Stability and $\mathcal{L}_2$ Gain Analysis of Switched Linear Discrete-Time Descriptor Systems

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1. Introduction

This article is focused on analyzing stability and $\mathcal{L}_2$ gain properties for switched systems composed of a family of linear discrete-time descriptor subsystems. Concerning descriptor systems, they are also known as singular systems or implicit systems and have high abilities in representing dynamical systems [1, 2]. Since they can preserve physical parameters in the coefficient matrices, and describe the dynamic part, static part, and even improper part of the system in the same form, descriptor systems are much superior to systems represented by state space models. There have been many works on descriptor systems, which studied feedback stabilization [1, 2], Lyapunov stability theory [2, 3], the matrix inequality approach for stabilization, $\mathcal{H}_2$ and/or $\mathcal{H}_\infty$ control [4–6].

On the other hand, there has been increasing interest recently in stability analysis and design for switched systems; see the survey papers [7, 8], the recent books [9, 10] and the references cited therein. One motivation for studying switched systems is that many practical systems are inherently multi-modal in the sense that several dynamical subsystems are required to describe their behavior which may depend on various environmental factors. Another important motivation is that switching among a set of controllers for a specified system can be regarded as a switched system, and that switching has been used in adaptive control to assure stability in situations where stability can not be proved otherwise, or to improve transient response of adaptive control systems. Also, the methods of intelligent control design are based on the idea of switching among different controllers.

We observe from the above that switched descriptor systems belong to an important class of systems that are interesting in both theoretic and practical sense. However, to the authors' best knowledge, there has not been much works dealing with such systems. The difficulty falls into two aspects. First, descriptor systems are not easy to tackle and there are not rich results available up to now. Secondly, switching between several descriptor systems makes the problem more complicated and even not easy to make clear the well-posedness of the solutions in some cases.

Next, let us review the classification of problems in switched systems. It is commonly recognized [9] that there are three basic problems in stability analysis and design of switched systems: (i) find conditions for stability under arbitrary switching; (ii) identify the limited but useful class of stabilizing switching laws; and (iii) construct a stabilizing switching law.

Specifically, Problem (i) deals with the case that all subsystems are stable. This problem seems trivial, but it is important since we can find many examples where all subsystems are stable but improper switchings can make the whole system unstable [11]. Furthermore, if we know that a switched system is stable under arbitrary switching, then we can consider higher control specifications for the system. There have been several works for Problem (i) with state space systems. For example, Ref. [12] showed that when all subsystems are stable and commutative pairwise, the switched linear system is stable under arbitrary switching. Ref. [13] extended this result from the commutation condition to a Lie-algebraic condition. Ref. [14, 15] and [16] extended the consideration to the case of $\mathcal{L}_2$ gain analysis and the case where both continuous-time and discrete-time subsystems exist, respectively. In the previous papers [17, 18], we extended the existing result of [12] to switched linear descriptor systems. In that context, we showed that in the case where all descriptor subsystems are stable, if the descriptor matrix and all subsystem matrices are commutative pairwise, then the switched system is stable under impulse-free arbitrary switching. However, since the commutation condition is quite restrictive in real systems, alternative conditions are desired for stability of switched descriptor systems under impulse-free arbitrary switching.

In this article, we propose a unified approach for both stability and $\mathcal{L}_2$ gain analysis of switched linear descriptor systems in discrete-time domain. Since the existing results for stability of switched state space systems suggest that the common Lyapunov functions condition should be less conservative than the commutation condition, we establish our approach based on common quadratic Lyapunov functions incorporated with linear matrix inequalities (LMIs). We show that if there is a common quadratic Lyapunov function for stability of all descriptor subsystems, then the switched system is stable under impulse-free arbitrary switching. This is a reasonable extension of the results in [17, 18], in the sense that if all descriptor subsystems are stable, and furthermore the descriptor matrix and all subsystem matrices are commutative pairwise, then there exists a common quadratic Lyapunov function for all subsystems, and thus the switched system is stable under impulse-free arbitrary switching. Furthermore, we show that if there is a common quadratic Lyapunov function for stability and certain $\mathcal{L}_2$ gain of all descriptor subsystems, then the switched system is stable and has the same $\mathcal{L}_2$ gain under impulse-free arbitrary switching. Since the results are consistent with those for switched state space systems when the descriptor matrix shrinks to an identity matrix, the results are natural but important extensions of the existing results.

The rest of this article is organized as follows. Section 2 gives some preliminaries on discrete-time descriptor systems, and then Section 3 formulates the problem under consideration. Section 4 states and proves the stability condition for the switched linear discrete-time descriptor systems under impulse-free arbitrary switching. The condition requires in fact a common quadratic Lyapunov function for stability of all the subsystems, and includes the existing commutation condition [17, 18] as a special case. Section 5 extends the results to $\mathcal{L}_2$ gain analysis of the switched system under impulse-free arbitrary switching, and the condition to achieve the same stability and $\mathcal{L}_2$ gain properties requires a common quadratic Lyapunov function for all the subsystems. Finally, Section 6 concludes the article.

