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Planar Dirac Fermions in External Electromagnetic Fields

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1. Introduction

We study the electron propagator in two spatial dimensions in the presence of external electromagnetic fields, this is, we focus in (2+1)-dimensional quantum electrodynamics (QED), where a third spatial dimension is suppressed. This is not a mere theoretical simplification, and we explain ourselves: back in time, some twenty years ago, it was shown that the low-energy effective theory of graphene in a tight-binding approach is the theory of two species of massless Dirac electrons in a (2+1)-dimensional Minkowski spacetime, each on a different irreducible representation of the Clifford algebra. The isolation of graphene samples in 2004 and 2005, has given rise to the new paradigm of relativistic condensed matter, bringing a new boost, both theoretical and experimental, to the matching of common interests of the condensed matter and high energy physics communities. Thus, the massless limit of our findings is of direct relevance in this subject. We assume the electrons moving in a magnetic field alone pointing perpendicularly to their plane of motion. We first develop the general case and then, we present a couple of examples: the motion of electrons in a uniform magnetic field, which is a canonical example to present the Ritus method and the case of a static magnetic field which decays exponentially along the x-axis (Murguía et al, 2010; Raya & Reyes, 2010).
There are many problems relating electrons in non-uniform magnetic fields of relevance in graphene. In particular, it has been established the possibility to confine quasiparticles in magnetic barriers (De Martino et al, 2007; Ramezani et al, 2009). This could be feasible creating spatially inhomogeneous, but constant in time, magnetic fields depositing ferromagnetic layers over the substrate of a graphene sample layer (Reijniers et al, 2001). The physical properties of graphene make it a promising novel material to control the transport properties in nanodevices. It has been considered to be used in electronics and spintronics applications, like in single-electron transistors (Ponomarenko et al, 2008; Wu et al, 2008), in the so called magnetic edge states (Park & Sim, 2008), which may play an important role in the spin-polarized currents along magnetic domains, and in quantum dots and antidots magnetically confined. Moreover, the quantum Hall effect in graphene has been observed at room temperature (Novoselov et al, 2007), evidence which confirms the great potential of graphene as the material to be used in carbon-based electronic devices. The effects of the exponentially decaying magnetic field can hardly be considered with other approaches,

but it can be straightforwardly studied within the Ritus method, which consist in the diagonalization of the electron propagator in external electromagnetic fields in the basis of the operator $(\not{\Pi})^2$ with $\Pi^\mu = p^\mu - eA^\mu$. Exploiting the Ritus formalism, we also derive the exact Foldy-Wouthuysen (FW) transformation for Dirac fermions in a time independent external electromagnetic field, where the transformation acquires a free form involving the dynamical quantum numbers induced by the field (Murguía & Raya, 2010). This is related with the fact that for some class of Hamiltonians of Dirac particles in presence of external static electromagnetic fields, it is possible to show a supersymmetric character, in the quantum mechanical sense, and in these cases the Ritus method provide a direct calculation of the exact FW transformation in arbitrary dimensions. Powerful applications of FW transformation in semiclassical calculations can be exploited in systems of other than (3+1) dimensions. The FW transformation has proven to be a favorite way to obtain the nonrelativistic limit of the Dirac equation, because it provides a block diagonal form representation of quantum operators and hence of the Dirac Hamiltonian itself. It has been widely used in both, gravitational and electromagnetic backgrounds, including the case of different stationary metrics. Going further in the applications of the Ritus formalism, we also explore other kind of useful transformations, like the Cini-Touschek transformation for the ultrarelativistic case.

2. Irreducible Dirac fermions

In order to describe planar Dirac electrons, let us start by defining the Dirac γ^μ-matrices, which satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{1}$$

The lowest dimensional representation of these matrices is 2×2. We can choose, for instance

$$\gamma^0 = \sigma^3, \qquad \gamma^1 = i\sigma^1, \qquad \gamma^2 = i\sigma^2. \tag{2}$$

It is straightforward to verify that these matrices satisfy the relations

$$[\gamma^\mu, \gamma^\nu] = -2i\epsilon^{\mu\nu\rho}\gamma_\rho, \qquad \text{and} \qquad \gamma^\mu\gamma^\nu = g^{\mu\nu} - i\epsilon^{\mu\nu\rho}\gamma_\rho. \tag{3}$$

This is the starting point to build up the planar Dirac theory. The free Dirac Lagrangian takes the form of its 4D counterpart, namely

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi\ , \tag{4}$$

where m is the mass of the electrons and we have used the γ matrices given in Eq. (2). The spectrum of solutions of the Dirac equation is

$$\begin{aligned} \psi^P(x) &= \begin{pmatrix} 1 \\ \frac{p_x - ip_y}{E+m} \end{pmatrix} e^{-ix\cdot p} \equiv u(p)e^{-ix\cdot p}\ , \\ \psi^N(x) &= \begin{pmatrix} \frac{p_x + ip_y}{E+m} \\ 1 \end{pmatrix} e^{ix\cdot p} \equiv v(p)e^{ix\cdot p}\ . \end{aligned} \tag{5}$$

These solutions verify the completeness relations

$$\sum u\bar{u} = \not{p} + m, \qquad \text{and} \qquad \sum v\bar{v} = \not{p} - m, \tag{6}$$

and the positive energy solution (labeled by P), describes a particle with spin up, whereas the negative energy one (labeled by N) describes an antiparticle with spin down (Anguiano

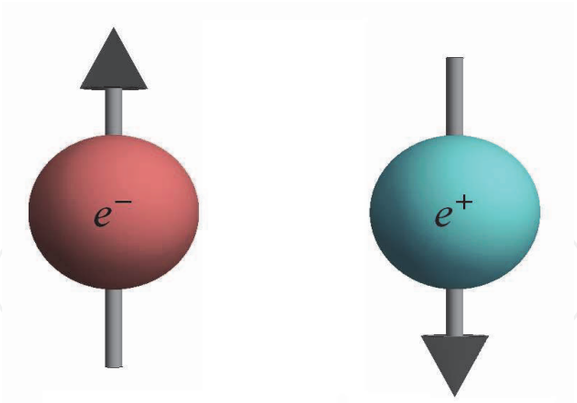


Fig. 1. The particle spectrum of solutions to the Dirac equation in Eq. (5). Adapted from Ref. (Hernández Ortíz, 2011).

