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Mechanics of Composite Beams

Mehdi Hajianmaleki and Mohammad S. Qatu
Mississippi State University
USA

1. Introduction

A structural element having one dimension many times greater than its other dimensions can be a rod, a bar, a column, or a beam. The definition actually depends on the loading conditions. A beam is a member mainly subjected to bending. The terms rod (or bar) and column are for those members that are mainly subjected to axial tension and compression, respectively.

Beams are one of the fundamental structural or machine components. Composite beams are lightweight structures that can be found in many diverse applications including aerospace, submarine, medical equipment, automotive and construction industries. Buildings, steel framed structures and bridges are examples of beam applications in civil engineering. In these applications, beams exist as structural elements or components supporting the whole structure. In addition, the whole structure can be modeled at a preliminary level as a beam. For example, a high rise building can be modeled as a cantilever beam, or a bridge modeled as a simply supported beam. In mechanical engineering, rotating shafts carrying pulleys and gears are examples of beams. In addition, frames in machines (e.g. a truck) are beams. Robotic arms in manufacturing are modeled as beams as well. In aerospace engineering, beams (curved and straight) are found in many areas of the plane or space vehicle. In addition, the whole wing of a plane is often modeled as a beam for some preliminary analysis. Innumerable other examples in these and other industries of beams exist.

This chapter is concerned with the development of the fundamental equations for the mechanics of laminated composite beams. Two classes of theories are developed for laminated beams. In the first class of theories, effects of shear deformation and rotary inertia are neglected. This class of theories will be referred to as thin beam theories or classical beam theories (CBT). This is typically accurate for thin beams and is less accurate for thicker beams. In the second class of theories, shear deformation and rotary inertia effects are considered. This class of theories will be referred to as thick beam theory or shear deformation beam theory (SDBT).

This chapter can be mainly divided into two sections. First, static analysis where deflection and stress analysis for composite beams are performed and second dynamic analysis where natural frequencies of them are assessed. In many applications deflection of the beam plays a key role in the structure. For example, if an aircraft wig tip deflection becomes high, in addition to potential structural failure, it may deteriorate the wing aerodynamic performance. In this and other applications, beams can be subjected to dynamic loads. Imbalance in driveline shafts, combustion in crank shaft applications, wind on a bridge or a structure, earthquake loading on a bridge or a structure, impact load when a vehicle goes over a pump are all examples of possible dynamic loadings that beam structures can be exposed to. All of these loads and others can excite the vibration of the beam structure. This can cause durability concerns or discomfort because of the resulting noise and vibration.

2. Stiffness of beams

Figure 1 shows a free body diagram of a differential beam element. Beams are considered as one dimensional (1D) load carriers and the main parameter for analysis of load carrier structures is stiffness.

Fig. 1. Free body diagram of a differential beam element

In general for composite laminates, stiffness matrix composed of ABD parameters is used to relate the stress resultants to strains.

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{0x} \\ \varepsilon_{0y} \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ 2\kappa_{xy} \end{bmatrix} \tag{1}$$

where regular ABD stiffness parameters for beams are defined as (Qatu, 2004).

$$A_{ij} = \sum_{k=1}^{N} b\overline{Q}_{ij}^{k}\left[\left(h_k - h_{k-1}\right)\right] \tag{2}$$

$$B_{ij} = \sum_{k=1}^{N} b\overline{Q}_{ij}^{k}\frac{\left(h_k^2 - h_{k-1}^2\right)}{2} \tag{3}$$

$$D_{ij} = \sum_{k=1}^{N} b\overline{Q}_{ij}^{k}\frac{\left(h_k^3 - h_{k-1}^3\right)}{3} \tag{4}$$

Note here that the above definitions are different from those used for general laminate analysis in the literature. The beam width is included in the definitions of these terms, while it is customary to leave this term out in general laminate analysis. In 1D analysis of beams, as we will see later, only parameters in x direction are considered and other parameters are ignored. So instead of 6X6 stiffness matrix for general laminate analysis we will have a 2X2 matrix for CBT and 3X3 matrix for SDBT. This formulation has the disadvantage of not accounting for any coupling. To overcome this problem, we propose that instead of normal definition of A_{11}, B_{11}, and D_{11}, one can use equivalent stiffness parameters that include couplings. That is why we will deal with stiffness parameters first.

2.1 Equivalent modulus

One approach for finding equivalent modulus for the whole laminate was proposed by finding the inverse of the ABD matrix (J matrix) (Kaw, 2005). The laminate modulus of elasticity is then defined as

$$E = \frac{b}{I\,J_{44}} \qquad J = [ABD]^{-1} \tag{5}$$

where J_{44} is the term in 4th row and 4th column of the inverse of the ABD matrix of the laminate and I is the moment of inertia. If one wants to use this approach for finding parameters A_{11}, B_{11}, and D_{11} the following formulas derived by authors should be used.

$$A_{11} = \frac{b}{J_{11}} \tag{6}$$

$$B_{11} = \frac{1}{J_{14}} \tag{7}$$

$$D_{11} = \frac{b}{J_{44}} \tag{8}$$

2.2 Equivalent stiffness parameters by Rios and Chan

Another approach using compliance matrix can be done by the following formulation (Rios and Chan, 2010).

$$A_{11} = \frac{1}{a_{11} - \dfrac{b_{11}^2}{d_{11}}} \tag{9}$$

$$B_{11} = \frac{1}{b_{11} - \dfrac{a_{11}d_{11}}{b_{11}}} \tag{10}$$

$$D_{11} = \frac{1}{d_{11} - \dfrac{b_{11}^2}{a_{11}}} \tag{11}$$

where a_{11}, b_{11}, and d_{11} are relevant compliance matrix terms. Similar to previous section we have $a_{11}=J_{11}$, $b_{11}=J_{14}$, $d_{11}=J_{44}$.

