - L_{ij}(T) T_{,i} \\ T_{,i} &= -\hat{\lambda}_{ij}(T) T \dot{\eta}_i - \hat{L}_{ij}(T) \dot{\xi}_i, \quad \mu_{,i} = -\hat{D}_{ij}(T) \dot{\xi}_i - \hat{L}_{ij}(T) T \dot{\eta}_i \end{aligned} \quad (31b)$$

Especially the coefficients  $\lambda_{ij}, L_{ij}, D_{ij}, \hat{\lambda}_{ij}, \hat{L}_{ij}, \hat{D}_{ij}$  in Eq. (31b) can all be considered as symmetric constants which are adopted in following sections. Eq. (31) is the extension of the Fourier's law and Fick's law.

Eq. (29) shows that in the equation of the heat flow the role of  $T\dot{s}$  is somewhat equivalent to  $\mu\dot{c}$ . So analogous to the inertial entropy  $s^{(a)}$  we can also introduce the inertial

concentration  $c^{(a)}$  and introduce a general inertial entropy theory of the thermal diffusion problem. Eq. (29) in the general inertial entropy theory is changed to (Kuang, 2010)

$$\begin{aligned} T(\dot{s} + \dot{s}^{(a)}) + \mu(\dot{c} + \dot{c}^{(a)}) &= \dot{r} - q_{i,i} = \dot{r} - (T\dot{\eta}_i)_{,i}; \quad \dot{c} + \dot{c}^{(a)} = -d_{i,i} \\ s^{(a)} &= \int_0^t \dot{s}^{(a)} d\tau, \quad \dot{s}^{(a)} = \rho_s \ddot{T}; \quad c^{(a)} = \int_0^t \dot{c}^{(a)} d\tau, \quad \dot{c}^{(a)} = \rho_c \ddot{\mu} \end{aligned} \quad (32)$$

where  $\rho_c$  is the inertial concentration coefficient. Applying the irreversible thermodynamics we can get the Gibbs free energy  $g_\mu$  and the complement dissipative energy  $h_\mu$  as

$$\begin{aligned} g_\mu(\varepsilon_{kl}, \vartheta, \mu) &= \frac{1}{2} C_{ijkl} \varepsilon_{ji} \varepsilon_{lk} - \alpha_{ij} \varepsilon_{ij} \vartheta - \frac{1}{2T_0} C \vartheta^2 - \frac{1}{2} b \mu^2 - b_{ij} \varepsilon_{ij} \mu - a \mu \vartheta \\ \delta h_\mu &= -T_{,i} \delta \eta_i - \mu_{,i} \delta \xi_i + \delta(T_{,i} \eta_i) + \delta(\mu_{,i} \xi_i) = \eta_j \delta \vartheta_{,j} + \xi_j \delta \mu_{,j} \\ &= -\delta \vartheta_{,j} \int_0^t (\lambda_{ij} T^{-1} \vartheta_{,i} + L_{ij} T^{-1} \mu_{,i}) d\tau - \delta \mu_{,j} \int_0^t (L_{ij} \vartheta_{,i} + D_{ij} \mu_{,i}) d\tau \end{aligned} \quad (33a)$$

where  $a, b, b_{ij}$  are also material constants. The constitutive and evolution equations are:

$$\begin{aligned} \sigma_{ij} &= \partial g_\mu / \partial \varepsilon_{ij} = C_{ijkl} \varepsilon_{kl} - \alpha_{ij} \vartheta - b_{ij} \mu, \\ s &= -\partial g_\mu / \partial \vartheta = \alpha_{ij} \varepsilon_{ij} + C \vartheta / T_0 + a \mu, \quad c = -\partial g_\mu / \partial \mu = b \mu + b_{ij} \varepsilon_{ij} + a \vartheta \\ \eta_i &= \partial h_\mu / \partial \vartheta_{,i} = -\int_0^t (\lambda_{ij} T^{-1} \vartheta_{,j} + L_{ij} T^{-1} \mu_{,j}) d\tau, \quad \xi_i = \partial h_\mu / \partial \mu_{,i} = -\int_0^t (L_{ij} \vartheta_{,j} + D_{ij} \mu_{,j}) d\tau \end{aligned} \quad (34)$$

Using Eq. (34)  $g_\mu$  in Eq. (33a) can also be rewritten as

$$g_\mu(\varepsilon_{kl}, \vartheta, \mu) = (1/2) C_{ijkl} \varepsilon_{ji} \varepsilon_{lk} + g_\mu^{(T)}, \quad g_\mu^{(T)} = -(1/2) (s \vartheta + c \mu + \alpha_{ij} \varepsilon_{ij} \vartheta + b_{ij} \varepsilon_{ij} \mu) \quad (35b)$$

where  $g_\mu^{(T)}$  is the energy containing the effects of temperature and concentration. Substituting Eq. (34) into Eq. (32) we get

$$\begin{aligned} T(\alpha_{ij} \varepsilon_{ij} + C \vartheta / T_0 + a \mu)^\cdot + T \rho_s \ddot{\vartheta} \\ + \mu(b \mu + b_{ij} \varepsilon_{ij} + a \vartheta)^\cdot + \mu \rho_s \ddot{\mu} &= \dot{r} + (\lambda_{ij} \vartheta_{,j} + L_{ij} \mu_{,j})_{,i} \\ (b \mu + b_{ij} \varepsilon_{ij} + a \vartheta)^\cdot + \rho_c \ddot{\mu} &= L_{ji} \vartheta_{,ij} + D_{ji} \mu_{,ij}; \quad \text{In medium} \end{aligned} \quad (35)$$

If we neglect the term in second order  $\mu d_{i,i}$  in Eq. (29), i.e. we take  $T \dot{s} = \dot{r} - q_{i,i}$  and assume that  $T_{,i}$  and  $\mu_{,j}$  are not dependent each other, i.e. in Eq. (31b) we assume  $\dot{\eta}_i = -\lambda_{ij} T^{-1} T_{,j}, \dot{\xi}_i = -D_{ij} \mu_{,j}$ , then for  $\dot{r} = 0$ , Eq. (35) becomes

$$\begin{aligned} T(\alpha_{ij} \dot{\mu}_{,j} + C \dot{\vartheta} / T_0 + a \dot{\mu} + \rho_s \ddot{\vartheta}) &= \lambda_{ij} \vartheta_{,ji} \\ b \dot{\mu} + b_{ij} \dot{\mu}_{,j} + a \dot{\vartheta} + \rho_c \ddot{\mu} &= D_{ij} \mu_{,ji}; \quad \text{In medium} \end{aligned} \quad (36)$$

The formulas in literatures analogous to Eq. (34) can be found, such as in Sherief, Hamza, and Saleh's paper (2004), where they used the Maxwell-Cattaneo formula.

The momentum equation is

$$\left( C_{ijkl} u_{k,l} - \alpha_{ij} \vartheta - b_{ij} \mu \right)_{,j} + f_i = \rho \ddot{u}_i \quad (37)$$

The above theory is easily extended to more complex materials.