& Bashir, 2005), as shown in Fig. 1. This fact is better seen from the definition of the spin operator

$$\Sigma_3 = \frac{\gamma_0}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

Thus *the Dirac particles have spin* $\pm 1/2$. These solutions, however, fail to incorporate important features of the ordinary Dirac spectrum familiar in relativistic quantum mechanics. For example, the solutions are not invariant under a Parity transformation $\mathcal{P}$, which for consistency with Lorentz symmetry in (2+1)-dimensions corresponds to the operation [1]

$$(t, x, y) \to (t, -x, y)^{\mathcal{P}}, \tag{8}$$

nor under a time reversal transformation $\mathcal{T}$. This is due to the fact that under $\mathcal{P}$, $\bar{\psi}\psi \to -\bar{\psi}\psi$. Furthermore, only one out of the two spin states of the physical electrons is present. A point of caution has to be raised here, in Condensed Matter Physics spin plays the role of flavor in High Energy Physics, thus one might be tempted to put by hand the spin factors of 2 whenever it is required. However, one cannot simply push this argument to the fully relativistic theory. The two spin states and symmetry features of the familiar spectrum of solutions to the Dirac equation can be recovered owing to the fact that there exists a second irreducible representation of the Dirac matrices. In graphene, the two representations describe two different electron species in each of the two triangular sub-lattices of the honeycomb lattice. The origin of the second irreducible representation is in direct connection with the fact that there is no chiral symmetry to be defined in (2+1)-dimensions with an irreducible representation of the Clifford algebra. Let γ be the product of all Dirac matrices, i.e, the would-be "γ_5" on the plane. From the properties of the σ matrices,

$$\gamma = \gamma^0\gamma^1\gamma^2 = \pm iI, \tag{9}$$

I being the 2×2 unit matrix. This allows us to define one of the matrices in terms of the other two. For instance $\gamma^2 = \pm i\gamma^0\gamma^1$. The second representation can then be chosen as

$$\gamma^0 = \sigma^3, \qquad \gamma^1 = i\sigma^1, \qquad \gamma^2 = -i\sigma^2, \tag{10}$$

[1] Parity is an improper Lorentz transformation, which should have determinant -1. If the usual definition of parity is employed, that would correspond to a rotation by an angle π of the plane.

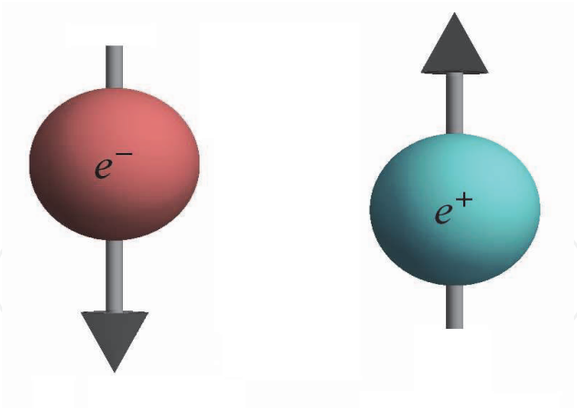


Fig. 2. The particle spectrum of solutions to the Dirac equation in Eq. (11). Adapted from Ref. (Hernández Ortíz, 2011).

with the property $\gamma^\mu\gamma^\nu = g^{\mu\nu} + i\epsilon^{\mu\nu\lambda}\gamma_\lambda$. Representations (2) and (10) are inequivalent, and hence correspond to physically different situations. This can be seen from the solutions of the Dirac equation in representation (10), which expanded in the representation (2) are

$$\psi^P(x) = \begin{pmatrix} \frac{p_x+ip_y}{E+m} \\ 1 \end{pmatrix} e^{-ix\cdot p} \equiv u(p)e^{-ix\cdot p} ,$$
$$\psi^N(x) = \begin{pmatrix} 1 \\ \frac{p_x-ip_y}{E+m} \end{pmatrix} e^{ix\cdot p} \equiv v(p)e^{ix\cdot p} , \tag{11}$$

and correspond to particle spinor with spin down and antiparticle with spin up (Anguiano & Bashir, 2005), as shown in Fig. 2. These solutions fulfill the completeness relations (6), but present also only one spin state for electron and yield a $\mathcal{P}$ and $\mathcal{T}$ non-invariant Lagrangian. Nevertheless, taking into account solutions for both representations, (2) and (10), labeled A and B, respectively, we recover two spin states for the electrons and their corresponding Lorentz conjugated positron states. The two "irreducible" fermion fields can be cast into the following *extended* form of the free Dirac Lagrangian (Anguiano & Bashir, 2005; Shimizu, 1985) [2]:

$$\mathcal{L} = \bar{\psi}_A(i\partial\!\!\!/ - m)\psi_A + \bar{\psi}_B(i\partial\!\!\!/ + m)\psi_B . \tag{12}$$

As we noticed before, neither under $\mathcal{P}$ nor under $\mathcal{T}$, the fields ψ_A and ψ_B transform onto themselves. In fact, under $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ transformations, these fields transform as

$$(\psi_A)^{\mathcal{C}} = \gamma^2 e^{i\eta_1}(\bar{\psi}_A)^T , \qquad (\psi_B)^{\mathcal{C}} = \gamma^2 e^{i\eta_2}(\bar{\psi}_B)^T ,$$
$$(\psi_A)^{\mathcal{P}} = -i\gamma^1 e^{i\phi_1}(\psi_B) , \qquad (\psi_B)^{\mathcal{P}} = -i\gamma^1 e^{i\phi_2}(\psi_A) ,$$
$$(\psi_A)^{\mathcal{T}} = i\gamma^0 e^{i\varphi_1}(\bar{\psi}_B)^T , \qquad (\psi_B)^{\mathcal{T}} = i\gamma^0 e^{i\varphi_2}(\bar{\psi}_A)^T . \tag{13}$$

where η_i, ϕ_i and φ_i, $i = 1, 2$ are constant phases. This shows that the extended Lagrangian (12) is $\mathcal{CPT}$ invariant (Shimizu, 1985). There are two chiral symmetries which can be defined. In infinitesimal form, these are

[2] Notice that only one irreducible representation of the Dirac matrices, say (2) is used.

Set I

$$\psi_A \to \psi'_A = \psi_A + \alpha\psi_B \ , \qquad \psi_B \to \psi'_B = \psi_B - \alpha\psi_A \ , \tag{14}$$

leading to the conserved current

$$j^\mu_I = \bar{\psi}_A\gamma^\mu\psi_B - \bar{\psi}_B\gamma^\mu\psi_A. \tag{15}$$

Set II

$$\psi_A \to \psi'_A = \psi_A + i\alpha\psi_B \ , \qquad \psi_B \to \psi'_B = \psi_B + i\alpha\psi_A \ , \tag{16}$$

leading to the conserved current

$$j^\mu_{II} = \bar{\psi}_A\gamma^\mu\psi_B + \bar{\psi}_B\gamma^\mu\psi_A. \tag{17}$$

The presence of two irreducible fermion fields in (12) naturally suggest that these can be merged into one reducible four-component spinor and hence we can make use of the ordinary 4×4 Dirac matrices. Such an issue is discussed below.