2. Preliminaries

Let us first give some preliminaries on linear discrete-time descriptor systems. Consider the descriptor system

$$\begin{cases} Ex(k+1) = Ax(k) + Bw(k) \\ z(k) = Cx(k)\,, \end{cases} \tag{2.1}$$

where the nonnegative integer k denotes the discrete time, $x(k) \in \mathcal{R}^n$ is the descriptor variable, $w(k) \in \mathcal{R}^p$ is the disturbance input, $z(k) \in \mathcal{R}^q$ is the controlled output, $E \in \mathcal{R}^{n\times n}$, $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times p}$ and $C \in \mathcal{R}^{q\times n}$ are constant matrices. The matrix E may be singular and we denote its rank by $r = \text{rank } E \leq n$.

Definition 1: Consider the linear descriptor system (2.1) with $w = 0$. The system has a unique solution for any initial condition and is called *regular*, if $|zE - A| \not\equiv 0$. The finite eigenvalues of the matrix pair (E, A), that is, the solutions of $|zE - A| = 0$, and the corresponding (generalized) eigenvectors define exponential modes of the system. If the finite eigenvalues lie in the open unit disc of z, the solution *decays exponentially*. The infinite eigenvalues of (E, A) with the eigenvectors satisfying the relations $Ex_1 = 0$ determine static modes. The infinite eigenvalues of (E, A) with generalized eigenvectors x_k satisfying the relations $Ex_1 = 0$ and $Ex_k = x_{k-1}$ $(k \geq 2)$ create *impulsive modes*. The system has no impulsive mode if and only if rank $E = \deg |sE - A|$ $(\deg |zE - A|)$. The system is said to be *stable* if it is regular and has only decaying exponential modes and static modes (without impulsive modes). ∎

Lemma 1 (Weiertrass Form)[1, 2] If the descriptor system (2.1) is regular, then there exist two nonsingular matrices M and N such that

$$MEN = \begin{bmatrix} I_d & 0 \\ 0 & J \end{bmatrix}, \quad MAN = \begin{bmatrix} \Lambda & 0 \\ 0 & I_{n-d} \end{bmatrix} \tag{2.2}$$

where $d = \deg |zE - A|$, J is composed of Jordan blocks for the finite eigenvalues. If the system (2.1) is regular and there is no impulsive mode, then (2.2) holds with $d = r$ and $J = 0$. If the system (2.1) is stable, then (2.2) holds with $d = r$, $J = 0$ and furthermore Λ is Schur stable. ∎

Let the singular value decomposition (SVD) of E be

$$E = U \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} V^T, \quad E_{11} = \text{diag}\{\sigma_1, \cdots, \sigma_r\} \tag{2.3}$$

where σ_i's are the singular values, U and V are orthonormal matrices ($U^T U = V^T V = I$). With the definitions

$$\bar{x} = V^T x \stackrel{\triangle}{=} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}, \quad U^T A V = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{2.4}$$

the difference equation in (2.1) (with $w = 0$) takes the form of

$$\begin{aligned} E_{11}\bar{x}_1(k+1) &= A_{11}\bar{x}_1(k) + A_{12}\bar{x}_2(k) \\ 0 &= A_{21}\bar{x}_1(k) + A_{22}\bar{x}_2(k)\,. \end{aligned} \tag{2.5}$$

It is easy to obtain from the above that the descriptor system is regular and has not impulsive modes if and only if A_{22} is nonsingular. Moreover, the system is stable if and only if A_{22} is nonsingular and furthermore $E_{11}^{-1}\left(A_{11}-A_{12}A_{22}^{-1}A_{21}\right)$ is Schur stable. This discussion will be used again in the next sections.

Definition 2: Given a positive scalar γ, if the linear descriptor system (2.1) is stable and satisfies

$$\sum_{j=0}^{k} z^T(j)z(j) \le \phi(x(0)) + \gamma^2 \sum_{j=0}^{k} w^T(j)w(j) \tag{2.6}$$

for any integer $k > 0$ and any l_2-bounded disturbance input w, with some nonnegative definite function $\phi(\cdot)$, then the descriptor system is said to be stable and have $\mathcal{L}_2$ gain less than γ. ∎

The above definition is a general one for nonlinear systems, and will be used later for switched descriptor systems.