2.3 Equivalent stiffness parameters by Vinson and Sierakowski

Finding equivalent modulus of elasticity of each lamina and using normal definition of ABDs leads to the following formulation (Vinson and Sierakowski, 2002).

$$\frac{1}{E_x^k}=\frac{\cos^4(\theta_k)}{E_{11}}+\left(\frac{1}{G_{12}}-\frac{2\upsilon_{12}}{E_{11}}\right)\cos^2(\theta_k)\sin^2(\theta_k)+\frac{\sin^4(\theta_k)}{E_{22}} \tag{12}$$

Equivalent A_{11}, B_{11} and D_{11} using these formulas would be

$$A_{11}=\sum_{k=1}^{N} bE_x^k\left(h_k-h_{k-1}\right) \tag{13}$$

$$B_{11}=\sum_{k=1}^{N} bE_x^k\frac{\left(h_k^2-h_{k-1}^2\right)}{2} \tag{14}$$

$$D_{11}=\sum_{k=1}^{N} bE_x^k\frac{\left(h_k^3-h_{k-1}^3\right)}{3} \tag{15}$$

3. Static analysis

In static analysis section we will consider composite beams loaded with classical loading condition and derive differential equations for displacements. Those equations would be solved with classical boundary conditions of both ends simply supported and both ends clamped. We will use the static analyses to find deflection and stress of composite beams under both CBT and SDBT.

3.1 Classical beam theory

Applying the traditional assumptions for thin beams (normals to the beam midsurface remain straight and normal, both rotary inertia and shear deformation are neglected), strains and curvature change at the middle surface are: (Qatu, 1993, 2004)

$$\varepsilon_0=\frac{\partial u_0}{\partial x}\,,\ \kappa=-\frac{\partial^2 w}{\partial x^2} \tag{16}$$

where u, w are displacements in x and z directions, respectively. Normal strain at any point would be

$$\varepsilon=\varepsilon_0+z\kappa \tag{17}$$

Force and moment resultants are calculated using

$$\begin{bmatrix} N \\ M \end{bmatrix}=\begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix}\begin{bmatrix} \varepsilon_0 \\ \kappa \end{bmatrix} \tag{18}$$

The equations of motion are

$$\frac{\partial^2 M}{\partial x^2}=-p_z \tag{19}$$

$$\frac{\partial N}{\partial x} = -p_x \tag{20}$$

where p_x and p_z are external forces per unit length in x and z direction, respectively.
The potential strain energy stored in a beam during elastic deformation is

$$PE = \frac{1}{2}\int_V \sigma\varepsilon dV = \frac{1}{2}\int_0^l \left(N\varepsilon_0 + M\kappa\right)dx \tag{21}$$

writing this expression for every lamina and summing for all laminate we have

$$PE = \frac{1}{2}\int_0^l \left(A_{11}\left(\varepsilon_0\right)^2 + 2B_{11}\varepsilon_0\kappa + D_{11}\kappa^2\right)dx \tag{22}$$

substituting kinematic relations to equation (22) it will become

$$PE = \frac{1}{2}\int_0^l \left(A_{11}\left(\frac{\partial u_0}{\partial x}\right)^2 + 2B_{11}\left(\frac{\partial u_0}{\partial x}\right)\left(-\frac{\partial^2 w}{\partial x^2}\right) + D_{11}\left(-\frac{\partial^2 w}{\partial x^2}\right)^2\right)dx \tag{23}$$

The work done by external forces on beam would be

$$W = \frac{1}{2}\int_0^l \left(p_x u_0 + p_z w\right)dx \tag{24}$$

The kinetic energy for each lamina is

$$KE = \frac{1}{2}b\rho^{(k)}\int_0^l\int_{z_{k-1}}^{z_k}\left(\left(\frac{\partial u_0}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right)dx \tag{25}$$

where $\rho^{(k)}$ is the lamina density per unit volume, and t is time. The kinetic energy of the entire beam is

$$KE = \frac{I_1}{2}\int_0^l\left(\left(\frac{\partial u_0}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right)dx \tag{26}$$

where I_1 is the average mass density of the beam per unit length. These energy expressions can be used in an energy-based analysis such as finite element or Ritz analyses.

3.1.1 Euler approach

Inserting displacement relations in equations of motion will result in (Vinson and Sierakowski, 2002)

$$A_{11}\frac{\partial^2 u}{\partial x^2} - B_{11}\frac{\partial^3 w}{\partial x^3} + p_x(x) = 0 \tag{27}$$

$$B_{11}\frac{\partial^3 u}{\partial x^3} - D_{11}\frac{\partial^4 w}{\partial x^4} + p_z(x) = 0 \tag{28}$$

Solving these two equations for u and w will result in the following differential equations.

$$\left[\frac{A_{11}D_{11}-B_{11}^2}{A_{11}}\right]\frac{\partial^4 w}{\partial x^4}=p_z(x)-\frac{B_{11}}{A_{11}}\frac{\partial p_x(x)}{\partial x} \tag{29}$$

$$\left[\frac{A_{11}D_{11}-B_{11}^2}{B_{11}}\right]\frac{\partial^3 u_0}{\partial x^3}=p_z(x)-\frac{D_{11}}{B_{11}}\frac{\partial p_x(x)}{\partial x} \tag{30}$$

Stress in the axial direction in any lamina can be found by the following equation

$$\sigma_x=Q_{11}\left(\varepsilon_0+z\kappa\right)=Q_{11}\left(\frac{\partial u_0}{\partial x}-z\frac{\partial^2 w}{\partial x^2}\right) \tag{31}$$

Different loading and boundary conditions can be applied to these equations in order to find equations for u and w. These boundary conditions are
Simply supported: $w=0,\ M=0$
Clamped: $w=0,\ \frac{dw}{dx}=0$
Free: $V=0,\ M=0$
where V and M are shear force and bending moment and are linearly dependent on third and second derivative of w respectively. Here, we propose solution for both ends simply supported and both ends clamped with constant loading q_0. The reader is urged to apply other boundary conditions and find the equations for deflection. For specific case of simply supported boundary conditions at both ends and assuming $u_0(0)=0$ we have

$$\left[\frac{A_{11}D_{11}-B_{11}^2}{A_{11}}\right]w(x)=\frac{q_0 l^4}{24}\left[\left(\frac{x}{l}\right)^4-2\left(\frac{x}{l}\right)^3+\left(\frac{x}{l}\right)\right] \tag{32}$$

$$\left[\frac{A_{11}D_{11}-B_{11}^2}{B_{11}}\right]u_0(x)=\frac{q_0 l^3}{24}\left[4\left(\frac{x}{l}\right)^3-6\left(\frac{x}{l}\right)^2\right] \tag{33}$$

$$\sigma_x=\frac{q_0 l^2}{2\left(A_{11}D_{11}-B_{11}^2\right)}\left[\left(\frac{x}{l}\right)^2-\left(\frac{x}{l}\right)\right]Q_{11}\left(B_{11}-zA_{11}\right) \tag{34}$$

$$\tau_{xz}=\frac{1}{b}\int_h^z\frac{\partial\sigma_x}{\partial x}dz=\frac{q_0 l}{2b\left(A_{11}D_{11}-B_{11}^2\right)}\left[2\left(\frac{x}{l}\right)-1\right]\int_h^z Q_{11}\left(B_{11}-zA_{11}\right)dz \tag{35}$$

