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Duration, Systems and Cosmology

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1. Introduction

In physics and other fields, the definition of duration is a fundamental problem. We are used to astronomical time based upon the idea that the rotation of Earth is perfectly regular, an assumption which nowadays we know to be slightly erroneous. A unit of time based upon the period of a chosen atomic vibration is preferred. In all cases we need a time which we accept as a standard, taken for granted. But there are cases where duration associated with such a universal time does not seem appropriate. At certain instants it seems to an individual that time elapses more slowly or more quickly. This psychological time is subjective, it depends upon the person concerned and the circumstances. For a given individual it also depends upon age; at the end of life a day seems shorter than in youth. Of course this "relativity" of duration has nothing to do with relativity of time met in special relativity and is not at all in opposition to it. Before starting our presentation we must warn that we shall abundantly make use of mathematics as we believe they may be a help for thinking, despite the fact that in many cases only the qualitative aspects of the conclusions must be retained.

If we have chosen a standard or reference time t , such as for example the astronomical one, what is the most general time θ we can derive from it as a function $\theta(t)$? We assume that $\theta(t)$ must be a continuous function of t (though a discrete time could be proposed) and add that $\theta(t)$ must not decrease when t increases. More precisely we may write, if $f(a,b)$ is the duration of interval (a,b) , and since $f(a,b)$ must increase in the large with b and decrease in the large with a

$$f(a,b) + f(b,c) = f(a,c)$$

$$f(a,b) \geq 0, b \geq a$$

$$f(a,a) = 0.$$

If we add the hypothesis that f is differentiable, we have

$$f(a+da, b) + f(b, c+dc) = f(a+da, c+dc).$$

Replacing $f(a+da,b)$ by $f(a,b) + \partial f(a,b)/\partial a da$ and the like for $f(b, c+dc)$ and $f(a+da, c+dc)$ we obtain

$$\partial f(a,b)/\partial a = \partial f(a,c)/\partial a.$$

So $\partial f(a,b)/\partial a$ is independent of b . Consequently, after integration with respect to a , we have

$$f(a,b) = F(a) + \text{Cst},$$

the integration constant being a function of b or $G(b)$. It gives

$$f(a,b) = F(a) + G(b).$$

But, since $f(a,a) = 0$, we have $G = -F$ and so

$$f(a,b) = G(b) - G(a)$$

or

$$f(a,b) = \theta(b) - \theta(a),$$

function $\theta(t)$ being obviously continuous at not decreasing with t , as required.

We consider now a dynamical system. This involves in its evolution equation a reference time t which, in a way, is impartial. But since it has nothing to do with the considered system it does not take into account its intrinsic behaviour. Metaphorically all reference instants have not the same value. If the system were conscious, some instants, or short intervals of reference time, would have a greater importance than others. In the extreme case of very profound sleep or, better, of a coma, duration is not felt. This is close to the point of view of Aristotle in chapter IV of his "Physics" (Hussey, 1983): "When we feel no change in our thought, or we are unconscious of this change, or when we feel it without being aware of it, then it seems to us that no time have elapsed". Augustine in book XI of his "Confessions" expresses an opinion not far from that of Aristotle (Warner, 1963): "What is time? If nobody asks, I know; but if I want to explain, I do not know! Nevertheless – I tell it confidently – I know that if nothing happened, there would be no time passed..." Finally we are inclined to propose as a first approach that the more rapidly the state of the system changes, the more important are the corresponding reference instants.

We choose, as an index of importance of reference instant t , the scalar square of the speed of evolution of the state at this instant, that is to say $(dX(t)/dt)^2$. Of course many other indexes are possible, given for example by a strictly increasing function of the modulus of the speed. So we propose as an intrinsic or "internal duration" $d(t_1, t_2)$ of reference interval (t_1, t_2) the integral (Vallée, 1996, 2005)

$$\int_{t_1, t_2} (dX(t)/dt)^2 dt,$$

the internal duration of infinitesimal interval $(t, t+dt)$ being $(dX(t)/dt)^2 dt$. An "internal time", coherent with this duration and defined up to an additive constant, is given by

$$\theta(t) = d(t_0, t)$$

and we have

$$d(t_1, t_2) = \theta(t_2) - \theta(t_1),$$

a result which is in accordance with what we expected from the most general time we can derive from a given standard or reference time t .

2. Explosions and implosions

This internal time may be used for any dynamical system defined by a differential equation. We have particularly considered what we have called “elliptic explosion-implosion”, “hyperbolic explosion” and, as an intermediary case, “parabolic explosion” (Vallée, 1996, 2005).

2.1 Elliptic explosion-implosion

In the case of an “elliptic explosion-implosion”, the equation of evolution is given by

$$dX(t)/dt = q/p \operatorname{sgn}(p-t) (q^2 - X^2(t))^{1/2} / X(t) \quad (1)$$

where the state $X(t)$ is a mere scalar with

$$X(0) = 0, \quad p > 0, \quad q > 0, \quad 0 \leq t \leq 2p.$$

It is easy to see that

$$X(t) = q/p (p^2 - (p-t)^2)^{1/2} \quad (2)$$

since by derivation it gives

$$dX(t)/dt = q/p (p-t) / (p^2 - (p-t)^2)^{1/2} \quad (3)$$

which is the expression obtained from (1) if we replace $X(t)$ by (2)

The graph of function $X(t)$, which represents the evolution of state $X(t)$ with reference time t , is the upper part of an ellipse of great axis $2p$ and small axis q . The abscissa of the center is p and its ordinate is q . $X(t)$ starts from 0 at $t=0$, increases to its maximum value q at $t=p$, then decreases and attains 0 at $t=2p$. The speed at $t=0$ is $+\infty$ and $-\infty$ at $t=2p$. That is why we have an explosion at the beginning and an implosion at the end, and so what we can call an “elliptic explosion-implosion”. The square of the speed of evolution is according to (3), after a very classical decomposition of

$$q^2/p^2 (t-p)^2 / (p^2 - (p-t)^2)^{1/2} = q^2/p^2 (p-t)^2 / t(2p-t),$$

given by

$$\begin{aligned} (d(X(t))/dt)^2 &= q^2/p^2 (q^2 - X^2(t)) / X^2(t) \\ &= q^2/2p (1/t - 2/p + 1/2p-t). \end{aligned}$$

The “internal time” we can obtain by integration is defined up to an additive constant we can choose freely. The most simple choice gives

$$\theta(t) = q^2/2p (\operatorname{Log} t - 2t/p - \operatorname{Log}(2p-t)) \quad (4)$$

We see that when the reference time t varies from 0 to $2p$, the “internal time” varies from $-\infty$ to $+\infty$. Obviously this circumstance, the push back of $t=0$ to $\theta=-\infty$ and the push forward

of $t = 2p$ to $+\infty$, is linked to the behaviour of the square of the speed near $t = 0$ and near $t = 2p$ which generates logarithms.

