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# Dynamics and Control for a Novel One-Legged Hopping Robot in Stance Phase

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## 1. Introduction

Legged locomotion systems are a class of biological robots by supporting on the ground with discrete points, and can traverse nature terrain in large range. This kind of robots is a class of important research objects in robotics community in a long time. In the early time, most of the legged robot systems are static balance locomotion systems, which show a slow moving speed and high requirement in energy dissipation. During the past twenty years, many scholars were interested in dynamic legged robots for improving the performance of the systems.

The one-legged hopping robots are a typical and the simplest systems of the dynamic legged robots, and can be found in many literatures, such as (Raibert, 1986; Francois & Samson, 1998; Gregorio et al, 1997; Ahmadi & Buehler, 1997, 2006; Ahmadi et al, 2007; Vakasi et al, 1991; Lapshin, 1992; Papadopoulos & Cherouvim, 2004; Hyon & Emura, 2002; Zeglin, 1999; Guang-Ping H. & Zhi-Yong G, 2008 etc.). As it pointed out that in (Papadopoulos & Cherouvim, 2004), most of researches about the one-legged hopping robots were limited to the systems with *Spring Loaded Inverted Pendulum* (SLIP) model, which is composed of a point mass attached on a telescopic spring that is free to rotate around its point of contact with the ground. The SLIP model hopping systems can be stabilized without much effort in control design since the simple decoupling dynamics (Raibert et al, 1984; Raibert, 1986; Francois & Samson, 1998; Gregorio et al, 1997; Ahmadi & Buehler, 1997, 2006; Ahmadi et al, 2007; Vakasi et al, 1991; Lapshin, 1992; Papadopoulos & Cherouvim, 2004; Hyon & Emura, 2002, 2004; Zeglin, 1999; Hodgins & Raibert, 1990; Hyon et al, 2004). It was shown that this class of hopping robots not only could stably hop but also realize some acrobatic motion such as somersaults (Raibert et al, 1984; Hodgins & Raibert, 1990). Nevertheless, the main limitations of the SLIP model systems are that the mechanical systems lose the biological characteristics and the leg of the robot commonly has translational telescopic joint. The translational telescopic legs are generally actuated by pneumatic or hydraulic actuators, and show small motion range. The pumping station of the hydraulic systems is not easy to be embedded into the robots such that the robots could not move in large range. Though (Gregorio et al, 1997; Ahmadi & Buehler, 1997, 2006; Ahmadi et al, 2007) realized the hopping control on an electrically actuated experimental robot, the

theory analysis and experimental methods have little help for a legged hopping system with fully articulated joints.

For overcoming the limitations of the hopping systems with SLIP model, several scholars studied the hopping systems with non-SLIP model (Zeglin, 1991; Hyon & Mita, 2002; Takahashi et al, 2006; Guang-Ping H. & Zhi-Yong G, 2008, 2009a, 2009b, 2010), such as Uniroo (Zeglin, 1991) and Kenken (Hyon & Mita, 2002). The hopping systems with non-SLIP model generally have more biological characteristics. Thus the mechanism is more complex, and control design for this class system is more intractable since the highly nonlinear and strong coupling in dynamics. For instance, the Uniroo and Kenken are experimental robots, whereas the Uniroo employed the control method for SLIP systems and the best experimental results is 40 times jumps before falling down (Zeglin, 1991), the Kenken was controlled based on accurate simulations of the dynamics, and both of the two robots are actuated by hydraulic systems.

For the sake of reducing the energy dissipation caused by impact, the one-legged hopping systems commonly has small foot such that the robot only contacts with ground on a point. On the assumption that the foot of the robot has no slip, the point contacting with the ground can be regarded as a passive rotational joint. Thus the one-legged hopping robots are generally underactuated mechanical systems. In the field of nonlinear control, the underactuated mechanical systems are a class of interesting nonlinear systems that has been given many attentions in recent years. The benchmark systems of them include the Cart-pole (Graichen, 2007), Acrobot (Lai, 2008), Pendubot (Spong & Block, 1995), Plate-Ball (Oriolo & Vendittelli, 2005), underactuated planar manipulators (Arai et al, 1998), and underactuated surface vessel (Reyhanoglu, 1997) etc. It had been proved that the underactuated mechanical systems are second-order nonholonomic systems in gravitational circumstance if the passive generalized coordinates are not cyclic (Oriolo & Nakamura, 1991), and the nonholonomic systems cannot be stabilized by smooth time-invariant state feedback (Kolmanivsky & McClamroch, 1995). Thus the control methods presented in the literatures introduced non-continuous feedback, time-varying feedback, or the combination of the two classes method, and the control plants are mainly limited to the nonholonomic systems with special differential geometric or differential algebraic properties, such as differentially flat (Nieuwstadt & Murray, 1995; Guang-Ping H. & Zhi-Yong G, 2008, 2009a) or nilpotent (Murray, 1994; Guang-Ping H. & Zhi-Yong G, 2009b) systems. By nonlinear coordinates and inputs (control) changes, the differentially flat nonholonomic systems can be transformed into high order linear systems, the nilpotent systems can be transformed into chained form system under certain conditions (Murray et al, 1993). For the second-order nonholonomic underactuated mechanical systems without these properties, the control problem was not discussed adequately in the nonlinear control field.

The hopping robots with SLIP model are generally second-order nonholonomic underactuated systems because of the small foot, whereas they can be stabilized to the periodical hopping orbits by smooth time-invariant state feedback (Raibert, 1986; Gregorio et al, 1997; Ahmadi & Buehler, 1997, 2006; Ahmadi et al, 2007; Hodgins & Raibert, 1990). The reason is that the special mechanical structure satisfies some conditions: ① the mass as well as the inertia of the leg are far less than them of the total system, swing the leg does not cause large orientation errors of the body; ② the position of the mass center (MC) of the robot is coincident to the hip joint, then most of the nonlinear force of the dynamics disappears; ③ the robot has linearly telescopic leg, the telescopic motion of the leg is approximately decoupled from the rotational motion of the systems. These dynamic

properties ensure that the SLIP model robot can be stabilized by controlling partial variables without considering the nonholonomic constraints. Nevertheless, a one-legged hopping system with non-SLIP model shows complex nonlinear dynamics, the nonholonomic constraints of the system cannot be ignored. So far there does not have systemic method for designing the control for this class of hopping robots.

Since the obstacle for controlling a general underactuated mechanical system, it is important to investigate the dynamic synthesis problem such that the dynamics of the underactuated systems can be effectively simplified while the underactuated systems holds the controllability, high energy-efficiency, and dexterous mobility. This is helpful for inventing a new underactuated mechanical system with feasible control method, investigating the new applications of underactuated mechanical systems, or simplifying the control problem of the existing underactuated mechanical systems. For instance, Franch investigated the design method of differentially flat planar space robots (Franch et al, 2003), Agrawal & Fattch studied the dynamics synthesis for planar biped robots (Agrawal & Fattch, 2006). In this paper, we propose a novel biological mechanism for the one-legged planar hopping robots on the basis of the dynamics synthesizing. The mechanism is similar to the skeleton of kangaroos, and the dynamics of the mechanism possesses kinetic symmetry with respect to the passive joint variable, then the nonlinear dynamics of the novel mechanism can be transformed into the so-called *strict feedback normal form*, which can be potentially stabilized by backstepping technique. Thus the novel mechanism can be used to compare with the SLIP model robot for more adequately understanding the dynamic balance principle, high energy-efficiency, and dexterous mobility of kangaroos.

In this chapter, section 2 introduces the novel mechanism, and the dynamics of it is presented in section 3. In section 4, the proposition that confirms the nonlinear dynamics can be transformed into the strict feedback normal form is proved. Then a sliding mode backstepping control is introduced in section 5 and the exponential stability is also proved. The motion planning method for the hopping system in stance phase is presented in section 6. The feasibility of the mechanism and the stability of the control are verified by some numerical simulations in section 7.

## 2. The new mechanism for one-legged planar hopping robots

Fig.1 shows the new mechanism for designing a biological one-legged hopping robot. The robot mechanism has four rigid bodies, of which the shank, thigh, body and tail have length, mass and inertia with respect to their MC are  $l_i$ ,  $m_i$  and  $I_i$ ,  $i=1,2,3,4$  respectively. Suppose the MC of every link deviates from the joint that nears to the ground, and the distance deviating from the joint is  $l_{ci}$ ,  $i=1,2,3,4$ . The angle of shank is denoted as  $\theta_1$ , the keen joint, hip joint, and tail joint are  $\theta_2$ ,  $\theta_3$ , and  $\theta_4$  respectively. To simplify the dynamics of the system, let  $l_{c3}=0$ , and the mechanism is designed to joint the body, the thigh, and the tail at the same axis. The turning of the tail joint and the keen joint is synchronous (by parallel four bars mechanism or synchronous belt) with constant phase angle  $\alpha_0$ . For improving the energy efficiency, a torsional linear spring with stiffness  $k$  is paralleled in the keen joint. The generalized coordinates of the mechanism in stance phase can be defined as  $\mathbf{q}=[x_0, y_0, \theta_1, \theta_2, \theta_3]^T$ , of which  $\theta_2, \theta_3$  are actuated joints,  $\theta_1$  is a passive joint, and  $(x_0, y_0)$  are constants in stance phase.