3. Reducible Dirac fermions

Planar Dirac fermions can also be described with the ordinary 4×4 matrices. Nevertheless, only three of them are required to describe the Dirac equation, for example $\{\gamma^0,\ \gamma^1,\ \gamma^2\}$, which in Euclidean space can be represented as

$$\gamma^0_E = \begin{pmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \quad \gamma^1_E = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}, \quad \gamma^2_E = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}. \tag{18}$$

In such a case, we have two other γ matrices (from now onwards we omit de subscript E for the Euclidean matrices) which commute with all the three matrices above, in such a fashion that the corresponding massless Dirac Lagrangian is invariant under the chiral-like transformations $\psi \to e^{i\alpha\gamma^3}\psi$, and $\psi \to e^{i\beta\gamma^5}\psi$, that is, it is invariant under a global $U(2)$ symmetry with generators 1, γ^3, γ^5 and $[\gamma^3, \gamma^5]$. Here

$$\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{19}$$

This symmetry is broken by an ordinary mass term $m_e\bar{\psi}\psi$. But there exists a second mass term, referred to as Haldane mass term (Haldane, 1988), which is invariant under the "chiral" transformations

$$m_o\bar{\psi}\frac{1}{2}[\gamma^3, \gamma^5]\psi \equiv m_o\bar{\psi}\tau\psi, \tag{20}$$

and has to be included in the complete Lagrangian when parity is allowed to be broken. If we write the 4-spinor as

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \ , \tag{21}$$

we observe that under $\mathcal{P}$ and $\mathcal{T}$, the components of spinors transform, up to a phase, as (Jackiw & Templeton, 1981)

$$\begin{aligned} &(\psi_A(t,x,y))^{\mathcal{P}} \to \sigma^1\psi_B(t,-x,y), \qquad &&(\psi_B(t,x,y))^{\mathcal{P}} \to \sigma^1\psi_A(t,-x,y), \\ &(\psi_A(t,x,y))^{\mathcal{T}} \to \sigma^2\psi_B(-t,x,y), \qquad &&(\psi_B(t,x,y))^{\mathcal{T}} \to \sigma^2\psi_A(-t,x,y). \end{aligned} \tag{22}$$

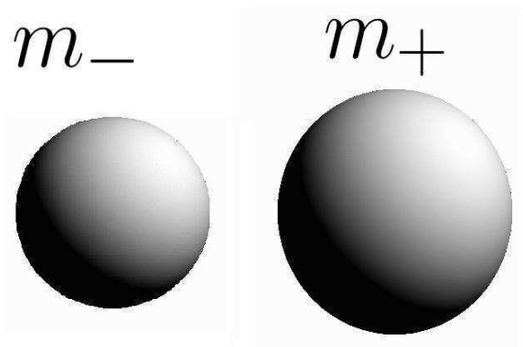


Fig. 3. Light and heavy fermion species of Lagrangian 25. Adapted from Ref. (Hernández Ortíz, 2011).

Thus, the term $m_e\bar{\psi}\psi$ is even under each of these transformations, but $m_o\bar{\psi}\tau\psi$ is not, although it is $\mathcal{PT}$ and thus $\mathcal{CPT}$ symmetric. The Euclidean space free *reducible* Dirac Lagrangian in this case has the form

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m_e - m_o\tau)\psi \ . \tag{23}$$

Written in this form, neither m_e nor m_o represent "physical" masses for electrons. In order to disentangle the species described by this Lagrangian, it is convenient to introduce the chiral-like projectors

$$\chi_\pm = \frac{1}{2}(1 \pm \tau) \ , \tag{24}$$

which verify (Kondo, 1996) $\chi_\pm^2 = \chi_\pm$, $\chi_+\chi_- = 0$, $\chi_+ + \chi_- = I$. The "right handed" ψ_+ and "left handed" ψ_- fermion fields in this case are given by $\psi_\pm = \chi_\pm\psi$. The $\chi_\pm$ project the upper and lower two component spinors, i.e, fermion species, out of the four-component ψ. In terms of the chiral fields, the Dirac Lagrangian acquires the form (see for instance (Jackiw & Pi, 2007))

$$\mathcal{L}_F = \bar{\psi}_+(i\partial\!\!\!/ - m_+)\psi_+ + \bar{\psi}_-(i\partial\!\!\!/ - m_-)\psi_-, \tag{25}$$

with $m_\pm = m_e \pm m_o$. This Lagrangian explicitly describes two fermion species of physical masses m_+ and m_-, respectively. For each species, the mass terms breaks chiral symmetry and parity at the same time. Moreover, the effect of the parity-violating mass is seen to remove the mass degeneracy between species. There is a light species and a heavy species, as illustrated in Fig. 3. Below we introduce interactions of fermions with a classical electromagnetic field within the Ritus formalism.

4. Propagator in magnetic fields

We start from the free Dirac equation derived from the Lagrangian (4)

$$(i\partial\!\!\!/ - m)\psi = 0 \ , \tag{26}$$

and work with representation (2). The extension to other representations is straightforward. In a background electromagnetic field, the Dirac equation takes the form

$$(\not{\Pi} - m)\,\Psi = 0, \tag{27}$$

where $\Pi_\mu = i\partial_\mu + eA_\mu$ and A_μ is the electromagnetic potential defining the external field. From now onwards, let us consider a magnetic field alone pointing perpendicularly to the plane of motion of the electrons. Moreover, let us work in a Landau-like gauge by choosing $A^\mu = (0, 0, W(x))$, where $W(x)$ is some function for which, in the general case, its derivative $W'(x) = \partial_x W(x)$ defines the profile of the field. We are interested in finding the Green's function or propagator for this equation, namely, the function $G(x, x')$ which satisfies

$$(\not{\Pi} - m)\, G(x, x') = \delta(x - x'). \tag{28}$$

Since $\not{\Pi}$ does not commute with the momentum operator, neither the wave function nor $G(x, x')$ can be expanded in plane-waves, and this does not allow to have a diagonal propagator in momentum space. The scheme we choose to deal with the external fields was developed by Ritus (Ritus, 1972; 1974; 1978). The crucial observation is that the Green's function above should be a combination of all scalar structures obtained by contracting the γ^μ-matrices, the canonical momentum Π_μ and the electromagnetic field strength tensor $F_{\mu\nu} = [\Pi_\mu, \Pi_\nu]/e \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, which are compatible with Lorentz symmetry, gauge invariance and charge conjugation, namely,

$$G(x, x') = G(\not{\Pi}, \sigma^{\mu\nu} F_{\mu\nu}, (\tilde{F}^\nu \Pi_\nu)^2), \tag{29}$$

where $\sigma_{\mu\nu} = i[\gamma_\mu, \gamma_\nu]/2$ and $\tilde{F}_\mu \equiv (1/2)\epsilon_{\mu\nu\alpha} F^{\nu\alpha}$ is the dual field strength tensor, which in (2+1)-dimensions is simply a vector. The key observation is that all the above structures commute with $(\not{\Pi})^2$, and thus

$$\Big[(\gamma \cdot \Pi)^2, G(x, x')\Big] = 0\,. \tag{30}$$

This fact allows us to expand the Green's function $G(x, x')$ in the same basis of eigenfunctions of $(\not{\Pi})^2$. Furthermore, if we perform a similarity transformation on $(\not{\Pi})^2$ in which it acquires a diagonal form in momentum space, then the same transformation makes the Green's function diagonal too. Such a similarity transformation is