2.2 Hyperbolic explosion

In the case of "hyperbolic explosion" the equation of evolution is

$$dX(t)/dt = q/p (q^2 + X^2(t)) / X(t), \quad (5)$$

with

$$X(0) = 0, \quad p > 0, \quad q > 0, \\ 0 \leq t$$

It is easy to verify that we have

$$X(t) = q/p ((p+t)^2 - p^2)^{1/2}, \quad (6)$$

since by derivation

$$dX(t)/dt = q/p (p+t) / ((p+t)^2 - p^2),$$

expression also obtained from (5) when we replace $X(t)$ by (6).

The graph of function $X(t)$ is the upper half right part of an hyperbola. The absciss of its center is $-p$ and its ordinate is 0. The asymptote associated with the graph of $X(t)$ has a slope equal to q/p . $X(t)$ starts from 0 at $t = 0$, then tends to $+\infty$ when t tends to $+\infty$. For great values of t , $X(t)$ behaves as $(q/p) t + q$. The square of the speed of evolution is

$$(dX(t)/dt)^2 = q^2/2p (1/t + 2/p - 1/2p+t).$$

This gives an "internal time" equal to

$$\theta(t) = q^2/2p (\text{Log} t + 2t/p - \text{Log}(2p+t)) \quad (7)$$

for which when reference time t varies from 0 to $+\infty$, "internal time" varies from $-\infty$ to $+\infty$.

2.3 Parabolic explosion

The "parabolic explosion" is an intermediary case, as parabola is "intermediary" between ellipse and hyperbola. Starting from equation (1), we shall make p tend to ∞ while keeping q^2/p equal to a constant h . We have

$$dX(t)/dt = (h/p)^{1/2} (hp - X^2(t))^{1/2} / X(t) \\ = (h^2 - h/p X^2(t))^{1/2} / X(t),$$

which gives, when p tends to ∞ , the new equation of evolution

$$dX(t)/dt = h/X(t) \quad (8)$$

with

$$X(0) = 0,$$

and

$$h > 0.$$

So

$$X(t) = (2ht)^{1/2}. \quad (9)$$

The graph of function $X(t)$ is the upper part of a parabola of summit at $t = 0$ and having t axis as axis. It is the limit of the half ellipse seen in the elliptic case. We have an explosion at $t = 0$ with initial speed $+\infty$. This speed decreases with time and tends to 0 while $X(t)$ tends to $+\infty$. The square of the speed is

$$(dX(t)/dt)^2 = h/2t,$$

giving the “internal time”

$$\theta(t) = h/2 \operatorname{Log} t, \quad (10)$$

which varies from $-\infty$ to $+\infty$ when t varies from 0 to $+\infty$.

3. Infinite internal duration

In the three cases seen above we have observed the possibility of an infinite “internal duration” linked to the push back (or forward) of a particular reference instant. Obviously this is linked to the behaviour of $(dX(t)/dt)^2$ near this reference instant. We choose, to simplify the presentation, reference instant $t = 0$, and suppose that $X(t)$ is an analytic function near this point. So $X(t)$ behaves near $t = 0$ as t^n , $dX(t)/dt$ as t^{n-1} , $(dX(t)/dt)^2$ as t^{2n-2} and so $\int (dX(t)/dt)^2 dt$ as $t^{2n-1}/2n-1$. If n is different from $1/2$, there is no singularity and no push back of $t = 0$ to $\theta = -\infty$. But if $n = 1/2$ $X(t)$ behaves as $t^{1/2}$, $dX(t)/dt$ as $t^{-1/2}$, $(dX(t)/dt)^2$ as t^{-1} and $\int (dX(t)/dt)^2 dt$ as $\operatorname{Log} t$. There is a push back of $t = 0$ to $\theta = -\infty$ and possibility of infinite interval time. A push forward of reference instant $t = a$ to $\theta = +\infty$ happens if $X(t)$ behaves as $(a-t)^{1/2}$ near $t = a$. In the case of “elliptic explosion-implosion”, $X(t)$ behaves as $t^{1/2}$ near $t = 0$ and as $(2p-t)^{1/2}$ near $t = 2p$. So, as we have seen, we have a push back and a push forward. For “hyperbolic explosion” as well as for “parabolic explosion” $X(t)$ behaves as $t^{1/2}$ near $t = 0$ and there is a push back.

4. Equation of evolution in term of “internal time”

It may be of interest, for a given evolution of a system described by function $X(t)$, to express $X(t)$ in term of “internal time” θ instead of reference time t . Let us take as an example the case of “parabolic explosion”. We have

$$X(t) = (2ht)^{1/2}$$

and

$$\theta(t) = h/2 \operatorname{Log} t,$$

$$t(\theta) = \exp(2\theta/h)$$

It gives

$$X(t(\theta)) = (2h)^{1/2} \exp(\theta/h).$$

So, in term of "internal time", the state varies exponentially from 0 to $+\infty$ while θ varies from $-\infty$ to $+\infty$, instead of growing as $t^{1/2}$ when t goes from 0 to $+\infty$.