For clamped boundary conditions at both ends we have

$$\left[\frac{A_{11}D_{11}-B_{11}^2}{A_{11}}\right]w(x)=\frac{q_0 l^4}{24}\left[\left(\frac{x}{l}\right)^4-2\left(\frac{x}{l}\right)^3+\left(\frac{x}{l}\right)^2\right] \tag{36}$$

$$\left[\frac{A_{11}D_{11}-B_{11}^2}{B_{11}}\right]u_0(x)=\frac{q_0 l^3}{24}\left[4\left(\frac{x}{l}\right)^3-6\left(\frac{x}{l}\right)^2+2\left(\frac{x}{l}\right)\right] \tag{37}$$

$$\sigma_x = \frac{q_0 l^2}{2\left(A_{11}D_{11} - B_{11}^2\right)}\left[\left(\frac{x}{l}\right)^2 - \left(\frac{x}{l}\right) + \frac{1}{6}\right]Q_{11}\left(B_{11} - zA_{11}\right) \tag{38}$$

$$\tau_{xz} = \frac{1}{b}\int_h^z \frac{\partial \sigma_x}{\partial x} dz = \frac{q_0 l}{2b\left(A_{11}D_{11} - B_{11}^2\right)}\left[2\left(\frac{x}{l}\right) - 1\right]\int_h^z Q_{11}\left(B_{11} - zA_{11}\right) dz \tag{39}$$

One should note that for simply supported boundary condition the maximum moment and consequently maximum stress occurs at middle of the beam, while for the clamped case maximum stress occurs at two ends.

3.1.2 Matrix approach

Inserting the strain and curvature relations in the force and moment resultants equations and using those in the equations of motion, one can express the equations of motion in terms of displacements. Expressing those equations in matrix form we have

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}\begin{bmatrix} u_0 \\ w_0 \end{bmatrix} + \begin{bmatrix} p_x \\ -p_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{40}$$

where $L_{11} = A_{11}\frac{\partial^2}{\partial x^2}$, $L_{22} = D_{11}\frac{\partial^4}{\partial x^4}$, $L_{12} = L_{21} = -B_{11}\frac{\partial^3}{\partial x^3}$.

The beam is supposed to have simply supported boundary condition. So we have on x=0, a.

$$w_0 = N_x = M_x = 0 \tag{41}$$

The above equations of motion as well boundary terms are satisfied if one chooses displacements functions as

$$\left[u, w\right] = \sum_{m=1}^{M}\left[A_m \cos(\alpha_m x), C_m \sin(\alpha_m x)\right] \tag{42}$$

where $\alpha_m = m\pi / a$ and a is the beam length. The external forces can be expanded in a Fourier series in x

$$\left[p_x, p_z\right] = \sum_{m=1}^{M}\left[p_{xm}\sin\left(\alpha_m x\right), p_{zm}\cos\left(\alpha_m x\right)\right] \tag{43}$$

Substituting these equations in the equations of motion we have the characteristic equation

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}\begin{bmatrix} A_m \\ C_m \end{bmatrix} + \begin{bmatrix} p_{xm} \\ -p_{zm} \end{bmatrix} = 0 \tag{44}$$

$$\begin{bmatrix} A_m \\ C_m \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^{-1}\begin{bmatrix} -p_{xm} \\ p_{zm} \end{bmatrix} \tag{45}$$

where $C_{11} = -\alpha_m^2 A_{11}$, $C_{22} = \alpha_m^4 D_{11}$, $C_{21} = -C_{12} = \alpha_m^3 B_{11}$. Stress in the axial direction would be found using the following procedure.

$$\begin{bmatrix} \varepsilon_0 \\ \kappa \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix}^{-1} \begin{bmatrix} N \\ M \end{bmatrix} \tag{46}$$

$$\sigma_x = Q_{11}\left(\varepsilon_0 + z\kappa\right) \tag{47}$$

3.2 Shear deformation beam theory

The inclusion of shear deformation in the analysis of beams was first made in early years of twentieth century (Timoshenko, 1921). A lot of models have been proposed based on this theory since then. In this chapter a first order shear deformation theory (FSDT) approach is presented to account for shear deformation and rotary inertia (Qatu, 1993, 2004).

$$u = u_0 + z\psi,\ w = w_0 \tag{48}$$

Strains and curvature changes at the middle surface are:

$$\varepsilon_0 = \frac{\partial u_0}{\partial x},\ \kappa = \frac{\partial \psi}{\partial x},\ \gamma = \frac{\partial w}{\partial x} + \psi \tag{49}$$

where ε_0 is middle surface strain, γ is the shear strain at the neutral axis and ψ is the rotation of a line element perpendicular to the original direction. Normal strain at any point can be found using equation 17. Force and moment resultants as well as shear forces are calculated using

$$\begin{bmatrix} N \\ M \\ Q \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} & 0 \\ B_{11} & D_{11} & 0 \\ 0 & 0 & A_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_0 \\ \kappa \\ \gamma \end{bmatrix} \tag{50}$$

where for A_{55} we have (Vinson and Sierakowski, 2002).

$$A_{55} = \frac{5}{4}\sum_{k=1}^{N} b\overline{Q}^{k}_{55}\left[\left(h_k - h_{k-1}\right) - \frac{4}{3h^2}\left(h_k^3 - h_{k-1}^3\right)\right] \tag{51}$$

The equations of motion considering rotary inertia and shear deformation would be

$$\frac{\partial N}{\partial x} = -p_x \tag{52}$$

$$\frac{\partial Q}{\partial x} = p_z \tag{53}$$

$$\frac{\partial M}{\partial x} - Q = 0 \tag{54}$$

The potential strain energy stored in a beam during elastic deformation is

$$PE = \frac{1}{2}\int_V \sigma\varepsilon dV = \frac{1}{2}\int_0^l \left(N\varepsilon_0 + M\frac{\partial \psi}{\partial x} + Q\gamma\right)dx \tag{55}$$