### 3. Physical variational principle

#### 3.1 General theory

Usually it is considered that the first law of thermodynamics is only a principle of the energy conservation. But we found that the first law of thermodynamics is also a physical variational principle (Kuang, 2007, 2008a, 2008b, 2009a, 2011a, 2011b). Therefore the first law of the classical thermodynamics includes two aspects: energy conservation law and physical variational principle:

$$\begin{aligned} \text{Classical Energy conservation: } & \int_V d\mathcal{U} dV - dW - dQ = 0 \\ \text{Classical physical variational principle: } & \delta \Pi = \int_V \delta \mathcal{U} dV - \delta W - \delta Q = 0 \end{aligned} \quad (38)$$

where  $\mathcal{U}$  is the internal energy per volume,  $W$  is the work applied on the body by the environment,  $Q$  is the heat supplied by the environment. According to Gibbs theory when the process is only slightly deviated from the equilibrium state  $dQ$  can be substituted by  $\int_V T ds dV$ . In practice we prefer to use the free energy  $g$ :

$$\begin{aligned} g &= \mathcal{U} - Ts, \quad dg = d\mathcal{U} - s dT - T ds \\ \text{Energy Principle: } & \int_V dg dV - dW^* - \int_V s dT dV = 0 \\ \text{Physical Variational Principle: } & \delta \Pi = \int_V \delta g dV - \delta W^* - \int_V s \delta T dV = 0 \end{aligned} \quad (39)$$

Here the physical variational principle is considered to be one of the fundamental physical law, which can be used to derive governing equations in continuum mechanics and other fields. We can also give it a simple explanation that the true displacement is one kind of the virtual displacement and obviously it satisfies the variational principle. Other virtual displacements cannot satisfy this variational principle, otherwise the first law is not objective. The physical variational principle is different to the usual mathematical variational method which is based on the known physical facts. In many problems the variation of a variable  $\phi$  different with displacement  $\mathbf{u}$ , should be divided into local variation and migratory variation, i.e. the variation  $\delta\phi = \delta_\phi\phi + \delta_u\phi$ , where the local variation  $\delta_\phi\phi$  of  $\phi$  is the variation due to the change of  $\phi$  itself and the migratory variation  $\delta_u\phi$  of  $\phi$  is the variation of change of  $\phi$  due to virtual displacements. In Eqs. (38) and (39) the new force produced by the migratory variation  $\delta_u\phi$  will enter the virtual work  $\delta W$  or  $\delta W^*$  as the same as the external mechanical force. But in the following sections we shall modify Eq. (39) or (38) to deal with this problem. The physical variational principle is inseparable with energy conservation law, so when the expressions of energies are given we get physical variational principle immediately. We need not to seek the variational functional as that in

usual mathematical methods. In the following sections we show how to derive the governing equations with the general Maxwell stress of some kind of materials by using the physical variational principle. From this physical variational principle all of the governing equations in the continuum mechanics and physics can be carried out and this fact can be considered as the indirect evidence of the physical variational principle.

### 3.2 Physical variational principle in thermo-elasticity

In the thermo-elasticity it is usually considered that only the thermal process is irreversible, but the elastic process is reversible. So the free energy  $g$  and the complement dissipative energy can be assumed as that in Eq. (9). The corresponding constitutive and evolution equations are expressed in Eq. (10). As shown in section 3.1, the variation of the virtual temperature  $\vartheta$  is divided into local variation  $\delta_g \vartheta$  due to the variation of  $\vartheta$  itself and the migratory variation  $\delta_u \vartheta$  due to  $\delta \mathbf{u}$ :

$$\delta \vartheta = \delta_g \vartheta + \delta_u \vartheta, \quad \delta_u \vartheta = \vartheta_{,i} \delta u_i \quad (40)$$

In previous paper (Kuang, 2011a) we showed that the migratory variation of virtual electric and magnetic potentials will produce the Maxwell stress in electromagnetic media, which is also shown in section 3.4 of this paper. Similarly the migratory variation  $\delta_u \vartheta$  will also produce the general Maxwell stress which is an external temperature stress. The effective general Maxwell stress can be obtained by the energy principle as that in electromagnetic media. Under assumptions that the virtual mechanical displacement  $\mathbf{u}$  and the virtual temperature  $\vartheta$  (or  $T$ ) satisfy their own boundary conditions  $u_i = u_i^*$ ,  $\vartheta = \vartheta^*$  on  $a_u$  and  $a_T$  respectively. The physical variational principle using the free energy in the inertial entropy theory for the thermo-elasticity can be expressed as:

$$\begin{aligned} \delta \Pi_T &= \int_V \delta(g + h) dV + \int_V g^{(T)} \delta u_{k,k} dV - \delta Q^* - \delta W = 0 \\ \delta Q^* &= - \int_V \int_0^t (\dot{\vartheta}/T) \delta \vartheta d\tau dV - \int_V \int_0^t \dot{s}^{(i)} \delta \vartheta d\tau dV + \int_{a_q} \int_0^t \dot{\eta}^* \delta \vartheta d\tau da + \int_V \int_0^t \rho_s \ddot{\vartheta} \delta \vartheta d\tau dV \\ \delta W &= \int_V (f_k - \rho \ddot{u}_k) \delta u_k dV + \int_{a_\sigma} T_k^* \delta u_k da \end{aligned} \quad (41)$$

where  $f_k, T_k^*$  and  $\dot{\eta}^* = \dot{\eta}_i^* n_i$  are the given mechanical body force, surface traction and surface entropy flow respectively. Eq. (41) is an alternative form of Eq. (39). In Eq. (41) the term  $-\int_0^t \dot{s}^{(i)} \delta \vartheta d\tau = \int_0^t \eta_i \delta T_{,i} d\tau$  is the complement dissipative heat rate per volume corresponding to the inner complement dissipation energy rate  $\delta h$ . The entropy  $s$  includes the contribution of  $-\int_0^t \dot{s}^{(i)} \delta \vartheta d\tau$ . The fact that the complement dissipation energy rate  $\int_V \delta h dV$  in  $\delta \Pi_T$  and the internal irreversible complement heat rate  $-\int_V \int_0^t \dot{s}^{(i)} \delta \vartheta d\tau dV$  in  $\delta Q^*$  are simultaneously included in Eq. (41) allows us to get the temperature wave equation and the boundary condition of the heat flow from the variational principle. In Eq. (41) there are two kinds of variational formulas. The first is  $\int_V \delta g dV + \int_V g^{(T)} \delta u_{k,k} dV - \delta W$ , in which the integrands contain variables themselves. The second is  $\int_V \delta h dV - \delta Q^*$ , in which the integrands contain the time derivatives of variables, so it needs integrate with time  $t$ . This is the common feature of the irreversible process because in the irreversible process the integral is dependent to the integral path.