5. Time and space

The system considered may, more generally, be defined by $X(t,x)$, a scalar function of "reference time" t and space point x , satisfying a partial derivative equation. We consider, as the index of importance of reference instant t , the integral, supposed to be convergent, extended to whole space S , of the square of the speed of evolution $(\partial X(t,x)/\partial t)^2$, that is to say

$$\int_S (\partial X(t,x)/\partial t)^2 dx. \quad (11)$$

So the "internal duration" of interval (t_1, t_2) is given by

$$d(t_1, t_2) = \int_{t_1, t_2} \int_S (\partial X(t,x)/\partial t)^2 dx dt,$$

and an « internal time » by

$$\theta(t) = d(t_0, t).$$

We shall apply this formalism to the dynamical system constituted by a space-time field of temperatures, in the case of heat diffusion, with $S = (-\infty, +\infty)$. Temperature at point x , at reference instant t , is $u(t,x)$. The partial derivative equation of evolution is

$$\partial u(t,x)/\partial t - \partial^2 u(t,x)/\partial x^2 = 0.$$

If the repartition of temperatures at $t = 0$ is given by function (more generally distribution) $u_0(x)$, the solution of the above equation is

$$u(x,t) = \int_{-\infty}^{+\infty} 1/2(\pi t)^{1/2} \exp(-(x-s)^2/4t) u_0(s) ds.$$

At initial reference instant $t = 0$, we suppose that the field of temperatures is given by $\delta(x)$ or Dirac distribution centered at $x = 0$ (in a rather simplified language it is equal to 0 everywhere except at $t = 0$ where it is infinite, the integral being nevertheless equal to 1). The repartition of temperatures at reference instant t is, according to the precedent equation and the properties of $\delta(x)$, given classically by the Laplace-Gauss function

$$u(t,x) = (4\pi t)^{1/2} \exp(-x^2/4t).$$

When reference instant t tends to $+\infty$, this function "flattens" and tends to $\varepsilon(x)$ or "epsilon distribution" (Vallée, 1992), in short it is equal to zero everywhere, the integral being nevertheless equal to 1. We have

$$(\partial u(t,x)/\partial t)^2 = 1/16\pi (1+x^2/2t)^2 / t^3 \exp(-x^2/2t)$$

and

$$\int_{\mathbb{R}} (\partial u(t,x)/\partial t)^2 dx = 3 (2\pi)^{1/2} / 16 t^{5/2}.$$

So the « internal duration » of reference interval (t_1, t_2) is, by integration from t_1 to t_2 ,

$$d(t_1, t_2) = (2\pi)^{1/2}/8 (t_1^{-3/2} - t_2^{-3/2})$$

and an “internal time” is given by the following function (increasing with t)

$$\theta(t) = - (2\pi)^{1/2}/8 t^{-3/2}. \quad (12)$$

When reference time t varies from 0 to $+\infty$, “internal time” θ varies from $-\infty$ to 0. Initial reference instant $t = 0$ is pushed back to $-\infty$. “Internal duration” from $t > 0$ to $+\infty$, is finite and equal to $(2\pi)^{1/2}/8 t^{-3/2}$.

6. Time and cosmology

We shall now interpret the notion of “internal time” in the field of cosmology. We consider models for which the state of the universe, at reference instant t , is given by the so called scalar factor $R(t)$. According to Lemaître, Friedman and Robertson (Berry, 1976) a possible equation of evolution is

$$(dR(t)/dt)^2 = 8\pi G/3 \rho(t) R^2(t) - kc^2 + \Lambda/3 R^2(t), \quad (13)$$

$$R(0) = 0,$$

G being the gravitational constant, c the speed of light, k the index of curvature ($k = -1$, space with negative curvature; $k = 0$, flat space; $k = +1$, space with positive curvature), Λ the cosmological constant, $\rho(t)$ the density of matter equal to $a/R^3(t)$ or its material equivalent $b/R^4(t)$ when there is only radiation, a and b being two constants. When $k = +1$, $R(t)$ is interpreted as the radius of the universe.

6.1 Radiation with null cosmological constant

If we consider the case of positive curvature with null cosmological constant and density of matter negligible compared to the equivalent density of matter of pure radiation ($k = +1$, $\Lambda = 0$, $\rho(t) = b/R^4(t)$), we have

$$(dR(t)/dt)^2 = 8\pi G/3 b/R^2(t) - c^2 \quad (14)$$

and

$$R(0) = 0.$$

This case corresponds to the « elliptic explosion-implosion » considered above where, after having taken the square of the two members of equation (1), we replace $X(t)$ by $R(t)$, choose

$$q = cp,$$

and

$$p = (b8\pi G/3)^{1/2} / c^2.$$

It gives, according to (2),

$$R(t) = q/p (2pt - t^2)^{1/2} = (2(8\pi Gb/3)^{1/2} t - c^2 t^2)^{1/2}$$

and the graph of function $R(t)$ is, as we know, elliptical.

The "internal time" of this cosmological system is, according to equation (3),

$$\theta(t) = c^2 p / 2 (\text{Log } t - 2t/p - \text{Log}(2p - t)),$$

or

$$\theta(t) = (2\pi G b / 3)^{1/2} (\text{Log } t - t c^2 / (2\pi G b / 3)^{1/2} - \text{Log}(2(2\pi G b / 3)^{1/2} - t)). \quad (15)$$

While reference time t goes from 0 (big bang) to $t = 2p$ (big crunch) "internal time" θ goes from $-\infty$ to $+\infty$. In that case I propose to call θ "generalized cosmological time" (Vallée, 1996, 2005) in remembrance of "cosmological time" (Milne, 1948) given by

$$c^2 p / 2 \text{Log } t$$

or

$$(2\pi G b / 3)^{1/2} \text{Log } t. \quad (16)$$

which is approximately valid for t "small".

If we consider now the case of flat space, null cosmological constant and pure radiation

$$k = 0, \Lambda = 0,$$

$$\rho(t) = b / R^4(t),$$

we have

$$(dR(t)/dt)^2 = 8\pi G / 3 \ b / R^2(t),$$

or

$$dR(t)/dt = 2 (2\pi G b / 3)^{1/2} / R(t).$$

We recognize, according to (9), a "parabolic explosion" with $h = 2(b2\pi G / 3)^{1/2}$. We have

$$R(t) = 2 (b2\pi G / 3)^{1/4} \ t^{1/2}$$

and, according to (10), the "internal time" is given by

$$\theta(t) = (2\pi G b / 3)^{1/2} \text{Log } t, \quad (17)$$

identical to the approximate formula (13). While t goes from 0 (big bang) to $+\infty$, θ varies from $-\infty$ to $+\infty$.