3. Problem formulation

In this article, we consider the switched system composed of $\mathcal{N}$ linear discrete-time descriptor subsystems described by

$$\begin{cases} Ex(k+1) = A_i x(k) + B_i w(k) \\ \qquad z(k) = C_i x(k)\,, \end{cases} \tag{3.1}$$

where the vectors x, w, z and the descriptor matrix E are the same as in (2.1), the index i denotes the i-th subsystem and takes value in the discrete set $\mathcal{I} = \{1, 2, \cdots, \mathcal{N}\}$, and thus the matrices A_i, B_i, C_i together with E represent the dynamics of the i-th subsystem.

For the above switched system, we consider the stability and $\mathcal{L}_2$ gain properties under the assumption that all subsystems in (3.1) are stable and have $\mathcal{L}_2$ gain less than γ. As in the case of stability analysis for switched linear systems in state space representation, such an analysis problem is well posed (or practical) since a switched descriptor system can be unstable even if all the descriptor subsystems are stable and there is no variable (state) jump at the switching instants. Additionally, switchings between two subsystems can even result in impulse signals, even if the subsystems do not have impulsive modes themselves. This happens when the variable vector $x(k_r)$, where k_r is a switching instant, does not satisfy the algebraic equation required in the subsequent subsystem. In order to exclude this possibility, Ref. [19] proposed an additional condition involving consistency projectors. Here, as in most of the literature, we assume for simplicity that there is no impulse occurring with the variable (state) vector at every switching instant, and call such kind of switching *impulse-free*.

Definition 3: Given a switching sequence, the switched system (3.1) with $w = 0$ is said to be *stable* if starting from any initial value the system's trajectories converge to the origin exponentially, and the switched system is said to have $\mathcal{L}_2$ gain less than γ if the condition (2.6) is satisfied for any integer $k > 0$. ∎

In the end of this section, we state two analysis problems, which will be dealt with in Section 4 and 5, respectively.

Stability Analysis Problem: Assume that all the descriptor subsystems in (3.1) are stable. Establish the condition under which the switched system is stable under impulse-free arbitrary switching.

$\mathcal{L}_2$ Gain Analysis Problem: Assume that all the descriptor subsystems in (3.1) are stable and have $\mathcal{L}_2$ gain less than γ. Establish the condition under which the switched system is also stable and has $\mathcal{L}_2$ gain less than γ under impulse-free arbitrary switching.

Remark 1: There is a tacit assumption in the switched system (3.1) that the descriptor matrix E is the same in all the subsystems. Theoretically, this assumption is restrictive at present. However, as also discussed in [17, 18], the above problem settings and the results later can be applied to switching control problems for linear descriptor systems. This is the main motivation that we consider the same descriptor matrix E in the switched system. For example, if for a single descriptor system $Ex(k+1) = Ax(k) + Bu(k)$ where $u(k)$ is the control input, we have designed two stabilizing descriptor variable feedbacks $u = K_1x$, $u = K_2x$, and furthermore the switched system composed of the descriptor subsystems characterized by $(E, A + BK_1)$ and $(E, A + BK_2)$ are stable (and have $\mathcal{L}_2$ gain less than γ) under impulse-free arbitrary switching, then we can switch arbitrarily between the two controllers and thus can consider higher control specifications. This kind of requirement is very important when we want more flexibility for multiple control specifications in real applications. ∎

4. Stability analysis

In this section, we first state and prove the common quadratic Lyapunov function (CQLF) based stability condition for the switched descriptor system (3.1) (with $w = 0$), and then discuss the relation with the existing commutation condition.

4.1 CQLF based stability condition

Theorem 1: The switched system (3.1) (with $w = 0$) is stable under impulse-free arbitrary switching if there are nonsingular symmetric matrices $P_i \in \mathcal{R}^{n\times n}$ satisfying for $\forall i \in \mathcal{I}$ that

$$E^T P_i E \geq 0 \tag{4.1}$$

$$A_i^T P_i A_i - E^T P_i E < 0 \tag{4.2}$$

and furthermore

$$E^T P_i E = E^T P_j E\,, \ \ \forall i, j \in \mathcal{I},\ i \neq j. \tag{4.3}$$

Proof: The necessary condition for stability under arbitrary switching is that each subsystem should be stable. This is guaranteed by the two matrix inequalities (4.1) and (4.2) [20].
Since the rank of E is r, we first find nonsingular matrices M and N such that

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.4}$$

Then, we obtain from (4.1) that

$$(N^T E^T M^T)(M^{-T} P_i M^{-1})(MEN) = \begin{bmatrix} P_{11}^i & 0 \\ 0 & 0 \end{bmatrix} \geq 0\,, \tag{4.5}$$

where

$$M^{-T} P_i M^{-1} \triangleq \begin{bmatrix} P_{11}^i & P_{12}^i \\ (P_{12}^i)^T & P_{22}^i \end{bmatrix}. \tag{4.6}$$

Since P_i (and thus $M^{-T}P_iM^{-1}$) is symmetric and nonsingular, we obtain $P_{11}^i > 0$.