Writing this expression for every lamina and summing for all laminate we have (Vinson and Sierakowski, 2002)

$$PE = \frac{1}{2}\int_0^l \left(A_{11}\left(\varepsilon_0\right)^2 + 2B_{11}\varepsilon_0\kappa + D_{11}\kappa^2 + A_{55}\gamma^2 \right) dx \tag{56}$$

substituting kinematic relations to equation (56) it will become

$$PE = \frac{1}{2}\int_0^l \left(A_{11}\left(\frac{\partial u_0}{\partial x}\right)^2 + 2B_{11}\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial \psi}{\partial x}\right) + D_{11}\left(\frac{\partial \psi}{\partial x}\right)^2 + A_{55}\left(\psi + \frac{\partial w}{\partial x}\right)^2 \right) dx \tag{57}$$

The work done by external forces on beam is found by equation (24). Finding the kinetic energy for each layer and then summing for all layers yield the kinetic energy of the entire beam.

$$KE = \int_0^l \left(I_1\left(\frac{\partial u_0}{\partial t}\right)^2 + I_1\left(\frac{\partial w}{\partial t}\right)^2 + 2I_2\left(\frac{\partial u_0}{\partial t}\right)\left(\frac{\partial \psi}{\partial t}\right) + I_3\left(\frac{\partial \psi}{\partial t}\right)^2 \right) dx \tag{58}$$

These energy expressions can be used in an energy-based analysis such as finite element or Ritz analyses.

3.2.1 Euler approach

Inserting displacement relations in equations of motion will result in

$$A_{11}\frac{\partial^2 u_0}{\partial x^2} + B_{11}\frac{\partial^2 \psi}{\partial x^2} + p_x(x) = 0 \tag{59}$$

$$A_{55}\left(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + p_z(x) = 0 \tag{60}$$

$$B_{11}\frac{\partial^2 u}{\partial x^2} + D_{11}\frac{\partial^2 \psi}{\partial x^2} - A_{55}\left(\psi + \frac{dw}{dx}\right) = 0 \tag{61}$$

Taking second derivative of equation (60) and solving for $\frac{\partial^3 \psi}{\partial x^3}$ from equations (59, 61) will result in following equations.

$$\frac{\partial^4 w}{\partial x^4} = \left[\frac{A_{11}}{A_{11}D_{11} - B_{11}^2}\right] p_z(x) - \frac{1}{A_{55}}\left[\frac{\partial^2 p_z(x)}{\partial x^2}\right] - \left[\frac{B_{11}}{A_{11}D_{11} - B_{11}^2}\right]\frac{\partial p_x(x)}{\partial x} \tag{62}$$

$$\frac{\partial^3 u_0}{\partial x^3} = \left[\frac{B_{11}}{A_{11}D_{11} - B_{11}^2}\right] p_z(x) - \frac{1}{A_{11}}\frac{\partial p_x(x)}{\partial x} \tag{63}$$

$$\frac{\partial^3 \psi}{\partial x^3} = \left[\frac{B_{11}}{A_{11}D_{11} - B_{11}^2}\right]\frac{\partial p_x(x)}{\partial x} - \left[\frac{A_{11}}{A_{11}D_{11} - B_{11}^2}\right] p_z(x) \tag{64}$$

For specific case of $p_z(x)=q_0$ with simply supported boundary conditions we have

$$w(x)=\frac{q_0 l^4}{24}\left(\frac{A_{11}}{A_{11}D_{11}-B_{11}^2}\right)\left[\left(\frac{x}{l}\right)^4-2\left(\frac{x}{l}\right)^3+\left(\frac{x}{l}\right)\right]+\frac{q_0 l^2}{2A_{55}}\left[\left(\frac{x}{l}\right)+\left(\frac{x}{l}\right)^2\right] \tag{65}$$

$$\psi(x)=\frac{q_0 l^3}{24}\left(\frac{A_{11}}{A_{11}D_{11}-B_{11}^2}\right)\left[1-4\left(\frac{x}{l}\right)^3+6\left(\frac{x}{l}\right)^2\right]+\frac{q_0 l}{2A_{55}} \tag{66}$$

$$\sigma_x=\frac{q_0 l^2}{2\left(A_{11}D_{11}-B_{11}^2\right)}\left[\left(\frac{x}{l}\right)^2-\left(\frac{x}{l}\right)\right]Q_{11}\left(B_{11}-zA_{11}\right)-\frac{2q_0 l^2 z}{A_{55}} \tag{67}$$

maximum deflection would occur at middle of the beam and it would be

$$w_{\max}=\frac{5q_0 l^4}{384}\left(\frac{A_{11}}{A_{11}D_{11}-B_{11}^2}\right)+\frac{q_0 l^2}{8A_{55}} \tag{68}$$

The first term in equation (68) is deflection due to bending and the second term is due to shear. For clamped boundary condition one can use the term due to bending from CBT analysis and add the term due to shear.

3.2.2 Matrix approach

Expressing equations of motion in terms of displacement we have in matrix form

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}\begin{bmatrix} u_0 \\ w_0 \\ \psi \end{bmatrix}+\begin{bmatrix} p_x \\ -p_z \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{69}$$

where $L_{11}=A_{11}\frac{\partial^2}{\partial x^2}$, $L_{22}=-A_{55}\frac{\partial^2}{\partial x^2}$, $L_{33}=D_{11}\frac{\partial^2}{\partial x^2}-A_{55}$, $L_{13}=L_{31}=B_{11}\frac{\partial^2}{\partial x^2}$, $L_{23}=L_{32}=-A_{55}\frac{\partial}{\partial x}$, $L_{12}=L_{21}=0$. The following simply supported boundary conditions are used on x=0, a

$$w_0=N_x=\frac{\partial\psi}{\partial x}=0 \tag{70}$$

The above equations would be satisfied if

$$\left[u_0,w_0,\psi\right]=\sum_{m=1}^{m}\left[A_m\cos(\alpha_m x),C_m\sin(\alpha_m x),B_m\cos(\alpha_m x)\right] \tag{71}$$

Substituting these equations in the equations of motion we have the characteristic equation

$$\begin{bmatrix} A_m \\ C_m \\ B_m \end{bmatrix}=\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}^{-1}\begin{bmatrix} -p_{xm} \\ p_{zm} \\ 0 \end{bmatrix} \tag{72}$$

where $C_{11} = -\alpha_m^2 A_{11}$, $C_{22} = \alpha_m^2 A_{55}$, $C_{33} = -\alpha_m^2 D_{11} - A_{55}$, $C_{31} = C_{13} = -\alpha_m^2 B_{11}$, $C_{23} = -C_{32} = A_{55}\alpha_m$, $C_{21} = C_{12} = 0$.