6.2 No matter nor radiation

There are other cases (Berry, 1989) for which we can introduce "internal time". Some of them may not be realistic, but due to the uncertainty concerning our conception of the universe and its evolution, they must not be discarded systematically.

For example we may have a universe with no matter and no radiation, at least as an approximation. As a first case we add that space has a negative curvature and a negative cosmological constant

$$k = -1, \Lambda < 0,$$

$$\rho(t) = 0.$$

We have

$$(dR(t)/dt)^2 = c^2 + \Lambda/3 R^2(t), \quad R(0) = 0,$$

$$dR(t)/dt = (c^2 + \Lambda/3 R^2(t))^{1/2},$$

which gives

$$R(t) = c (3/|\Lambda|)^{-1/2} \sin(t (|\Lambda|/3)^{1/2}).$$

$R(t)$ starts from 0 (big bang) reaches its maximum $c(3/|\Lambda|)^{1/2}$, decreases and attains 0 at $t = 2\pi (|\Lambda|/3)^{-1/2}$ (big crunch). We have

$$(dR(t)/dt)^2 = c^2 \cos^2(t (|\Lambda|/3)^{1/2}) = c^2/2 (1 + \cos 2t (|\Lambda|/3)^{1/2})$$

which give after integration

$$\theta(t) = c^2/2 (t + \sin 2t (|\Lambda|/3)^{1/2}) / 2 (|\Lambda|/3)^{1/2}. \quad (18)$$

So θ varies from 0 to $c^2\pi/4 (|\Lambda|/3)^{-1/2}$ as t varies from 0 to $\pi/2 (|\Lambda|/3)^{-1/2}$. A finite reference duration gives here a finite "internal duration".

Another possibility is the case of a universe with no matter nor radiation as above but with positive curvature and positive cosmological constant

$$k = +1, \Lambda > 0,$$

$$\rho(t) = 0.$$

We have

$$(dR(t)/dt)^2 = -c^2 + \Lambda/3 R^2(t),$$

$$dR(t)/dt = (-c^2 + \Lambda/3 R^2(t))^{1/2},$$

and so

$$R(t) = c (3/\Lambda)^{1/2} \cosh(t(\Lambda/3)^{1/2}).$$

$R(t)$ starts from $c(3/\Lambda)^{1/2}$ at $t = 0$ and tends, in a way closer and closer to an exponential to $+\infty$ as t tends to $+\infty$. We have

$$(dR(t)/dt)^2 = c^2 \sinh^2(t(\Lambda/3)^{1/2}),$$

and after integration

$$\theta(t) = c^2/2 (-t + \cosh 2t(\Lambda/3)^{1/2}/2(\Lambda/3)^{1/2}), \quad (19)$$

which varie from $c^2/2$ to $+\infty$ when t varies from 0 to $+\infty$ (not forgetting that θ is defined up to an arbitrary constant). An infinite reference duration gives an infinite "internal duration".

6.3 Matter but no radiation

We shall consider some other cases, also presented by Berry, where radiation is negligible. We start with the hypothesis of a flat space and a negative cosmological constant

$$k=0, \Lambda < 0,$$

$$\rho(t) = a/R^3(t).$$

We have

$$(dR(t)/dt)^2 = 8\pi G a/3/R(t) + \Lambda/3 R^2(t),$$

which gives, Λ being a constant,

$$R(t) = A \sin^{2/3} (t/2 (3|\Lambda|)^{1/2}). \quad (20)$$

So $R(t)$ starts from 0 at $t = 0$ with infinite speed (big bang) reaches its maximum, decreases and attains 0 again for $t = 2\pi/(3|\Lambda|)^{1/2}$ (big crunch). Near $t = 0$ $R(t)$ behaves as $t^{2/3}$ which is different from $t^{1/2}$. So, as we have seen, there is no push back of reference instant $t = 0$ to $-\infty$ and, for analogous reasons, no push forward of instant $t = 2\pi/(3|\Lambda|)^{1/2}$ to $+\infty$.

We have now the intermediary case where the cosmological constant is equal to zero ($k=0$, $\Lambda=0$, $\rho(t) = a/R^3(t)$), which gives

$$(dR(t)/dt)^2 = 8\pi G a/3 / R(t)$$

and

$$R^{1/2}(t) dR(t)/dt = (8\pi G a/3)^{1/2}$$

or

$$R(t) = (8\pi G a/3)^{1/3} t^{2/3}$$

When t varies from 0 to $+\infty$, $R(t)$ increases from 0 with infinite speed (big bang) to $+\infty$. We have

$$(dR(t)/dt)^2 = 4/3 (8\pi G a/3)^{2/3} t^{-2/3},$$

which gives after integration

$$\theta(t) = 3 (8\pi G a/3)^{2/3} (2/3)^{2/3} t^{1/3}. \quad (21)$$

There is no push back of reference instant $t = 0$.

Now we must see the case where the cosmological constant is positive ($k=0$, $\Lambda>0$, $\rho(t) = a/R$). We have

$$(dR(t)/dt)^2 = 8\pi G/3 a/R(t) + \Lambda/3 R^2(t)$$

and B being a constant

$$R(t) = B \sinh^{2/3}(t/2 (3\Lambda)^{1/2}). \quad (22)$$

$R(t)$ starts from zero at $t=0$ with infinite speed (big bang) and tends to $+\infty$ exponentially. There is no push back of reference instant $t=0$ to $-\infty$.

6.4 Null cosmological constant and no radiation

There are other interesting cases with negligible radiation and null cosmological constant. We start with a space of negative curvature ($k=-1$, $\Lambda=0$, $\rho(t) = a/R^3(t)$). We have

$$(dR(t)/dt)^2 = 8\pi G a/3 / R(t) + c^2. \quad (23)$$

It is easier to represent the graph of function $R(t)$ parametrically than explicitly. This gives with $0 \leq u$

$$R(t) = 8\pi G a / 6 (\cosh u - 1),$$

$$t = 8\pi G a / 6 c (\sinh u - u).$$

$R(t)$ starts from 0 at reference instant $t = 0$ with infinite speed (big bang), then tends to $+\infty$ asymptotically as ct . Near $t = 0$, $R(t)$ behaves as $t^{1/3}$. It proves as we have already seen that there is no push back of reference instant $t = 0$ to $-\infty$.