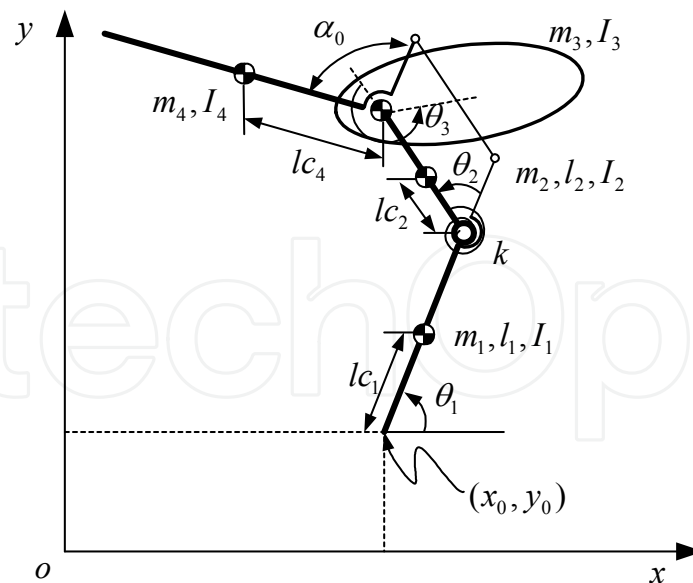


Fig. 1. A novel mechanism for one-legged hopping robot

The characteristics of the hopping robot mechanism can be summarized as follows:

- The mechanism is underactuated because of the passive joint  $\theta_1$ . In the following section, it is shown that the single passive generalized coordinate  $\theta_1$  does not appear in the kinetic energy but appears in the potential energy of the robot system. Thus the robot system is similar to an Acrobot system (Lai et al, 2008; Spong, 1995) in dynamics. This property in dynamics makes the nonlinear dynamics of the hopping robot can be transformed into the strict feedback normal form, which belongs to a special class of nonlinear system that can be stabilized by backstepping control.
- The turning of the knee joint and the tail joint is synchronous, so the actuator of the knee joint can be installed on the tail. This special design can reduce the mass as well as the inertia of the leg. As proved by (Ahmadi & Buehler, 2006), a leg with less mass and inertia is helpful for improving the energy-efficiency of the hopping robot systems.
- The MC of the body is coincident with the hip joint, i.e.  $l_{c3} = 0$ , such that the Coriolis and centrifugal forces about  $l_{c3}$  disappear, then the dynamics of the robot system is considerably simplified. In section 4, it will be shown that this property in dynamics makes the novel hopping system has the analytical coordinate transformations, which is necessary for designing the nonlinear controller.
- The robot mechanism has articulate knee joint, so the leg can provide a larger clearance from the ground than it provided by a linearly telescopic leg in continuous hopping. This is beneficial for leaping over different size obstacles with the same energy cost.

### 3. The dynamics of the mechanism

In stance phase, the foot of the robot is contacting with the ground, thus the coordinates  $(x_0, y_0)$  are constants, and then the generalized coordinates of the hopping system shown in Fig.1 is reduced to  $\mathbf{q}_1 = [\theta_1 \ \theta_2 \ \theta_3]^T$ . If  $L = T - U$  denotes the Lagrangian of the hopping system, where  $T$  is kinetic energy,  $U$  is potential energy of the system, then they can be given as form

$$T = \frac{1}{2} \sum_{i=1}^{i=4} m_i (\dot{x}_{ci}^2 + \dot{y}_{ci}^2) + \frac{1}{2} [I_1 \dot{\theta}_1^2 + I_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + I_3 (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2 + I_4 \dot{\theta}_1^2],$$

$$U = g \sum_{i=1}^4 (m_i y_{ci}) + \frac{1}{2} k (\theta_2 - \theta_{20})^2,$$

where  $(x_{ci}, y_{ci}), i = 1, 2, 3, 4$  is the position of MC of every bodies of the mechanism, and can be written as

$$x_{c1} = x_0 + l_{c1} \cos \theta_1,$$

$$y_{c1} = y_0 + l_{c1} \sin \theta_1,$$

$$x_{c2} = x_0 + l_1 \cos \theta_1 + l_{c2} \cos(\theta_1 + \theta_2),$$

$$y_{c2} = y_0 + l_1 \sin \theta_1 + l_{c2} \sin(\theta_1 + \theta_2),$$

$$x_{c3} = x_0 + l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2),$$

$$y_{c3} = y_0 + l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2),$$

$$x_{c4} = x_0 + l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_{c4} \cos(\theta_1 + \alpha_0),$$

$$y_{c4} = y_0 + l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_{c4} \sin(\theta_1 + \alpha_0).$$

$g$  is the gravitational acceleration,  $k$  is the stiffness of the spring in keen joint,  $\theta_{20}$  is the position of keen joint with spring free.

The Euler-Lagrange dynamics of the hopping system in stance phase can be written as

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \tau_i, i = 1, 2, 3 \quad (1)$$

For more clearly, the dynamics (1) has form

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}_1} \right) + \frac{\partial U}{\partial \theta_1} &= 0 \\ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}_2} \right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial U}{\partial \theta_2} &= \tau_2 \\ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}_3} \right) &= \tau_3 \end{aligned} \quad (2)$$

where  $\tau_1 = 0, \partial T / \partial \theta_1 = 0$  and  $\partial L / \partial \theta_3 = 0$  are considered. In other words, the kinetic energy is not depended on the generalized coordinates  $\theta_1$  and  $\theta_3$ , and potential energy is not depended on  $\theta_3$ . Olfati-Saber (Olfati-Saber, 2002) defines a coordinate to be *kinetic symmetry* if the coordinate does not appear in the kinetic energy of a mechanical system. The kinetic



symmetry is different from the well-known Lagrangian symmetry in classical mechanics. The existence of kinetic symmetries does not lead to the existence of conserved quantities in potential field. This is important to preserve the controllability of an underactuated mechanism (2). Since the relationship  $\partial U / \partial \theta_1 \neq 0$  is satisfied outside the unstable balance point, the first equation of (2) cannot be integrated to a first-order differential equation, then it can be regarded as a second-order differential constraints (or second-order nonholonomic constraints (Oriolo & Nakamura, 1991)) of the actuated subsystem given by the last two equations of (2).

Further more, the dynamics (2) can be written as matrix form

$$\mathbf{M}_1(\mathbf{q}_1)\ddot{\mathbf{q}}_1 + \mathbf{C}_1(\mathbf{q}_1, \dot{\mathbf{q}}_1) + \mathbf{H}_1(\mathbf{q}_1) = \mathbf{Q}_1 \quad (3)$$

where  $\mathbf{Q}_1 = [0 \quad \tau_2 \quad \tau_3]^T$  is the generalized force vector,  $\mathbf{M}_1$  is the inertia matrix of the robot in stance phase and can be given by

$$\mathbf{M}_1 = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

where

$$m_{11} = (m_2 + m_3 + m_4)l_1^2 + (m_3 + m_4)l_2^2 + \sum_{i=1}^4 (m_i l_{ci}^2 + I_i) + 2m_2 l_1 l_{c2} \cos \theta_2 + 2m_3 l_1 l_2 \cos \theta_2, \\ + 2m_4 [l_1 l_2 \cos \theta_2 + l_1 l_{c4} \cos(\alpha_0) + l_2 l_{c4} \cos(\theta_2 - \alpha_0)]$$

$$m_{12} = m_2 l_{c2}^2 + (m_3 + m_4)l_2^2 + \sum_{i=2}^4 I_i + m_2 l_1 l_{c2} \cos \theta_2 + m_3 l_1 l_2 \cos \theta_2, \\ + m_4 [l_1 l_2 \cos \theta_2 + l_1 l_{c4} \cos(\alpha_0) + l_2 l_{c4} \cos(\theta_2 - \alpha_0)]$$

$$m_{13} = I_3,$$

$$m_{21} = m_{12},$$

$$m_{22} = m_2 l_{c2}^2 + (m_3 + m_4)l_2^2 + \sum_{i=2}^4 I_i,$$

$$m_{23} = I_3,$$

$$m_{31} = m_{13}, m_{32} = m_{23}, \text{ and } m_{33} = I_3.$$

$\mathbf{C}_1(\mathbf{q}_1, \dot{\mathbf{q}}_1)$  denotes the Coriolis and centrifugal forces, and has form

$$\mathbf{C}_1(\mathbf{q}_1, \dot{\mathbf{q}}_1) = [c_1 \quad c_2 \quad c_3]^T$$

where

$$c_1 = \dot{m}_{11}\dot{\theta}_1 + \dot{m}_{12}\dot{\theta}_2 + \dot{m}_{13}\dot{\theta}_3,$$

$$c_2 = \dot{m}_{21}\dot{\theta}_1 - \frac{1}{2}\dot{\mathbf{q}}_1^T \left( \frac{\partial}{\partial \theta_2} M_1 \right) \dot{\mathbf{q}}_1 \text{ and}$$

$$c_3 = 0.$$

$\mathbf{H}_1(\mathbf{q}_1)$  is the potential forces including both gravity and elastic force of the coil spring in keen joint, and can be written as

$$\mathbf{H}_1(\mathbf{q}_1) = [h_1 \quad h_2 \quad h_3]^T$$

where

$$h_1 = m_1 g l_{c1} \cos \theta_1 + (m_2 + m_3 + m_4) g l_1 \cos \theta_1 + (m_2 l_{c2} + m_3 l_2 + m_4 l_2) g \cos(\theta_1 + \theta_2), \\ + m_4 l_{c4} g \cos(\theta_1 + \alpha_0)$$