4. Dynamic analysis

To the knowledge of authors, there is no simple approach for dynamic analysis of composite beams considering all kinds of couplings. A review was conducted on advances in analysis of laminated beams and plates vibration and wave propagation (Kapania and Raciti, 1989. Another review was done on the published literature of vibrations of curved bars, beams, rings and arches of arbitrary shape which lie in a plane (Chidamparam and Leissa, 1993). Among FSDT works, some were validated for symmetric cross-ply laminates that have no coupling (Chandrashekhara et al., 1990; Krishnaswamy et al., 1992; Abramovich et al., 1994). In some other models, symmetric beams having fibers in one direction (only bending-twisting coupling) were considered (Teboub and Hajela, 1995; Banerjee 1995, 2001; Lee at al., 2004). Some FSDT models were validated for cross-ply laminates that have only bending-stretching coupling (Eisenberger et al. 1995; Qatu 1993, 2004).

Higher order shear deformation theories (HSDT) were also developed for composite beams to address issues of cross sectional warping and transverse normal strains. Some were validated for cross-ply laminates (Khdier and Reddy, 1997; Kant et al., 1998; Matsunaga, 2001; Subramanian, 2006). Other theories like zigzag theory (Kapuria et al. 2004) were used to satisfy continuity of transverse shear stress through the laminate and showed to be accurate for natural frequency calculations of beams with specific geometry and lay-up (symmetric or cross-ply laminates). Another theory was global-local higher order theory (Zhen and Wanji, 2008) that was validated for cross-ply laminates.

In this section, classic and FSDT beam models will be evaluated for their accuracy in a vibration analysis using different approaches for stiffness parameters calculation. Their results will be compared with those obtained using a 3D finite element model for different laminates (unidirectional, symmetric and asymmetric cross ply and symmetric and asymmetric angle-ply). The accurate model presented would then be verified for composite shafts.

4.1 Classical beam theory

Equations of motion for dynamic analysis of laminated beams would be

$$\frac{\partial^2 M}{\partial x^2} = I_1 \frac{\partial^2 w}{\partial t^2} - p_z \tag{73}$$

$$\frac{\partial N}{\partial x} = I_1 \frac{\partial^2 u}{\partial t^2} - p_x \tag{74}$$

where $I_1 = \sum_{k=1}^{N} b\rho^{(k)}\left(h_k - h_{k-1}\right)$. Expressing those equations in matrix form we have for free vibration

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} u_0 \\ w_0 \end{bmatrix} + \begin{bmatrix} -I_1 & 0 \\ 0 & I_1 \end{bmatrix} \frac{\partial^2}{\partial t^2} \begin{bmatrix} u_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{75}$$

The equations of motion as well as simply supported boundary terms are satisfied if one chooses displacements as

$$[u,w]=\sum_{m=1}^{M}[A_m\cos(\alpha_m x),C_m\sin(\alpha_m x)]\sin(\omega t) \tag{76}$$

Substituting these equations in the equations of motion we have the characteristic equation

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}\begin{bmatrix} A_m \\ C_m \end{bmatrix}+\omega^2\begin{bmatrix} I_1 & 0 \\ 0 & -I_1 \end{bmatrix}\begin{bmatrix} A_m \\ C_m \end{bmatrix}+\begin{bmatrix} p_{xm} \\ -p_{zm} \end{bmatrix}=0 \tag{77}$$

The nontrivial solution for natural frequency can be found by setting the determinant of characteristic equation of matrix to zero.

One should note here that if the laminate is symmetric, the B_{11} term vanishes and the bending frequencies are totally decoupled from axial ones. As a result, the following well-known formula for the natural frequencies of a symmetrically laminated simply supported composite beam can be applied:

$$\omega_n=\left(\frac{n\pi}{\ell}\right)^2\sqrt{\frac{D_{11}}{\rho A}} \tag{78}$$

where ρ is density, ℓ is length and A is the cross section area of the beam. As we will see later it cannot be used for thick laminates and those that have any kind of coupling.

4.2 Shear deformation beam theory

The equations of motion considering rotary inertia and shear deformation would be (Qatu, 1993, 2004)

$$\frac{\partial N}{\partial x}=I_1\frac{\partial^2 u}{\partial t^2}+I_2\frac{\partial^2\psi}{\partial t^2}-p_x \tag{79}$$

$$-\frac{\partial Q}{\partial x}=p_z-I_1\frac{\partial^2 w}{\partial t^2} \tag{80}$$

$$\frac{\partial M}{\partial x}-Q=I_2\frac{\partial^2 u}{\partial t^2}+I_3\frac{\partial^2\psi}{\partial t^2} \tag{81}$$

where $(I_1,I_2,I_3)=\sum_{k=1}^{N}b\rho^{(k)}\left((h_k-h_{k-1}),\frac{1}{2}(h_k^2-h_{k-1}^2),\frac{1}{3}(h_k^3-h_{k-1}^3)\right)$. So by expressing equations of motion in terms of displacement we have in matrix form (for free vibration)

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}\begin{bmatrix} u_0 \\ w_0 \\ \psi \end{bmatrix}-\begin{bmatrix} I_1 & 0 & I_2 \\ 0 & -I_1 & 0 \\ I_2 & 0 & I_3 \end{bmatrix}\frac{\partial^2}{\partial t^2}\begin{bmatrix} u_0 \\ w_0 \\ \psi \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{82}$$

Equations of motion as long as simply supported boundary condition would be satisfied if

$$\left[u_0, w_0, \psi\right] = \sum_{m=1}^{M} \left[A_m \cos(\alpha_m x), C_m \sin(\alpha_m x), B_m \cos(\alpha_m x)\right] \sin(\omega t) \tag{83}$$

Substituting these equations in the equations of motion we have the characteristic equation

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} A_m \\ C_m \\ B_m \end{bmatrix} + \omega^2 \begin{bmatrix} I_1 & 0 & I_2 \\ 0 & -I_1 & 0 \\ I_2 & 0 & I_3 \end{bmatrix} \begin{bmatrix} A_m \\ C_m \\ B_m \end{bmatrix} + \begin{bmatrix} p_{xm} \\ -p_{zm} \\ 0 \end{bmatrix} = 0 \tag{84}$$

The nontrivial solution for natural frequency can be found by setting the determinant of characteristic equation matrix to zero.

