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Robust Control Design of Uncertain Discrete-Time Systems with Delays

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1. Introduction

When we consider control problems of physical systems, we often see time-delay in the process of control algorithms and the transmission of information. Time-delay often appear in many practical systems and mathematical formulations such as electrical system, mechanical system, biological system, and transportation system. Hence, a system with time-delay is a natural representation for them, and its analysis and synthesis are of theoretical and practical importance. In the past decades, research on continuous-time delay systems has been active. Difficulty that arises in continuous time-delay system is that it is infinite dimensional and a corresponding controller can be a memory feedback. This class of controllers may minimize a certain performance index, but it is difficult to implement it to practical systems due to a memory feedback. To overcome such a difficulty, a memoryless controller is used for time-delay systems. In the last decade, sufficient stability conditions have been given via linear matrix inequalities (LMIs), and stabilization methods by memoryless controllers have been investigated by many researchers. Since Li and de Souza considered robust stability and stabilization problems in (8), less conservative robust stability conditions for continuous time-delay systems have been obtained ((7), (11)). Recently, H_∞ disturbance attenuation conditions have also been given ((10), (15), (16)).

On the other hand, research on discrete-time delay systems has not attracted as much attention as that of continuous-time delay systems. In addition, most results have focused on state feedback stabilization of discrete-time systems with time-varying delays. Only a few results on observer design of discrete-time systems with time-varying delays have appeared in the literature(for example, (9)). The results in (3), (12), (14), (18) considered discrete-time systems with time-invariant delays. Gao and Chen (4), Hara and Yoneyama (5), (6) gave robust stability conditions. Fridman and Shaked (1) solved a guaranteed cost control problem. Fridman and Shaked (2), Yoneyama (17), Zhang and Han (19) considered the H_∞ disturbance attenuation. They have given sufficient conditions via LMIs for corresponding control problems. Nonetheless, their conditions still show the conservatism. Hara and Yoneyama (5) and Yoneyama (17) gave least conservative conditions but their conditions require many LMI slack variables, which in turn require a large amount of computations. Furthermore, to authors' best knowledge, few results on robust observer design problem for uncertain discrete-time systems with time-varying delays have given in the literature.

In this paper, we consider the stabilization for a nominal discrete-time system with time-varying delay and robust stabilization for uncertain system counterpart. The system under consideration has time-varying delays in state, control input and output measurement. First, we obtain a stability condition for a nominal time-delay system. To this end, we define

a Lyapunov function and use Leibniz-Newton formula and free weighting matrix method. These methods are known to reduce the conservatism in our stability condition, which are expressed as linear matrix inequality. Based on such a stability condition, a state feedback design method is proposed. Then, we extend our stabilization result to robust stabilization for uncertain discrete-time systems with time-varying delay. Next, we consider observer design and robust observer design. Similar to a stability condition, we obtain a condition such that the error system, which comes from the original system and its observer, is asymptotically stable. Using a stability condition of the error system, we proposed an observer design method. Furthermore, we give a robust observer design method for an uncertain time-delay system. Finally, we give some numerical examples to illustrate our results and to compare with existing results.

2. Time-delay systems

Consider the following discrete-time system with a time-varying delay and uncertainties in the state and control input.

$$x(k+1) = (A + \Delta A)x(k) + (A_d + \Delta A_d)x(k - d_k) + (B + \Delta B)u(k)$$
$$+ (B_d + \Delta B_d)u(k - d_k) \tag{1}$$

where $x(k) \in \Re^n$ is the state and $u(k) \in \Re^m$ is the control. A, A_d, B and B_d are system matrices with appropriate dimensions. d_k is a time-varying delay and satisfies $0 \le d_m \le d_k \le d_M$ and $d_{k+1} \le d_k$ where d_m and d_M are known constants. Uncertain matrices are of the form

$$\begin{bmatrix} \Delta A & \Delta A_d & \Delta B & \Delta B_d \end{bmatrix} = HF(k) \begin{bmatrix} E & E_d & E_1 & E_b \end{bmatrix} \tag{2}$$

where $F(k) \in \Re^{l \times j}$ is an unknown time-varying matrix satisfying $F^T(k)F(k) \le I$ and H, E, E_d, E_1 and E_b are constant matrices of appropriate dimensions.

Definition 2.1. *The system (1) is said to be robustly stable if it is asymptotically stable for all admissible uncertainties (2).*

When we discuss a nominal system, we consider the following system.

$$x(k+1) = Ax(k) + A_d x(k - d_k) + Bu(k) + B_d u(k - d_k). \tag{3}$$

Our problem is to find a control law which makes the system (1) or (3) robustly stable. Let us now consider the following memoryless feedback:

$$u(k) = Kx(k) \tag{4}$$

where K is a control gain to be determined. Applying the control (4) to the system (1), we have the closed-loop system

$$x(k+1) = ((A + \Delta A) + (B + \Delta B)K)x(k) + ((A_d + \Delta A_d) + (B_d + \Delta B_d)K)x(k - d_k). \tag{5}$$

For the nominal case, we have

$$x(k+1) = (A + BK)x(k) + (A_d + B_d K)x(k - d_k). \tag{6}$$

In the following section, we consider the robust stability of the closed-loop system (5) and the stability of the closed-loop system (6).
The following lemma is useful to prove our results.

Lemma 2.2. *((13)) Given matrices $Q = Q^T$, H, E and $R = R^T > 0$ with appropriate dimensions.*

$$Q + HF(k)E + E^T F^T(k)H^T < 0$$

for all $F(k)$ satisfying $F^T(k)F(k) \leq R$ if and only if there exists a scalar $\varepsilon > 0$ such that

$$Q + \frac{1}{\varepsilon}HH^T + \varepsilon E^T RE < 0.$$

3. Stability analysis

This section analyzes the stability and robust stability of discrete-time delay systems. Section 3.1 gives a stability condition for nominal systems and Section 3.2 extends the stability result to a case of robust stability.

3.1 Stability for nominal systems

Stability conditions for discrete-time delay system (6) are given in the following theorem.

Theorem 3.1. *Given integers d_m and d_M, and control gain K. Then, the time-delay system (6) is asymptotically stable if there exist matrices $P_1 > 0$, $P_2 > 0$, $Q_1 > 0$, $Q_2 > 0$, $S > 0$, $M > 0$,*

$$L = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{bmatrix}, \ N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \end{bmatrix}, \ T = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix}$$

satisfying

$$\Phi = \begin{bmatrix} \Phi_1 + \Xi_L + \Xi_L^T + \Xi_N + \Xi_N^T + \Xi_T + \Xi_T^T & \sqrt{d_M}Z \\ \sqrt{d_M}Z^T & -S \end{bmatrix} < 0 \tag{7}$$

where

$$\Phi_1 = \begin{bmatrix} P_1 & 0 & 0 & 0 & 0 \\ 0 & \Phi_{22} & 0 & 0 & 0 \\ 0 & 0 & -Q_1 - M & 0 & 0 \\ 0 & 0 & 0 & \Phi_{44} & -P_2 \\ 0 & 0 & 0 & -P_2 & P_2 - Q_2 \end{bmatrix},$$

$$\Phi_{22} = -P_1 + Q_1 + (d_M - d_m + 1)M,$$

$$\Phi_{44} = P_2 + Q_2 + d_M S,$$

$$Z = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -P_2 \\ P_2 \end{bmatrix} + N,$$

$$\Xi_L = \begin{bmatrix} L & -L & 0 & -L & 0 \end{bmatrix},$$

$$\Xi_N = \begin{bmatrix} 0 & N & -N & 0 & 0 \end{bmatrix},$$

$$\Xi_T = \begin{bmatrix} T & -T(A + BK) & -T(A_d + B_d K) & 0 & 0 \end{bmatrix}.$$