Again, we obtain from (4.3) that

$$(N^T E^T M^T)(M^{-T} P_i M^{-1})(MEN) = (N^T E^T M^T)(M^{-T} P_j M^{-1})(MEN), \tag{4.7}$$

and thus

$$\begin{bmatrix} P_{11}^i & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_{11}^j & 0 \\ 0 & 0 \end{bmatrix} \tag{4.8}$$

which leads to $P_{11}^i = P_{11}^j$, $\forall i, j \in \mathcal{I}$. From now on, we let $P_{11}^i = P_{11}$ for notation simplicity. Next, let

$$MA_iN = \begin{bmatrix} \bar{A}_{11}^i & \bar{A}_{12}^i \\ \bar{A}_{21}^i & \bar{A}_{22}^i \end{bmatrix} \tag{4.9}$$

and substitute it into the equivalent inequality of (4.2) as

$$(N^T A_i^T M^T)(M^{-T} P_i M^{-1})(MA_iN) - (N^T E^T M^T)(M^{-T} P_i M^{-1})(MEN) < 0 \tag{4.10}$$

to reach

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix} < 0, \tag{4.11}$$

where

$$\begin{aligned} \Lambda_{11} &= (\bar{A}_{11}^i)^T P_{11} \bar{A}_{11}^i - P_{11} + (\bar{A}_{21}^i)^T (P_{12}^i)^T \bar{A}_{11}^i + (\bar{A}_{11}^i)^T P_{12}^i \bar{A}_{21}^i + (\bar{A}_{21}^i)^T P_{22}^i \bar{A}_{21}^i \\ \Lambda_{12} &= (\bar{A}_{11}^i)^T P_{11} \bar{A}_{12}^i + (\bar{A}_{11}^i)^T P_{12}^i \bar{A}_{22}^i + (\bar{A}_{21}^i)^T (P_{12}^i)^T \bar{A}_{12}^i + (\bar{A}_{21}^i)^T P_{22}^i \bar{A}_{22}^i \\ \Lambda_{22} &= (\bar{A}_{12}^i)^T P_{11} \bar{A}_{12}^i + (\bar{A}_{22}^i)^T (P_{12}^i)^T \bar{A}_{12}^i + (\bar{A}_{12}^i)^T P_{12}^i \bar{A}_{22}^i + (\bar{A}_{22}^i)^T P_{22}^i \bar{A}_{22}^i . \end{aligned} \tag{4.12}$$

At this point, we declare $\bar{A}_{22}^i$ is nonsingular from $\Lambda_{22} < 0$. Otherwise, there is a nonzero vector v such that $\bar{A}_{22}^i v = 0$. Then, $v^T \Lambda_{22} v < 0$. However, by simple calculation,

$$v^T \Lambda_{22} v = v^T (\bar{A}_{12}^i)^T P_{11} \bar{A}_{12}^i v \geq 0 \tag{4.13}$$

since P_{11} is positive definite. This results in a contradiction.

Multiplying the left side of (4.11) by the nonsingular matrix $\begin{bmatrix} I & -(\bar{A}_{21}^i)^T (\bar{A}_{22}^i)^{-T} \\ 0 & I \end{bmatrix}$ and the right side by its transpose, we obtain

$$\begin{bmatrix} (\tilde{A}_{11}^i)^T P_{11} \tilde{A}_{11}^i - P_{11} & * \\ (*)^T & \Lambda_{22} \end{bmatrix} < 0, \tag{4.14}$$

where $\tilde{A}_{11}^i = \bar{A}_{11}^i - \bar{A}_{12}^i (\bar{A}_{22}^i)^{-1} \bar{A}_{21}^i$.
With the same nonsingular transformation $\bar{x}(k) = N^{-1} x(k) = [\bar{x}_1^T(k) \;\; \bar{x}_2^T(k)]^T$, $\bar{x}_1(k) \in \mathcal{R}^r$, all the descriptor subsystems in (3.1) take the form of

$$\begin{aligned} \bar{x}_1(k+1) &= \bar{A}_{11}^i \bar{x}_1(k) + \bar{A}_{12}^i \bar{x}_2(k) \\ 0 &= \bar{A}_{21}^i \bar{x}_1(k) + \bar{A}_{22}^i \bar{x}_2(k), \end{aligned} \tag{4.15}$$

which is equivalent to

$$\bar{x}_1(k+1) = \tilde{A}^i_{11}\bar{x}_1(k) \tag{4.16}$$

with $\bar{x}_2(k) = -(\bar{A}^i_{22})^{-1}\bar{A}^i_{21}\bar{x}_1(k)$. It is seen from (4.14) that

$$(\tilde{A}^i_{11})^T P_{11}\tilde{A}^i_{11} - P_{11} < 0\,, \tag{4.17}$$

which means that all $\tilde{A}^i_{11}$'s are Schur stable, and a common positive definite matrix P_{11} exists for stability of all the subsystems in (4.16). Therefore, $\bar{x}_1(k)$ converges to zero exponentially under impulse-free arbitrary switching. The $\bar{x}_2(k)$ part is dominated by $\bar{x}_1(k)$ and thus also converges to zero exponentially. This completes the proof. ∎