5. Case studies

5.1 Rectangular beam

A rectangular cross section beam model having 1 m length, 0.025 m width, and 0.05 m height was considered and modeled in ANSYS® finite element code. Solid elements were used to apply 3D elasticity. A convergence study was done and the convergent model had 8 elements in thickness, 4 elements in width direction and 160 elements in length direction. Ratio of length to height of 20 was selected to be at the border of thin beams. Figure 2 shows the model.

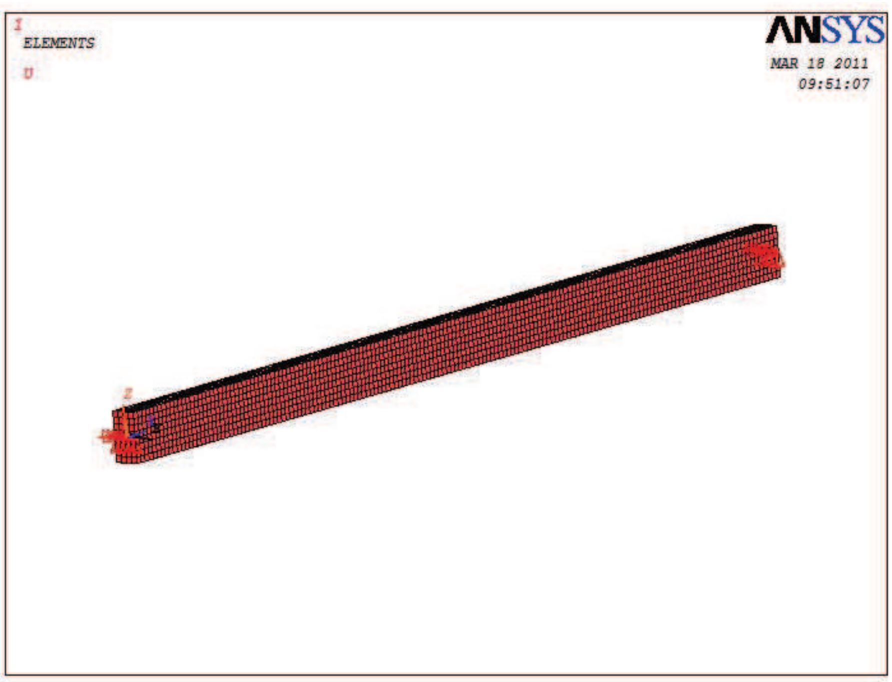

Fig. 2. 3D finite element model in ANSYS

The simply supported boundary condition was modeled by applying constraint on z direction at middle line of end faces. The material properties are E_1 = 138 GPa, E_2 =8.96 GPa, υ_{12}=0.3, G_{12}= 7.1 GPa, ρ =1580 kg/m^3.

Both static and modal analyses are done and the results of CBT and SDBT with different stiffness parameters are compared with 3D FEM in order to find the most accurate model.

5.1.1 Static analysis

A load of 250000 N/m were applied to the beam and the resulting deflection for cross-ply and angle-ply laminates were assessed using different models. The simply supported beam maximum deflection using Euler approach and matrix approach are given in Tables 1 and 2. Maximum normal stress is also presented in Table 3. Since the stress due to shear is low the results for CBT is not presented. The maximum deflection and stress in clamped beam are presented in Tables 4 and 5.

Laminate	CBT				SDBT				FEM
	(S_{11})	$(S_{11})_{VS}$	$(S_{11})_{Kaw}$	$(S_{11})_{Chan}$	(S_{11})	$(S_{11})_{VS}$	$(S_{11})_{Kaw}$	$(S_{11})_{Chan}$	3D
$[0]_4$	0.0901	0.0906	0.0906	0.0906	0.0988	0.0993	0.0993	0.0993	0.1000
$[0/90]_s$	0.1019	0.1026	0.1022	0.1022	0.1107	0.1113	0.1109	0.1109	0.1116
$[45]_4$	0.2753	0.7980	0.7980	0.7980	0.2840	0.8067	0.8067	0.8067	0.7824

Table 1. Maximum deflection of a SS beam (Euler aproach)

Laminate	CBT				SDBT				FEM
	(S_{11})	$(S_{11})_{VS}$	$(S_{11})_{Kaw}$	$(S_{11})_{Chan}$	(S_{11})	$(S_{11})_{VS}$	$(S_{11})_{Kaw}$	$(S_{11})_{Chan}$	3D
$[0]_4$	0.0908	0.0913	NA*	0.0913	0.1002	0.1008	NA	0.1008	0.1000
$[0/90]_s$	0.1028	0.1034	NA	0.1031	0.1123	0.1129	NA	0.1125	0.1116
$[45]_4$	0.2776	0.8046	NA	0.8046	0.2870	0.8140	NA	0.8140	0.7824

* Ill conditioning observed

Table 2. Maximum deflection of a SS beam (Matrix aproach)

Laminate	Maximum Stress			
	(S_{11})	$(S_{11})_{VS}$	matrix	FEM
$[0]_4$	3.000E+08	3.000E+08	3.000E+08	3.04E+08
$[0/90]_s$	3.397E+08	3.397E+08	3.397E+08	3.42E+08
$[45]_4$	3.000E+08	3.000E+08	3.000E+08	3.12E+08

Table 3. Maximum axial stress of a SS beam (at middle point)

Laminate	CBT				SDBT				FEM
	(S_{11})	$(S_{11})_{VS}$	$(S_{11})_{Kaw}$	$(S_{11})_{Chan}$	(S_{11})	$(S_{11})_{VS}$	$(S_{11})_{Kaw}$	$(S_{11})_{Chan}$	3D
$[0]_4$	0.01801	0.01812	0.01812	0.01812	0.02673	0.02684	0.02684	0.02684	0.02658
$[0/90]_s$	0.02039	0.02051	0.02044	0.02044	0.02911	0.02923	0.02916	0.02916	0.0286
$[45]_4$	0.05506	0.1596	0.1596	0.1596	0.06378	0.1683	0.1683	0.1683	0.1654