If space is flat ($k=0$, $\Lambda=0$, $\rho(t) = a/R^3$) we find a case already studied above, we have

$$(dR(t)/dt)^2 = 8\pi G a / 3 / R(t)$$

and

$$\theta(t) = 3 (8\pi G a / 3)^{2/3} (2/3)^{2/3} t^{1/3}.$$

We also have the case of a space of positive curvature ($k=+1$, $\Lambda=0$, $\rho(t)=a/R^3$). We have

$$(dR(t)/dt)^2 = 8\pi G a / 3 / R(t) - c^2. \quad (24)$$

The graph of function $R(t)$ is a cycloid represented parametrically by

$$R(t) = 8\pi G a / 6 (1 - \cos v),$$

$$t = 8\pi G a / 6 c (v - \sin v),$$

$$0 \leq v \leq \pi.$$

$R(t)$ starts from 0, at reference instant $t=0$, with an infinite speed (big bang). It increases up to $8\pi G a / 6$ attained at $t = 8\pi G a / 6 c (\pi/2 - 1)$, then decreases to 0 attained at $t = 8\pi G a / 6 c (\pi - 2)$. Near $t = 0$, $R(t)$ behaves as $t^{2/3}$ and so there is no push back of instant $t = 0$ to $-\infty$, and for similar reasons no push forward of instant $t = 8\pi G a / 6 c (\pi - 2)$ to $+\infty$.

7. Another approach to “internal time”

This new approach will put, metaphorically speaking, emphasis on perception. The purpose being to propose a modelling of the perception duration. First we consider the linear differential equation

$$dx(t)/dt = -a(t)x(t) + v(t), \quad (25)$$

where t is reference time, $x(t)$ and $v(t)$ two scalar functions. We have classically

$$x(t) = \varphi(t, t_0) x(t_0) + \int_{t_0, t} \varphi(t, \tau) v(\tau) d\tau \quad (26)$$

where

$$\varphi(t, t_0) = \exp \left(- \int_{t_0, t} a(s) ds \right) \quad (27)$$

with the hypothesis that $\varphi(t, t_0)$ tends to 0 if t tends to $+\infty$. When $a(t)$ is a mere constant a , it means obviously that a is strictly positive. The sign $-$ has been placed before the integral to

make more evident that the positivity of a has this consequence. Let us remark that we have the following property of "transitivity"

$$\varphi(t', t) \varphi(t, \tau) = \varphi(t', \tau). \quad (28)$$

We interpret $v(t)$ as an external influence. In the most simple case we have

$$v(t) = b(t) u(t),$$

more generally we could have

$$v(t) = b_0(t) u(t) + b_1(t) du(t)/dt + \dots$$

involving derivatives of $u(t)$. This formulation is not unrealistic and is well adapted to the modelling of a tachymeter or an accelerometer if we consider mainly the first or the second derivatives the other terms being rather negligible. This formula which we may also write with the help of the Dirac distribution δ and its derivatives

$$v(t) = \int_{-\infty, +\infty} (b_0(\tau) \delta(t-\tau) + b_1(\tau) \delta'(t-\tau) + \dots) u(\tau) d\tau$$

Each $b_i(t)$ is a "factor of attention" concerning a particular derivative. The passage of function u to function v is made by what we call "observation operator" (Vallée, 1951, 2002). Here this operator acts in an instantaneous way, being purely local. The first factor $b_0(t)$, or more simply $b(t)$, may be considered positive (when it is null there is no attention and so no perception at the considered instant t). This factor of attention has been pointed out (Condillac, 1754): "...it remains an impression more or less strong according to the fact that the attention has been more or less intense". More generally the passage of u to v through $v = O(u)$, O being an "observation operator", is not instantaneous but hereditary, that is to say involving the past and present of u . This has been observed (Bergson, 1939): "In fact 'pure' perception, that is to say instantaneous, is only an ideal, a limit. Every perception fills a certain length of duration, extends the past in the present...". For example we may have a convolution

$$v(t) = \int_{t_0, t} k(t-\tau) u(\tau) d\tau,$$

where $v(t)$ depends upon the values of u on interval (t_0, t) . More generally, if we do not leave the case of linear "observation operators" we have a Volterra composition giving

$$v(t) = \int_{t_0, t} k(t, \tau) d\tau.$$

The formalism of "observation operators" permits to see in which cases such an operator does not alter the observed function u . We must have

$$O(u) = \lambda u,$$

so u must be an eigen function of operator O and λ is the associated eigen value which may be complex. If $\lambda = 1$, we have a fixed point. We give these details about "observation operators" because we shall meet them in several circumstances, the problem of appreciation of time having to do with observation and more generally with what we could call "mathematical epistemology" (Vallée, 2002).

Let us come back to the differential equation and its solution (26). We interpret $\varphi(t, \tau) v(\tau)$ as what remains at instant t of the perception $v(\tau)$ felt at anterior instant τ , it is the result of the

transfer by memorization of $v(\tau)$ from τ to t . The property of transitivity (27) makes this transfer coherent. According to the hypothesis that $\varphi(t, \tau)$ tends to 0 if $t - \tau$ tends to $+\infty$, we may say that the transferred perception $\varphi(t, \tau) v(\tau)$ tends to 0 if τ tends to $-\infty$. In other words we may conclude that the more ancient is a perception the more feeble is its remembrance. If our differential system starts with the null state we have according to (26) a Volterra composition which reduces to convolution when $a(s)$ is a constant a

$$x(t) = \int_{t_0, t} \varphi(t, \tau) v(\tau) d\tau.$$

We may interpret $x(t)$ as the result of the superposition of all the successive perceptions, from t_0 to t , as they are transferred to t by memorization. The passage from $v(t)$ to $x(t)$ is given by a special type of linear "observation operator" which we call "memorization operator". It may be compared to the factor of forgetfulness (Vogel, 1965) or the memory coefficient (Allais, 1972).