Remark 2: When $E = I$ and all the subsystems are Schur stable, the condition of Theorem 1 actually requires a common positive definite matrix P satisfying $A_i^T PA_i - P < 0$ for $\forall i \in \mathcal{I}$, which is exactly the existing stability condition for switched linear systems composed of $x(k+1) = A_i x(k)$ under arbitrary switching [12]. Thus, Theorem 1 is an extension of the existing result for switched linear state space subsystems in discrete-time domain. ∎

Remark 3: It can be seen from the proof of Theorem 1 that $\bar{x}_1^T P_{11}\bar{x}_1$ is a common quadratic Lyapunov function for all the subsystems (4.16). Since the exponential convergence of $\bar{x}_1$ results in that of $\bar{x}_2$, we can regard $\bar{x}_1^T P_{11}\bar{x}_1$ as a common quadratic Lyapunov function for the whole switched system. In fact, this is rationalized by the following equation.

$$\begin{aligned}
x^T E^T P_i E x &= (N^{-1}x)^T (MEN)^T (M^{-T} P_i M^{-1})(MEN)(N^{-1}x) \\
&= \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}^T \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P^i_{12} \\ (P^i_{12})^T & P^i_{22} \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \\
&= \bar{x}_1^T P_{11}\bar{x}_1
\end{aligned} \tag{4.18}$$

Therefore, although $E^T P_i E$ is not positive definite and neither is $V(x) = x^T E^T P_i E x$, we can regard this $V(x)$ as a common quadratic Lyapunov function for all the descriptor subsystems in discrete-time domain. ∎

Remark 4: The LMI conditions (4.1)-(4.3) include a nonstrict matrix inequality, which may not be easy to solve using the existing LMI Control Toolbox in Matlab. As a matter of fact, the proof of Theorem 1 suggested an alternative method for solving it in the framework of strict LMIs: (a) decompose E as in (4.4) using nonsingular matrices M and N; (b) compute MA_iN for $\forall i \in \mathcal{I}$ as in (4.9); (c) solve the strict LMIs (4.11) for $\forall i \in \mathcal{I}$ simultaneously with respect to $P_{11} > 0$, P^i_{12} and P^i_{22}; (d) compute the original P_i with $P_i = M^T \begin{bmatrix} P_{11} & P^i_{12} \\ (P^i_{12})^T & P^i_{22} \end{bmatrix} M$. ∎

Although we assumed in the above that the descriptor matrix is the same for all the subsystems (as mentioned in Remark 1), it can be seen from the proof of Theorem 1 that what we really need is the equation (4.4). Therefore, Theorem 1 can be extended to the case where the subsystem descriptor matrices are different as in the following corollary.

Corollary 1: Consider the switched system composed of $\mathcal{N}$ linear descriptor subsystems

$$E_i x(k+1) = A_i x(k)\,, \tag{4.19}$$

where E_i is the descriptor matrix of the ith subsystem and all the other notations are the same as before. Assume that all the descriptor matrices have the same rank r and there are common nonsingular matrices M and N such that

$$ME_iN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \ \forall i \in \mathcal{I}. \tag{4.20}$$

Then, the switched system (4.19) is stable under impulse-free arbitrary switching if there are symmetric nonsingular matrices $P_i \in \mathcal{R}^{n\times n}$ $(i = 1, \cdots, \mathcal{N})$ satisfying for $\forall i \in \mathcal{I}$

$$E_i^T P_i E_i \geq 0, \ A_i^T P_i A_i - E_i^T P_i E_i < 0 \tag{4.21}$$

and furthermore

$$E_i^T P_i E_i = E_j^T P_j E_j, \ \forall i, j \in \mathcal{I}, \ i \neq j. \tag{4.22}$$

∎

4.2 Relation with existing commutation condition

In this subsection, we consider the relation of Theorem 1 with the existing commutation condition proposed in [17].

Lemma 2:([17]) If all the descriptor subsystems are stable, and furthermore the matrices E, $A_1, \cdots, A_{\mathcal{N}}$ are commutative pairwise, then the switched system is stable under impulse-free arbitrary switching. ∎

The above lemma establishes another sufficient condition for stability of switched linear descriptor systems in the name of pairwise commutation. It is well known [12] that in the case of switched linear systems composed of the state space subsystems

$$x(k+1) = A_i x(k), \ \ i \in \mathcal{I}, \tag{4.23}$$

where all subsystems are Schur stable and the subsystem matrices commute pairwise ($A_iA_j = A_jA_i$, $\forall i, j \in \mathcal{I}$), there exists a common positive definite matrix P satisfying

$$A_i^T P A_i - P < 0. \tag{4.24}$$

One then tends to expect that if the commutation condition of Lemma 2 holds, then a common quadratic Lyapunov function $V(x) = x^T E^T P_i E x$ should exist satisfying the condition of Theorem 1. This is exactly established in the following theorem.