$$h_2 = (m_2 l_{c2} + m_3 l_2 + m_4 l_2) g \cos(\theta_1 + \theta_2) + m_4 l_{c4} \cos(\theta_1 + \alpha_0) g + k(\theta_2 - \theta_{20}),$$

and  $h_3 = 0.$

The dynamics (3) can also be partitioned as following form according to the passive and actuated coordinates

$$\begin{aligned} \mathbf{M}_{pp}(\theta_2) \ddot{\mathbf{q}}_p + \mathbf{M}_{pa}(\theta_2) \ddot{\mathbf{q}}_a + \mathbf{C}_p + \mathbf{H}_p(\theta_1, \theta_2) &= 0 \\ \mathbf{M}_{pa}^T(\theta_2) \ddot{\mathbf{q}}_p + \mathbf{M}_{aa}(\theta_2) \ddot{\mathbf{q}}_a + \mathbf{C}_a + \mathbf{H}_a(\theta_1, \theta_2) &= \boldsymbol{\tau} \end{aligned} \quad (4)$$

where  $\mathbf{q}_p = \theta_1$ ,  $\mathbf{q}_a = [\theta_2 \quad \theta_3]^T$ , and the subscript  $p$  and  $a$  denote “passive” and “actuated” respectively. Define  $\mathbf{u} = \ddot{\mathbf{q}}_a$  to be a new input, and by a control change

$$\boldsymbol{\tau} = -(\mathbf{M}_{aa} - \mathbf{M}_{pa}^T \mathbf{M}_{pp}^{-1} \mathbf{M}_{pa}) \mathbf{u} + [\mathbf{C}_a + \mathbf{H}_a - \mathbf{M}_{pa}^T \mathbf{M}_{pp}^{-1} (\mathbf{C}_p + \mathbf{H}_p)]$$

the dynamics Eq.(4) can be transformed into the *partial feedback linearization form* that is due to (Spong, 1995).

$$\begin{aligned} \ddot{\mathbf{q}}_p &= -\mathbf{M}_{pp}^{-1} (\mathbf{C}_p + \mathbf{H}_p) - \mathbf{M}_{pp}^{-1} \mathbf{M}_{pa} \mathbf{u} \\ \ddot{\mathbf{q}}_a &= \mathbf{u} \end{aligned} \quad (5)$$

The main property of the underactuated system in (5) is the new control  $\mathbf{u}$  appears in both the subsystems  $\mathbf{q}_p$  and  $\mathbf{q}_a$  of dynamics (5). This leads the control design for the system (5) is very difficult. In following section, we prove that the dynamics (5) can be further transformed into a special cascade nonlinear system, which simplifies the problem of designing a feasible control for the underactuated mechanical system (5) under certain additional conditions.



#### 4. The strict feedback normal form of the dynamics

Olfati-Saber had been studied the normal form of underactuated mechanical systems in his excellent paper (Olfati-Saber, 2002). He presented three classes of cascade nonlinear systems in strict feedback form, feedforward form, and nontriangular quadratic form. As to the robot system considered in this paper, the following proposition ensures that the dynamics (5) can be transformed into the strict feedback normal form.

**Proposition 1: (strict feedback form of (5))** The following global change of coordinates:

$$\begin{aligned}\mathbf{q}_r &= \mathbf{q}_p + \boldsymbol{\psi}(\mathbf{q}_a) \\ \mathbf{p}_r &= \partial T / \partial \dot{\mathbf{q}}_p\end{aligned}\quad (6)$$

transforms the dynamics of (5) into a cascade nonlinear system in strict feedback form

$$\begin{aligned}\dot{\mathbf{q}}_r &= \mathbf{M}_{pp}^{-1}(\mathbf{q}_a)\mathbf{p}_r \\ \dot{\mathbf{p}}_r &= -\mathbf{H}_p(\mathbf{q}_r - \boldsymbol{\psi}(\mathbf{q}_a), \mathbf{q}_a) \\ \dot{\mathbf{q}}_a &= \mathbf{p}_a \\ \dot{\mathbf{p}}_a &= \mathbf{u}\end{aligned}\quad (7)$$

where  $\boldsymbol{\psi}(\mathbf{q}_a) = \int_0^{\mathbf{q}_a} \mathbf{M}_{pp}^{-1}(\boldsymbol{\sigma})\mathbf{M}_{pa}(\boldsymbol{\sigma})d\boldsymbol{\sigma}$ .

**Proof:** Considering the second equation of (6), it follows that

$$\mathbf{p}_r = \partial T / \partial \dot{\mathbf{q}}_p = \mathbf{M}_{pp}(\mathbf{q}_a)\dot{\mathbf{q}}_p + \mathbf{M}_{pa}(\mathbf{q}_a)\dot{\mathbf{q}}_a$$

thus

$$\mathbf{M}_{pp}^{-1}(\theta_2)\mathbf{p}_r = \dot{\mathbf{q}}_p + \mathbf{M}_{pp}^{-1}(\theta_2)\mathbf{M}_{pa}(\theta_2)\dot{\mathbf{q}}_a.$$

With considering the first equation of (6), then it can be obtained  $\dot{\mathbf{q}}_r = \mathbf{M}_{pp}^{-1}(\theta_2)\mathbf{p}_r$ . The first equation of (7) is proved.

Once more, by the second equation of (6), we have

$$\dot{\mathbf{p}}_r = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\mathbf{q}}_p} \right) = \frac{\partial T}{\partial \mathbf{q}_p} - \frac{\partial U}{\partial \mathbf{q}_p} \quad (8)$$

since  $\mathbf{q}_p = \theta_1$  and  $\partial T / \partial \mathbf{q}_p = 0$ , then

$$\dot{\mathbf{p}}_r = -\frac{\partial U}{\partial \mathbf{q}_p} = -\mathbf{H}_p(\mathbf{q}_p, \mathbf{q}_a)$$

and

$$\dot{\mathbf{p}}_r = -\mathbf{H}_p(\mathbf{q}_r - \boldsymbol{\psi}(\mathbf{q}_a), \mathbf{q}_a)$$

is following, with considering the first equation of (6). The second equation of (7) is proved.

Let  $\mathbf{p}_a = \dot{\mathbf{q}}_a$ , then the last two equation of (7) follows. This completes the proof.

**Remark 1:** An obstacle of using the Proposition 1 is that the integral  $\Psi(\mathbf{q}_a) = \int_0^{\mathbf{q}_a} \mathbf{M}_{pp}^{-1}(\sigma) \mathbf{M}_{pa}(\sigma) d\sigma$  should be given in an explicit form. This is not always available for general underactuated mechanical system with satisfying the condition  $\partial T / \partial \mathbf{q}_p = 0$ . As to the robot system shown by Fig.1, thanks to the special design  $l_{c3} = 0$ , the integral can be explicitly obtained with slight errors. If let

$$A_1 = (m_2 + m_3 + m_4)l_1^2 + (m_3 + m_4)l_2^2 + \sum_{i=1}^4 (m_i l_{ci}^2 + I_i) + 2m_4 l_1 l_{c4} \cos \alpha_0,$$

$$A_2 = m_2 l_{c2}^2 + (m_3 + m_4)l_2^2 + \sum_{i=2}^4 I_i + m_4 l_1 l_{c4} \cos \alpha_0,$$

$$B_2 = m_2 l_1 l_{c2} + (m_3 + m_4)l_1 l_2 + m_4 l_2 l_{c4} \cos \alpha_0,$$

$$C_2 = m_4 l_2 l_{c4} \sin \alpha_0,$$

$$a = A_1 - 2B_2,$$

$$b = 4C_2, \text{ and}$$

$$c = A_1 + 2B_2,$$

then the integral has form

$$\begin{aligned} \Psi(\mathbf{q}_a) &= \int_0^{\mathbf{q}_a} \mathbf{M}_{pp}^{-1}(\sigma) \mathbf{M}_{pa}(\sigma) d\sigma \\ &= \int_0^{\theta_2} \frac{m_{12}(\sigma_1)}{m_{11}(\sigma_1)} d\sigma_1 + \frac{m_{13}}{m_{11}(\theta_2)} \int_0^{\theta_3} d\sigma_2 \\ &\approx \int \frac{A_2 + B_2 \cos \theta_2 + C_2 \sin \theta_2}{A_1 + 2B_2 \cos \theta_2 + 2C_2 \sin \theta_2} d\theta_2 + \frac{m_{13}}{m_{11}(\theta_{20})} \theta_3 \end{aligned}$$

Since  $b^2 < 4ac$ ,  $m_{13} = I_3$  is constant, and  $m_{11}(\theta_2) - m_{11}(\theta_{20}) > 0$ , then the integral can be written as

$$\begin{aligned} \Psi(\mathbf{q}_a) &\approx (2A_2 - A_1) \frac{2}{\sqrt{4ac - b^2}} \arctan \left( \frac{2a \cdot \tan \frac{\theta_2}{2} + b}{\sqrt{4ac - b^2}} \right) + \frac{\theta_2}{2} \\ &\quad - (2A_2 - A_1) \frac{2}{\sqrt{4ac - b^2}} \arctan \left( \frac{b}{\sqrt{4ac - b^2}} \right) + \frac{m_{13}}{m_{11}(\theta_{20})} \theta_3 \end{aligned}$$

**Remark 2:** Since  $\partial U / \partial \mathbf{q}_p \neq 0$  is satisfied outside the unstable balance point, then  $\partial U / \partial \mathbf{q}_p = \mathbf{H}_p(\mathbf{q}_p, \mathbf{q}_a)$  can be regarded as the control input of the subsystem  $(\mathbf{q}_r, \mathbf{p}_r)$  of (7). Otherwise, the subsystem  $(\mathbf{q}_r, \mathbf{p}_r)$  is not controllable.