We shall consider now differential equation (25), or more precisely its solution (26) where we replace $x(t)$ by $\theta(t)$, as a model of perception and memorization of duration valid for a dynamical system considering t as "reference time" and $\theta(t)$ as a subjective or "internal time", even if these expressions are to be understood metaphorically. Taking into account (27) we have

$$\theta(t) = \int_{t_0, t} \exp\left(-\int_{\tau, t} a(s) ds\right) v(\tau) d\tau, \quad (29)$$

to which we add

$$a(t) \geq 0, \quad v(t) \geq 0.$$

While $t - t_0$ is the reference duration of interval (t_0, t) , $\theta(t) - \theta(t_0) = \theta(t)$ is its subjective or internal duration. It depends upon the way the considered system perceives through $v(t)$ and memorizes, not upon the way the state of the system evolves as it was the case in the first approach to "internal time".

Of course t is a good parametrisation of time, in the sense that two distinct instants are represented by two different values of t . If $\theta(t)$ is to be a good parametrisation of time, it must be a strictly increasing function of t , a condition realized if $v(t)$ never vanishes on an interval not reduced to a mere instant. The hypothesis that $a(t) \geq 0$ has for consequence that the factor of memorization $\varphi(t, \tau)$ decreases in the large when τ diminishes.

We must now interpret function $v(t)$ in the most simple case where

$$v(t) = b(t) u(t),$$

with

$$b(t) \geq 0, \quad u(t) \geq 0.$$

Of course more elaborated cases could be considered, making a full use of « observation operators ».

We consider that function $u(t)$ gives the evolution of the weight of importance of reference instant t . The very simple "observation operator" represented by $b(t)$ appears as a "factor of attention" given to $u(t)$ which represents the intrinsic importance of reference instant t itself. Since t is by definition an objective time or "reference time", all instants t have an equal intrinsic importance which we may decide to be equal to 1. So we may write simply

$$v(t) = b(t)$$

as the weight attributed to instant t is equal to the "factor of attention" $b(t)$ at this instant.

7.1 Perception of duration with imperfect memorization

In our model we have an imperfect memorization when $a(t)$ is not identical to 0. The factor of memorization

$$\varphi(t, \tau) = \exp \left(- \int_{\tau, t} a(s) ds \right)$$

is strictly inferior to 1 and tends to 0 when $t - \tau$ tends to $+\infty$, so when τ tends to $-\infty$. The remembrance of past perceptions vanishes with time. Since, as we have seen, $v(t)$ is equal to $b(t)$ we may write according to (28)

$$\theta(t) = \int_{t_0, t} \exp \left(- \int_{\tau, t} a(s) ds \right) b(\tau) d\tau,$$

or

$$d\theta(t) = (-a(t) \theta(t) + b(t)) dt,$$

explaining the antagonistic roles of $b(t) > 0$ and $-a(t) \theta(t) < 0$, representing attention on one side and oblivion on the other. Or, if we consider the case where $a(t)$ reduces to constant $a > 0$,

$$\theta(t) = \exp(-at) \int_{t_0, t} \exp at b(\tau) d\tau.$$

It is interesting to see what happens when $b(t)$ is a Dirac distribution $\delta(t-\gamma)$, centered on instant γ . It is an idealisation which gives

$$\theta(t) = \exp(-a(t-\gamma)),$$

$$t > \gamma.$$

We see that the reference duration of instant γ , obviously equal to 0, is perceived just after this instant, due to the infinite attention implied, as having a finite subjective or internal duration; It is perceived later has having a decreasing value tending to 0. If we replace the Dirac δ by a flash of attention, things are not so sharply defined but are of the same nature: a very short interval of reference time is perceived as middle sized interval of "internal time" and this impression diminishes with time and disappears. The reference time, $1/a$, necessary to see the perceived duration divided by e is a measure of what we may call the "subjective duration of the present" or the "thickness of an instant", in accordance with Bergson's remark quoted above and close to the concept of constant of time familiar in dynamics.

7.2 Perception of duration with perfect memorization

Perfect memorization is obtained when the factor of memorization is always equal to 1, that is to say when $a(t)$ is identical to 0. Then

$$d\theta(t)/dt = b(t),$$

with

$$b(t) \geq 0, \theta(t_0) = 0.$$

We have

$$\theta(t) = \int_{t_0, t} b(\tau) d\tau, \quad (30)$$

it is the subjective or “internal duration” of interval (t_0, t) . In other terms it is the subjective duration of instants of consciousness in this interval, instants where $b(t) = 0$ having no impact (metaphorically or not : deep sleep or coma). If $b(t)$ takes only values 1 or 0, $\theta(t)$ is equal to the internal duration (as well as reference duration) of instants of consciousness. When $b(t)$ takes value 1 at discrete instants it acts as a stroboscopic selecting device. When $b(t)$ is constant $\theta(t)$ is proportional to $t - t_0$. If for example, the “factor of attention” decreases exponentially with time, that is to say if

$$b(t) = b \exp(-\lambda t),$$

with

$$b > 0, \lambda > 0.$$

We have, with $t_0 = 0$,

$$\theta(t) = 1 - \exp(-\lambda t).$$

The “internal duration” or perceived duration since $t_0 = 0$, increases from 0 to 1 while “reference duration” tends to $+\infty$. This circumstance may be found during the observation of a disintegrating radioactive material if the factor of attention $b(t)$ is proportional to disintegration activity which decreases exponentially.

8. Another approach to cosmology and other problems

First we consider the case of perfect memorization for a conscious being which may be human. It has often been remarked that the perceived or “internal duration” of the same interval of “reference time” diminishes with age. It may be interpreted by saying that the intrinsic importance of an instant in early age is much greater than later. Birth may be compared to a kind of biological big bang, particularly if we consider that life starts at the very moment of conception. Another argument is that an interval of reference time measured by comparison to the length life already elapsed, seems shorter and shorter. In other words we may say that the “factor of attention” given to a reference instant is proportional to the intrinsic importance of this instant. So if $t = 0$ is the reference instant of conception, an acceptable “factor of attention” $b(t)$ adapted to this case may be given by a function decreasing with time and starting with a great value, we may even have $b(0) = +\infty$. The most simple example is given by

$$b(t) = b/t, \quad (31)$$

$$b > 0.$$

Then we have

$$\theta(t) - \theta(t_0) = b \operatorname{Log} (t/t_0) = b \operatorname{Log} t - b \operatorname{Log} t_0,$$

but since $\theta(t)$ is defined up to an arbitrary constant, we write

$$\theta(t) = b \text{Log } t \quad (32)$$

We find here a logarithmic psychological time already proposed (Lecomte du Noüy, 1936) on the basis of the speed of cicatrisation of wounds which decreases with time. With this "internal time", the beginning of life is pushed back to $-\infty$, a feeling frequent among human beings. Considerations formally similar may be developed in the case of some thermodynamical systems where entropy decreases as $1/t$, generating an "entropic time" (Prigogine, 1947).