Theorem 2: If all the descriptor subsystems in (3.1) are stable, and furthermore the matrices E, $A_1, \cdots, A_{\mathcal{N}}$ are commutative pairwise, then there are nonsingular symmetric matrices P_i's $(i = 1, \cdots, \mathcal{N})$ satisfying (4.1)-(4.3), and thus the switched system is stable under impulse-free arbitrary switching.

Proof: For notation simplicity, we only prove the case of $\mathcal{N} = 2$. Since (E, A_1) is stable, according to Lemma 1, there exist two nonsingular matrices M, N such that

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \ MA_1N = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} \tag{4.25}$$

where Λ_1 is a Schur stable matrix. Here, without causing confusion, we use the same notations M, N as before. Defining

$$N^{-1}M^{-1} = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} \tag{4.26}$$

and substituting it into the commutation condition $EA_1 = A_1E$ with

$$(MEN)(N^{-1}M^{-1})(MA_1N) = (MA_1N)(N^{-1}M^{-1})(MEN)\,, \tag{4.27}$$

we obtain

$$\begin{bmatrix} W_1\Lambda_1 & W_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \Lambda_1 W_1 & 0 \\ W_3 & 0 \end{bmatrix}. \tag{4.28}$$

Thus, $W_1\Lambda_1 = \Lambda_1 W_1$, $W_2 = 0$, $W_3 = 0$.
Now, we use the same nonsingular matrices M, N for the transformation of A_2 and write

$$MA_2N = \begin{bmatrix} \Lambda_2 & X_1 \\ X_2 & X \end{bmatrix}. \tag{4.29}$$

According to another commutation condition $EA_2 = A_2E$,

$$\begin{bmatrix} W_1\Lambda_2 & W_1X_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \Lambda_2 W_1 & 0 \\ X_2W_1 & 0 \end{bmatrix} \tag{4.30}$$

holds, and thus $W_1\Lambda_2 = \Lambda_2 W_1$, $W_1X_1 = 0$, $X_2W_1 = 0$. Since NM is nonsingular and $W_2 = 0, W_3 = 0$, W_1 has to be nonsingular. We obtain then $X_1 = 0, X_2 = 0$. Furthermore, since (E, A_2) is stable, Λ_2 is Schur stable and X has to be nonsingular.
The third commutation condition $A_1A_2 = A_2A_1$ results in

$$\begin{bmatrix} \Lambda_1 W_1 \Lambda_2 & 0 \\ 0 & W_4X \end{bmatrix} = \begin{bmatrix} \Lambda_2 W_1 \Lambda_1 & 0 \\ 0 & XW_4 \end{bmatrix}. \tag{4.31}$$

We have $\Lambda_1 W_1 \Lambda_2 = \Lambda_2 W_1 \Lambda_1$. Combining with $W_1\Lambda_1 = \Lambda_1 W_1$, $W_1\Lambda_2 = \Lambda_2 W_1$, we obtain that

$$W_1\Lambda_1\Lambda_2 = \Lambda_1 W_1 \Lambda_2 = \Lambda_2 W_1 \Lambda_1 = W_1 \Lambda_2 \Lambda_1 \tag{4.32}$$

which implies Λ_1 and Λ_2 are commutative ($\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$) since W_1 is nonsingular.
To summarize the above discussion, we get to

$$MA_2N = \begin{bmatrix} \Lambda_2 & 0 \\ 0 & X \end{bmatrix}, \tag{4.33}$$

where Λ_2 is Schur stable, X is nonsingular, and $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$. According to the existing result [12], there is a common positive definite matrix P_{11} satisfying $\Lambda_i^T P_{11} \Lambda_i - P_{11} < 0$, $i = 1, 2$. Then, with the definition

$$P_1 = P_2 = M^T \begin{bmatrix} P_{11} & 0 \\ 0 & -I \end{bmatrix} M\,, \tag{4.34}$$

it is easy to confirm that

$$(MEN)^T(M^{-T}P_iM^{-1})(MEN) = \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \tag{4.35}$$

and

$$\begin{aligned} &(MA_1N)^T(M^{-T}P_1M^{-1})(MA_1N) - (MEN)^T(M^{-T}P_1M^{-1})(MEN) \\ &\quad = \begin{bmatrix} \Lambda_1^T P_{11}\Lambda_1 - P_{11} & 0 \\ 0 & -I \end{bmatrix} < 0\,, \\ &(MA_2N)^T(M^{-T}P_2M^{-1})(MA_2N) - (MEN)^T(M^{-T}P_2M^{-1})(MEN) \\ &\quad = \begin{bmatrix} \Lambda_2^T P_{11}\Lambda_2 - P_{11} & 0 \\ 0 & -X^TX \end{bmatrix} < 0\,. \end{aligned} \tag{4.36}$$