Proof: First, we note from Leibniz-Newton formula that

$$2\xi^T(k)L[x(k+1) - x(k) - e(k)] = 0, \tag{8}$$

$$2\xi^T(k)N[x(k) - x(k-d_k) - \sum_{i=k-d_k}^{k-1} e(i)] = 0 \tag{9}$$

where $e(k) = x(k+1) - x(k)$ and

$$\xi^T(k) = [x^T(k+1) \; x^T(k) \; x^T(k-d_k) \; e^T(k) \; e^T(k-d_k)].$$

It is also true that

$$2\xi^T(k)T[x(k+1) - (A+BK)x(k) - (A_d + B_dK)x(k-d_k)] = 0. \tag{10}$$

Now, we consider a Lyapunov function

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k)$$

where

$$V_1(k) = x^T(k)P_1x(k) + \sum_{i=k-d_k}^{k-1} e^T(i)P_2 \sum_{i=k-d_k}^{k-1} e(i),$$

$$V_2(k) = \sum_{i=k-d_k}^{k-1} x^T(i)Q_1x(i) + \sum_{i=k-d_k}^{k-1} e^T(i)Q_2e(i),$$

$$V_3(k) = \sum_{i=-d_k}^{-1} \sum_{j=k+i}^{k-1} e^T(j)Se(j),$$

$$V_4(k) = \sum_{j=-d_M}^{-d_m} \sum_{i=k+j}^{k-1} x^T(i)Mx(i),$$

and P_1, P_2, Q_1, Q_2, S and M are positive definite matrices to be determined. Then, we calculate the difference $\Delta V = V(k+1) - V(k)$ and add the left-hand-side of equations (8)-(10). Since $\Delta V_i(k)$, $i = 1, \cdots, 4$ are calculated as follows;

$$\Delta V_1(k) = x^T(k+1)P_1x(k+1) + \sum_{i=k+1-d_{k+1}}^{k} e^T(i)P_2 \sum_{i=k+1-d_{k+1}}^{k} e(i)$$

$$-x^T(k)P_1x(k) - \sum_{i=k-d_k}^{k-1} e^T(i)P_2 \sum_{i=k-d_k}^{k-1} e(i)$$

$$\leq x^T(k+1)P_1x(k+1) - x^T(k)P_1x(k) + e^T(k)P_2e(k)$$

$$-2e^T(k)P_2e(k-d_k) + 2e^T(k)P_2 \sum_{i=k-d_k}^{k-1} e(i)$$

$$+e^T(k-d_k)P_2e(k-d_k) - 2e^T(k-d_k)P_2 \sum_{i=k-d_k}^{k-1} e(i),$$

$$\Delta V_2(k) = \sum_{i=k+1-d_{k+1}}^{k} x^T(i)Q_1 x(i) + \sum_{i=k+1-d_{k+1}}^{k} e^T(i)Q_2 e(i)$$
$$- \sum_{i=k-d_k}^{k-1} x^T(i)Q_1 x(i) - \sum_{i=k-d_k}^{k-1} e^T(i)Q_2 e(i)$$
$$\leq x^T(k)Q_1 x(k) + e^T(k)Q_2 e(k) - x^T(k-d_k)Q_1 x(k-d_k)$$
$$- e^T(k-d_k)Q_2 e(k-d_k),$$
$$\Delta V_3(k) = d_{k+1}e^T(k)Se(k) - \sum_{i=k-d_{k+1}}^{k-1} e^T(i)Se(i)\cdots - \sum_{i=k-d_k}^{k-1} e^T(i)Se(i)$$
$$\leq d_M e^T(k)Se(k) - \sum_{i=k-d_k}^{k-1} e^T(i)Se(i),$$
$$\Delta V_4(k) = (d_M - d_m + 1)x^T(k)Mx(k) - \sum_{i=k-d_M+1}^{k-d_m} x^T(i)Mx(i)$$
$$\leq (d_M - d_m + 1)x^T(k)Mx(k) - x^T(k-d_k)Mx(k-d_k),$$

we have

$$\Delta V(k) = \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + \Delta V_4(k)$$
$$\leq \xi^T(k)[\Phi_1 + \Xi_L + \Xi_L^T + \Xi_N + \Xi_N^T + \Xi_T + \Xi_T^T]\xi(k) + \sum_{i=k-d_k}^{k-1} \xi^T(k)ZS^{-1}Z^T\xi(k)$$
$$- \sum_{i=k-d_k}^{k-1} (\xi^T(k)Z + e^T(i)S)S^{-1}(Z^T\xi(k) + Se(i))$$
$$\leq \xi^T(k)[\Phi_1 + \Xi_L + \Xi_L^T + \Xi_N + \Xi_N^T + \Xi_T + \Xi_T^T + d_M ZS^{-1}Z^T]\xi(k)$$

If (7) is satisfied, by Schur complement formula, we have $\Phi_1 + \Xi_L + \Xi_L^T + \Xi_N + \Xi_N^T + \Xi_T + \Xi_T^T + d_M ZS^{-1}Z^T < 0$. It follows that $\Delta V(k) < 0$ and hence the proof is completed.

Remark 3.2. *We employ* $\sum_{i=k-d_k}^{k-1} (\star)$ *in our Lyapunov function instead of* $\sum_{i=k-d_M}^{k-1} (\star)$. *This gives a less conservative stability condition.*

3.2 Robust stability for uncertain systems
By extending Theorem 3.1, we obtain a condition for robust stability of uncertain system (5).

Theorem 3.3. *Given integers d_m and d_M, and control gain K. Then, the time-delay system (5) is robustly stable if there exist matrices $P_1 > 0$, $P_2 > 0$, $Q_1 > 0$, $Q_2 > 0$, $S > 0$, $M > 0$,*

$$L = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{bmatrix}, \ N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \end{bmatrix}, \ T = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix}$$

and a scalar $\lambda > 0$ satisfying

$$\Pi = \begin{bmatrix} \Phi + \lambda \bar{E}^T \bar{E} & \bar{H}^T \\ \bar{H} & -\lambda I \end{bmatrix} < 0, \tag{11}$$

where Φ is given in Theorem 3.1, and

$$\bar{H} = \begin{bmatrix} -H^T T_1^T & -H^T T_2^T & -H^T T_3^T & -H^T T_4^T & -H^T T_5^T & 0 \end{bmatrix},$$

and

$$\bar{E} = \begin{bmatrix} 0 & E + E_1 K & E_d + E_b K & 0 & 0 & 0 \end{bmatrix}.$$

Proof: Replacing A, A_d, B and B_d in (7) with $A + HF(k)E$, $A_d + HF(k)E_d$, $B + HF(k)E_1$ and $B + HF(k)E_b$, respectively, we obtain a robust stability condition for the system (5).

$$\Phi + \bar{H}^T F(k) \bar{E} + \bar{E}^T F^T(k) \bar{H} < 0 \tag{12}$$

By Lemma 2.2, a necessary and sufficient condition that guarantees (12) is that there exists a scalar $\lambda > 0$ such that

$$\Phi + \lambda \bar{E}^T \bar{E} + \frac{1}{\lambda} \bar{H}^T \bar{H} < 0 \tag{13}$$

Applying Schur complement formula, we can show that (13) is equivalent to (11).