It is noted that

$$\begin{aligned}
 \int_V \delta g dV &= \int_V (C_{ijkl} \varepsilon_{kl} - \alpha_{ij} \vartheta) \delta u_{i,j} dV - \int_V (\alpha_{ij} \varepsilon_{ij} + C \vartheta / T_0) \delta \vartheta dV \\
 &= \int_a \sigma_{ij} n_j \delta u_i da - \int_V \sigma_{ij,j} \delta u_i dV - \int_V s \delta \vartheta dV \\
 \int_V g^{(T)} \delta u_{k,k} dV &= -(1/2) \int_V (s + \alpha_{ij} \varepsilon_{ij}) \vartheta \delta u_{k,k} dV \\
 &= -(1/2) \int_a (s + \alpha_{ij} \varepsilon_{ij}) \vartheta n_k \delta u_k dV + (1/2) \int_V [(s + \alpha_{ij} \varepsilon_{ij}) \vartheta]_{,k} \delta u_k dV \\
 \int_V \delta h dV &= - \int_a \left( \int_0^t \lambda_{ij} T^{-1} \vartheta_{,i} n_j d\tau \right) \delta \vartheta da + \int_V \left[ \int_0^t (\lambda_{ij} T^{-1} \vartheta_{,i})_{,j} d\tau \right] \delta \vartheta dV
 \end{aligned} \quad (42)$$

Finishing the variational calculation, we have

$$\begin{aligned}
 \delta \Pi &= \int_{a_\sigma} (\tilde{\sigma}_{ij} n_j - T_i^*) \delta u_i da \\
 &\quad - \int_V (\tilde{\sigma}_{ij,j} + f_i - \rho \ddot{u}_i) \delta u_i dV - \int_{a_q} \left[ \int_0^t T^{-1} \lambda_{ij} \vartheta_{,i} n_j d\tau + \eta^* \right] \delta \vartheta da \\
 &\quad - \int_V \{ s + \int_0^t [-T^{-1} \dot{\vartheta} - (\lambda_{ij} T^{-1} \vartheta_{,i})_{,j} - \dot{s}^{(i)}] d\tau \} \delta \vartheta dV - \int_V \int_0^t \rho_s \ddot{\vartheta} d\tau dV \} \delta \vartheta dV = 0 \\
 \tilde{\sigma}_{ij} &= \sigma_{ij} + \sigma_{ij}^T, \quad \sigma_{ij}^T = -(1/2) (s + \alpha_{ij} \varepsilon_{ij}) \vartheta \delta_{ij} \approx -(1/2) s \vartheta \delta_{ij}
 \end{aligned} \quad (43)$$

where  $\sigma^T$  is the effective or equivalent general Maxwell stress which is the external equal axial normal temperature stress. This general Maxwell stress is first introduced and its rationality should be proved by experiments. Obviously  $\sigma^T$  can be neglected for the case of the small strain and small change of temperature. In Eq. (43) it is seen that  $\delta \vartheta = \delta_g \vartheta + \delta_u \vartheta$  is appeared in a whole. Using

$$-T(\lambda_{ij} T^{-1} \vartheta_{,i})_{,j} - T \dot{s}^{(i)} = T \dot{\eta}_{j,j} - T \dot{s}^{(i)} = T \dot{\eta}_{j,j} + \dot{\eta}_i T_{,i} = (T \dot{\eta}_i)_{,i} = q_{i,i}$$

and the arbitrariness of  $\delta u_i$  and  $\delta \vartheta$ , from Eq. (43) we get

$$\begin{aligned}
 \tilde{\sigma}_{ij,j} + f_i &= \rho \ddot{u}_i; \quad T(\dot{s} + \rho_s \ddot{\vartheta}) = \dot{\vartheta} - q_{i,i} \quad \text{in medium} \\
 \tilde{\sigma}_{kl} n_l &= T_k^*, \quad \text{on } a_\sigma; \quad \dot{\eta}_i = -T^{-1} \lambda_{ij} \vartheta_{,i}, \quad \dot{\eta}_i n_i = \dot{\eta}_i^*, \quad \text{or } q_n = q_n^*, \quad \text{on } a_q
 \end{aligned} \quad (44)$$

Here  $\nabla \cdot \sigma^T$  is the external temperature body force and  $\mathbf{n} \cdot \sigma^T$  is the surface traction. The above variational principle requests prior that displacements and the temperature satisfy the boundary conditions, so in governing equations the following equations should also be added

$$\mathbf{u} = \mathbf{u}^*, \quad \text{on } a_u; \quad \vartheta = \vartheta^* (\text{or } T = T^*), \quad \text{on } a_T \quad (45)$$

Eqs. (44) and (45) are the governing equations of the thermo-elasticity derived from the physical variational principle.

### 3.3 Physical variational principle in thermo-diffusion theory

The electro-chemical Gibbs free energy  $g_\mu$  and the complement dissipative energy  $h_\mu$  are expressed in Eq. (33) and the constitutive and evolution equations are expressed in Eq. (34).

Under assumptions that the mechanical displacement  $\mathbf{u}$ , the temperature  $\vartheta$  and the chemical potential  $\mu$  satisfy their own boundary conditions  $\mathbf{u} = \mathbf{u}^*$ ,  $\vartheta = \vartheta^*$  and  $\mu = \mu^*$  on  $a_u$ ,  $a_T$  and  $a_\mu$  respectively. When the variation of temperature is not large the physical variational principle for the thermo-elasto-diffusive problem is

$$\begin{aligned}\delta \Pi_\mu &= \int_V \delta(g_\mu + h_\mu) dV + \int_V g_\mu^{(T)} \delta u_{k,k} dV - \delta Q^* - \delta \Phi^* - \delta W = 0 \\ \delta Q^* &= - \int_V \left( \int_0^t T^{-1} \dot{r} d\tau \right) \delta \vartheta dV + \int_V s^{(a)} \delta \vartheta dV + \int_{a_q} \eta^* \delta \vartheta da \\ &\quad + \int_V \int_0^t T^{-1} (T_{,i} \dot{\eta}_i + \mu_{,i} \dot{\xi}_i) \delta \vartheta d\tau dV - \int_a \int_0^t T^{-1} \mu \dot{\xi}_i n_i \delta \vartheta d\tau da \\ \delta \Phi^* &= \int_V c^{(a)} \delta \mu dV + \int_{a_d} \xi^* \delta \mu da \\ \delta W &= \int_V (f_k - \rho \ddot{u}_k) \delta u_k dV + \int_{a_\sigma} T_k^* \delta u_k da\end{aligned}\quad (46)$$

In Eqs. (46)  $f_k, T_k^*, \dot{\eta}^* = \dot{\eta}_i^* n_i$  and  $\dot{\xi}^* = \dot{\xi}_i^* n_i$  are given values. In Eq. (46)  $\delta Q^*$  is related to heat (including the heat produced by the irreversible process in the material),  $\delta \Phi^*$  is related to the diffusion energy. Eq. (46) shows that there is no term in  $\int_V \delta h_\mu dV$  corresponding to the term  $-\int_a \int_0^t T^{-1} \mu \dot{\xi}_i n_i \delta \vartheta d\tau da$ , so it should not be included in  $\delta Q^*$  and  $\int_V \int_0^t T^{-1} \mu_{,i} \dot{\xi}_i \delta \vartheta d\tau dV - \int_a \int_0^t T^{-1} \mu \dot{\xi}_i n_i \delta \vartheta d\tau da = - \int_V \int_0^t T^{-1} \mu \dot{\xi}_{i,i} \delta \vartheta d\tau dV$ .