Since P_{11} is common for $i = 1, 2$ and N is nonsingular, (4.35) and (4.36) imply that the matrices in (4.34) satisfy the conditions (4.1)-(4.3). ∎

It is observed from (4.34) that when the conditions of Theorem 2 hold, we can further choose $P_1 = P_2$, which certainly satisfies (4.3). Since the actual Lyapunov function for the stable descriptor system $Ex[k+1] = A_ix[k]$ takes the form of $V(x) = x^TE^TP_iEx$ (as mentioned in Remark 3), the commutation condition is more conservative than the LMI condition in Theorem 1. However, we state for integrity the above observation as a corollary of Theorem 2.

Corollary 2: If all the descriptor subsystems in (3.1) are stable, and furthermore the matrices $E, A_1, \cdots, A_\mathcal{N}$ are commutative pairwise, then there is a nonsingular symmetric matrix P satisfying

$$E^TPE \geq 0 \tag{4.37}$$

$$A_i^TPA_i - E^TPE < 0\,, \tag{4.38}$$

and thus the switched system is stable under impulse-free arbitrary switching. ∎

5. $\mathcal{L}_2$ gain analysis

In this section, we extend the discussion of stability to $\mathcal{L}_2$ gain analysis fro the switched linear descriptor system under consideration.

Theorem 3: The switched system (3.1) is stable and the $\mathcal{L}_2$ gain is less than γ under impulse-free arbitrary switching if there are nonsingular symmetric matrices $P_i \in \mathcal{R}^{n\times n}$ satisfying for $\forall i \in \mathcal{I}$ that

$$E^TP_iE \geq 0 \tag{5.1}$$

$$\begin{bmatrix} A_i^TP_iA_i - E^TP_iE + C_i^TC_i & A_i^TP_iB_i \\ B_i^TP_iA_i & B_i^TP_iB_i - \gamma^2 I \end{bmatrix} < 0 \tag{5.2}$$

together with (4.3).

Proof: Since (5.1) is the same as (4.1) and (5.2) includes (4.2), we conclude from Theorem 1 that the switched descriptor system is exponentially stable under impulse-free arbitrary switching. What remains is to prove the $\mathcal{L}_2$ gain property.

Consider the Lyapunov function candidate $V(x) = x^T E^T P_i E x$, which is always nonnegative due to (5.1) and always continuous due to (4.3). Then, on any discrete-time interval where the i-th subsystem is activated, the difference of $V(x)$ along the system's trajectories satisfies

$$\begin{aligned} V(x(k+1)) - V(x(k)) &= x^T(k+1)E^T P_i E x(k+1) - x^T(k)E^T P_i E x(k) \\ &= (Ex(k+1))^T P_i (Ex(k+1)) - x^T(k)E^T P_i^T E x(k) \\ &= (A_i x(k) + B_i w(k))^T P_i (A_i x(k) + B_i w(k)) - x^T(k)E^T P_i^T E x(k) \\ &= \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^T \begin{bmatrix} A_i^T P_i A_i - E^T P_i E & A_i^T P_i B_i \\ B_i^T P_i A_i & B_i^T P_i B_i \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \\ &\le \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^T \begin{bmatrix} -C_i^T C_i & 0 \\ 0 & \gamma^2 I \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \\ &= -z^T(k)z(k) + \gamma^2 w^T(k)w(k), \end{aligned} \tag{5.3}$$

where the condition (5.2) was used in the inequality.

Now, for an impulse-free arbitrary piecewise constant switching signal and any given $k > 0$, suppose $k_1 < k_2 < \cdots < k_r$ ($r \ge 1$) be the switching points of the switching signal on the discrete-time interval $[0, k)$. Then, according to (5.3), we obtain

$$\begin{aligned} V(x(k+1)) - V(x(k_r^+)) &\le \sum_{j=k_r}^{k} \left\{ -z^T(j)z(j) + \gamma^2 w^T(j)w(j) \right\} \\ V(x(k_r^-)) - V(x(k_{r-1}^+)) &\le \sum_{j=k_{r-1}}^{k_r-1} \left\{ -z^T(j)z(j) + \gamma^2 w^T(j)w(j) \right\} \\ &\cdots\cdots \\ V(x(k_2^-)) - V(x(k_1^+)) &\le \sum_{j=k_1}^{k_2-1} \left\{ -z^T(j)z(j) + \gamma^2 w^T(j)w(j) \right\} \\ V(x(k_1^-)) - V(x(0)) &\le \sum_{j=0}^{k_1-1} \left\{ -z^T(j)z(j) + \gamma^2 w^T(j)w(j) \right\}, \end{aligned} \tag{5.4}$$

where

$$V(x(k_j^+)) = \lim_{k \to k_j + 0} V(x(k)), \quad V(x(k_j^-)) = \lim_{k \to k_j - 0} V(x(k)). \tag{5.5}$$