$$\mathbb{E}_p^{-1} (\not{\Pi})^2 \mathbb{E}_p = p^2 \mathbb{I}\,, \tag{31}$$

where $\mathbb{E}_p$ are the transformation matrices, $\mathbb{I}$ is the unit matrix and p^2 can be any real number. Therefore, when we apply $\mathbb{E}_p$ functions to the propagator, it will become diagonal in momentum space. It is important to notice that in the fermionic case, the spin operator is realized in terms of the γ^μ-matrices, and thus the $\mathbb{E}_p$ functions inherit its matrix form. For different charged particles, the spin operator is realized in a different ways. For example, for scalar particles, the $\mathbb{E}_p$ functions are simply scalars (Ginzburg, 1995), whereas in the case of charged gauge bosons, the spin structure is embedded in a Lorentz tensor, and therefore the $\mathbb{E}_p$ functions also comply a Lorentz tensor structure (Elizalde et al, 2002). Our goal in this work is to study the structure of the $\mathbb{E}_p$ matrices for the case of Dirac fermions in (2+1)-dimensions. The similarity transformation (31) can be written as

$$(\not{\Pi})^2 \mathbb{E}_p = p^2 \mathbb{E}_p\,, \tag{32}$$

which is an eigenvalue equation for the matrices $\mathbb{E}_p$, which are referred to as the Ritus eigenfunctions in the specialized literature. Now,

$$(\not{\Pi})^2 = \Pi^2 + \frac{e}{2} \sigma^{\mu\nu} F_{\mu\nu}\,. \tag{33}$$

The only non-vanishing elements of the field strength tensor are $F_{12} = -F_{21} = W'(x)$, and because $\sigma^{12} = \sigma_3$, the $\mathbb{E}_p$ functions satisfy

$$\left(\Pi^2 + e\sigma_3 W'(x)\right) \mathbb{E}_p = p^2 \, \mathbb{E}_p \ . \tag{34}$$

Moreover, because

$$[\not{\Pi}, i\partial_t] = \left[\not{\Pi}, -i\partial_y\right] = [\not{\Pi}, \mathcal{H}] = 0 \ , \tag{35}$$

with

$$\mathcal{H} = -\not{\Pi}^2 + \Pi_0^2, \tag{36}$$

the functions $\mathbb{E}_p$ are eigenfunctions of these operators with eigenvalues

$$i\partial_t \mathbb{E}_p = p_0 \mathbb{E}_p \ , \qquad i\partial_y \mathbb{E}_p = -p_2 \mathbb{E}_p \ , \qquad \mathcal{H}\mathbb{E}_p = k\mathbb{E}_p \ , \tag{37}$$

which label the solutions to the massless Dirac equation in the background field. Notice that $p^2 = p_0^2 - k$, and hence, the $\mathbb{E}_p$ functions verify

$$\left(-\Pi_1^2 - \Pi_2^2 + e\sigma_3 W'(x)\right) \mathbb{E}_p = -k\mathbb{E}_p \ . \tag{38}$$

The first two terms of the operator on the l.h.s. of this equation act on the orbital degrees of freedom of the eigenfunctions $\mathbb{E}_p$, whereas the last term acts only in its spin degrees of freedom. Hence we can make the ansatz

$$\mathbb{E}_p = E_{p,\sigma} \, \omega_\sigma \ , \tag{39}$$

where ω_σ is the matrix of eigenvectors of σ_3 with eigenvalues $\sigma = \pm 1$, respectively, and

$$E_{p,\sigma} = N_\sigma e^{-i(p_o t - p_2 y)} F_{k,p_2,\sigma} \ , \tag{40}$$

with N_σ being the corresponding normalization constant. Substituting the ansätze (39) and (40) into Eq. (38), we arrive at

$$\left(\frac{d^2}{dx^2} - (-p_2 + eW(x))^2 + e\sigma W'(x)\right) F_{k,p_2,\sigma} = -kF_{k,p_2,\sigma} \ . \tag{41}$$

For the squared-integrability of the solutions, the eigenvalue k must be discrete. The above expression has the form of the Pauli Hamiltonian with the constrained vector potential, mass $m = 1/2$ and gyromagnetic factor $g = 2$, and turns out to be supersymmetric in the Quantum Mechanical sense (SUSY-QM) (Cooper et al, 1995; 2001). From the solutions to the above equation, we can construct the Ritus eigenfunctions $\mathbb{E}_p$ as

$$\mathbb{E}_p = \begin{pmatrix} E_{p,1}(z) & 0 \\ 0 & E_{p,-1}(z) \end{pmatrix} \ , \tag{42}$$

where the label $p = (p_0, p_2, k)$ and $z = (t, x, y)$. Being a complete set, the eigenfunctions $\mathbb{E}_p$ given in Eq. (42), satisfy

$$\begin{aligned} \int dz \, \overline{\mathbb{E}}_{p'}(z) \mathbb{E}_p(z) &= \mathbb{I}\delta(p - p') \ , \\ \int dp \, \mathbb{E}_p(z) \overline{\mathbb{E}}_p(z') &= \mathbb{I}\delta(z - z') \ , \end{aligned} \tag{43}$$

with $\overline{\mathbb{E}}_p(z) = \gamma^0 \mathbb{E}_p^*(z)\gamma^0$ and $\mathbb{I}$ is the 2×2 unit matrix. Let us introduce the bar-momentum $\overline{p}_\mu = (p_0, 0, \sqrt{k})$, which plays an important role in the method. Its definition involves the dynamical quantum numbers p_0 and k, but not p_2, which merely fixes the origin of the x coordinate. In other words, in the Ritus method the propagator is written only in terms of the eigenvalues of the dynamical operators commuting with $\not{\Pi}$. Notice that the bar-momentum verifies $\overline{p}^2 = p_0^2 - k = p^2$, and it is defined through the relation

$$\not{\Pi}\mathbb{E}_p = \mathbb{E}_p \not{\overline{p}} \, . \tag{44}$$

This relation will become important in the discussion of unitary transformations of the Dirac Hamiltonian. With the $\mathbb{E}_p$ functions, we can consider the Green's function method to obtain the propagator in the presence of the field. From Eq. (28), we define the Green's function in momentum space as

$$G(z, z') = \int dp \, \mathbb{E}_p(z) S_F(p) \overline{\mathbb{E}}_p(z') \, . \tag{45}$$