It is noted that we have the following relations

$$\begin{aligned}\int_V \delta g_\mu dV &= \int_a \sigma_{ij} n_j \delta u_i da - \int_V \sigma_{ij,j} \delta u_i dV - \int_V s \delta \vartheta dV - \int_V c \delta \mu dV \\ \int_V g_\mu^{(T)} \delta u_{k,k} dV &= -(1/2) \int_V (s \vartheta + c \mu + \alpha_{ij} \varepsilon_{ij} \vartheta + b_{ij} \varepsilon_{ij} \mu) \delta u_{k,k} dV \\ &= -(1/2) \int_a (s \vartheta + c \mu + \alpha_{ij} \varepsilon_{ij} \vartheta + b_{ij} \varepsilon_{ij} \mu) n_k \delta u_k da \\ &\quad + (1/2) \int_V (s \vartheta + c \mu + \alpha_{ij} \varepsilon_{ij} \vartheta + b_{ij} \varepsilon_{ij} \mu)_{,k} \delta u_k dV \\ \delta h_\mu &= - \int_V \left\{ \delta \vartheta_{,i} \int_0^t (\lambda_{ij} T^{-1} T_{,i} + L_{ij} T^{-1} \mu_{,i}) d\tau + \delta \mu_{,i} \int_0^t (L_{ij} T_{,i} + D_{ij} \mu_{,i}) d\tau \right\} dV\end{aligned}\quad (47)$$

The further derivation is fully similar to that in the thermo-elasticity. Combining Eqs. (46) and (47) we get

$$\begin{aligned}\delta \Pi_\mu &= \int_{a_\sigma} (\tilde{\sigma}_{ij} n_j - T_i^*) \delta u_i da - \int_V (\tilde{\sigma}_{ij,j} + f_i - \rho \ddot{u}_i) \delta u_i dV - \int_V s \delta \vartheta dV - \int_V c \delta \mu dV \\ &\quad - \int_a \left\{ \delta \vartheta n_j \int_0^t (\lambda_{ij} T^{-1} T_{,i} + L_{ij} T^{-1} \mu_{,i}) d\tau + \delta \mu n_j \int_0^t (L_{ij} T_{,i} + D_{ij} \mu_{,i}) d\tau \right\} da \\ &\quad + \int_V \left\{ \delta \vartheta \int_0^t (\lambda_{ij} T^{-1} T_{,i} + L_{ij} T^{-1} \mu_{,i})_{,j} d\tau + \delta \mu \int_0^t (L_{ij} T_{,i} + D_{ij} \mu_{,i})_{,j} d\tau \right\} dV \\ &\quad + \int_V \left( \int_0^t T^{-1} \dot{r} d\tau \right) \delta \vartheta dV - \int_V s^{(a)} \delta \vartheta dV - \int_V c^{(a)} \delta \mu dV \\ &\quad - \int_{a_q} \eta^* \delta \vartheta da - \int_{a_d} \xi^* \delta \mu da - \int_V \int_0^t (T^{-1} T_{,i} \dot{\eta}_i - T^{-1} \mu \dot{\xi}_{i,i}) \delta \vartheta d\tau dV = 0\end{aligned}\quad (48)$$



where

$$\tilde{\sigma}_{ij} = \sigma_{ij} + \sigma_{ij}^{T\mu}, \quad \sigma_{ij}^{T\mu} = -(1/2)(s\vartheta + c\mu + \alpha_{ij}\varepsilon_{ij}\vartheta + b_{ij}\varepsilon_{ij}\mu)\delta_{ij} \approx -(1/2)(s\vartheta + c\mu) \quad (49)$$

Due to the arbitrariness of  $\delta\mathbf{u}$ ,  $\delta\vartheta$  and  $\delta\mu$ , from Eq. (48) we get

$$\tilde{\sigma}_{kl,l} + f_k = \rho\ddot{u}_k, \quad \text{in medium}; \quad \tilde{\sigma}_{kl}n_l = T_k^*, \quad \text{on } a_\sigma \quad (50)$$

and

$$\begin{aligned} \int_V \int_0^t (\dot{s} + \rho_s \ddot{\vartheta}) d\tau dV &= \int_V \int_0^t T^{-1} (\dot{r} - q_{j,j} + \mu \dot{\xi}_{j,j}) d\tau dV \\ \int_V \int_0^t (\dot{c} + \rho_c \ddot{\mu}) d\tau dV &= - \int_V \int_0^t \dot{\xi}_{j,j} d\tau dV \quad \text{In medium} \\ \dot{\eta}_j &= -T^{-1} (\lambda_{ij} T_{,i} + L_{ij} \mu_{,i}), \quad \dot{\eta}_j n_j = \dot{\eta}^*, \quad \text{or } q_n = q_n^* \quad \text{on } a_q \\ \dot{\xi}_j &= - (L_{ij} T_{,i} + D_{ij} \mu_{,i}), \quad \dot{\xi}_j n_j = \dot{\xi}^*, \quad \text{or } d_n = d_n^* \quad \text{on } a_d \end{aligned} \quad (51)$$

where  $T^{-1} (\lambda_{ij} T_{,i} + L_{ij} \mu_{,i})_{,j} - T^{-1} (T_{,i} \dot{\eta}_i - \mu \dot{\xi}_{i,i}) = -T^{-1} (q_{j,j} + \mu \dot{\xi}_{i,i})$  has been used.

The first two formulas in Eq. (51) can be rewritten as

$$\begin{aligned} T(\dot{s} + \rho_s \ddot{\vartheta}) &= \dot{r} - q_{j,j} + \mu \dot{\xi}_{j,j}; \quad \dot{c} + \rho_c \ddot{\mu} = -\dot{\xi}_{j,j} \\ T(\dot{s} + \rho_s \ddot{\vartheta}) + \mu(\dot{c} + \rho_c \ddot{\mu}) &= \dot{r} - q_{j,j}; \quad \text{In medium} \end{aligned} \quad (52)$$

The last equation in Eq. (52) is just the same as that in Eq. (32).

The above variational principle requests prior that the  $\mathbf{u}$ ,  $\vartheta$  and  $\mu$  satisfy their own boundary conditions, so in governing equations the following equations should also be added

$$\mathbf{u} = \mathbf{u}^*, \quad \text{on } a_u; \quad \vartheta = \vartheta^*, \quad \text{on } a_T; \quad \mu = \mu^*, \quad \text{on } a_\mu \quad (53)$$

Eqs. (49)-(53) are the governing equations of the generalized thermodiffusion theory.