4. State feedback sabilization

This section proposes a state feedback stabilization method for the uncertain discrete-time delay system (1). First, stabilization of nominal system is considered in Section 4.1. Then, robust stabilization is proposed in Section 4.2

4.1 Stabilization

First, we consider stabilization for the nominal system (3). Our problem is to find a control gain K such that the closed-loop system (6) is asymptotically stable. Unfortunately, Theorem 3.1 does not give LMI conditions to find K. Hence, we must look for another method.

Theorem 4.1. *Given integers d_m and d_M, and scalars ρ_i, $i = 1, \cdots, 5$. Then, the controller (4) asymptotically stabilizes the time-delay system (3) if there exist matrices $\bar{P}_1 > 0$, $\bar{P}_2 > 0$, $\bar{Q}_1 > 0$, $\bar{Q}_2 > 0$, $\bar{S} > 0$, $\bar{M} > 0$, G, Y*

$$\bar{L} = \begin{bmatrix} \bar{L}_1 \\ \bar{L}_2 \\ \bar{L}_3 \\ \bar{L}_4 \\ \bar{L}_5 \end{bmatrix}, \quad \bar{N} = \begin{bmatrix} \bar{N}_1 \\ \bar{N}_2 \\ \bar{N}_3 \\ \bar{N}_4 \\ \bar{N}_5 \end{bmatrix},$$

satisfying

$$\Psi = \begin{bmatrix} \Psi_1 + \Theta_L + \Theta_L^T + \Theta_N + \Theta_N^T + \Theta_T + \Theta_T^T & \sqrt{d_M} Z \\ \sqrt{d_M} \bar{Z}^T & -\bar{S} \end{bmatrix} < 0 \tag{14}$$

where

$$\Psi_1 = \begin{bmatrix} \bar{P}_1 & 0 & 0 & 0 & 0 \\ 0 & \Psi_{22} & 0 & 0 & 0 \\ 0 & 0 & -\bar{Q}_1 - \bar{M} & 0 & 0 \\ 0 & 0 & 0 & \Psi_{44} & -\bar{P}_2 \\ 0 & 0 & 0 & -\bar{P}_2 & \bar{P}_2 - \bar{Q}_2 \end{bmatrix},$$

$$\Psi_{22} = -\bar{P}_1 + \bar{Q}_1 + (d_M - d_m + 1)\bar{M},$$
$$\Psi_{44} = \bar{P}_2 + \bar{Q}_2 + d_M \bar{S},$$

$$\bar{Z} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\bar{P}_2 \\ \bar{P}_2 \end{bmatrix} + \bar{N},$$

$$\Theta_L = \begin{bmatrix} \bar{L} & -\bar{L} & 0 & -\bar{L} & 0 \end{bmatrix},$$
$$\Theta_N = \begin{bmatrix} 0 & \bar{N} & -\bar{N} & 0 & 0 \end{bmatrix},$$

$$\Theta_T = \begin{bmatrix} \rho_1 Y^T & -\rho_1(AY^T + BG) & -\rho_1(A_d Y^T + B_d G) & 0 & 0 \\ \rho_2 Y^T & -\rho_2(AY^T + BG) & -\rho_2(A_d Y^T + B_d G) & 0 & 0 \\ \rho_3 Y^T & -\rho_3(AY^T + BG) & -\rho_3(A_d Y^T + B_d G) & 0 & 0 \\ \rho_4 Y^T & -\rho_4(AY^T + BG) & -\rho_4(A_d Y^T + B_d G) & 0 & 0 \\ \rho_5 Y^T & -\rho_5(AY^T + BG) & -\rho_5(A_d Y^T + B_d G) & 0 & 0 \end{bmatrix}.$$

In this case, a controller gain in the controller (4) is given by

$$K = GY^{-T} \tag{15}$$

Proof: Let $T_i = \rho_i Y^{-1}$, $i = 1, \cdots, 5$ where each ρ_i is given. We substitute them into (7). Then, we calculate $\Psi = \Sigma \Phi \Sigma^T$ with $\Sigma = \text{diag}[Y \ Y \ Y \ Y \ Y]$. Defining $\bar{P}_i = YP_i Y^T$, $\bar{Q}_i = YQ_i Y^T$, $i = 1, 2$, $\bar{S} = YSY^T$, $\bar{M} = YMY^T$, $\bar{L} = YLY^T$, $\bar{N} = YNY^T$, we obtain $\Theta < 0$ in (14) where we let $G = KY^T$. If the condition (14) hold, state feedback control gain matrix K is obviously given by (15).

Remark 4.2. *Should Y be singular, Let $\bar{L}_1 = 0$. In this case, it follows from $(1,1)$-block of Ψ that $\bar{P}_1 + \rho_1(Y + Y^T) < 0$. Then, if (14) holds, Y must be nonsingular.*

4.2 Robust stabilization
In a similar way to robust stability, we extend a stabilization result in the previous section to robust stabilization for uncertain discrete-time delay system (1).

Theorem 4.3. *Given integers d_m and d_M, and scalars ρ_i, $i = 1, \cdots, 5$. Then, the controller (4) robustly stabilizes the time-delay system (1) if there exist matrices $\bar{P}_1 > 0$, $\bar{P}_2 > 0$, $\bar{Q}_1 > 0$, $\bar{Q}_2 > 0$, $\bar{S} > 0$, $\bar{M} > 0$, G, Y*

$$\bar{L} = \begin{bmatrix} \bar{L}_1 \\ \bar{L}_2 \\ \bar{L}_3 \\ \bar{L}_4 \\ \bar{L}_5 \end{bmatrix}, \ \bar{N} = \begin{bmatrix} \bar{N}_1 \\ \bar{N}_2 \\ \bar{N}_3 \\ \bar{N}_4 \\ \bar{N}_5 \end{bmatrix},$$

and a scalar $\lambda > 0$ satisfying

$$\Lambda = \begin{bmatrix} \Psi + \lambda \hat{H}^T \hat{H} & \hat{E}^T \\ \hat{E} & -\lambda I \end{bmatrix} < 0, \tag{16}$$

where

$$\hat{H} = \begin{bmatrix} -\rho_1 H^T & -\rho_2 H^T & -\rho_3 H^T & -\rho_4 H^T & -\rho_5 H^T & 0 \end{bmatrix},$$

and

$$\hat{E} = \begin{bmatrix} 0 & EY^T + E_1 G & E_d Y^T + E_b G & 0 & 0 & 0 \end{bmatrix}.$$

In this case, a controller gain in the controller (4) is given by (15).