Here, the integral sign might as well represent a sum, depending upon the continuous or discrete nature of the components of the momentum. Applying the Dirac operator $(\not{\Pi} - m)$ to $G(z, z')$, we have that

$$\begin{aligned} (\not{\Pi} - m)\, G(z, z') &= \int dp \, \mathbb{E}_p(z) (\not{\overline{p}} - m) S_F(p) \overline{\mathbb{E}}_p(z') \\ &= \int dp \, \mathbb{E}_p(z) \overline{\mathbb{E}}_p(z') \, , \end{aligned} \tag{46}$$

where in the last step we have used the representation of the δ-function in the $\mathbb{E}_p$ basis. Hence we notice that, in this basis, the Ritus propagator takes the form of a free propagator, namely,

$$S_F(p) = \frac{1}{\not{\overline{p}} - m} \, , \tag{47}$$

with $\overline{p}_\mu$ defined through Eq. (44). On physical grounds, the $\mathbb{E}_p$ functions correspond to the states of electrons with momentum $\overline{p}$ in the background of the external field under consideration. With the help of these functions and the property (44), we can find the solutions of the Dirac equation (27) in a straightforward manner. To this end, we propose

$$\Psi = \mathbb{E}_p u_{\overline{p}}, \tag{48}$$

where $u_{\overline{p}}$ is a spinor of momentum $\overline{p}$. Then

$$(\not{\Pi} - m)\mathbb{E}_p u_{\overline{p}} = \mathbb{E}_p (\not{\overline{p}} - m) u_{\overline{p}} = 0, \tag{49}$$

and thus we see that $u_{\overline{p}}$ is simply a free spinor describing an electron with momentum $\overline{p}$. Notice that with this form of Ψ, the information concerning the interaction with the background magnetic field has been factorized into the $\mathbb{E}_p$ functions and throughout the $\overline{p}$ dependence of $u_{\overline{p}}$. Several relevant physical observables can then be found immediately, such as the probability density, the transmission and reflection coefficients between magnetic domains, and the density matrix, which are all useful, for example, in graphene applications as those which were mentioned in the Introduction, we discuss below, where the Ritus method plays useful.

5. Examples

5.1 Uniform magnetic field

Consider in the first place the case of a uniform magnetic field (Khalilov, 1999). This corresponds to the choice $W(x) = B_0 x$. To simplify the calculations, we rename the quantum number $k \to 2|eB_0|k$ in Eq. (37). In this case, Eq. (41) simplifies to

$$\left[\frac{d^2}{dx^2} - (-p_2 + eB_0 x)^2 + \sigma eB_0\right] F_{k,p_2,\sigma}(x) = -2|eB_0|kF_{k,p_2,\sigma}(x) \ . \tag{50}$$

Letting $\eta = \sqrt{2|eB_0|}[x - p_2/(eB_0)]$, the above expression acquires the form

$$\left[\frac{d^2}{d\eta^2} + k + \frac{\sigma}{2}\mathrm{sgn}(eB_0) - \frac{\eta^2}{4}\right] F_{k,p_2,\sigma}(\eta) = 0 \ , \tag{51}$$

that is, the equation for a quantum harmonic oscillator, with center of oscillation in $x_0 = p_2/(eB_0)$ and cyclotron frequency $w_c = 2eB_0$. Thus, the normalized functions $E_{p,\sigma}$ acquire the form

$$\begin{aligned} E_{p,1} &= \frac{(\pi|eB_0|)^{1/4}}{2\pi^{3/2}k!^{1/2}} e^{-ip_0 t + ip_2 y} D_k(\eta) \ , \\ E_{p,-1} &= \frac{(\pi|eB_0|)^{1/4}}{2\pi^{3/2}(k-1)!^{1/2}} e^{-ip_0 t + ip_2 y} D_{k-1}(\eta) \ , \end{aligned} \tag{52}$$

where

$$D_n(x) = 2^{-n/2} e^{-x^2/4} H_n\left(x/\sqrt{2}\right) \tag{53}$$

is the parabolic cylinder function of order

$$n = k + \frac{\sigma}{2}\mathrm{sgn}(eB_0) - \frac{1}{2} \ , \tag{54}$$

and $H_n(x)$ are the Hermite's polynomials. Expectedly, the uniform magnetic field renders the $(n-1)$-th state with spin down with the same energy of the n-th state with spin up. Inserting these functions into Eq. (42), we obtain the Ritus eigenfunctions which render the propagator diagonal in momentum space. Alternative forms of this propagator were recently reported (Rusin & Zawadzki, 2011).

Observe that physical observables like probability densities are linear combinations of $|E_p|^2$. These functions have the profile shown Fig. 4. The dashed curve enveloping these solutions corresponds to the potential

$$y = \bar{W}^2(x) - \bar{W}'(x) \ , \tag{55}$$

where $\bar{W} = eW - p_2$ is referred to as the superpotential in the SUSY-QM literature (Cooper et al, 1995; 2001).

5.2 Exponential magnetic field

In this section we study the electron propagator in a background static magnetic field which has an exponentially decaying spatial profile along one direction, described through the function $W(x) = -B_0[\exp\{-\alpha x\} - 1]/\alpha$. In this case, Eq. (41) simplifies to

$$\left[\frac{d^2}{dx^2} - \left(-p_2 - \frac{eB_0}{\alpha}(\exp\{-\alpha x\} - 1)\right)^2 + \sigma eB_0 \exp\{-\alpha x\}\right] F_{k,p_2,\sigma}(x) = -kF_{k,p_2,\sigma}(x) \ . \tag{56}$$

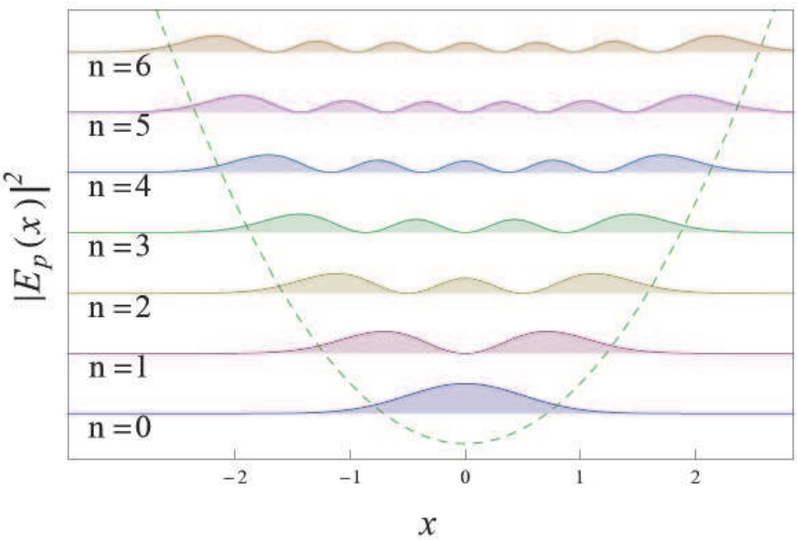


Fig. 4. The $|E_p|^2$ function from Eq. (52) in arbitrary units along the dynamical direction for various Landau levels. The dashed curve $y = x^2 - 1$ corresponds to the potential (55) for this field configuration. The scale of the graphic is set by $eB_0 = 1$ and $p_2 = 0$.