If we neglect the term  $\mu(\dot{c} + \dot{c}^{(a)})$  in Eq. (32), or  $T(\dot{s} + \dot{s}^{(a)}) = \dot{r} - q_{i,i}$  is adopted, then we easily get

$$\begin{aligned} T(\dot{s} + \rho_s \ddot{\vartheta}) &= \dot{r} - q_{j,j}, \quad \dot{c} + \rho_c \ddot{\mu} = -\dot{\xi}_{j,j}; \quad \text{In medium} \\ \dot{\eta}_j n_j &= \dot{\eta}^*, \quad \text{or } q_n = q_n^* \quad \text{on } a_q \\ \dot{\xi}_j n_j &= \dot{\xi}^*, \quad \text{or } d_n = d_n^* \quad \text{on } a_d \text{ and } a_q \end{aligned} \quad (54)$$

If we also assume that  $T_{,i}$  and  $\mu_{,j}$  are not dependent each other, then for  $\dot{r} = 0$ , the Eq. (54) becomes Eq. (36), i.e.

$$\begin{aligned} T(\alpha_{ij} \dot{u}_{i,j} + C \dot{\vartheta} / T_0 + a \dot{\mu} + \rho_s \ddot{\vartheta}) &= \lambda_{ij} \vartheta_{,j} \\ b \dot{\mu} + b_{ij} \dot{u}_{i,j} + a \dot{\vartheta} + \rho_c \ddot{\mu} &= D_{ij} \mu_{,j}; \quad \text{In medium} \end{aligned} \quad (55)$$



### 3.4 Physical variational principle in electro-magneto-elastic analysis

In this section we discuss the nonlinear electro-magneto-elastic media. Here we extend the theory in previous paper (Kuang, 2011) to the material with the electromagnetic body couple. Because the asymmetric part of the stress is introduced by the electromagnetic body couple, the specific electromagnetic Gibbs free energy  $g_{em}$  is taken as

$$g_{em}(\varepsilon_{kl}, E_k, H_k) = (1/2)C_{ijkl}\varepsilon_{ji}\varepsilon_{lk} - (e_{kij}^e E_k + e_{kij}^m H_k)\varepsilon_{ij} - (1/2)(\varepsilon_{ij} E_i E_j + \mu_{ij} H_i H_j) \\ - (1/2)(l_{ijkl}^e E_i E_j + l_{ijkl}^m H_i H_j)\varepsilon_{kl} - (\varepsilon_{km} E_m E_l + \mu_{km} H_m H_l)\varepsilon_{kl} \quad (56a) \\ (C_{ijkl}, l_{ijkl}^e, l_{ijkl}^m) = (C_{jikl}, l_{jikl}^e, l_{jikl}^m) = (C_{ijlk}, l_{ijlk}^e, l_{ijlk}^m) = (C_{klij}, l_{klij}^e, l_{klij}^m), \quad e_{kij}^e = e_{kji}^e, \quad e_{kij}^m = e_{kji}^m$$

where  $l_{ijkl}^e$  and  $l_{ijkl}^m$  are the electrostrictive and magnetostrictive constants respectively;  $\varepsilon$  and  $\mu$  may be asymmetric. The corresponding constitutive equations are

$$\sigma_{kl} = \partial g_{em} / \partial \varepsilon_{kl} = C_{ijkl}\varepsilon_{ij} - e_{jkl}^e E_j - e_{jkl}^m H_j - (1/2)l_{ijkl}^e E_i E_j - (1/2)l_{ijkl}^m H_i H_j \\ - \varepsilon_{km} E_m E_l - \mu_{km} H_m H_l \quad (57) \\ D_k = -\partial g_{em} / \partial E_k = [\varepsilon_{kl} + l_{ijkl}^e \varepsilon_{ij} + (\varepsilon_{ml} \varepsilon_{mk} + \varepsilon_{mk} \varepsilon_{ml})] E_l + e_{kij}^e \varepsilon_{ij} \approx \varepsilon_{kl} E_l \\ B_k = -\partial g_{em} / \partial H_k = [\mu_{kl} + l_{ijkl}^m \varepsilon_{ij} + (\alpha_{ml}^m \varepsilon_{mk} + \alpha_{mk}^m \varepsilon_{ml})] H_l + e_{kij}^m \varepsilon_{ij} \approx \mu_{kl} H_l$$

Let  $\sigma^s$  and  $\sigma^a$  be the symmetric and asymmetric parts of  $\sigma$  respectively, we have

$$\sigma_{kl}^s = (1/2)(\sigma_{kl} + \sigma_{lk}) = C_{ijkl}\varepsilon_{ji} - e_{jkl}^e E_j - (1/2)l_{ijkl}^e E_i E_j - e_{jkl}^m H_j - (1/2)l_{ijkl}^m H_i H_j \\ - (1/2)(\varepsilon_{km} E_l + \varepsilon_{lm} E_k) E_m - (1/2)(\mu_{km} H_l + \mu_{lm} H_k) H_m \\ \sigma_{kl}^a = (1/2)(\sigma_{kl} - \sigma_{lk}) = -(1/2)(\varepsilon_{km} E_l - \varepsilon_{lm} E_k) E_m - (1/2)(\mu_{km} H_l - \mu_{lm} H_k) H_m \quad (58) \\ D_k E_l - D_l E_k = P_k E_l - P_l E_k \approx (\varepsilon_{km} E_l - \varepsilon_{lm} E_k) E_m \\ B_k H_l - B_l H_k = \mu_0 (M_k H_l - M_l H_k) \approx (\mu_{km} H_l - \mu_{lm} H_k) H_m$$

where  $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ ,  $\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$  have been used,  $\mathbf{P}$  and  $\mathbf{M}$  are the polarization density and magnetization density,  $\varepsilon_0$  and  $\mu_0$  are the dielectric constant and magnetic permeability in vacuum respectively. The terms containing  $\varepsilon$  in  $\mathbf{D}$  and  $\mathbf{B}$  in Eq. (58) have been neglected. In the usual electromagnetic theory the electromagnetic body couple is  $\mathbf{P} \times \mathbf{E} + \mu_0 \mathbf{M} \times \mathbf{H}$ . From Eq. (58) it is seen that  $2\sigma_{kl}^a + (D_k E_l - D_l E_k) + (B_k H_l - B_l H_k) = 0$  or the electromagnetic body couple is balanced by the moment produced by the asymmetric stresses.