## 5. Design the control for the robot in stance phase

As analyzed in the introduction, there are feasible controls for underactuated mechanical systems to date only concentrated on a few classes of systems with special mathematical property in dynamics, such as the differentially flat or nilpotent. Whereas, there is still no a sufficient and necessary condition that makes certain a nonlinear systems to be differentially flat (Sira-Ramirez & Agrawal, 2004, Ch.8), and a general nilpotent nonlinear system is also difficult in control, with the major exception of the systems that can be transformed into the *chained form* (Kolmanovsky & McClamroch, 1995; Murray & Sastry, 1993; Murray, 1994). Despite that a systemic control method was proposed by (De Luca et al, 2001), for the general nilpotent systems, the convergent speed of control is slow and is hard to be applied to multi-inputs systems. The three classes of cascade nonlinear systems presented by Olfati-Saber (Olfati-Saber, 2002) are additional underactuated systems that exist feasible nonlinear control approaches. Olfati-Saber also proposed a globally stable control for two degrees of freedom (DOF) underactuated mechanical systems in (Olfati-Saber, 2000), nevertheless the suggested control is only applicable for stabilizing the system to it's origin but a trajectory tracking task is not. (Qaiser et al, 2007) also investigated the control problem for a class of underactuated mechanical systems with two DOF based on the result of (Olfati-Saber, 2000), and realized the globally exponential stabilization by Dynamic Surface Control, whereas he also didn't consider the trajectory tracking. For a nonholonomic nonlinear system, it is well known that the stabilization of the origin is not equal to it of a trajectory, since the former is a control problem for driftless nonlinear system under certain conditions, while the later is always a control problem with drift term for a nonlinear system.

To design a control for strict feedback form system (7), we define  $\mathbf{z}_1 = \mathbf{q}_r - \mathbf{q}_r^d$ ,  $\mathbf{z}_2 = \mathbf{p}_r - \mathbf{p}_r^d$ ,  $\xi_1 = \mathbf{q}_a - \mathbf{q}_a^d$ , and  $\xi_2 = \mathbf{p}_a - \mathbf{p}_a^d$ , where the superscript "d" denotes the desired or planned trajectory. Then the error system of (7) can be written as

$$\begin{aligned}\dot{\mathbf{z}}_1 &= \mathbf{f}_1 + \mathbf{g}_1 \mathbf{z}_2 \\ \dot{\mathbf{z}}_2 &= \mathbf{f}_2 + \mathbf{g}_2 \xi_1 + \varepsilon \\ \dot{\xi}_1 &= \mathbf{f}_3 + \mathbf{g}_3 \xi_2 \\ \dot{\xi}_2 &= \mathbf{f}_4 + \mathbf{g}_4 \mathbf{u}\end{aligned}\tag{9}$$

where  $\mathbf{f}_1 = -\dot{\mathbf{q}}_r^d + \mathbf{M}_{pp}^{-1} \mathbf{p}_r^d$ ,  $\mathbf{g}_1 = \mathbf{M}_{pp}^{-1}$ ,

$$\mathbf{f}_2 = -\dot{\mathbf{p}}_r^d - \left[ \mathbf{H}_p(\mathbf{z}_1 - \boldsymbol{\psi}(\xi_1), \xi_1) \right]_{\xi_1=0} = 0,$$

$$\mathbf{g}_2(\mathbf{z}_1, \xi_1) = - \left( \frac{\partial}{\partial \xi_1} \mathbf{H}_p(\mathbf{z}_1 - \boldsymbol{\psi}(\xi_1), \xi_1) \right)_{\xi_1=0},$$

$$\mathbf{f}_3 = -\dot{\mathbf{q}}_a^d + \mathbf{p}_a^d = 0, \quad \mathbf{g}_3 = \mathbf{I},$$

$$\mathbf{f}_4 = -\dot{\mathbf{p}}_a^d, \quad \mathbf{g}_4 = \mathbf{I},$$

$$\Psi(\xi_1) = \int_0^{(q_a^d + \xi_1)} \mathbf{M}_{pp}^{-1}(\theta_2) \mathbf{M}_{pa}(\theta_2) d\sigma,$$

$\varepsilon \sim \xi_1^2$  is the errors item because of the affine approximation, and  $\mathbf{I}$  is the identity matrix. To design a control for the affine nonlinear system (9) with strict feedback normal form, the following proposition can be proved.

**Proposition 2: (Sliding mode backstepping control)**

Consider the system (9), given  $\|\varepsilon\| < \Gamma$ ,  $\Gamma > 0$  is a constant, and  $\forall \eta > 0$ , then there exist a set of positive real numbers  $k_i > 0, i = 1, 2, \dots, 5$ , and control

$$\begin{aligned} \mathbf{u} = & -\mathbf{g}_4^{-1} [k_4(\xi_2 - \alpha_3) + k_5 \text{sign}(\xi_2 - \alpha_3)] \\ & -\mathbf{g}_4^{-1} \left[ \mathbf{g}_3^T(\xi_1 - \alpha_2) + \mathbf{f}_4 - \frac{\partial \alpha_3}{\partial \xi_1}(\mathbf{f}_3 + \mathbf{g}_3 \xi_2) - \frac{\partial \alpha_3}{\partial \mathbf{z}_2}(\mathbf{f}_2 + \mathbf{g}_2 \xi_1) - \frac{\partial \alpha_3}{\partial \mathbf{z}_1}(\mathbf{f}_1 + \mathbf{g}_1 \mathbf{z}_2) \right] \end{aligned} \quad (10)$$

where

$$\alpha_3(\mathbf{z}_1, \mathbf{z}_2, \xi_1) = -\mathbf{g}_3^{-1} \left[ k_3(\xi_1 - \alpha_2) + \mathbf{g}_2^T(\mathbf{z}_2 - \alpha_1) + \mathbf{f}_3 - \frac{\partial \alpha_2}{\partial \mathbf{z}_2}(\mathbf{f}_2 + \mathbf{g}_2 \xi_1) - \frac{\partial \alpha_2}{\partial \mathbf{z}_1}(\mathbf{f}_1 + \mathbf{g}_1 \mathbf{z}_2) \right],$$

$$\alpha_2(\mathbf{z}_1, \mathbf{z}_2) = -\mathbf{g}_2^+(k_2 + \eta \Gamma^2)(\mathbf{z}_2 - \alpha_1) - \mathbf{g}_2^+ \left[ \mathbf{g}_1^T \mathbf{z}_1 + \mathbf{f}_2 - \frac{\partial \alpha_1}{\partial \mathbf{z}_1}(\mathbf{f}_1 + \mathbf{g}_1 \mathbf{z}_2) \right],$$

$$\alpha_1(\mathbf{z}_1) = -\mathbf{g}_1^{-1}(\mathbf{f}_1 + k_1 \mathbf{z}_1), \text{ and}$$

$$\text{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases},$$

renders the system (9) exponentially stabilize to the origin  $(\mathbf{z}, \xi) = (0, 0)$ .

Proof: Consider the subsystem  $\mathbf{z}_1$ , and select  $V_1(\mathbf{z}_1) = \frac{1}{2} \mathbf{z}_1^T \mathbf{z}_1$  to be the candidate Lyapunov function, then one has

$$\dot{V}_1(\mathbf{z}_1) = \mathbf{z}_1^T (\mathbf{f}_1 + \mathbf{g}_1 \mathbf{z}_2).$$

Let  $\mathbf{z}_2 = \alpha_1(\mathbf{z}_1) = -\mathbf{g}_1^{-1}(\mathbf{f}_1 + k_1 \mathbf{z}_1)$ , then it follows that

$$\dot{V}_1(\mathbf{z}_1) = -k_1 \mathbf{z}_1^T \mathbf{z}_1.$$

Further consider the subsystem  $(\mathbf{z}_1, \mathbf{z}_2)$ , select  $V_2(\mathbf{z}_1, \mathbf{z}_2) = V_1(\mathbf{z}_1) + \frac{1}{2}(\mathbf{z}_2 - \alpha_1)^T(\mathbf{z}_2 - \alpha_1)$

to be a new candidate Lyapunov function, and let  $\mathbf{e}_{z_2} = \mathbf{z}_2 - \alpha_1$ ,  $\forall \eta > 0$ . By the Young's inequality  $2ab \leq a^2 + b^2$  and Cauchy-Schwarz inequality  $|\mathbf{x}^T \mathbf{y}| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$ , we have