Proof: Replacing A, A_d, B and B_d in (14) with $A + HF(k)E$, $A_d + HF(k)E_d$, $B + HF(k)E_1$ and $B + HF(k)E_b$, respectively, we obtain robust stability conditions for the system (1):

$$\Psi + \bar{H}^T F(k) \bar{E} + \bar{E}^T F^T(k) \bar{H} < 0 \tag{17}$$

By Lemma 2.2, a necessary and sufficient condition that guarantees (17) is that there exists a scalar $\lambda > 0$ such that

$$\Psi + \lambda \bar{H}^T \bar{H} + \frac{1}{\lambda} \bar{E}^T \bar{E} < 0 \tag{18}$$

Applying Schur complement formula, we can show that (18) is equivalent to (16).

5. State estimation

All the information on the state variables of the system is not always available in a physical situation. In this case, we need to estimate the values of the state variables from all the available information on the output and input. In the following, we make analysis of the existence of observers. Section 5.1 analyzes the observer of a nominal system, and Section 5.2 considers the robust observer analysis of an uncertain system.

5.1 Observer analysis

Using the results in the previous sections, we consider an observer design for the system (1), which estimates the state variables of the system using measurement outputs.

$$x(k+1) = (A + \Delta A)x(k) + (A_d + \Delta A_d)x(k - d_k), \tag{19}$$

$$y(k) = (C + \Delta C)x(k) + (C_d + \Delta C_d)x(k - d_k) \tag{20}$$

where uncertain matrices are of the form:

$$\begin{bmatrix} \Delta A & \Delta A_d \\ \Delta C & \Delta C_d \end{bmatrix} = \begin{bmatrix} H \\ H_2 \end{bmatrix} F(k) \begin{bmatrix} E & E_d \end{bmatrix}$$

where $F(k) \in \Re^{l \times j}$ is an unknown time-varying matrix satisfying $F^T(k)F(k) \leq I$ and H, H_2, E and E_d are constant matrices of appropriate dimensions.

We consider the following system to estimate the state variables:

$$\hat{x}(k+1) = A\hat{x}(k) + \bar{K}(y(k) - C\hat{x}(k)) \tag{21}$$

where $\hat{x}$ is the estimated state and $\bar{K}$ is an observer gain to be determined. It follows from (19), (20) and (21) that

$$x_c(k+1) = (\tilde{A} + \tilde{H}F(k)\tilde{E})x_c(k) + (\tilde{A}_d + \tilde{H}F(k)\tilde{E}_d)x_c(k-d_k).\qquad(22)$$

where $x_c^T = [x^T\ e^T]^T$, $e(k) = x(k) - \hat{x}(k)$ and

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & A - \bar{K}C \end{bmatrix},\ \tilde{A}_d = \begin{bmatrix} A_d & 0 \\ A_d - \bar{K}C_d & 0 \end{bmatrix},$$

$$\tilde{H} = \begin{bmatrix} H \\ H - \bar{K}H_2 \end{bmatrix},\ \tilde{E} = \begin{bmatrix} E & 0 \end{bmatrix},\ \tilde{E}_d = \begin{bmatrix} E_d & 0 \end{bmatrix}.$$

We shall find conditions for (22) to be robustly stable. In this case, the system (21) becomes an observer for the system (19) and (20).

For nominal case, we have

$$x_c(k+1) = \tilde{A}x_c(k) + \tilde{A}_d x_c(k-d_k).\qquad(23)$$

We first consider the asymptotic stability of the system (23). The following theorem gives conditions for the system (23) to be asymptotically stable.

Theorem 5.1. *Given integers d_m and d_M, and observer gain $\bar{K}$. Then, the system (23) is asymptotically stable if there exist matrices $0 < \tilde{P}_1 \in \Re^{2n\times 2n}$, $< \tilde{P}_2 \in \Re^{2n\times 2n}$, $0 < \check{Q}_1 \in \Re^{2n\times 2n}$, $0 < \check{Q}_2 \in \Re^{2n\times 2n}$, $0 < \tilde{S} \in \Re^{2n\times 2n}$, $0 < \tilde{M} \in \Re^{2n\times 2n}$,*

$$\check{L} = \begin{bmatrix} \check{L}_1 \\ \check{L}_2 \\ \check{L}_3 \\ \check{L}_4 \\ \check{L}_5 \end{bmatrix} \in \Re^{10n\times 2n},\ \tilde{N} = \begin{bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \\ \tilde{N}_3 \\ \tilde{N}_4 \\ \tilde{N}_5 \end{bmatrix} \in \Re^{10n\times 2n},\ \tilde{T} = \begin{bmatrix} \tilde{T}_1 \\ \tilde{T}_2 \\ \tilde{T}_3 \\ \tilde{T}_4 \\ \tilde{T}_5 \end{bmatrix} \in \Re^{10n\times 2n}$$

satisfying

$$\Phi = \begin{bmatrix} \check{\Phi}_1 + \tilde{\Xi}_L + \tilde{\Xi}_L^T + \tilde{\Xi}_N + \tilde{\Xi}_N^T + \tilde{\Xi}_T + \tilde{\Xi}_T^T & \sqrt{d_M}\tilde{Z} \\ \sqrt{d_M}\tilde{Z}^T & -\tilde{S} \end{bmatrix} < 0\qquad(24)$$

where

$$\check{\Phi}_1 = \begin{bmatrix} \tilde{P}_1 & 0 & 0 & 0 & 0 \\ 0 & \check{\Phi}_{22} & 0 & 0 & 0 \\ 0 & 0 & -\check{Q}_1 - \tilde{M} & 0 & 0 \\ 0 & 0 & 0 & \check{\Phi}_{44} & -\tilde{P}_2 \\ 0 & 0 & 0 & -\tilde{P}_2 & \tilde{P}_2 - \check{Q}_2 \end{bmatrix},$$

$$\check{\Phi}_{22} = -\tilde{P}_1 + \check{Q}_1 + (d_M - d_m + 1)\tilde{M},$$

$$\check{\Phi}_{44} = \tilde{P}_2 + \check{Q}_2 + d_M\tilde{S},$$

$$\tilde{Z} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\tilde{P}_2 \\ \tilde{P}_2 \end{bmatrix} + \tilde{N},$$

$$\tilde{\Xi}_L = \begin{bmatrix} \check{L} & -\check{L} & 0 & -\check{L} & 0 \end{bmatrix},$$

$$\tilde{\Xi}_N = \begin{bmatrix} 0 & \tilde{N} & -\tilde{N} & 0 & 0 \end{bmatrix},$$

$$\tilde{\Xi}_T = \begin{bmatrix} \tilde{T} & -\tilde{T}\tilde{A} & -\tilde{T}\tilde{A}_d & 0 & 0 \end{bmatrix}.$$

Proof: We follow similar lines of proof of Theorem 3.1 for the stability of the system (23). Then, the result is straightforward.

5.2 Robust observer analysis

Now, we extend the result for the uncertain system (23).