Using Eq. (57), Eq. (56a) can be reduced to

$$g_{em} = (1/2)C_{ijkl}\varepsilon_{ji}\varepsilon_{lk} + g^{em}, \quad g^{em} = -(1/2)(D_k E_k + B_k H_k + \Delta_{kl}\varepsilon_{lk}) \approx -(1/2)(D_k E_k + B_k H_k) \quad (56b) \\ \Delta_{kl} = e_{mkl}^e E_m + e_{mkl}^m H_m = -e_{mkl}^e \varphi_{,m} - e_{mkl}^m \psi_{,m}$$

Because the value of the term  $\Delta : \varepsilon$  is much less than that of other terms, it can be neglected. In the nonlinear electro-magneto-elastic analysis the medium and its environment should be considered together as shown in Fig. 1 (Kuang, 2011a, 2011b), because the electromagnetic field exists in all space. Under the assumption that  $\mathbf{u}, \varphi, \psi, \mathbf{u}^{env}, \varphi^{env}, \psi^{env}$  satisfy their

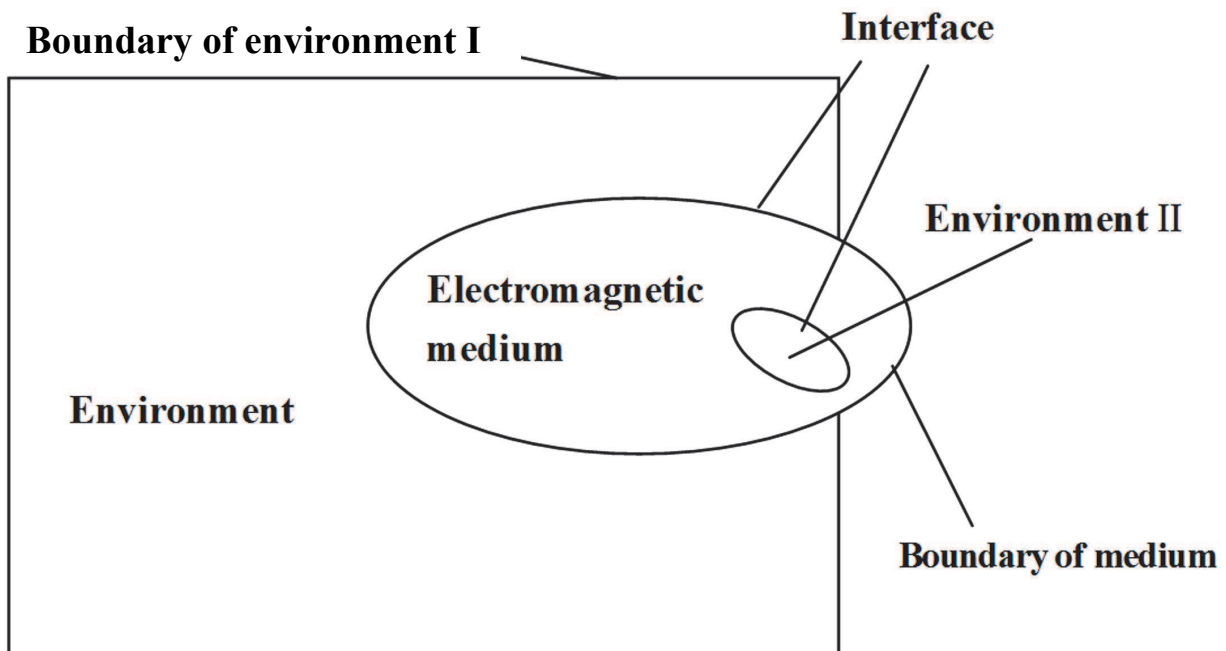


Fig. 1. Electromagnetic medium and its environment

boundary conditions on their own boundaries  $a_u, a_\varphi, a_\psi, a_u^{env}, a_\varphi^{env}, a_\psi^{env}$  and the continuity conditions on the interface  $a^{int}$ . The Physical variational principle in the nonlinear electro-magneto-elastic analysis is

$$\begin{aligned}
 \delta\Pi &= \delta\Pi_1 + \delta\Pi_2 - \delta W^{*int} = 0 \\
 \delta\Pi_1 &= \int_V \delta g_{em} dV + \int_V g^{em} \delta u_{i,i} dV - \delta W^* \\
 \delta\Pi_2 &= \int_{V^{env}} \delta g_{em}^{env} dV + \int_{V^{env}} g^{em\,env} \delta u_{i,i}^{env} dV - \delta W^{*env} \\
 \delta W^* &= \int_V (f_k - \rho \ddot{u}_k) \delta u_k dV - \int_V \rho_e \delta \varphi dV + \int_{a_\sigma} T_k^* \delta u_k da - \int_{a_q} \sigma^* \delta \varphi da + \int_{a_\mu} B_i^* n_i \delta \psi da \\
 \delta W^{*int} &= \int_{a^{int}} T_k^{*int} \delta u_k da - \int_{a^{int}} \sigma^{*int} \delta \varphi da + \int_{a_\mu^{int}} B_i^{*int} n_i \delta \psi^{env} da
 \end{aligned} \tag{59}$$

$g_{em}^{env}, g^{em\,env}, \delta W^{*env}$  are similar and omitted here

where the superscript “env” means the variable in environment, “int” means the variable on the interface,  $f_k, T_k^*, \sigma^*, B_n^* = B_i^* n_i; T_k^{*env}, \sigma^{*env}, B_n^{*env}; T_k^{*int}, \sigma^{*int}, B_n^{*int}$  are the given values on the corresponding surfaces. Eq. (59) is an alternative form of Eq. (39) and the electromagnetic force is directly enclosed in the formula (Kuang, 2008a, 2009a).

As shown in previous paper (Kuang, 2011a, 2011b) and in section 3.1 the variations of  $\varphi, \psi, \mathbf{E}, \mathbf{H}$  will be distinguished into local and migratory variations, i.e.

$$\begin{aligned}
 \delta(\varphi, \psi, E_i, H_i) &= \delta_{(\varphi, \psi, E_i, H_i)}(\varphi, \psi, E_i, H_i) + \delta_u(\varphi, \psi, E_i, H_i) \\
 \delta_u(\varphi, \psi) &= (\varphi, \psi)_{,p} \delta u_p = -(E_p, H_p) \delta u_p \\
 \delta_u(E_i, H_i) &= -\delta_u(\varphi, \psi)_{,i} = -(\varphi, \psi)_{,ip} \delta u_p = (E_{p,i}, H_{p,i}) \delta u_p = (E_{i,p}, H_{i,p}) E_{i,p} \delta u_p
 \end{aligned} \tag{60}$$

Noting that in Eq. (59) we have

$$\begin{aligned}
& \int_V \delta g_{em} dV + \int_V g^{em} \delta u_{k,k} dV = \int_V \sigma_{ij} \delta \varepsilon_{ij} dV - \int_V D_i \delta E_i dV - \int_V B_i \delta H_i dV \\
& - (1/2) \int_V (D_k E_k + B_k H_k) \delta u_{j,j} dV = \int_a [\sigma_{ij} - (1/2)(D_k E_k + B_k H_k) \delta_{ij}] n_j \delta u_i da \\
& - \int_V [\sigma_{ij} - (1/2)(D_k E_k + B_k H_k) \delta_{ij}]_{,j} \delta u_i dV + \int_a D_i n_i \delta \varphi da - \int_V D_{i,i} \delta \varphi dV \\
& - \int_V D_i E_{p,i} \delta u_p dV + \int_a B_i n_i \delta \psi da - \int_V B_{i,i} \delta \psi dV - \int_V B_i H_{p,i} \delta u_p dV
\end{aligned}$$