Let $\varrho = (2eB_0/\alpha^2)\exp\{-\alpha x\}$ and $s = -(p_2 - eB/\alpha)/\alpha$, then, the above expression is equivalent to

$$\left[\varrho^2\frac{d^2}{d\varrho^2} + \varrho\frac{d}{d\varrho} - \left(s - \frac{1}{2}\varrho\right)^2 + \frac{\sigma}{2}\varrho + \frac{k}{\alpha^2}\right] F_{k,p_2,\sigma}(\varrho) = 0\,. \tag{57}$$

This equation has the normalized solutions $E_{p,\sigma}$ given as

$$\begin{aligned} E_{p,1} &= \frac{1}{2\pi}\left(\frac{2\alpha n!(s-n)}{\Gamma(2s-n+1)}\right)^{1/2} e^{-ip_0t+ip_2y}e^{-\varrho/2}\varrho^{(s-n)}L_n^{2(s-n)}(\varrho), \\ E_{p,-1} &= \frac{1}{2\pi}\left(\frac{2\alpha(n-1)!(s-n)}{\Gamma(2s-n)}\right)^{1/2} e^{-ip_0t+ip_2y}e^{-\varrho/2}\varrho^{(s-n)}L_{n-1}^{2(s-n)}(\varrho), \end{aligned} \tag{58}$$

where $L_a^b(x)$ are the associate Laguerre polynomials with

$$n = s - \sqrt{-\frac{k}{\alpha^2} + s^2}. \tag{59}$$

The quantum number n is the principal quantum number, whereas s a center of oscillation weighted by the damping factor α. Fig. 5 we show $|E_p|^2$ for various values of n at fixed $s = 8$. Notice that in this case the potential (55) also envelops the squares of the solutions.

6. Non relativistic and ultrarelativistic forms of the Dirac equation

The study of semiclassical and nonrelativistic limits of the Dirac equation is a useful method to understand some effects on fermions coupled to external fields. In both, gravitational (Goncalves et al, 2007; Obukhov et al, 2009) and electromagnetic backgrounds (Barducci et al, 2009; Silenko, 2008), the Foldy-Wouthuysen (FW) transformation (Foldy & Wouthuysen, 1950) has proven to be a favorite way to obtain

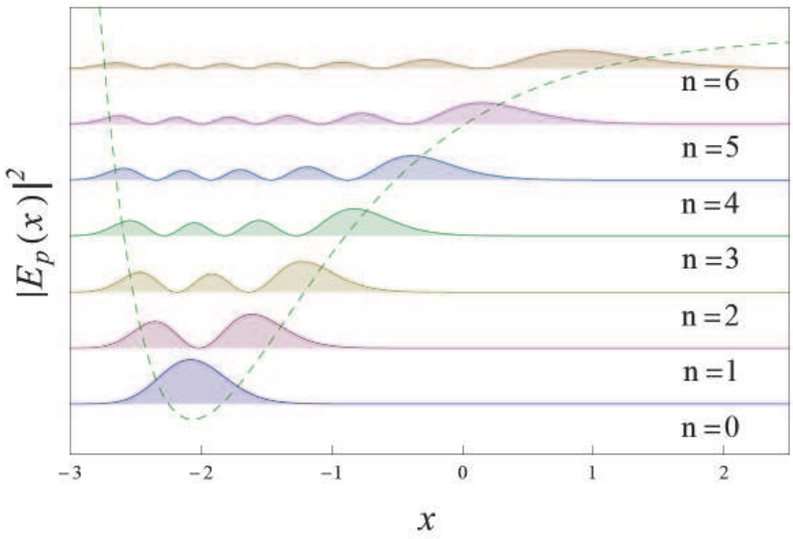


Fig. 5. The $|E_p|^2$ functions from Eq. (58) in arbitrary units along the dynamical direction for various values of n at fixed $s = 8$. The dashed curve $y = (8 - e^{-x})^2 + e^{-x}$ corresponds to the potential (55) for this field configuration. The scale of the graphic is set by $eB_0 = \alpha = 1$.

the nonrelativistic limit of the Dirac equation, because it provides a block diagonal form representation of quantum operators and hence of the Dirac Hamiltonian itself. Powerful applications of FW transformation in semiclassical calculations can be exploited in systems of other than (3+1) dimensions (Moreno & Méndez-Moreno, 1992) due to its relation with the supersymmetric character, in the quantum mechanical sense (Cooper et al, 1995; 2001), of some class of Hamiltonians, as well as in different stationary metrics (Buhl et al, 2008; Heidenreich et al, 2006). In the non-relativistic domain, the components of the Dirac spinors in either eq. (5) or eq. (11) are such that $u_1 \gg u_2$ and $v_1 \ll v_2$, thus the Dirac equation reduces to the *Pauli equation*, which is a first non-relativistic approximation of the Dirac equation for an electron in an external electromagnetic field,

$$i\frac{\partial}{\partial t}\psi = e\Phi\psi + \alpha^j \Pi^j \psi + \beta m\psi, \tag{60}$$

where $\alpha^j = \gamma^0\gamma^j$, $\beta = \gamma^0$ and $A^\mu = (\Phi, \mathbf{A})$. Now we write $\psi = \begin{pmatrix} \phi' \\ \chi' \end{pmatrix}$ and substitute it into (60):

$$\begin{aligned} i\frac{\partial}{\partial t}\phi' &= e\Phi\phi' + (\Pi^1 - i\Pi^2)\chi' + m\phi', \\ i\frac{\partial}{\partial t}\chi' &= e\Phi\chi' + (\Pi^1 + i\Pi^2)\phi' - m\chi'. \end{aligned} \tag{61}$$

Writing

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} e^{-imt}$$

in order to subtract the relativistic rest energy, the coupled system of equations (61) now takes the form:

$$i\frac{\partial}{\partial t}\phi = e\Phi\phi + (\Pi^1 - i\Pi^2)\chi,$$
$$i\frac{\partial}{\partial t}\chi = e\Phi\chi + (\Pi^1 + i\Pi^2)\phi - 2m\chi. \tag{62}$$

In the weak external electromagnetic field, $m \gg e\Phi$ and $m \gg e|\mathbf{A}|$, and so, from the second equation in (62):

$$\chi \approx \frac{1}{2m}(\Pi^1 + i\Pi^2)\phi. \tag{63}$$

Substituting the last expression into the first equation (62), we obtain

$$i\frac{\partial}{\partial t}\phi = e\Phi\phi + \frac{1}{2m}(\Pi^1 - i\Pi^2)(\Pi^1 + i\Pi^2)\phi. \tag{64}$$