Theorem 5.2. *Given integers d_m and d_M, and observer gain $\bar{K}$. Then, the system (22) is robustly stable if there exist matrices $0 < \tilde{P}_1 \in \Re^{2n \times 2n}$, $< \tilde{P}_2 \in \Re^{2n \times 2n}$, $0 < \tilde{Q}_1 \in \Re^{2n \times 2n}$, $0 < \tilde{Q}_2 \in \Re^{2n \times 2n}$, $0 < \tilde{S} \in \Re^{2n \times 2n}$, $0 < \tilde{M} \in \Re^{2n \times 2n}$,*

$$\tilde{L} = \begin{bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \\ \tilde{L}_3 \\ \tilde{L}_4 \\ \tilde{L}_5 \end{bmatrix} \in \Re^{10n \times 2n}, \ \tilde{N} = \begin{bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \\ \tilde{N}_3 \\ \tilde{N}_4 \\ \tilde{N}_5 \end{bmatrix} \in \Re^{10n \times 2n}, \ \tilde{T} = \begin{bmatrix} \tilde{T}_1 \\ \tilde{T}_2 \\ \tilde{T}_3 \\ \tilde{T}_4 \\ \tilde{T}_5 \end{bmatrix} \in \Re^{10n \times 2n}$$

and a scalar $\lambda > 0$ satisfying

$$\tilde{\Pi} = \begin{bmatrix} \tilde{\Phi} + \lambda \hat{E}^T \hat{E} & \hat{H}^T \\ \hat{H} & -\lambda I \end{bmatrix} < 0$$

where $\tilde{\Phi}$ is given in Theorem 5.1, and

$$\hat{H} = \begin{bmatrix} -\tilde{H}^T \tilde{T}_1^T & -\tilde{H}^T \tilde{T}_2^T & -\tilde{H}^T \tilde{T}_3^T & -\tilde{H}^T \tilde{T}_4^T & -\tilde{H}^T \tilde{T}_5^T & 0 \end{bmatrix},$$
$$\hat{E} = \begin{bmatrix} 0 & \tilde{E} & \tilde{E}_d & 0 & 0 & 0 \end{bmatrix}.$$

Proof: Replacing $\tilde{A}$ and $\tilde{A}_d$ in (24) with $\tilde{A} + \tilde{H}F(k)\tilde{E}$ and $\tilde{A}_d + \tilde{H}F(k)\tilde{E}_d$, respectively, and following similar lines of proof of Theorem 3.3, we have the desired result.

6. Observer design

This section gives observer design methods for discrete-time delay systems. Section 6.1 provides an observer design method for a nominal delay system, and Section 6.2 proposes for an uncertain delay system.

6.1 Nominal observer

Similar to Theorem 3.1, Theorem 5.1 does not give a design method of finding an observer gain $\bar{K}$. Hence, we obtain another theorem below.

Theorem 6.1. *Given integers d_m and d_M, and scalars ρ_i and $\hat{\rho}_i$, $i = 1, \cdots, 5$. Then, (21) becomes an observer for the system (19) and (20) with $\Delta A = \Delta A_d = 0$, $\Delta C = \Delta C_d = 0$ if there exist matrices $0 < \tilde{P}_1 \in \Re^{2n \times 2n}$, $0 < \tilde{P}_2 \in \Re^{2n \times 2n}$, $0 < \tilde{Q}_1 \in \Re^{2n \times 2n}$, $0 < \tilde{Q}_2 \in \Re^{2n \times 2n}$, $0 < \tilde{S} \in \Re^{2n \times 2n}$, $0 < \tilde{M} \in \Re^{2n \times 2n}$, $\tilde{G} \in \Re^{n \times n}$, $Y \in \Re^{n \times n}$*

$$\tilde{L} = \begin{bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \\ \tilde{L}_3 \\ \tilde{L}_4 \\ \tilde{L}_5 \end{bmatrix} \in \Re^{10n \times 2n}, \ \tilde{N} = \begin{bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \\ \tilde{N}_3 \\ \tilde{N}_4 \\ \tilde{N}_5 \end{bmatrix} \in \Re^{10n \times 2n}, \ T = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} \in \Re^{5n \times n}, \ \hat{T} = \begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \\ \hat{T}_3 \\ \hat{T}_4 \\ \hat{T}_5 \end{bmatrix} \in \Re^{5n \times n}$$

satisfying

$$\tilde{\Psi} = \begin{bmatrix} \check{\Psi}_1 + \tilde{\Theta}_L + \tilde{\Theta}_L^T + \tilde{\Theta}_N + \tilde{\Theta}_N^T + \tilde{\Theta}_T + \tilde{\Theta}_T^T & \sqrt{d_M}\check{Z} \\ \sqrt{d_M}\check{Z}^T & -\tilde{S} \end{bmatrix} < 0 \tag{25}$$

where

$$\check{\Psi}_1 = \begin{bmatrix} \tilde{P}_1 & 0 & 0 & 0 & 0 \\ 0 & \check{\Psi}_{22} & 0 & 0 & 0 \\ 0 & 0 & -\tilde{Q}_1 - \tilde{M} & 0 & 0 \\ 0 & 0 & 0 & \check{\Psi}_{44} & -\tilde{P}_2 \\ 0 & 0 & 0 & -\tilde{P}_2 & \tilde{P}_2 - \tilde{Q}_2 \end{bmatrix},$$

$$\check{\Psi}_{22} = -\tilde{P}_1 + \tilde{Q}_1 + (d_M - d_m + 1)\tilde{M},$$

$$\check{\Psi}_{44} = \tilde{P}_2 + \tilde{Q}_2 + d_M\tilde{S},$$

$$\check{Z} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\tilde{P}_2 \\ \tilde{P}_2 \end{bmatrix} + \tilde{N},$$

$$\tilde{\Theta}_L = \begin{bmatrix} \tilde{L} & -\tilde{L} & 0 & -\tilde{L} & 0 \end{bmatrix},$$

$$\tilde{\Theta}_N = \begin{bmatrix} 0 & \tilde{N} & -\tilde{N} & 0 & 0 \end{bmatrix},$$

$$\tilde{\Theta}_T = \begin{bmatrix} T_1 & \rho_1 Y & -T_1 A & -\rho_1(YA - \tilde{G}C) & -T_1 A_d & -\rho_1(YA_d - \tilde{G}C_d) & 0 & 0 & 0 & 0 & 0 \\ \hat{T}_1 & \hat{\rho}_1 Y & -\hat{T}_1 A & -\hat{\rho}_1(YA - \tilde{G}C) & -\hat{T}_1 A_d & -\hat{\rho}_1(YA_d - \tilde{G}C_d) & 0 & 0 & 0 & 0 & 0 \\ T_2 & \rho_2 Y & -T_2 A & -\rho_2(YA - \tilde{G}C) & -T_2 A_d & -\rho_2(YA_d - \tilde{G}C_d) & 0 & 0 & 0 & 0 & 0 \\ \hat{T}_2 & \hat{\rho}_2 Y & -\hat{T}_2 A & -\hat{\rho}_2(YA - \tilde{G}C) & -\hat{T}_2 A_d & -\hat{\rho}_2(YA_d - \tilde{G}C_d) & 0 & 0 & 0 & 0 & 0 \\ T_3 & \rho_3 Y & -T_3 A & -\rho_3(YA - \tilde{G}C) & -T_3 A_d & -\rho_3(YA_d - \tilde{G}C_d) & 0 & 0 & 0 & 0 & 0 \\ \hat{T}_3 & \hat{\rho}_3 Y & -\hat{T}_3 A & -\hat{\rho}_3(YA - \tilde{G}C) & -\hat{T}_3 A_d & -\hat{\rho}_3(YA_d - \tilde{G}C_d) & 0 & 0 & 0 & 0 & 0 \\ T_4 & \rho_4 Y & -T_4 A & -\rho_4(YA - \tilde{G}C) & -T_4 A_d & -\rho_4(YA_d - \tilde{G}C_d) & 0 & 0 & 0 & 0 & 0 \\ \hat{T}_4 & \hat{\rho}_4 Y & -\hat{T}_4 A & -\hat{\rho}_4(YA - \tilde{G}C) & -\hat{T}_4 A_d & -\hat{\rho}_4(YA_d - \tilde{G}C_d) & 0 & 0 & 0 & 0 & 0 \\ T_5 & \rho_5 Y & -T_5 A & -\rho_5(YA - \tilde{G}C) & -T_5 A_d & -\rho_5(YA_d - \tilde{G}C_d) & 0 & 0 & 0 & 0 & 0 \\ \hat{T}_5 & \hat{\rho}_5 Y & -\hat{T}_5 A & -\hat{\rho}_5(YA - \tilde{G}C) & -\hat{T}_5 A_d & -\hat{\rho}_5(YA_d - \tilde{G}C_d) & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case, an observer gain in the observer (21) is given by