We had found the same result with "parabolic explosion" and with the case, seen at the end of section 6.1, which introduces Milne's cosmological time. Near reference instant $t=0$ (big bang) the events affecting the universe are important to an extreme, it is quite acceptable to admit that $b(t)$, as a physical "factor of importance" (very metaphorically a "factor of attention" adapted to it), has a pole at $t=0$ and may be represented by function $1/t$. Reference instant $t=0$ is pushed back to "internal instant" $\theta = -\infty$. From the "internal time" point of view this universe has no beginning, the instant of big bang is not an "internal instant".

We may imagine now a more general case, of explosive-implosive type. The reference instants near the beginning of explosion ($t=0$) or near the end of implosion ($t=\sigma$) have a tremendous importance and we may admit that function $b(t)$ has a pole at each of these points. The most simple example, if we affect the two poles of the same coefficient b , assuming that they are of the same importance, is given by

$$b(t) = b(1/t + 1/(\sigma-t)), \quad (33)$$

$$b > 0, \sigma > 0,$$

and then we have

$$\theta(t) = b \text{Log } (t/t_0) - b \text{Log } (\sigma-t/\sigma-t_0)$$

or, $\theta(t)$ being defined up to a constant,

$$\theta(t) = b (\text{Log } t - \text{Log } (\sigma-t)), \quad (34)$$

which varies from $-\infty$ (at $t=0$) to $+\infty$ (at $t=\sigma$) and is equal to 0 for $t=\sigma/2$. It is the reciprocal of the logistic function

$$t(\theta) = \sigma \exp \theta/b / 1 + \exp \theta/b .$$

We already had a result close to (34) with the case of elliptic explosion-implosion which gave for "internal time"

$$\theta(t) = q^2/2p (\text{Log } t - 2t/p - \text{Log}(2p-t)) \quad (4)$$

met again in the cosmological case of radiation with null cosmological constant (15). According to (34) reference instants $t=0$ and $t=\sigma$ are pushed back to $-\infty$ for the first one and to $+\infty$ for the second. There is no reference instants for big bang nor for big crunch So the "life of the universe" whose reference duration is finite and equal to σ has an infinite "internal duration". From the "internal time" point of view this universe has no beginning

nor end as well as in the elliptic explosion-implosion and cosmological case with no radiation and null cosmological constant.

The presence of term $2t/p$ may be surprising. In fact it does not exist in the new formulation (34). If we identify σ with $2p$ and $q^2/2p$ to b , to make the comparison more clear, (4) becomes

$$\theta(t) = b (\text{Log} t - 4t/\sigma - \text{Log}(\sigma-t)).$$

In the case of elliptic « explosion-implosion », according to the following expression given in section 2.1

$$(d(X(t)/dt)^2 = q^2/2p (1/t - 2/p + 1/2p-t),$$

We see that the index of importance of reference instant t contains, apart from the terms involving $1/t$ and $1/2p-t$, a constant negative term. The result is that for $t = p$, the index of importance is equal to 0. This is quite normal since at instant $t = p$ the speed of evolution of the system is null. Since equation (33) gives

$$b(\sigma/2) = 4/\sigma,$$

we may change (33) into

$$b(t) = b (1/t - 4/\sigma + 1/\sigma-t), \quad (35)$$

the factor of importance or of attention $b(t)$, always positive, being considered as a whole and not decomposable into parts.

We may also consider that the behaviour of the system near the end of its evolution is not necessarily symmetrical of its behaviour near the beginning, and propose

$$b(t) = b_1 1/t + b_2 1/\sigma-t,$$

$$b_1 > 0, \quad b_2 > 0,$$

or even, if we want to have

$$b(\sigma/2) = 0,$$

$$b(t) = b_1 1/t - 2(b_1 + b_2)/\sigma + b_2 1/\sigma-t. \quad (36)$$

Since we made a comparison between an explosive system with an infinite speed of evolution at the beginning, which is the case of universe in certain modellisations, and the evolution of a human being (or another sort of conscious creature) as far as "internal time" is concerned. We could also make the same comparison between an explosive-implosive system with an infinite positive speed of evolution at the beginning and an infinite negative speed at the end, also a possible case for universe or a conscious creature.

9. Conclusion

The measure of time has always been a problem to philosophers and scientists. Early human beings of the paleolithic age founded it on the regular movements of moon and sun and the apparent rotation of heavens, opening the way to scientific thinking. It has been said that if

the sky of earth had been extremely cloudy this sort of thinking would have been delayed or even would never have started. Space seems to have been less mysterious to human mind up to the point that the standard representation of time is a straight line, even if attempts to introduce n dimensional time have been made (Vallée, 1991). But aside from attempts to measure time in a way acceptable to all, time as it is felt by each individual is another problem. Each of us, by an "inverse transfer" (Vallée, 1974) is unconsciously ready to attribute his own intimate structures, also his feeling of time and duration to universe itself. A behaviour which, despite its obvious defect of giving a distorted image of the environment has the advantage to render time more familiar, introducing a kind of taming of the external world.

Influenced by the subjective apprehension of time we evoked, our aim is to propose, for a dynamical system, this expression being taken with its broadest meaning, a concept of "internal time". The system may be inanimate, in the sense that, for it, consciousness is meaningless, or it may be a conscious being. So in many cases certain concepts will be used metaphorically, particularly when applied to the inanimate. We used the concept of importance of an instant, giving two kinds of definitions. In the first case the degree of importance is directly linked to the intensity of change of the state of the system at this instant, a point of view which is more or less akin to that of Aristotle and Augustine. In the second case, relatively close to that of Condillac or even of Lecomte du Noüy, this importance is more or less "felt". It seems that this aspect of the problem of time has mostly interested philosophers and not scientists with the exception of economist Allais and cosmologist Milne.