Table 4. Maximum deflection of a clamped beam (Euler aproach)

Laminate	Maximum Stress			
	(S_{11})	$(S_{11})_{VS}$	matrix	FEM
$[0]_4$	1.000E+08	1.00E+08	1.000E+08	1.00E+08
$[0/90]_s$	1.132E+08	1.132E+08	1.132E+08	1.16E+08
$[45]_4$	1.000E+08	1.00E+08	1.000E+08	1.00E+08

Table 5. Axial stress at middle of a clamped beam

The results show that Euler and matrix approaches have very close results. In general, using SDBT along normal ABD parameters can cause problems in laminates where coupling exists. However using equivalent ABDs from Vinson and Sierakowski or Chan's formulation one can get the most accurate results for deflection. This formulation is valid for any laminate having bending-twisting coupling.

5.1.2 Dynamic analysis

Different approaches for calculating the natural frequencies of the first 5 modes were evaluated. Five different stacking sequences were selected to cover different kinds of composite beams. These include unidirectional, symmetric cross-ply, asymmetric cross-ply, angle-ply and general laminates. The results are given in Table 6.

The results show that the classic beam model using normal ABD parameters is only valid for 1st mode of cross-ply laminates. The effective length becomes less on higher modes and the thin beam assumption no longer applies leading to inaccurate results. Although the $[45]_4$ laminate is symmetric; it has bending twisting coupling and using the normal ABD formulation leads to inaccurate results. The equivalent ABDs by equivalent stiffness parameters improve the classic approach for unsymmtric laminates but still not accurate enough for higher modes since the shear deformation is not included.

Using FSDT approach for thick beams (Qatu, 1993, 2004) along Vinson and Sierakowski equivalent modulus of elasticity for calculation of ABD parameters (Eqs. 13-15) one can reach accurate results for higher modes. This approach does not have coupling problems and accurate results for all laminate is achieved. The overall range of error is about 1 percent. The other equivalent parameters defined by compliance matrix are not as accurate as Vinson and Sierakowski and even do not have real results in some cases.

5.2 Tubular beam

Experimental results of a tubular boron/epoxy beam (Zinberg and Symonds, 1970) are used in this section to verify the accuracy of the model for tubular cross section. The laminate was $[90/45/-45/0_6/90]$ from inner to outer layers. The following equations were used for stiffness parameters.

$$A_{11} = \pi \sum_{k=1}^{N} E_x^k \left[\left(r_k^2 - r_{k-1}^2 \right) \right] \tag{85}$$

$$D_{11} = \frac{\pi}{4} \sum_{k=1}^{N} E_x^k \left[\left(r_k^4 - r_{k-1}^4 \right) \right] \tag{86}$$

[0_4]									
n	CBT				SDBT				FEM
	(S_{11})*	**(S_{11})vs**	**(S_{11})Kaw**	**(S_{11})Chan**	**(S_{11})**	**(S_{11})vs**	**(S_{11})Kaw**	**(S_{11})Chan**	3D
1	9.898	9.869	9.869	9.869	9.431	9.406	9.406	9.406	9.373
2	39.593	39.477	39.477	39.477	33.413	33.343	33.343	33.343	32.978
3	89.084	88.824	88.824	88.824	64.529	64.428	64.428	64.428	63.28
4	158.372	157.909	157.909	157.909	98.109	97.996	97.996	97.996	95.77
5	247.457	246.733	246.733	246.733	132.196	132.082	132.082	132.082	128.67
[0/90]s									
n	CBT				SDBT				FEM
	(S_{11})	**(S_{11})vs**	**(S_{11})Kaw**	**(S_{11})Chan**	**(S_{11})**	**(S_{11})vs**	**(S_{11})Kaw**	**(S_{11})Chan**	3D
1	9.302	9.275	9.291	9.291	8.910	8.886	8.901	8.901	8.873
2	37.207	37.098	37.163	37.163	31.931	31.861	31.903	31.903	31.651
3	83.716	83.472	83.618	83.618	62.369	62.266	62.327	62.327	61.51
4	148.829	148.394	148.653	148.653	95.659	95.540	95.611	95.611	93.91
5	232.546	231.865	232.271	232.271	129.705	129.583	129.656	129.656	126.88
[0_2/90_2]									
n	CBT				SDBT				FEM
	(S_{11})	**(S_{11})vs**	**(S_{11})Kaw**	**(S_{11})Chan**	**(S_{11})**	**(S_{11})vs**	**(S_{11})Kaw**	**(S_{11})Chan**	3D
1	4.688	4.674	NA	4.680	4.637	4.624	NA	4.630	4.609
2	18.718	18.663	NA	18.688	17.950	17.901	NA	17.924	17.651
3	41.990	41.867	NA	41.924	38.413	38.319	NA	38.362	37.251
4	74.337	74.120	NA	74.219	64.164	64.025	NA	64.088	61.354
5	115.526	115.188	NA	115.343	93.493	93.316	NA	93.397	88.302
[45_4]									
n	CBT				SDBT				FEM
	(S_{11})	**(S_{11})vs**	**(S_{11})Kaw**	**(S_{11})Chan**	**(S_{11})**	**(S_{11})vs**	**(S_{11})Kaw**	**(S_{11})Chan**	3D
1	5.6613	3.3251	4.5402	4.5402	5.5659	3.3033	4.4890	4.4890	3.3540
2	22.6450	18.1610	13.3005	18.1610	21.2317	12.9626	17.3840	17.3840	12.970
3	50.9513	40.8622	29.9262	40.8622	44.5592	28.3035	37.2377	37.2377	28.316
4	90.5801	72.6439	53.2021	72.6439	72.9229	48.4101	62.2926	62.2926	48.321
5	141.5314	113.5062	83.1283	113.5062	104.2553	72.3163	90.9410	90.9410	71.93
[30_2/60_2]									
n	CBT				SDBT				FEM
	(S_{11})	**(S_{11})vs**	**(S_{11})Kaw**	**(S_{11})Chan**	**(S_{11})**	**(S_{11})vs**	**(S_{11})Kaw**	**(S_{11})Chan**	3D
1	5.1433	3.4403	NA	4.8010	5.0735	3.4173	NA	4.7449	3.4830
2	20.5549	13.7558	NA	19.1791	19.5074	13.3998	NA	18.3329	13.458
3	46.1795	30.9295	NA	43.0585	41.3688	29.2239	NA	39.1346	29.222
4	81.9230	54.9322	NA	76.3072	68.4318	49.9103	NA	65.1941	49.857
5	127.6498	85.7212	NA	118.7286	98.7990	74.4320	NA	94.7571	73.97