$$\bar{K} = Y^{-1}\tilde{G}. \tag{26}$$

Proof: Proof is similar to that of Theorem 4.1. Let

$$T_i = \begin{bmatrix} T_i & \rho_i Y \\ \hat{T}_i & \hat{\rho}_i Y \end{bmatrix}, \ i = 1, \cdots, 5$$

where ρ_i and $\hat{\rho}_i$, $i = 1, \cdots, 5$ are given. We substitute them into (24). Defining $\tilde{G} = Y\bar{K}$, we obtain $\check{\Psi} < 0$ in (25). If the condition (25) hold, observer gain matrix $\bar{K}$ is obviously given by (26).

6.2 Robust observer

Extending Theorem 4.1, we have the following theorem, which proposes a robust observer design for an uncertain delay system.

Theorem 6.2. *Given integers d_m and d_M, and scalars ρ_i, $\hat{\rho}_i$, $i = 1, \cdots, 5$. Then, (21) becomes an observer for the system (19) and (20) if there exist matrices $0 < \tilde{P}_1 \in \Re^{2n \times 2n}$, $0 < \tilde{P}_2 \in \Re^{2n \times 2n}$, $0 < \tilde{Q}_1 \in \Re^{2n \times 2n}$, $0 < \tilde{Q}_2 \in \Re^{2n \times 2n}$, $0 < \tilde{S} \in \Re^{2n \times 2n}$, $0 < \tilde{M} \in \Re^{2n \times 2n}$, $G \in \Re^{n \times n}$, $Y \in \Re^{n \times n}$*

$$\tilde{L} = \begin{bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \\ \tilde{L}_3 \\ \tilde{L}_4 \\ \tilde{L}_5 \end{bmatrix} \in \Re^{10n \times 2n}, \ \tilde{N} = \begin{bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \\ \tilde{N}_3 \\ \tilde{N}_4 \\ \tilde{N}_5 \end{bmatrix} \in \Re^{10n \times 2n}, \ T = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} \in \Re^{5n \times n}, \ \hat{T} = \begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \\ \hat{T}_3 \\ \hat{T}_4 \\ \hat{T}_5 \end{bmatrix} \in \Re^{5n \times n},$$

and a scalar $\lambda > 0$ satisfying

$$\tilde{\Lambda} = \begin{bmatrix} \tilde{\Psi} + \lambda \tilde{E}^T \tilde{E} & \tilde{H}^T \\ \tilde{H} & -\lambda I \end{bmatrix} < 0 \tag{27}$$

where $\tilde{\Psi}$ is given in Theorem 6.1, and

$$\tilde{H} = - \left[H^T T_1^T + \rho_1 (YH - \tilde{G}H_2)^T \ \ H^T \hat{T}_1^T + \hat{\rho}_1 (YH - \tilde{G}H_2)^T \ \ H^T T_2^T + \rho_2 (YH - \tilde{G}H_2)^T \right.$$
$$H^T \hat{T}_2^T + \hat{\rho}_2 (YH - \tilde{G}H_2)^T \ \ H^T T_3^T + \rho_3 (YH - \tilde{G}H_2)^T \ \ H^T \hat{T}_3^T + \hat{\rho}_3 (YH - \tilde{G}H_2)^T$$
$$H^T T_4^T + \rho_4 (YH - \tilde{G}H_2)^T \ \ H^T \hat{T}_4^T + \hat{\rho}_4 (YH - \tilde{G}H_2)^T \ \ H^T T_5^T + \rho_5 (YH - \tilde{G}H_2)^T$$
$$\left. H^T \hat{T}_5^T + \hat{\rho}_5 (YH - \tilde{G}H_2)^T \ 0 \ 0 \right],$$
$$\tilde{E} = \begin{bmatrix} 0 \ 0 \ E \ 0 \ E_d \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \end{bmatrix}.$$

In this case, an observer gain in the observer (21) is given by (26).

Proof: Replacing A and A_d in (27) with $A + HF(k)E$ and $A_d + HF(k)E_d$, respectively, and following similar lines of proof of Theorem 4.3, we have the desired result.

7. Examples

In this section, the following examples are provided to illustrate the proposed results. First example shows stabilization and robust stabilization. Second one gives observer design and robust observer design.

Example 7.1. *Consider the following discrete-time delay system:*

$$x(k+1) = \begin{bmatrix} 1.1 + \alpha & 0 \\ 0 & 0.97 \end{bmatrix} x(k) + \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix} x(k - d_k)$$
$$+ \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix} u(k) + \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix} u(k - d_k)$$

where α satisfies $|\alpha| \leq \bar{\alpha}$ for $\bar{\alpha}$ is an upper bound of $\alpha(k)$. First, we consider the stabilization for a nominal time-delay system with $\alpha(k) = 0$ by Theorem 4.1. Table 1 shows control gains for different time-invariant delay d_k, while Table 2 gives control gains for different time-varying delay d_k.
Next, we consider the robust stabilization for the uncertain time-delay system with $\alpha(k) \neq 0$. In this case, system matrices can be represented in the form of (1) with matrices given by

$$A = \begin{bmatrix} 1.1 & 0 \\ 0 & 0.97 \end{bmatrix}, \ A_d = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \ E = \begin{bmatrix} 1 \ 0 \end{bmatrix}, \ E_d = E_1 = \begin{bmatrix} 0 \ 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}, \ B_d = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, \ H = \begin{bmatrix} \bar{\alpha} \\ 0 \end{bmatrix}, \ F(k) = \frac{\alpha(k)}{\bar{\alpha}}.$$

d_k	$\rho's$	K
1	$[0.1\ 0.1\ -0.1\ 0.5\ 0.1]$	$[-1.1316\ -0.1360]$
2	$[0.1\ 0.1\ -0.1\ 0.5\ 0.1]$	$[-0.9690\ -0.0976]$
3	$[0.1\ 0.1\ -0.1\ 0.5\ 0.1]$	$[-0.7908\ -0.0545]$
4	$[0.09\ 0.05\ -0.1\ 0.55\ 0.1]$	$[-0.5815\ -0.0306]$