However, due to the condition (4.3), we obtain $V(x(k_j^+)) = V(x(k_j^-))$ at all switching instants. Therefore, summing up all the inequalities of (5.4) results in

$$V(x(k+1)) - V(x(0)) \le \sum_{j=0}^{k} \left\{ -z^T(j)z(j) + \gamma^2 w^T(j)w(j) \right\}. \tag{5.6}$$

Since $V(x(k+1)) \geq 0$, we obtain that

$$\sum_{j=0}^{k} z^T(j)z(j) \leq V(x(0)) + \gamma^2 \sum_{j=0}^{k} w^T(j)w(j)\,, \tag{5.7}$$

which implies the $\mathcal{L}_2$ gain of the switched system is less than γ. ∎

Remark 5: When $E = I$, the conditions (5.1)-(5.2) and (4.3) require a common positive definite matrix P satisfying

$$\begin{bmatrix} A_i^T P_i A_i - P_i + C_i^T C_i & A_i^T P_i B_i \\ B_i^T P_i A_i & B_i^T P_i B_i - \gamma^2 I \end{bmatrix} < 0 \tag{5.8}$$

for all $\forall i \in \mathcal{I}$, which is the same as in [15]. Thus, Theorem 3 extended the $\mathcal{L}_2$ gain analysis result from switched time space systems to switched descriptor systems in discrete-time domain. In addition, it can be seen from the proof that $V(x) = x^T E^T P_i E x$ plays the important role of a common quadratic Lyapunov function for stability and $\mathcal{L}_2$ gain γ of all the descriptor subsystems. ∎

6. Concluding remarks

We have established a unified approach to stabilility and $\mathcal{L}_2$ gain analysis for switched linear discrete-time descriptor systems under impulse-free arbitrary switching. More precisely, we have shown that if there is a common quadratic Lyapunov function for stability of all subsystems, then the switched system is stable under impulse-free arbitrary switching. Furthermore, we have extended the results to $\mathcal{L}_2$ gain analysis of the switched descriptor systems, also in the name of common quadratic Lyapunov function approach. As also mentioned in the remarks, the common quadratic Lyapunov functions proposed are not positive definite with respect to all states, but they actually play the role of a Lyapunov function as in classical Lyapunov stability theory. The approach in this article is unified in the sense that it is valid for both continuous-time [21] and discrete-time systems, and it takes almost the same form in both stability and $\mathcal{L}_2$ gain analysis.

7. Acknowledgment

The authors would like to thank Prof. Xuping Xu with California Baptist University, USA, Mr. Ryuuen Kou, Prof. Joe Imae and Prof. Tomoaki Kobayashi with Osaka Prefecture University, Japan, for valuable discussions. This research has been supported in part by the Japan Ministry of Education, Sciences and Culture under Grant-in-Aid for Scientific Research (C) 21560471.

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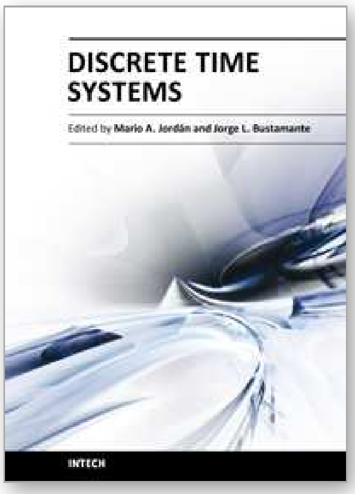

Discrete Time Systems
Edited by Dr. Mario Alberto JordÃ¡n

ISBN 978-953-307-200-5
Hard cover, 526 pages
Publisher InTech
Published online 26, April, 2011
Published in print edition April, 2011

Discrete-Time Systems comprehend an important and broad research field. The consolidation of digital-based computational means in the present, pushes a technological tool into the field with a tremendous impact in areas like Control, Signal Processing, Communications, System Modelling and related Applications. This book attempts to give a scope in the wide area of Discrete-Time Systems. Their contents are grouped conveniently in sections according to significant areas, namely Filtering, Fixed and Adaptive Control Systems, Stability Problems and Miscellaneous Applications. We think that the contribution of the book enlarges the field of the Discrete-Time Systems with signification in the present state-of-the-art. Despite the vertiginous advance in the field, we also believe that the topics described here allow us also to look through some main tendencies in the next years in the research area.

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