The second term on the r.h.s. of the last equation may be rewritten as

$$\frac{1}{2m}\left((\Pi^1)^2 + (\Pi^2)^2 + i[\Pi^1, \Pi^2]\right) = \frac{1}{2m}\left((\mathbf{p} - e\mathbf{A})^2 - e\left(\frac{\partial A^2}{\partial x^1} - \frac{\partial A^1}{\partial x^2}\right)\right),$$

and recalling that $\mathbf{B} = \nabla \times \mathbf{A}$, we reach at the well-known Pauli equation

$$i\frac{\partial}{\partial t}\phi = e\Phi\phi + \frac{(\mathbf{p} - e\mathbf{A})^2}{2m}\phi + \frac{e}{2m}B\phi, \tag{65}$$

which is a *one component* equation for an electron in an external electromagnetic field. We can obtain the non-relativistic form of the Dirac equation through the FW transformation. Notice that in the free Dirac equation,

$$i\frac{\partial\Psi}{\partial t} = (\boldsymbol{\alpha}\cdot p + \beta m)\Psi, \tag{66}$$

the large and small components of the Dirac spinor Ψ, labeled by the momentum $\mathbf{p}$, get mixed by the odd operator $\boldsymbol{\alpha}\cdot\mathbf{p}$ involving off-diagonal elements. The FW is a canonical transformation which, by removing such an operator from the Dirac Hamiltonian,

$$H_{\text{free}} = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m, \tag{67}$$

allows to decouple these large and small components of Ψ. The free FW transformation,

$$U_{\text{free}}(p) = e^{iS_{\text{free}}(\mathbf{p})} = \cos|\mathbf{p}|\theta + \frac{\boldsymbol{\gamma}\cdot\mathbf{p}}{|\mathbf{p}|}\sin|\mathbf{p}|\theta, \tag{68}$$

with θ given through

$$\tan(2|\mathbf{p}|\theta) = \frac{|\mathbf{p}|}{m}, \tag{69}$$

is exact, and renders the free Hamiltonian in the form

$$H^{FW}_{\text{free}} = \gamma^0\sqrt{\mathbf{p}^2 + m^2}. \tag{70}$$

In the presence of an external electromagnetic field, the FW transformation can be ordinarily obtained through successive approximations as an expansion in powers of $1/m$ (Bjorken & Drell, 1984). For example, at $O(1/m^3)$,

$$S(\Pi) = -i\left(\frac{\gamma^0}{2m}\right)^2\left([\mathcal{O}',\mathcal{E}'] + i\dot{\mathcal{O}}'\right), \tag{71}$$

with

$$\mathcal{O}' = \tfrac{\gamma^0}{2m}[\mathcal{O},\mathcal{E}] - \tfrac{\mathcal{O}^3}{3m^2} + i\tfrac{\gamma^0\dot{\mathcal{O}}}{2m}, \tag{72}$$

$$\mathcal{E}' = \mathcal{E} + \gamma^0\left(\tfrac{\mathcal{O}}{2m} - \tfrac{\mathcal{O}^4}{8m^3}\right) - \tfrac{1}{8m^2}[\mathcal{O},[\mathcal{O},\mathcal{E}]] - i\tfrac{1}{8m^2}[\mathcal{O},\dot{\mathcal{O}}]. \tag{73}$$

In the expressions above, the even (diagonal) and odd (off-diagonal) operators correspond to $\mathcal{E} = e\Phi$ and $\mathcal{O} = \gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\Pi} \equiv \tilde{\boldsymbol{\Pi}}$, respectively, and the dot represents the time derivative. To this order, the FW transformation renders the Dirac Hamiltonian to its leading non-relativistic form. For an external static inhomogeneous magnetic field the series can be written in closed form as

$$H_\Pi^{FW} = \gamma^0\sqrt{(\tilde{\boldsymbol{\Pi}})^2 + m^2}, \tag{74}$$

where the transformation matrix for this case is

$$S(\Pi) = -i(\boldsymbol{\gamma}\cdot\boldsymbol{\Pi})\theta, \tag{75}$$

with θ given through

$$\tan(2|\tilde{\boldsymbol{\Pi}}|\theta) = \frac{|\tilde{\boldsymbol{\Pi}}|}{m}. \tag{76}$$

Here, $|\tilde{\boldsymbol{\Pi}}| = \sqrt{(\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\Pi})^2}$ plays the role of the momentum $|\mathbf{p}|$ in the free case and $(\tilde{\boldsymbol{\Pi}})^2 = \mathcal{H}$ as it was defined by Eq. (36). With the decomposition (48) of the Dirac wave function, the stationary Schrödinger form of the Dirac equation becomes

$$E_D\mathbb{E}_p u_{\overline{p}} = \gamma^0(\boldsymbol{\gamma}\cdot\boldsymbol{\Pi} + m)\mathbb{E}_p u_{\overline{p}} \tag{77}$$

which with the aid of property (44), simplifies to

$$E_D\mathbb{E}_p u_{\overline{p}} = \mathbb{E}_p\gamma^0(\boldsymbol{\gamma}\cdot\overline{\boldsymbol{p}} + m)u_{\overline{p}}. \tag{78}$$

In the above expressions E_D represent the eigenenergies of the Dirac equation. Moreover, the Hamiltonian on the r.h.s. of Eq. (78) acquires a free form involving $\overline{p}$ alone. Thus, it is straightforward to convince oneself that the Ritus eigenfunctions map the FW transformation in external fields to a free transformation which depends on $\overline{p}$, namely

$$e^{iS(\Pi)}\mathbb{E}_p = \mathbb{E}_p e^{iS_{\text{free}}(\overline{p})}. \tag{79}$$

So, the $\mathbb{E}_p$ functions not only render the fermion propagator in external fields diagonal in momentum space, with a free form involving the quantum numbers induced by the field. These also allow to express the exact FW transformation in the presence of the fields in a free form. To see the usefulness of Eq. (79), we first apply the Hamiltonian in Eq. (74) to the Ritus eigenfunctions $\mathbb{E}_p$,

$$H_\Pi^{FW}\mathbb{E}_p = \left(\gamma^0\sqrt{(\tilde{\boldsymbol{\Pi}})^2 + m^2}\right)\mathbb{E}_p, \tag{80}$$

which has to be evaluated expanding the square-root operator in a power series of $(\tilde{\mathbf{\Pi}}/m)^2$. This procedure leads to an expression in terms of the eigenvalues k of the operator $(\tilde{\mathbf{\Pi}})^2$ given through Eqs. (31) and (37) with $p^2 = \overline{p}^2$. Since $p_0 = E_D$ are the eigenvalues of $\Pi_0 = i\partial_t$, from the Dirac equation $p_0^2 = k + m^2$, thus $\sqrt{k}$ correspond to the energy eigenvalues of a particle on-shell. From Eq. (36), $\overline{p}^2 = p_0^2 - k$, and it can be fulfilled with the choice of $\overline{p}^\mu = (p_0, 0, \sqrt{k})$, in accordance to our selection of gauge. Hence, Eq. (80) simplifies to

$$H_{\Pi}^{FW}\mathbb{E}_p = \mathbb{E}_p \left(\gamma^0 \sqrt{E_D^2 + m^2}\right). \tag{81}$$

On the other hand, notice that under the FW transformation, the Hamiltonian $H = \gamma^0(\boldsymbol{\gamma}\cdot\overline{\boldsymbol{p}} + m)$ on the r.h.s. of Eq. (78) transforms in a free form, as in Eq. (70), but involving $\overline{\mathbf{p}}^2$ alone. Thus the FW transformed Hamiltonian (78) can be written directly:

$$H_{\text{free}}^{FW}\mathbb{E}_p = \mathbb{E}_p \left(\gamma^0 \sqrt{E_D^2 + m^2}\right). \tag{82}$$