One of the most interesting results obtained is that in certain cases the "reference duration" of the life (or a part of it) of the system is finite while its "internal duration" is infinite. Of course this is not the general case, but permits in some cases, mainly certain "explosion-implosions" to eliminate apparent paradoxes. Many people when a big bang or big crunch theory is presented to them do not see that the "instant" of the big bang or the "instant" of the big crunch are not instants. They do not belong to the set of instants of time as well as the degree 0 of Kelvin temperature scale is not a degree of temperature in fact unaccessible. Allusions made to psychological interpretation of some sort of "biological" explosion-implosion are obviously extremely controversial, particularly when a sort of big crunch is considered. These considerations must be seen from a rather metaphorical point of point of view, remembering that in many cases only the qualitative aspects of mathematical results must be retained., a viewpoint which is rather new, despite the progress of ideas such as fuzzy sets and fuzzy logiics.

Indeed in recent cosmological models with inflation, big bang is excluded. So what we have presented concerning big bang seems to loose a part of its interest. Nevertheless a universe starting with $R(0)$ strictly positive may have an "internal time" pushing back $t = 0$ to $\theta = -\infty$, since it is only the behaviour of $(dR(t)/dt)^2$ which is important. Moreover we do not know what will be the models of universe even in near future with the problem of "dark matter". So the idea of "internal time" remains, in cosmology. But another problem arises: before the so called Planck's time there are difficulties concerning laws of physics, and so the definition of time itself.

What about the future of "internal time" in cosmology and other fields? A first point is that, for the sake of simplicity, we considered only systems with scalar state. It would be good to

generalize to vector state of dimension n . The definition of the index of importance of reference instant permits it, since it is the scalar square of a vector. But other indexes are possible, they just need to be a norm or any positive increasing function of a norm. The choice we made, the square of the Euclidian norm, has many advantages but others are eligible. Another direction of research would be to introduce some kind of "internal time" adapted to a quantum mechanical systems, despite obvious difficulties, or even to the conception of a discrete time having something to do with Planck's time as a unit of duration.

More generally a scientific introduction of subjectivities affecting systems, in certain cases metaphorically speaking, has a great interest. The consideration of concepts intrinsically linked to a system, instead of being imposed from the outside, is also very desirable. Equations of evolution or values of some traits expressed in term of such subjective or intrinsic features may be suggestive. An example is given by the length of life for certain species of animals measured by the number of their heartbeats, length which appears as of the same order of magnitude.

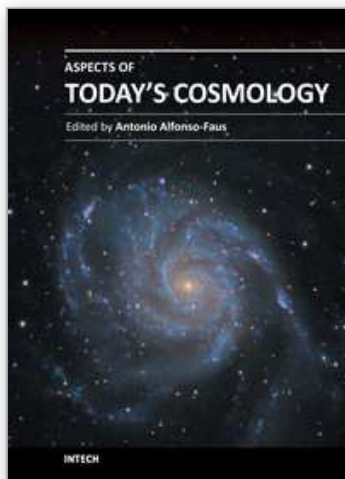
10. References

- Allais, M (1972), Forgetfulness and interest, *Journal of Money, Credit and Banking*, (Feb.) pp.46-76.
- Bergson, H (1939), *Matière et mémoire*, Presses Universitaires de France, Paris.
- Berry, M (1989), *Principles of Cosmology and Gravitation*, Institute of Physics, ISBN 0-85274-037-9, Bristol.
- Condillac, E Bonnot de (1754), *Traité des sensations*, Paris.
- Hussey, E (1983), *Physics*, books III and IV, Oxford Press.
- Lecomte du Noüy, P (1936), *Le temps et la vie*, Gallimard, Paris.
- Milne, E A (1948), *Kinematic Relativity*, Clarendon Press, Oxford.
- Prigogine, I (1947), *Étude thermodynamique des processus irréversibles*, Dunod, Paris.
- Vallée, R (1951), Sur deux classes d'"opérateurs d'observation", *Comptes Rendus de l'Académie des Sciences*, Vol. 233, pp. 1350-1351.
- Vallée, R (1975), Observation, decision and structure transfer in systems theory, In: *Progress in Cybernetics and Systems Research*, R. Trappl, (Ed.), Vol.1, pp.15-20, Hemisphere Publishing Corporation, Washington.
- Vallée, R (1991), Perception, memorisation and multidimensional time; *Kybernetes*, Vol. 20, No.6, pp. 15-28.
- Vallée, R (1992), The 'epsilon distribution' or the antithesis of Dirac's delta, In: *Cybernetics and Systems Research*, R. Trappl, (Ed.), 97-102, World Scientific, Singapore.
- Vallée, R (1996), Temps propre d'un système dynamique, cas d'un système explosif-implosif, *Actes du 3ème Congrès International de Systémique*, E. Pessa & M P Penna (Eds.), pp.967-970, Edizioni Kappa, Rome
- Vallée, R (2002), Mathematical and formalized epistemologies, In: *Quantum Mechanics, Mathematics, Cognition and Action, Proposals for a Formalized Epistemology*, M. Mugur-Schachter, A. van der Merwe (Eds.), pp. 309-324, Kluwer Academic Publishers, Dordrecht.
- Vogel, T (1965), *Théorie des systèmes évolutifs*, Gauthier-Villars, Paris.

Warner, R (1963), The Confessions of St. Augustine, Penguin Books, ISBN 0-451-62474-2, New York.

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This book presents some aspects of the cosmological scientific odyssey that started last century. The chapters vary with different particular works, giving a versatile picture. It is the result of the work of many scientists in the field of cosmology, in accordance with their expertise and particular interests. Is a collection of different research papers produced by important scientists in the field of cosmology. A sample of the great deal of efforts made by the scientific community, trying to understand our universe. And it has many challenging subjects, like the possible doomsday to be confirmed by the next decade of experimentation. May be we are now half way in the life of the universe. Many more challenging subjects are not present here: they will be the result of further future work. Among them, we have the possibility of cyclic universes, and the evidence for the existence of a previous universe.

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