$$\begin{aligned}
\dot{V}_2(\mathbf{z}_1, \mathbf{z}_2) &= \dot{V}_1(\mathbf{z}_1) + \mathbf{e}_{z_2}^T (\dot{\mathbf{z}}_2 - \dot{\boldsymbol{\alpha}}_1) \\
&= \mathbf{z}_1^T \left[ \mathbf{f}_1 + \mathbf{g}_1(\mathbf{e}_{z_2} + \boldsymbol{\alpha}_1) \right] + \mathbf{e}_{z_2}^T \left[ \mathbf{f}_2 + \mathbf{g}_2 \boldsymbol{\xi}_1 + \varepsilon - \frac{\partial \boldsymbol{\alpha}_1}{\partial \mathbf{z}_1} (\mathbf{f}_1 + \mathbf{g}_1 \mathbf{z}_2) \right] \\
&\leq \mathbf{z}_1^T [\mathbf{f}_1 + \mathbf{g}_1 \boldsymbol{\alpha}_1] \\
&\quad + \mathbf{e}_{z_2}^T \left[ \mathbf{g}_1^T \mathbf{z}_1 + \mathbf{f}_2 + \mathbf{g}_2 \boldsymbol{\xi}_1 - \frac{\partial \boldsymbol{\alpha}_1}{\partial \mathbf{z}_1} (\mathbf{f}_1 + \mathbf{g}_1 \mathbf{z}_2) \right] + \eta \|\mathbf{e}_{z_2}^T\|^2 \|\boldsymbol{\varepsilon}\|^2 + \frac{1}{4\eta} \\
&\leq -k_1 \mathbf{z}_1^T \mathbf{z}_1 \\
&\quad + \mathbf{e}_{z_2}^T \left[ \mathbf{g}_1^T \mathbf{z}_1 + \mathbf{f}_2 + \mathbf{g}_2 \boldsymbol{\alpha}_2 + \mathbf{g}_2 (\boldsymbol{\xi}_1 - \boldsymbol{\alpha}_2) - \frac{\partial \boldsymbol{\alpha}_1}{\partial \mathbf{z}_1} (\mathbf{f}_1 + \mathbf{g}_1 \mathbf{z}_2) + \eta \Gamma^2 \mathbf{e}_{z_2} \right] + \frac{1}{4\eta}
\end{aligned}$$

Let  $\boldsymbol{\alpha}_2(\mathbf{z}_1, \mathbf{z}_2) = -\mathbf{g}_2^+ (k_2 + \eta \Gamma^2) \mathbf{e}_{z_2} - \mathbf{g}_2^+ \left[ \mathbf{g}_1^T \mathbf{z}_1 + \mathbf{f}_2 - \frac{\partial \boldsymbol{\alpha}_1}{\partial \mathbf{z}_1} (\mathbf{f}_1 + \mathbf{g}_1 \mathbf{z}_2) \right]$ , then the last inequality can be written as

$$\dot{V}_2(\mathbf{z}_1, \mathbf{z}_2) \leq -k_1 \mathbf{z}_1^T \mathbf{z}_1 - k_2 \mathbf{e}_{z_2}^T \mathbf{e}_{z_2} + \mathbf{e}_{z_2}^T \mathbf{g}_2 (\boldsymbol{\xi}_1 - \boldsymbol{\alpha}_2) + \frac{1}{4\eta}.$$

For the subsystem  $(\mathbf{z}_1, \mathbf{z}_2, \boldsymbol{\xi}_1)$ , select  $V_3(\mathbf{z}_1, \mathbf{z}_2, \boldsymbol{\xi}_1) = V_2(\mathbf{z}_1, \mathbf{z}_2) + \frac{1}{2}(\boldsymbol{\xi}_1 - \boldsymbol{\alpha}_2)^T (\boldsymbol{\xi}_1 - \boldsymbol{\alpha}_2)$  as the candidate Lyapunov function, and let  $\mathbf{e}_{\xi_1} = \boldsymbol{\xi}_1 - \boldsymbol{\alpha}_2$ , then

$$\begin{aligned}
\dot{V}_3(\mathbf{z}_1, \mathbf{z}_2, \boldsymbol{\xi}_1) &= \dot{V}_2(\mathbf{z}_1, \mathbf{z}_2) + (\boldsymbol{\xi}_1 - \boldsymbol{\alpha}_2)^T (\dot{\boldsymbol{\xi}}_1 - \dot{\boldsymbol{\alpha}}_2) \\
&= -k_1 \mathbf{z}_1^T \mathbf{z}_1 - k_2 \mathbf{e}_{z_2}^T \mathbf{e}_{z_2} + \mathbf{e}_{z_2}^T \mathbf{g}_2 \mathbf{e}_{\xi_1} \\
&\quad + \mathbf{e}_{\xi_1}^T \left[ \mathbf{f}_3 + \mathbf{g}_3 \boldsymbol{\alpha}_3 + \mathbf{g}_3 (\boldsymbol{\xi}_2 - \boldsymbol{\alpha}_3) - \frac{\partial \boldsymbol{\alpha}_2}{\partial \mathbf{z}_2} (\mathbf{f}_2 + \mathbf{g}_2 \boldsymbol{\xi}_1) - \frac{\partial \boldsymbol{\alpha}_2}{\partial \mathbf{z}_1} (\mathbf{f}_1 + \mathbf{g}_1 \mathbf{z}_2) \right] + \frac{1}{4\eta}
\end{aligned}$$

Select

$$\boldsymbol{\alpha}_3(\mathbf{z}_1, \mathbf{z}_2, \boldsymbol{\xi}_1) = -\mathbf{g}_3^{-1} \left[ k_3 \mathbf{e}_{\xi_1} + \mathbf{g}_2^T \mathbf{e}_{z_2} + \mathbf{f}_3 - \frac{\partial \boldsymbol{\alpha}_2}{\partial \mathbf{z}_2} (\mathbf{f}_2 + \mathbf{g}_2 \boldsymbol{\xi}_1) - \frac{\partial \boldsymbol{\alpha}_2}{\partial \mathbf{z}_1} (\mathbf{f}_1 + \mathbf{g}_1 \mathbf{z}_2) \right],$$

then it follows that

$$\dot{V}_3(\mathbf{z}_1, \mathbf{z}_2, \boldsymbol{\xi}_1) \leq -k_1 \mathbf{z}_1^T \mathbf{z}_1 - k_2 \mathbf{e}_{z_2}^T \mathbf{e}_{z_2} - k_3 \mathbf{e}_{\xi_1}^T \mathbf{e}_{\xi_1} + \mathbf{e}_{\xi_1}^T \mathbf{g}_3 (\boldsymbol{\xi}_2 - \boldsymbol{\alpha}_3) + \frac{1}{4\eta}.$$

For the system (9), let  $V_4(\mathbf{z}_1, \mathbf{z}_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = V_3(\mathbf{z}_1, \mathbf{z}_2, \boldsymbol{\xi}_1) + \frac{1}{2}(\boldsymbol{\xi}_2 - \boldsymbol{\alpha}_3)^T (\boldsymbol{\xi}_2 - \boldsymbol{\alpha}_3)$  be the candidate Lyapunov function and let  $\mathbf{e}_{\xi_2} = \boldsymbol{\xi}_2 - \boldsymbol{\alpha}_3$ , we have

$$\begin{aligned}\dot{V}_4(\mathbf{z}_1, \mathbf{z}_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) &= \dot{V}_3(\mathbf{z}_1, \mathbf{z}_2, \boldsymbol{\xi}_1) + \mathbf{e}_{\xi_2}^T (\dot{\boldsymbol{\xi}}_2 - \dot{\boldsymbol{\alpha}}_3) \\ &\leq -k_1 \mathbf{z}_1^T \mathbf{z}_1 - k_2 \mathbf{e}_{z_2}^T \mathbf{e}_{z_2} - k_3 \mathbf{e}_{\xi_1}^T \mathbf{e}_{\xi_1} + \mathbf{e}_{\xi_1}^T \mathbf{g}_3 (\boldsymbol{\xi}_2 - \boldsymbol{\alpha}_3) + \\ &\quad \mathbf{e}_{\xi_2}^T \left[ \mathbf{f}_4 + \mathbf{g}_4 \mathbf{u} - \frac{\partial \boldsymbol{\alpha}_3}{\partial \boldsymbol{\xi}_1} (\mathbf{f}_3 + \mathbf{g}_3 \boldsymbol{\xi}_2) - \frac{\partial \boldsymbol{\alpha}_3}{\partial \mathbf{z}_2} (\mathbf{f}_2 + \mathbf{g}_2 \boldsymbol{\xi}_1) - \frac{\partial \boldsymbol{\alpha}_3}{\partial \mathbf{z}_1} (\mathbf{f}_1 + \mathbf{g}_1 \mathbf{z}_2) \right] + \frac{1}{4\eta}\end{aligned}$$

If we select the control to be

$$\begin{aligned}\mathbf{u} &= -\mathbf{g}_4^{-1} \left[ k_4 \mathbf{e}_{\xi_2} + k_5 \text{sign}(\mathbf{e}_{\xi_2}) \right] \\ &\quad - \mathbf{g}_4^{-1} \left[ \mathbf{g}_3^T \mathbf{e}_{\xi_1} + \mathbf{f}_4 - \frac{\partial \boldsymbol{\alpha}_3}{\partial \boldsymbol{\xi}_1} (\mathbf{f}_3 + \mathbf{g}_3 \boldsymbol{\xi}_2) - \frac{\partial \boldsymbol{\alpha}_3}{\partial \mathbf{z}_2} (\mathbf{f}_2 + \mathbf{g}_2 \boldsymbol{\xi}_1) - \frac{\partial \boldsymbol{\alpha}_3}{\partial \mathbf{z}_1} (\mathbf{f}_1 + \mathbf{g}_1 \mathbf{z}_2) \right].\end{aligned}$$

Obviously, it follows that

$$\dot{V}_4(\mathbf{z}_1, \mathbf{z}_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \leq -k_1 \mathbf{z}_1^T \mathbf{z}_1 - k_2 \mathbf{e}_{z_2}^T \mathbf{e}_{z_2} - k_3 \mathbf{e}_{\xi_1}^T \mathbf{e}_{\xi_1} - k_4 \mathbf{e}_{\xi_2}^T \mathbf{e}_{\xi_2} - k_5 \|\mathbf{e}_{\xi_2}\| + \frac{1}{4\eta},$$

then if one selects  $k_5 > 0$  and  $k_5 \|\mathbf{e}_{\xi_2}\| > \frac{1}{4\eta}$ , the following inequality is satisfied

$$\dot{V}_4(\mathbf{z}_1, \mathbf{z}_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) < -k_1 \mathbf{z}_1^T \mathbf{z}_1 - k_2 \mathbf{e}_{z_2}^T \mathbf{e}_{z_2} - k_3 \mathbf{e}_{\xi_1}^T \mathbf{e}_{\xi_1} - k_4 \mathbf{e}_{\xi_2}^T \mathbf{e}_{\xi_2} < 0,$$

thus the origin of the affine nonlinear system with drift term is asymptotically stable. If we select  $k_i = \frac{\lambda}{2} > 0, i = 1, 2, 3, 4$ , then the last inequality can be rewritten as

$$\dot{V}_4 < -\lambda V_4.$$

The solution of the differential inequality is given by

$$V_4(t) < V_4(t_0) e^{-\lambda(t-t_0)}.$$

Thus the control (10) will render the system (9) exponentially stabilize to the origin  $(\mathbf{z}, \boldsymbol{\xi}) = (0, 0)$ . This completes the proof.