Table 1. The stabilization for time-invariant delay d_k

d_k	$\rho's$	K
$0 \leq d_k \leq 1$	$[0.1\ 0.1\ -0.1\ 0.5\ 0.1]$	$[-1.1209\ -0.1174]$
$0 \leq d_k \leq 2$	$[0.1\ 0.1\ -0.1\ 0.5\ 0.1]$	$[-0.9429\ -0.0839]$
$0 \leq d_k \leq 3$	$[0.1\ 0.1\ -0.1\ 0.5\ 0.1]$	$[-0.7950\ -0.0469]$
$0 \leq d_k \leq 4$	$[0.09\ 0.05\ -0.1\ 0.55\ 0.1]$	$[-0.5586\ -0.0253]$

Table 2. The stabilization for time-varying delay d_k

d_k	$\bar{\alpha}$	$\rho's$	K
3	0.05	$[0.1\ 0.1\ -0.1\ 0.5\ 0.1]$	$[-0.8622\ -0.0059]$
3	0.10	$[0.1\ 0.1\ -0.1\ 0.5\ 0.1]$	$[-0.6243\ -0.0000]$
2	0.15	$[0.12\ 0.12\ -0.1\ 0.5\ 0.05]$	$[-1.2515\ -0.0115]$

Table 3. The robust stabilization for time-invariant delay d_k

d_k	$\bar{\alpha}$	$\rho's$	K
$0 \leq d_k \leq 3$	0.05	$[0.1\ 0.1\ -0.1\ 0.5\ 0.1]$	$[-0.8394\ -0.0047]$
$0 \leq d_k \leq 3$	0.10	$[0.12\ 0.1\ -0.1\ 0.5\ 0.1]$	$[-1.2539\ -0.0108]$
$0 \leq d_k \leq 2$	0.15	$[0.12\ 0.12\ -0.1\ 0.5\ 0.05]$	$[-1.1740\ -0.0015]$

Table 4. The robust stabilization for time-varying delay d_k

For time-invariant delay d_k, Theorem 4.3 gives control gains for different $\bar{\alpha}$ in Table 3. Table 4 provides the result for time-varying delay d_k.

Example 7.2. *Consider the following discrete-time delay system:*

$$x(k+1) = \begin{bmatrix} 0.85 + 0.1\alpha & 0 \\ 0 & 0.97 \end{bmatrix} x(k) + \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix} x(k-d_k),$$
$$y(k) = \begin{bmatrix} 0.5\ 0.2 \end{bmatrix} x(k) + \begin{bmatrix} 0.1\ 0.1 \end{bmatrix} x(k-d_k)$$

where α satisfies $|\alpha| \leq \bar{\alpha}$ for $\bar{\alpha}$ is an upper bound of $\alpha(k)$. We first consider the observer design for a nominal time-delay system with $\alpha(k) = 0$ by Theorem 6.1. Table 5 shows observer gains for different time-invariant delay d_k, while Table 6 gives observer gains for different time-varying delay d_k. In the following observer design, all $\rho's$ are set to be zero for simplicity.

Next, we consider the robust observer design for the uncertain time-delay system with $\alpha(k) \neq 0$. In this case, system matrices can be represented in the form of (1) with matrices given by

$$A = \begin{bmatrix} 0.85 & 0 \\ 0 & 0.97 \end{bmatrix}, A_d = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, E = \begin{bmatrix} \bar{\alpha}\ 0 \end{bmatrix}, E_d = E_1 = \begin{bmatrix} 0\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.5\ 0.2 \end{bmatrix}, C_d = \begin{bmatrix} 0.1\ 0.1 \end{bmatrix}, H = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, F(k) = \frac{\alpha(k)}{\bar{\alpha}}.$$

d_k	$\hat{\rho}'$s	$\bar{K}$
1	$[-22.6\ 2.5\ -2.1\ -1.9\ -1.9]$	$\begin{bmatrix} 0.4835 \\ 0.3622 \end{bmatrix}$
2	$[-21.9\ 6.7\ -0.3\ -2.6\ -1.9]$	$\begin{bmatrix} 0.5372 \\ 0.3348 \end{bmatrix}$
3	$[-23.4\ 8.6\ 0.2\ -1.9\ -1.9]$	$\begin{bmatrix} 0.5174 \\ 0.3910 \end{bmatrix}$
4	$[-9.0\ 26.4\ 0.2\ -2.5\ -1.9]$	$\begin{bmatrix} -3.2281 \\ 0.0459 \end{bmatrix}$
5	$[-9.0\ 25.9\ 0.2\ -2.5\ -1.9]$	$\begin{bmatrix} -2.3508 \\ -0.7232 \end{bmatrix}$

Table 5. The observer design for time-invariant delay d_k

d_k	$\hat{\rho}'$s	$\bar{K}$
$0 \leq d_k \leq 1$	$[-20.2\ 6.5\ -2.1\ -2.5\ -1.9]$	$\begin{bmatrix} 0.6295 \\ 0.3777 \end{bmatrix}$
$0 \leq d_k \leq 2$	$[-21.9\ 6.7\ -0.3\ -2.6\ -1.9]$	$\begin{bmatrix} 0.5817 \\ 0.3475 \end{bmatrix}$
$0 \leq d_k \leq 3$	$[-22.0\ 8.6\ 0.2\ -1.9\ -1.9]$	$\begin{bmatrix} 0.5490 \\ 0.3037 \end{bmatrix}$
$0 \leq d_k \leq 4$	$[-22.0\ 8.6\ 0.2\ -2.5\ -1.9]$	$\begin{bmatrix} 0.5157 \\ 0.2921 \end{bmatrix}$
$0 \leq d_k \leq 5$	$[-22.5\ 8.6\ 0.2\ -1.9\ -3.1]$	$\begin{bmatrix} 0.5170 \\ 0.2956 \end{bmatrix}$

Table 6. The observer design for time-varying delay d_k

d_k	$\bar{\alpha}$	$\hat{\rho}'$s	$\bar{K}$
1	0.5	$[-22.6\ 2.5\ -2.1\ -1.9\ -1.9]$	$\begin{bmatrix} 0.5047 \\ 0.3813 \end{bmatrix}$
2	0.4	$[-21.9\ 6.7\ -0.3\ -2.6\ -1.9]$	$\begin{bmatrix} 0.5625 \\ 0.3633 \end{bmatrix}$
3	0.3	$[-23.4\ 8.6\ 0.2\ -1.9\ -1.9]$	$\begin{bmatrix} 0.5264 \\ 0.3641 \end{bmatrix}$
4	0.3	$[-9.0\ 26.4\ 0.2\ -2.5\ -1.9]$	$\begin{bmatrix} -2.6343 \\ -1.6768 \end{bmatrix}$
5	0.2	$[-9.0\ 25.9\ 0.2\ -2.5\ -1.9]$	$\begin{bmatrix} -2.5959 \\ -1.5602 \end{bmatrix}$

Table 7. The observer design for time-invariant delay d_k

For time-invariant delay d_k, Theorem 6.2 gives observer gains for different $\bar{\alpha}$ in Table 7. Table 8 provides observer gains for time-varying delay d_k by the same theorem.