Table 6. Nondimensional natural frequencies $\Omega = \omega a^2 \sqrt{12\rho / E_1 h^2}$ of rectangular simply supported beams. a/h = 20, b/h =0.5, Graphite/Epoxy, E_1/E_2 = 15.4, G_{12}/E_2 = 0.79, υ_{12} = 0.3 (subscript stands for formulation in deriving ABDs)

A number of researchers have worked on this beam with different beam and shell models and their results are shown in Table 7.

Author	Method used	Frequency (Hz)
Zinberg, Symonds, 1970	Measured experimentally	91.67
dos Reis et al., 1987	Bernoulli–Euler beam theory. Stiffness determined by shell finite elements	82.37
Kim and Bert, 1993	Sanders shell theory	97.87
	Donnell shallow shell theory	106.65
Bert and Kim, 1995	Bresse–Timoshenko beam theory	96.47
Singh and Gupta, 1996	Effective Modulus Beam Theory	95.78
Chang et al. 2004	Continuum based Timoshenko Beam	96.03
Qatu and Iqbal, 2010	Finite element analysis using ABAQUS	95.4
	Euler–Bernoulli beam theory	102.47
present study	Finite element analysis using ANSYS	95.89
	CBT using V-S	96.12
	SDBT using V-S	94.71

Table 7. Tubular Boron-epoxy beam fundamental natural frequencies (Hz) by different authors (E_{11} = 211 GPa, E_{22} = 24 GPa, $G_{12} = G_{13} = G_{23}$ = 6.9 GPa, υ = 0.36, density = 1967 kg/m^3), length = 2470 mm, mean diameter = 126.9 mm, thickness = 1.321 mm. (90, 45, -45,0,0,0,0,0,0,90) laminate (from inner to outer)

The results show that most of the models can predict the natural frequency of this beam with good accuracy. Only the models by dos Reis et al. predicted results that are far from those obtained by experiment. However the FSDT used in this paper is the most accurate model for this case.
The effect of ply orientation on reduction of stiffness and consequently natural frequency of a graphite-epoxy tube is presented in Table 8 (Bert and Kim, 1995).

Theory	Lamination angle						
	0	15	30	45	60	75	90
Sanders Shell	92.12	72.75	50.13	39.77	35.33	33.67	33.28
Bernoulli-Euler	107.08	89.88	71.15	52.85	38.20	31.42	30.22
Bresse-Timoshenko	101.20	86.82	69.95	52.38	37.97	32.90	30.05
Present FEM analysis	100.28	68.80	45.51	35.90	31.96	30.57	30.27
Present CBT approach	108.42	71.12	46.05	36.15	32.17	30.78	30.50
Present SDBT approach	104.43	70.50	45.91	36.06	32.09	30.70	30.36

Table 8. Effect of lamination angle on fundamental natural frequencies of tubular a graphite-epoxy beam. (E_{11} = 139 GPa, E_{22} = 11 GPa, $G_{12} = G_{13}$ = 6.05 GPa, G_{23} = 3.78 GPa, v = 0.313, density = 1478 kg/m^3)

These results are for the first natural frequency of a graphite epoxy tubular beam with the same geometry of the previous one. Results of the present CBT, SDBT and FEM using shell elements are presented.
Results show a good agreement between this study and the previous ones. It shows the decrease in natural frequency by lowering stiffness and also bending twisting coupling.

6. Conclusion

Different approaches for static and dynamic analysis of composite beams were proposed and a modified FSDT model that accounts for various laminate couplings and shear deformation and rotary inertia was validated. The method was verified using 3D FEM model. The results showed good accuracy of the model for rectangular beams in static analysis for laminates having bending-twisting coupling and in dynamic analysis for all kinds of laminates. Also the model was verified for dynamic analysis of tubular cross section beams (or shafts) and the results were accurate compared to experimental ones and other models. This model provides an accurate approach for calculating the natural frequencies of beams and shafts with arbitrary laminate for engineers and scientists.

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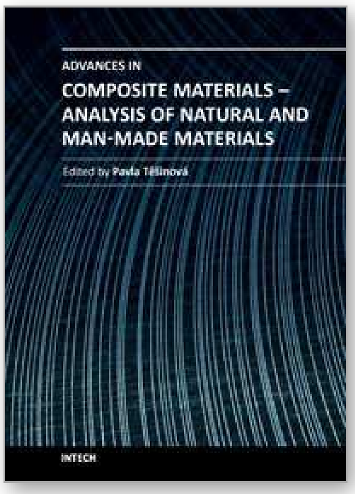

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Composites are made up of constituent materials with high engineering potential. This potential is wide as wide is the variation of materials and structure constructions when new updates are invented every day. Technological advances in composite field are included in the equipment surrounding us daily; our lives are becoming safer, hand in hand with economical and ecological advantages. This book collects original studies concerning composite materials, their properties and testing from various points of view. Chapters are divided into groups according to their main aim. Material properties are described in innovative way either for standard components as glass, epoxy, carbon, etc. or biomaterials and natural sources materials as ramie, bone, wood, etc. Manufacturing processes are represented by moulding methods; lamination process includes monitoring during process. Innovative testing procedures are described in electrochemistry, pulse velocity, fracture toughness in macro-micro mechanical behaviour and more.

InTech Europe
University Campus STeP Ri
Slavka Krautzeka 83/A
51000 Rijeka, Croatia
Phone: +385 (51) 770 447
Fax: +385 (51) 686 166
www.intechopen.com

InTech China
Unit 405, Office Block, Hotel Equatorial Shanghai
No.65, Yan An Road (West), Shanghai, 200040, China
中国上海市延安西路65号上海国际贵都大饭店办公楼405单元
Phone: +86-21-62489820
Fax: +86-21-62489821