So  $\delta \Pi_1$  in Eq. (59) is reduced to

$$\begin{aligned}
\delta \Pi_1 &= \delta \Pi'_1 + \delta \Pi''_1 \\
\delta \Pi'_1 &= \int_{a_\sigma} (\sigma_{kj} n_j - T_k^*) \delta u_k da + \int_{a_{\text{int}}} \sigma_{kj} n_j \delta u_k da \\
& - \int_V (\sigma_{kj,j} + f_k - \rho \ddot{u}_k) \delta u_k dV - \int_V (D_{i,i} - \rho_e) \delta \varphi dV + \int_{a_D} (D_i n_i + \sigma^*) \delta \varphi da \\
& + \int_{a_{\text{int}}} D_i n_i \delta \varphi da - \int_V B_{i,i} \delta \psi dV + \int_{a_\mu} (B_i - B_i^*) n_i \delta \psi da + \int_{a_{\text{int}}} B_i n_i \delta \psi da \\
\delta \Pi''_1 &= (1/2) \int_{a_\sigma} (D_m \varphi_{,m} + B_m \psi_{,m}) n_k \delta u_k da + (1/2) \int_{a_{\text{int}}} (D_m \varphi_{,m} + B_m \psi_{,m}) n_k \delta u_k da \\
& - (1/2) \int_V (D_m \varphi_{,m} + B_m \psi_{,m})_{,k} \delta u_k dV - \int_V D_i E_{p,i} \delta u_p dV - \int_V \rho_e E_p \delta u_p dV \\
& - \int_{a_D} \sigma^* E_p \delta u_p da - \int_V B_i H_{p,i} \delta u_p dV + \int_{a_{\mu}^{env}} B_i^* n_i H_p \delta u_p da
\end{aligned} \tag{61}$$

where  $\delta \Pi'_1$  is the part of  $\delta \Pi_1$  due to the local variations of  $\mathbf{u}, \varphi, \psi$ ;  $\delta \Pi''_1$  is the part of  $\delta \Pi_1$  due to the migratory variations of  $\varphi, \psi$ . Substituting the following identity

$$\begin{aligned}
& - \int_V D_i E_{p,i} \delta u_p dV - \int_V \rho_e E_p \delta u_p dV - \int_{a_D} \sigma^* E_p \delta u_p da - \int_V B_i H_{p,i} \delta u_p dV \\
& + \int_{a_\mu} B_i^* n_i H_p \delta u_p da = \int_{a_D} (D_i n_i + \sigma^*) \delta \varphi da + \int_{a_{\text{int}}} D_i n_i \delta \varphi da + \int_a (D_i E_p) n_i \delta u_p da \\
& - \int_V (D_i E_p)_{,i} \delta u_p dV - \int_V (D_{i,i} - \rho_e) \delta \varphi dV + \int_{a_\mu} (B_i - B_i^*) n_i \delta \psi da \\
& + \int_{a_{\text{int}}} B_i n_i \delta \psi da + \int_a (B_i H_p) n_i \delta u_p da - \int_V (B_i H_p)_{,i} \delta u_p dV - \int_V B_{i,i} \delta \psi dV
\end{aligned} \tag{62}$$

into  $\delta \Pi''_1$  in Eq. (61) we get

$$\begin{aligned}
\delta \Pi''_1 &= - \int_V (D_{i,i} - \rho_e) \delta \varphi dV + \int_{a_D} (D_i n_i + \sigma^*) \delta \varphi da + \int_{a_{\text{int}}} D_i n_i \delta \varphi da \\
& - \int_V B_{i,i} \delta \psi dV + \int_{a_\mu} (B_i - B_i^*) n_i \delta \psi da + \int_{a_{\text{int}}} B_i n_i \delta \psi da \\
& + \int_{a_\sigma} \sigma_{ik}^M n_i \delta u_k da + \int_{a_{\text{int}}} \sigma_{ik}^M n_i \delta u_k da - \int_V \sigma_{ik,i}^M \delta u_k dV
\end{aligned} \tag{63}$$

where  $\sigma^M$  is the Maxwell stress:

$$\sigma_{ik}^M = D_i E_k + B_i H_k - (1/2)(D_n E_n + B_n H_n) \delta_{ik} \tag{64}$$

Substituting Eq. (63) into Eq. (61) we get

$$\begin{aligned}
\delta \Pi_1 = & \int_{a_\sigma} (\tilde{\sigma}_{ij} n_i - T_j^*) \delta u_j da + \int_{a^{\text{int}}} \tilde{\sigma}_{ij} n_i \delta u_j da - \int_V (\tilde{\sigma}_{ij,i} + f_j - \rho \ddot{u}_j) \delta u_j dV \\
& + \int_{a_D} (D_{i,i} + \sigma^*) \delta \varphi da + \int_{a^{\text{int}}} D_i n_i \delta \varphi da - \int_V (D_{i,i} - \rho_e) \delta \varphi dV \\
& + \int_{a_\mu} (B_i - B_i^*) n_i \delta \psi da + \int_{a^{\text{int}}} B_i n_i \delta \psi da - \int_V B_{i,i} \delta \psi dV \\
\tilde{\sigma}_{kl} = & \sigma_{kl} + \sigma_{kl}^M = C_{ijkl} \varepsilon_{ij} - e_{jkl}^e E_j - e_{jkl}^m H_j - (1/2) l_{ijkl}^e E_i E_j - (1/2) l_{ijkl}^m H_i H_j \\
& - (1/2) (D_m E_m + B_m H_m) \delta_{kl}
\end{aligned} \quad (65)$$

where  $\tilde{\sigma}$  is the pseudo total stress (Jiang and Kuang, 2004), which is not the true stress in electromagnetic media. From the expression of  $\tilde{\sigma}$  it is known that  $\tilde{\sigma}$  is symmetric though  $\sigma$  and  $\sigma^M$  are asymmetric. Due to the arbitrariness of  $\delta u_i, \delta \varphi$  and  $\delta \psi$ , from Eq. (65) we get

$$\begin{aligned}
\tilde{\sigma}_{jk,j} + f_k &= \rho \ddot{u}_k, \quad D_{i,i} = \rho_e, \quad B_{i,i} = 0, \quad \text{in } V \\
\tilde{\sigma}_{jk} n_j &= T_k^*, \quad \text{on } a_\sigma; \quad D_i n_i = -\sigma^*, \quad \text{on } a_D; \quad (B_i - B_i^*) n_i = 0, \quad \text{on } a_\mu \\
\delta \Pi_1 &= \int_{a^{\text{int}}} \tilde{\sigma}_{ij} n_i \delta u_j da + \int_{a^{\text{int}}} D_i n_i \delta \varphi da + \int_{a^{\text{int}}} B_i n_i \delta \psi da
\end{aligned} \quad (66)$$