8. Conclusions

In this paper, we proposed stabilization and robust stabilization method for discrete-time systems with time-varying delay. Our conditions were obtained by introducing new Lyapunov function and using Leibniz-Newton formula and free weighting matrix method.

d_k	$\bar{\alpha}$	$\hat{\rho}'$s	$\bar{K}$
$0 \le d_k \le 1$	0.5	$[-20.2 \; 6.5 \; -2.1 \; -2.5 \; -1.9]$	$\begin{bmatrix} 0.6345 \\ 0.3520 \end{bmatrix}$
$0 \le d_k \le 2$	0.4	$[-21.9 \; 6.7 \; -0.3 \; -2.6 \; -1.9]$	$\begin{bmatrix} 0.5870 \\ 0.3153 \end{bmatrix}$
$0 \le d_k \le 3$	0.3	$[-22.0 \; 8.6 \; 0.2 \; -1.9 \; -1.9]$	$\begin{bmatrix} 0.5675 \\ 0.3003 \end{bmatrix}$
$0 \le d_k \le 4$	0.3	$[-22.0 \; 8.6 \; 0.2 \; -2.5 \; -1.9]$	$\begin{bmatrix} 0.5375 \\ 0.3444 \end{bmatrix}$
$0 \le d_k \le 5$	0.2	$[-22.5 \; 8.6 \; 0.2 \; -1.9 \; -3.1]$	$\begin{bmatrix} 0.5077 \\ 0.3425 \end{bmatrix}$

Table 8. The observer design for time-varying delay d_k

Similarly, we also gave observer design and robust observer design methods. Numerical examples were given to illustrate our proposed design method.

9. References

[1] Fridman, E. & Shaked, U. (2005). Stability and guaranteed cost control of uncertain discrete delay systems, *International Journal of Control*, Vol.78, 235-246.

[2] Fridman, E. & Shaked, U. (2005). Delay-dependent H$_\infty$ control of uncertain discrete delay systems, *European Journal of Control*, Vol.11, 29-37.

[3] Gao, H.; Lam, J.:Wang, C. & Wang, Y. (2004). delay-dependent output feedback stabilization of discrete-time systems with time-varying state delay, *IEE Proc. Control Theory Appl.*, Vol.151, 691-698.

[4] Gao, H. & Chen, T. (2007). New results on stability of discrete-time systems with time-varying state delay, *IEEE Transactions on Automatic Control*, Vol.52, 328-334.

[5] Hara, M. & Yoneyama, J. (2008), New robust stability condition for uncertain discrete-time systems with time-varying delay, in *SICE Annual Conference 2008*, 743-747, Tokyo, August 2008.

[6] Hara, M. & Yoneyama, J. (2009). An improved robust stability condition for uncertain discrete time-varying delay systems, *Journal of Cybernetics and Systems*, Vol.2, 23-27.

[7] He, Y.; Wang, Q.; Xie, L. & Lin, C. (2007). Further improvement of free-weighting matrices technique for systems with time-varying delay, *IEEE Transactions on Automatic Control*, Vol.52, 293-299.

[8] Li, X. & de Souza, C. E. (1997). Delay dependent robust stability and stabilization of uncertain linear delay systems: a linear matrix inequality approach, *IEEE Transactions on Automatic Control*, Vol.42, 1144-1148.

[9] Liu, Y.; Wang, Z. & Liu, X. (2008). Robust H$_\infty$ filtering for discrete nonlinear stochastic systems with time-varying delay, *Journal of Mathematical Analysis and Applications*, Vol.341, 318-336.

[10] Ma, S.; Zhang, C. & Cheng, Z. (2008). Delay-dependent Robust H$_\infty$ Control for Uncertain Discrete-Time Singular Systems with Time-Delays, *Journal of Computational and Applied Mathematics*, Vol.217, 194-211.

[11] Mahmoud, M.S. (2000). *Robust Control and Filtering for Time-Delay Systems*, New York: Marcel Dekker, Inc.

[12] Palhares, R.M.; Campos, C.D.; Ekel, P. Ya.; Leles, M.C.R. & D'Angelo, M.F.S.V. (2005). Delay-dependent robust H_∞ control of uncertain linear systems with lumped delays, *IEE Proc. Control Theory Appl.*, Vol.152, 27-33

[13] Xie, L. (1996). Output Feedback H_∞ Control of systems with parameter uncertainty, *International Journal of Control*, Vol.63, 741-750.

[14] Xu, S.; Lam, J. & Zou, Y. (2005). Improved conditions for delay-dependent robust stability and stabilization of uncertain discrete-time systems, *Asian Journal of Control*, Vol.7, 344-348.

[15] Xu, S.; Lam, J. & Zou, Y. (2006). New results on delay-dependent robust H_∞ control for systems with time-varying delays, *Automatica*, Vol.42, 343-348.

[16] Ye, D. & Yang, G. H. (2008). Adaptive robust H_∞ state feedback control for linear uncertain systems with time-varying delay, *International Journal of Adaptive Control and Signal Processing*, Vol.22, 845-858.

[17] Yoneyama, J. (2008). H_∞ disturbance attenuation for discrete-time systems with time varying delays, *SICE Transactions*, Vol.44, 285-287(in Japanese).

[18] Yoneyama, J. & Tsuchiya, T. (2008). New delay-dependent conditions on robust stability and stabilisation for discrete-time systems with time-delay, *International Journal of Systems Science*, Vol.39, 1033-1040.

[19] Zhang, X.-M. & Han, Q.-L. (2008). A new finite sum inequality approach to delay-dependent H_∞ control of discrete-time systems with time-varying delay, *International Journal of Robust and Nonlinear Control*, Vol.18, 630-647.
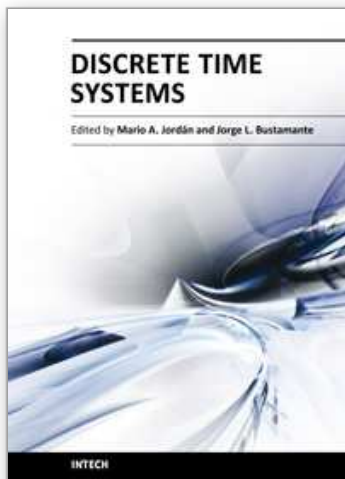
Discrete Time Systems

Edited by Dr. Mario Alberto Jordán

Discrete-Time Systems comprehend an important and broad research field. The consolidation of digital-based computational means in the present, pushes a technological tool into the field with a tremendous impact in areas like Control, Signal Processing, Communications, System Modelling and related Applications. This book attempts to give a scope in the wide area of Discrete-Time Systems. Their contents are grouped conveniently in sections according to significant areas, namely Filtering, Fixed and Adaptive Control Systems, Stability Problems and Miscellaneous Applications. We think that the contribution of the book enlarges the field of the Discrete-Time Systems with signification in the present state-of-the-art. Despite the vertiginous advance in the field, we also believe that the topics described here allow us also to look through some main tendencies in the next years in the research area.

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