**Remark 3:** As  $\mathbf{g}_2 \in R^{1 \times 2}$  is not a square matrix for the robot system considered in this paper, the inverse matrix of  $\mathbf{g}_2$  is calculated by Moore-Penrose pseudo-inverse  $\mathbf{g}_2^+ = \mathbf{g}_2^T (\mathbf{g}_2 \mathbf{g}_2^T)^{-1}$ .

**Remark 4:** Different from the most of control plants of nonholonomic systems in literatures, the affine nonlinear system (9) with drift terms  $\mathbf{f}_i \neq 0$  is considered in this paper, thus the control presented by (10) can both stabilize the unstable balance point and track a feasible trajectory of the hopping robot system (2). The nonlinear control methods suggested by (Olfati-Saber, 2000, 2002) and (Qaiser, 2007) can not be utilized to the trajectory tracking problem.

## 6. Motion planning for the hopping robot in stance phase

Motion planning for a hopping robot with non-SLIP model is not intuitional. Fig.2 shows a sketch of motion of the hopping robot in stance phase. In the figure, MC denotes the mass



center of the robot,  $x_s$  indicates the moving distance of MC of the robot in stance phase,  $\beta$  is the angle of MC deviating from the vertical line that traverses the point of foot contacting with the ground. For an underactuated mechanical system, not arbitrary motions are feasible, the planned motion must satisfy the second-order nonholonomic constraints of the system. As to the robot system in Fig.1, the second-order nonholonomic constraints are given by the first equation of (4).

Denote  $\mathbf{X}_c = [x_c \ y_c]^T$  to be the position of MC of the robot in stance phase, the kinematics of the MC of the robot can be formulated as

$$\mathbf{X}_c = \mathbf{F}(\mathbf{q}_1). \quad (11)$$

The acceleration equation of (11) has form

$$\ddot{\mathbf{X}}_c = \mathbf{J}\ddot{\mathbf{q}}_1 + \dot{\mathbf{J}}\dot{\mathbf{q}}_1, \quad (12)$$

where  $\mathbf{J} = \partial \mathbf{F} / \partial \mathbf{q}_1$ . With considering the second-order nonholonomic constraints, the feasible motion in joint space can be calculated by

$$\ddot{\mathbf{q}}_1^d = \begin{bmatrix} \mathbf{J} \\ \mathbf{M}_{pp} & \mathbf{M}_{pa} \end{bmatrix}^{-1} \left( \begin{bmatrix} \ddot{\mathbf{X}}_c^d \\ 0 \end{bmatrix} - \begin{bmatrix} \dot{\mathbf{J}}\dot{\mathbf{q}}_1 \\ \mathbf{C}_p + \mathbf{H}_p \end{bmatrix} \right), \quad (13)$$

where  $\mathbf{X}_c^d$  is the desired trajectory of MC that is planned in Cartesian space.

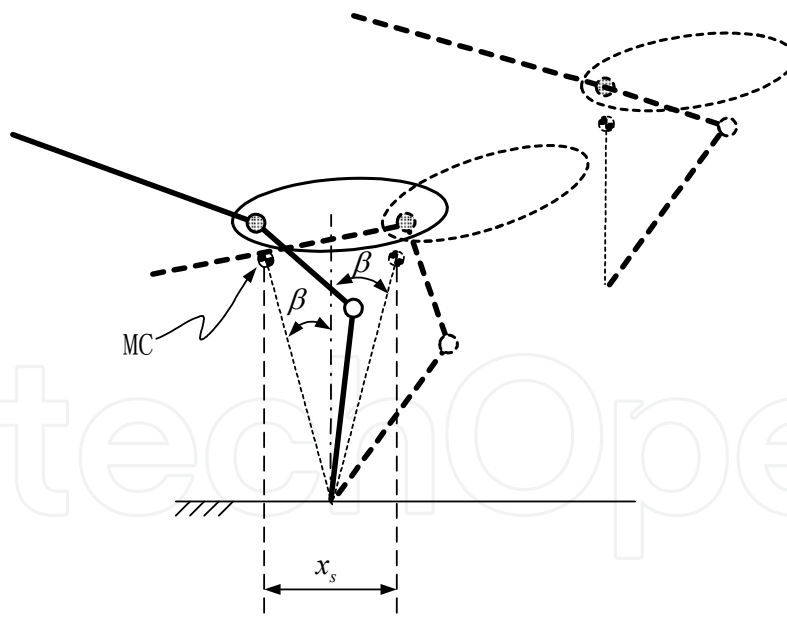


Fig. 2. Motion sketch of the robot in stance phase

The configuration with balance potential forces is an important point in joint space of the robot in stance phase. This point is both the control target of the robot for static balance and the reference point of planning a feasible trajectory of MC of the robot in every stance phase for continuous hopping. For searching the static balance configuration, one has to employ a numerical method since the complexity of the robot dynamics. For instance, define an optimization measure as follow

$$\mu(\mathbf{q}_1) = \mathbf{H}_1^T \mathbf{H}_1 + x_c^2, \quad (14)$$

where  $\mathbf{H}_1$  is the potential force term in equation (3), and  $x_c$  is the coordinate of MC of the robot along horizontal direction. If the measure  $\mu(\mathbf{q}_1) > 0$  is minimized such that it has  $\mu(\mathbf{q}_1^*) \approx 0$ , then the corresponding optimized variables  $\mathbf{q}_1^*$  is the searching configuration that satisfies the static balance condition.

Many algorithms can be employed to search for the optimal solution. One simple algorithm can be given as

$$\begin{cases} \mathbf{q}_1(i+1) = \mathbf{q}_1(i) - \lambda \frac{\nabla_{\mathbf{q}_1} \mu}{(\nabla_{\mathbf{q}_1} \mu)^T \nabla_{\mathbf{q}_1} \mu}, \\ \mathbf{q}_1(0) = \mathbf{q}_1^0 \end{cases} \quad (15)$$

where  $\lambda > 0$  is the iterative step length,  $\nabla_{\mathbf{q}_1} \mu = \frac{\partial \mu}{\partial \mathbf{q}_1}$  is the grads of the measure  $\mu(\mathbf{q}_1)$  along the smooth vector field  $\mathbf{q}_1$ , and  $i$  is the iterative times.

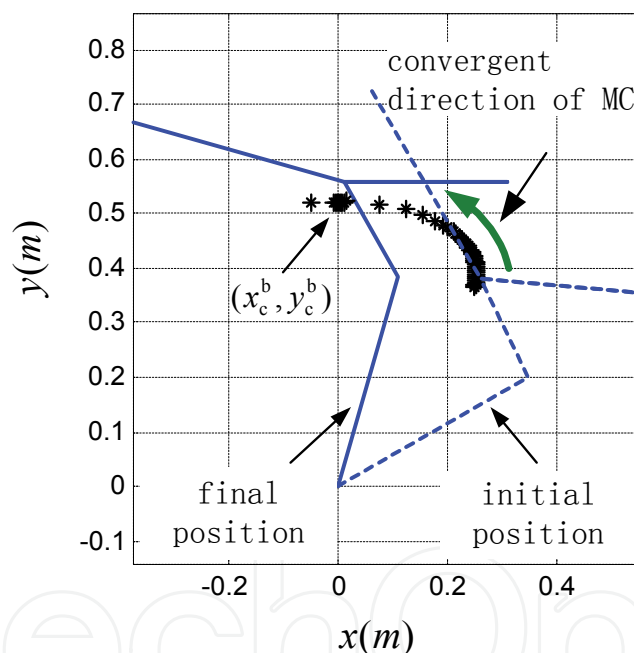


Fig. 3. Simulation for searching the balance configuration of the robot in stance phase

The static balance configuration of the robot can be calculated to be  $\mathbf{q}_1^* = (\theta_1, \theta_2, \theta_3) \approx (72.763^\circ, 45.980^\circ, -120^\circ)$  when the physical parameters of the robot is selected as

- The gravitational acceleration is  $g = 9.8 \text{ N/kg}$ ;
- The stiffness of spring in keen joint is  $k = 46.19 \text{ Nm/rad}$ ;
- The phase angle between the tail and thigh is  $\alpha_0 = 90^\circ$ ;
- The mass, length and location of MC, and the inertia of every body are given by

$$m_1 = 0.6 \text{ kg}, l_1 = 0.4 \text{ m}, l_{c1} = \frac{1}{2} l_1, I_1 = \frac{1}{12} m_1 l_{c1}^2,$$

$$m_2 = 1.5\text{kg}, l_2 = 0.2\text{m}, l_{c2} = \frac{1}{2}l_2, I_2 = \frac{1}{12}m_2l_{c2}^2,$$

$$m_3 = 5.0\text{kg}, l_3 = 0.3\text{m}, l_{c3} = 0, I_3 = \frac{1}{2}m_3l_3^2,$$

$$m_4 = 1.5\text{kg}, l_4 = 0.4\text{m}, l_{c4} = \frac{1}{2}l_4, I_4 = \frac{1}{12}m_4l_{c4}^2.$$

Fig.3 shows the convergent procedure for finding the balance configuration, and  $(x_c^b, y_c^b) \approx (0\text{m}, 0.5219\text{m})$  denotes the position of MC when the robot is static balance.