For the environment we have the similar formula:

$$\begin{aligned}
\tilde{\sigma}_{ij,i}^{\text{env}} + f_j^{\text{env}} &= \rho^{\text{env}} \ddot{u}_j^{\text{env}}, \quad D_{i,i}^{\text{env}} = \rho_e^{\text{env}}, \quad B_{i,i} = 0, \quad \text{in } V^{\text{env}} \\
\tilde{\sigma}_{ij}^{\text{env}} n_i^{\text{env}} &= T_j^{*\text{env}}, \quad \text{on } a_\sigma^{\text{env}}; \quad D_i^{\text{env}} n_i^{\text{env}} = -\sigma^{*\text{env}}, \quad \text{on } a_D^{\text{env}}; \quad B_i^{\text{env}} n_i^{\text{env}} = B_i^{*\text{env}} n_i^{\text{env}}, \quad \text{on } a_\mu^{\text{env}} \\
\delta \Pi_2 &= \int_{a^{\text{int}}} \tilde{\sigma}_{ij}^{\text{env}} n_i^{\text{env}} \delta u_j^{\text{env}} da + \int_{a^{\text{int}}} D_i^{\text{env}} n_i^{\text{env}} \delta \varphi^{\text{env}} da + \int_{a^{\text{int}}} B_i^{\text{env}} n_i^{\text{env}} \delta \psi^{\text{env}} da \\
\tilde{\sigma}_{jk}^{\text{env}} &= \sigma_{jk}^{\text{env}} + \sigma_{jk}^{M\text{env}}
\end{aligned} \quad (67)$$

Using  $n_i = -n_i^{\text{env}}, u_i = u_i^{\text{env}}, \varphi = \varphi^{\text{env}}, \psi = \psi^{\text{env}}$  and  $\delta \Pi_1 + \delta \Pi_2 = \delta W^{*\text{int}}$  on the boundary surface we get

$$(\tilde{\sigma}_{ij} - \tilde{\sigma}_{ij}^{\text{env}}) n_i = T_j^{*\text{int}}, \quad (D_i - D_i^{\text{env}}) n_i = -\sigma^{*\text{int}}, \quad (B_i - B_i^{\text{env}}) n_i = B_i^{*\text{int}} n_i, \quad \text{on } a^{\text{int}} \quad (68)$$

The above variational principle requests prior that the displacements, the electric potential and the magnetic potential satisfy their own boundary conditions and the continuity conditions on the interface, so the following equations should also be added to governing equations

$$\begin{aligned}
u_i &= u_i^*, \quad \text{on } a_u; \quad \varphi = \varphi^*, \quad \text{on } a_\varphi; \quad \psi = \psi^*, \quad \text{on } a_\mu \\
u_i^{\text{env}} &= u_i^{*\text{env}}, \quad \text{on } a_u^{\text{env}}; \quad \varphi^{\text{env}} = \varphi^{*\text{env}}; \quad \text{on } a_\varphi^{\text{env}}; \quad \psi^{\text{env}} = \psi^{*\text{env}}, \quad \text{on } a_\mu^{\text{env}} \\
u_i &= u_i^{\text{env}}, \quad \varphi = \varphi^{\text{env}}, \quad \psi = \psi^{\text{env}}; \quad \text{on } a^{\text{int}}
\end{aligned} \quad (69)$$

Eqs. (66)–(69) are the governing equations. It is obvious that the above physical variational principle is easy to extend to other materials.

### 3.5 Materials with static magnetoelectric coupling effect

In this section we discuss the electro-magneto-elastic media with static magnetoelectric coupling effect shortly. For these materials the constitutive equations are

$$\begin{aligned}\sigma_{kl} &= C_{ijkl}\varepsilon_{ij} - e_{jkl}^e E_j - e_{jkl}^m H_j - (1/2)l_{ijkl}^e E_i E_j - (1/2)l_{ijkl}^m H_i H_j \\ &\quad - \epsilon_{km} E_m E_l - \mu_{km} H_m H_l - \beta_{km} H_m E_l - \beta_{km} E_m H_l \\ D_k &= \left[ \epsilon_{kl} + l_{ijkl}^e \varepsilon_{ij} + (\epsilon_{ml} \varepsilon_{mk} + \epsilon_{mk} \varepsilon_{ml}) \right] E_l + e_{kij}^e \varepsilon_{ij} + \beta_{kl} H_l \\ B_k &= \left[ \mu_{kl} + l_{ijkl}^m \varepsilon_{ij} + 2(\alpha_{ml}^m \varepsilon_{mk} + \alpha_{mk}^m \varepsilon_{ml}) \right] H_l + e_{kij}^m \varepsilon_{ij} + \beta_{kl} E_l\end{aligned}\quad (70)$$

where  $\beta_{ij} = \beta_{ji}$  is the static magnetoelectric coupling coefficient. The electromagnetic body couple is still balanced by the asymmetric stress, i.e.

$$\begin{aligned}D_k E_l - D_l E_k + B_k H_l - B_l H_k &= \left[ (\epsilon_{km} E_l - \epsilon_{lm} E_k) E_m + (\mu_{km} H_l - \mu_{lm} H_k) H_m \right] \\ &\quad + \left[ (\beta_{km} E_l - \beta_{lm} E_k) H_m + (\beta_{km} H_l - \beta_{lm} H_k) E_m \right] = -2\sigma_{kl}^a\end{aligned}$$

In this case though the constitutive equations are changed, but the electromagnetic Gibbs free energy  $g_e$  in Eq. (56b), governing equations (66)–(69) and the Maxwell stress (64) are still tenable.

## 4. Conclusions

In this chapter some advances of thermodynamics in continuum mechanics are introduced. We advocate that the first law of the thermodynamics includes two contents: one is the energy conservation and the other is the physical variational principle which is substantially the momentum equation. For the conservative system the complete governing equations can be obtained by using this theory and the classical thermodynamics. For the non-conservative system the complete governing equations can also be obtained by using this theory and the irreversible thermodynamics when the system is only slightly deviated from the equilibrium state. Because the physical variational principle is tensely connected with the energy conservation law, so we write down the energy expressions, we get the physical variational principle immediately and do not need to seek the variational functional as that in usual mathematical methods.

In this chapter we also advocate that the accelerative variation of temperature needs extra heat and propose the general inertial entropy theory. From this theory the temperature wave and the diffusion wave with finite propagation velocities are easily obtained. It is found that the coupling effect in elastic and temperature waves attenuates the temperature wave, but enhances the elastic wave. So the theory with two parameters by introducing the viscous effect in this problem may be more appropriate.

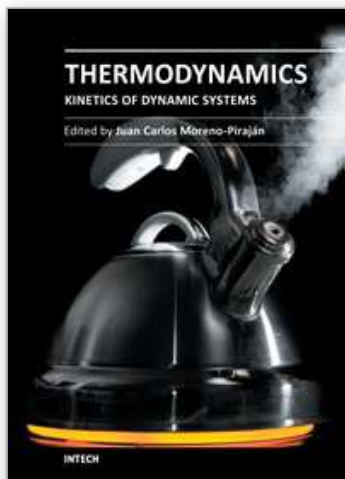
Some explanation examples for the physical variational principle and the inertial entropy theory are also introduced in this chapter, which may indirectly prove the rationality of these theories. These theories should still be proved by experiments.

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Thermodynamics is one of the most exciting branches of physical chemistry which has greatly contributed to the modern science. Being concentrated on a wide range of applications of thermodynamics, this book gathers a series of contributions by the finest scientists in the world, gathered in an orderly manner. It can be used in post-graduate courses for students and as a reference book, as it is written in a language pleasing to the reader. It can also serve as a reference material for researchers to whom the thermodynamics is one of the area of interest.

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