The r.h.s. of this last equation precisely corresponds to the r.h.s. of Eq. (81). This last was obtained transforming the Dirac Hamiltonian of Eq. (77) with a magnetic filed in the *usual* way. As comparison, with the aid of Eq. (48), the corresponding FW transformed Hamiltonian was obtained directly from a free one, Eq. (78), given in terms of the tri-momentum $\overline{p}^\mu$ which contains all the dynamics induced by the external magnetic field. It is then straightforward to prove the relationship between the FW transformations $S(\Pi)$ and $S_{\text{free}}(\overline{p})$ established by Eq. (79) in terms of the Ritus eigenfunctions $\mathbb{E}_p$. The same idea can be generalized to the case on which the Dirac Hamiltonian (66) is expressed in its ultra-relativistic form through the Cini-Touschek (CT) transformation (Cini & Touschek, 1958). In the free case, the parameter (69) for the CT transformation acquires the form

$$\tan(2|\mathbf{p}|\theta) = -\frac{m}{|\mathbf{p}|}\,, \tag{83}$$

and correspondingly the hamiltonian takes the form

$$H_{\text{free}}^{CT} = \frac{\sqrt{\mathbf{p}^2 + m^2}}{|\mathbf{p}|}\boldsymbol{\alpha}\cdot\mathbf{p}\,, \tag{84}$$

which is precisely the ultra-relativistic form of the Dirac Hamiltonian.

7. Graphene hamiltonian in diagonal form

Let us consider the "free" Hamiltonian of graphene

$$H_g = \frac{E_F}{|\mathbf{p}_F|}\boldsymbol{\gamma}\cdot\mathbf{p}. \tag{85}$$

Now, the free Dirac hamiltanian can be written in its non-relativistic and ultrarelativistic forms through the FW and CT transformations, respectively:

$$\begin{aligned} U_{FW}(m,\mathbf{p}) &= \cos\left(\frac{1}{2}\arctan\frac{1}{x}\right) + \frac{\boldsymbol{\gamma}\cdot\mathbf{p}}{|\mathbf{p}|}\sin\left(\frac{1}{2}\arctan\frac{1}{x}\right), \\ U_{CT}(m,\mathbf{p}) &= \cos\left(\frac{1}{2}\arctan x\right) + \frac{\boldsymbol{\gamma}\cdot\mathbf{p}}{|\mathbf{p}|}\sin\left(\frac{1}{2}\arctan x\right)\,, \end{aligned} \tag{86}$$

with $x = m/|\mathbf{p}|$. This means that graphene can be described through a CT-transformed Dirac Hamiltonian,

$$H_g = U_{CT}(\mu, \mathbf{p}_F) H_D U_{CT}^{\dagger}(\mu, \mathbf{p}_F) \ , \tag{87}$$

with μ some mass parameter of the transformation. In the free case, the CT transformation is exact, thus

$$H_D = U_{CT}^{\dagger}(\mu, \mathbf{p}_F) H_g U_{CT}(\mu, \mathbf{p}_F) \ . \tag{88}$$

Moreover, we know that

$$H_{\Pi}^{FW} = \gamma^0 E_k \equiv U_{FW}(\mu, \mathbf{p}_F) H_D U_{FW}^{\dagger}(\mu, \mathbf{p}_F) \ , \tag{89}$$

with $E_k = \sqrt{E_D^2 + \mu^2} = E_F$, therefore

$$H_{FW} = U_{FW}(\mu, \mathbf{p}_F) U_{CT}^{\dagger}(\mu, \mathbf{p}_F) H_g U_{CT}(\mu, \mathbf{p}_F) U_{FW}^{\dagger}(\mu, \mathbf{p}_F) \ . \tag{90}$$

Now, since

$$\arctan(z) \ \arctan\left(\frac{1}{z}\right) = \mathrm{sgn}(z)\frac{\pi}{2} \ , \tag{91}$$

we have that

$$U_{FW}(\mu, \mathbf{p}_F) U_{CT}^{\dagger}(\mu, \mathbf{p}_F) = \frac{1}{\sqrt{2}}\left(1 + \mathrm{sgn}(\mu)\frac{\boldsymbol{\gamma}\cdot\mathbf{p}}{|\mathbf{p}|}\right) . \tag{92}$$

Now, consider the Hamiltonian of graphene in a static magnetic field. We learnt that for the FW and CT transformations

$$U(\Pi)\mathbb{E}_p = \mathbb{E}_p U_{\text{free}}(\overline{p}) \ , \tag{93}$$

in such a manner that

$$H_g = \gamma^0 E_F(\mu) \tag{94}$$

in the Ritus basis, with $E_F(\mu) = \sqrt{\mathbf{p}^2 + \mu^2}$.

8. Concluding remarks

Summarizing, we studied the electron propagator in (2+1)-dimensions in the presence of external electromagnetic fields under the Ritus formalism. We have seen that the Ritus method offers an alternative way to study the electron propagator in the presence of external magnetic fields. Within the framework of this method the electron propagator acquires a free form involving only the dynamical quantum numbers induced by the external field when it is spanned in the Ritus functions E_p, the eigenfunctions of the operator $(\gamma \cdot \Pi)^2$. We have also shown that the Ritus eigenfunctions provide a direct connection with the non-relativistic and the ultra-relativistic limit of the Dirac equation. In the non-relativistic limit case we showed that, in the Ritus basis, the exact Foldy-Wouthuysen (FW) transformation of the Dirac Hamiltonian in presence of an external and time independent electromagnetic field can be expressed in a closed form in terms of a free transformation which only depends on the dynamical quantities induced by the field. In the ultra-relativistic limit, we have shown that the Cini-Touschek (CT) transformed Dirac Hamiltonian leads into the corresponding for graphene. We have shown the relationship between the Ritus eigenfunctions and the FW and CT transformations which let us write down the solutions of the graphene Hamiltonian only in terms of dynamical quantities induced by the external fields, namely, only on terms of $\overline{p}$.

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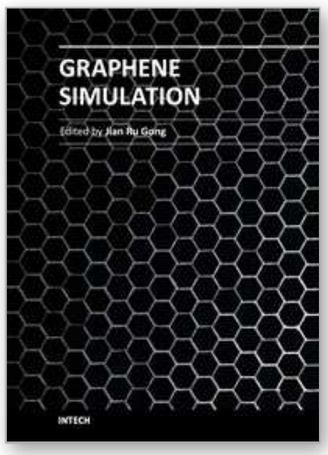
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Graphene, a conceptually new class of materials in condensed-matter physics, has been the interest of many theoretical studies due to the extraordinary thermal, mechanical and electrical properties for a long time. This book is a collection of the recent theoretical work on graphene from many experts, and will help readers to have a thorough and deep understanding in this fast developing field.

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