## 7. Numerical simulations

In this section, some numerical simulations are provided for verifying the feasibility of the robot mechanism and the nonlinear controller proposed in the former sections. The physical parameters of the robot mechanism are listed in section 6. Fig.4 shows the simulation results for stance balance control. In this simulation, the initial configuration errors of the robot is given by  $\mathbf{e}_{q_1} = [10^\circ \ -15^\circ \ -50^\circ]^T$  and the target configuration is  $(\theta_1, \theta_2, \theta_3) \approx (72.763^\circ, 45.980^\circ, -120^\circ)$ , which is obtained in section 6. One can find the configuration errors converge to zero (Fig.4 (b)) and finally stable at the balance configuration (Fig. 4 (a)). Fig. 4 (c) shows the trajectory of MC of the robots during the stabilizing procedure. Fig.4 (d) shows the corresponding torques of the actuators. For more intuitively, Fig.5 also shows the configurations snapshots of the robot during the stance balance control.

The general stance phase motion for the hopping robot is commonly a continuous trajectory that nears to the stance balance configuration. Given the desired motion of MC of the robot is

$$\ddot{\mathbf{x}}_c^d(t) = \begin{cases} \begin{bmatrix} -0.07t\omega_n^2 \sin(\omega_n t) & 0 \end{bmatrix}^T & t < 1\text{s} \\ \begin{bmatrix} 0.07\omega_n^2 \sin(\omega_n t) & 0 \end{bmatrix}^T & t \geq 1\text{s} \end{cases},$$

where  $\omega_n$  is the nature angular frequency of the robot. The desired motion is a periodical motion of MC moving along direction  $x$ . Fig.6 depicts the simulation results and one can find that the desired motion is approximately realized (see Fig.6 (b)). In Fig.6 (a)-(c), the curve in dashed indicates the desired motion, and the solid curve is the controlled motion of the corresponding variable. The large fluctuation of MC along direction  $y$  is induced by the special synchronous transmission system of the keen joint and tail joint.

Fig.7 shows another simulation result for trajectory control of MC moving along the direction  $y$ . In this simulation, the planned motion of MC is given by

$$\mathbf{x}_c^d(t) = \begin{cases} \begin{bmatrix} 0 & 0.03t\omega_n^2 \sin(\omega_n t) \end{bmatrix}^T & t < 1\text{s} \\ \begin{bmatrix} 0 & 0.03\omega_n^2 \sin(\omega_n t) \end{bmatrix}^T & t \geq 1\text{s} \end{cases}.$$

In Fig.7 (a)-(c), the curve in dashed also indicates the desired motion, and the solid curve indicates the controlled motion of the corresponding variable. There is no capsizal torque

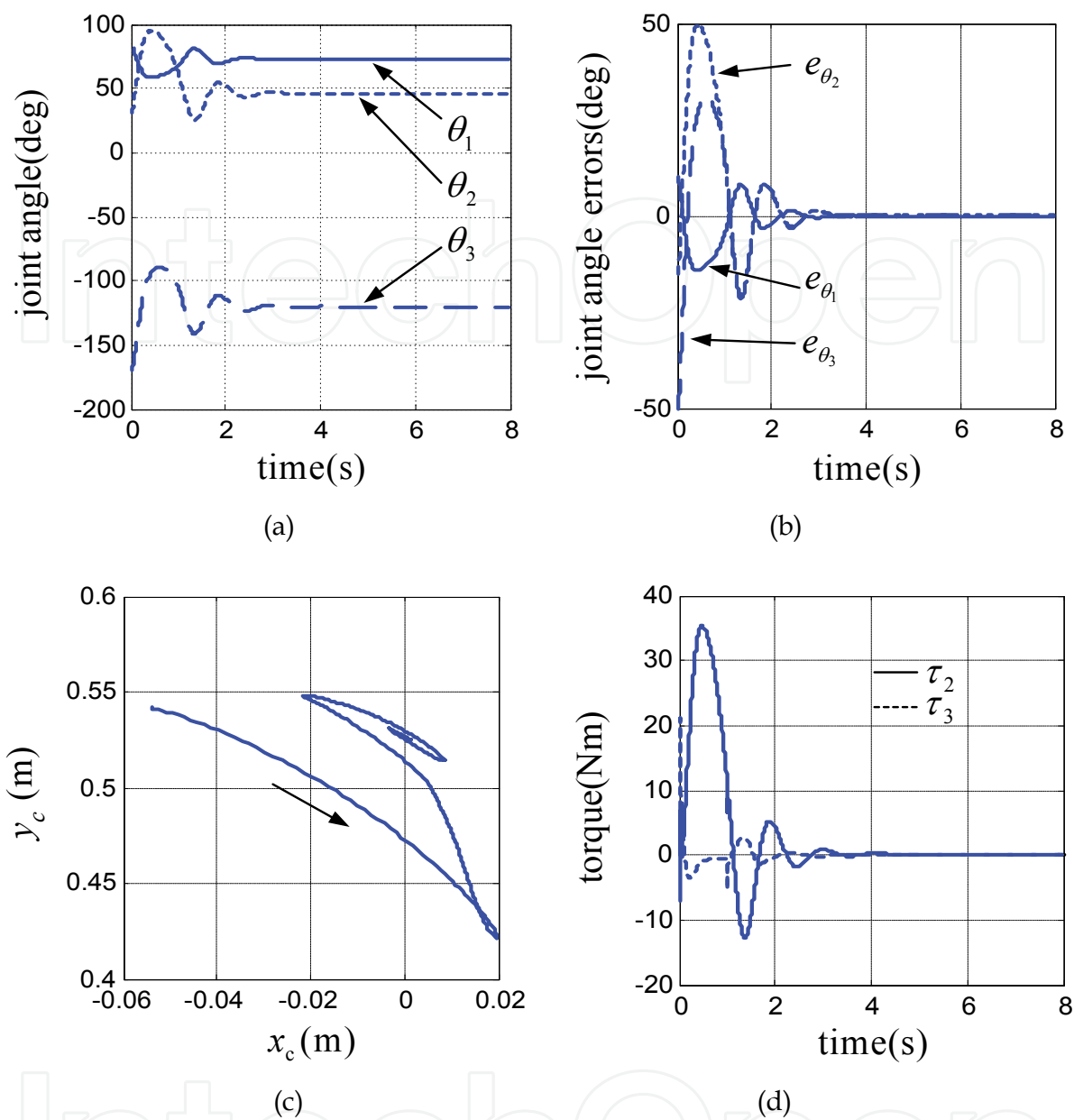


Fig. 4. Simulation for the stance balance control

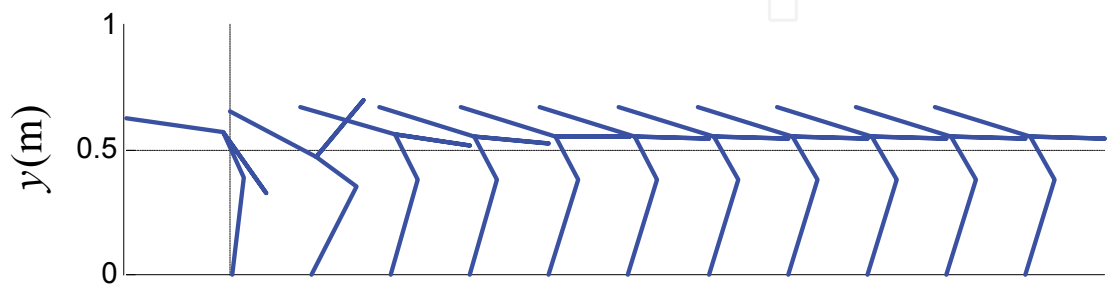


Fig. 5. Some configuration snapshots of the hopping robot during stance balance control shown by Fig.4

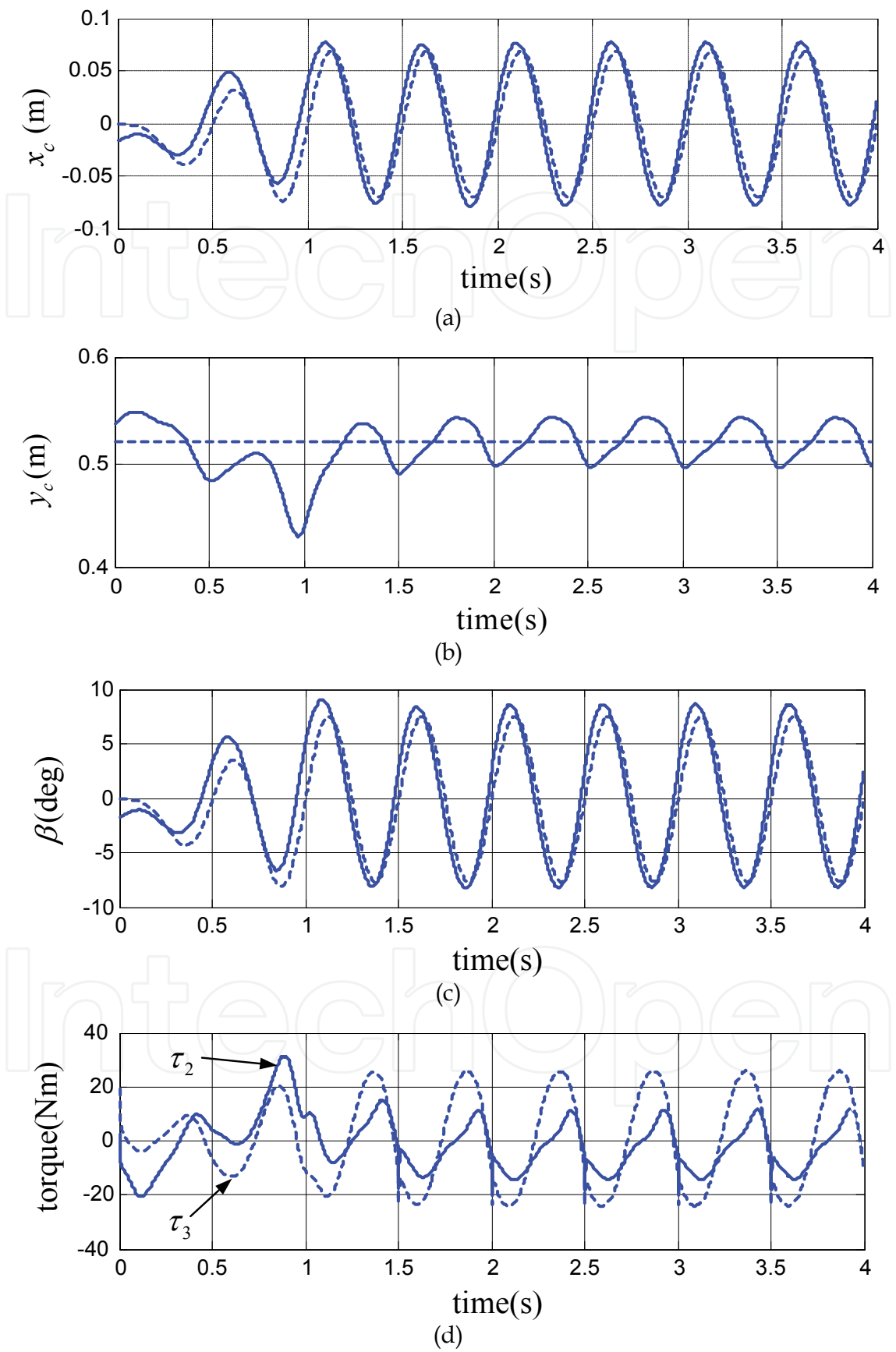


Fig. 6. Simulation for a periodical motion of MC along direction  $x$  .

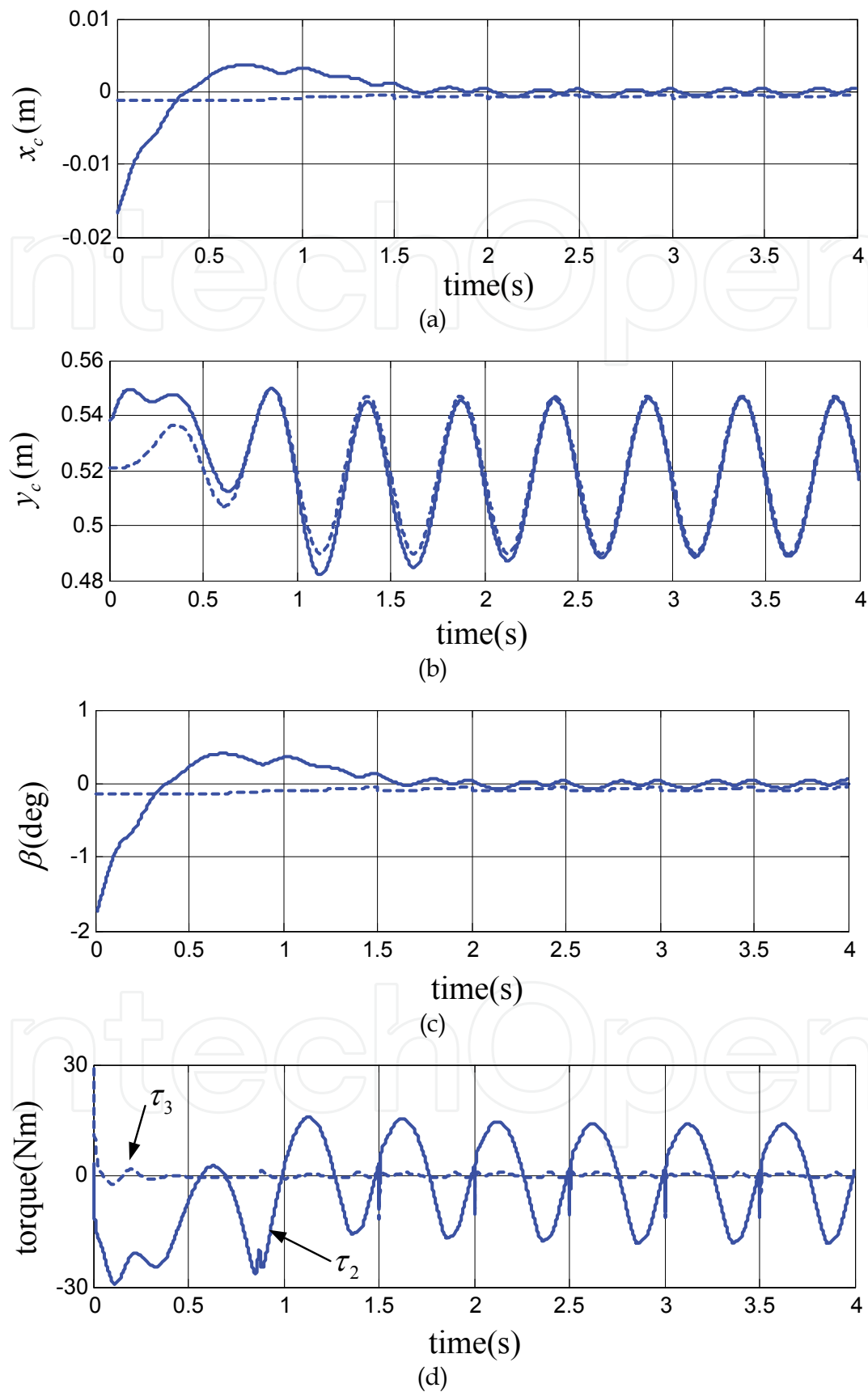


Fig. 7. Simulation for a periodical motion of MC along direction  $y$  .



need to be balanced since that the desired motion follows the vertical direction through the supporting point, one can find that the motion is realized with less error than it in Fig.6. Both Fig.6 (d) and Fig. 7(d) illustrate the torque of actuators during the corresponding controlled motion in two different directions of MC.

It is well known that the angular momentum of a hopping robot system in flight phase is conserved, and the angular momentum is generally unintegrable, thus the hopping robot systems in flight phase are first order nonholonomic system. For the first order nonholonomic systems with two inputs, there is a theorem that confirms the system can be transformed into the so-called chained form (Murray & Sastry, 1993; Murray, 1994). As to the chained from systems, there are many feasible control method in literatures. Therefore, one can expect that the novel hopping system presented in this paper will stable hopping under appropriate motion planning. The optimal motion plan and control problems for the novel hopping system with considering the energy-efficiency, and comparing the presented mechanism with the SLIP model system from the point of view of energy-efficiency and mobility, are interesting works in the future.

## 8. Conclusions

On the basis of dynamic synthesis, a novel mechanism for one-legged hopping robot is proposed. Different from the most of relative researches in literature, the proposed hopping robot mechanism is a non-SLIP model system, which generally shows more biological characteristics while the control problem of it is intractable, due to the complex nonlinear dynamics and the second-order nonholonomic constraints. Thanks to the special design, it is proved that the dynamics of the presented hopping robot mechanism can be transformed into the non-affine strict feedback normal form. Further more, it is shown that the normal form can also be rewritten as affine system with slightly approximation. Then a sliding model backstepping control is proposed for stabilizing the nonlinear dynamic system to its origin as well as a given trajectory around the balance configuration of the robot in stance phase. The stability of the presented control is proved, and verified by some numerical simulations.

## 9. Acknowledgments

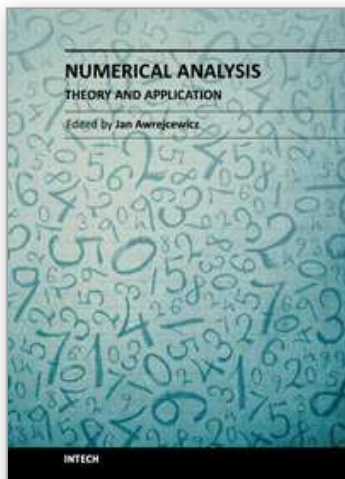
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