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Observers Design for a Class of Lipschitz Discrete-Time Systems with Time-Delay

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1. Introduction

The observer design problem for nonlinear time-delay systems becomes more and more a subject of research in constant evolution Germani et al. (2002), Germani & Pepe (2004), Aggoune et al. (1999), Raff & Allgöwer (2006), Trinh et al. (2004), Xu et al. (2004), Zemouche et al. (2006), Zemouche et al. (2007). Indeed, time-delay is frequently encountered in various practical systems, such as chemical engineering systems, neural networks and population dynamic model. One of the recent application of time-delay is the synchronization and information recovery in chaotic communication systems Cherrier et al. (2005). In fact, the time-delay is added in a suitable way to the chaotic system in the goal to increase the complexity of the chaotic behavior and then to enhance the security of communication systems. On the other hand, contrary to nonlinear continuous-time systems, little attention has been paid toward discrete-time nonlinear systems with time-delay. We refer the readers to the few existing references Lu & Ho (2004a) and Lu & Ho (2004b), where the authors investigated the problem of robust H_∞ observer design for a class of Lipschitz time-delay systems with uncertain parameters in the discrete-time case. Their method show the stability of the state of the system and the estimation error simultaneously.

This chapter deals with observer design for a class of Lipschitz nonlinear discrete-time systems with time-delay. The main result lies in the use of a new structure of the proposed observer inspired from Fan & Arcak (2003). Using a *Lyapunov-Krasovskii* functional, a new nonrestrictive synthesis condition is obtained. This condition, expressed in term of LMI, contains more degree of freedom than those proposed by the approaches available in literature. Indeed, these last use a simple Luenberger observer which can be derived from the general form of the observer proposed in this paper by neglecting some observer gains.

An extension of the presented result to H_∞ performance analysis is given in the goal to take into account the noise which affects the considered system. A more general LMI is established. The last section is devoted to systems with differentiable nonlinearities. In this case, based on the use of the Differential Mean Value Theorem (DMVT), less restrictive synthesis conditions are proposed.

Notations : The following notations will be used throughout this chapter.

- $\|\cdot\|$ is the usual Euclidean norm;

- (\star) is used for the blocks induced by symmetry;
- A^T represents the transposed matrix of A ;
- I_r represents the identity matrix of dimension r ;
- for a square matrix S , $S > 0$ ($S < 0$) means that this matrix is positive definite (negative definite);
- $z_t(k)$ represents the vector $x(k-t)$ for all z ;
- The notation $\|x\|_{\ell_2^s} = \left(\sum_{k=0}^{\infty} \|x(k)\|^2 \right)^{\frac{1}{2}}$ is the ℓ_2^s norm of the vector $x \in \mathbb{R}^s$. The set ℓ_2^s is defined by

$$\ell_2^s = \left\{ x \in \mathbb{R}^s : \|x\|_{\ell_2^s} < +\infty \right\}.$$

2. Problem formulation and observer synthesis

In this section, we introduce the class of nonlinear systems to be studied, the proposed state observer and the observer synthesis conditions.

2.1 Problem formulation

Consider the class of systems described in a detailed forme by the following equations :

$$x(k+1) = Ax(k) + A_d x_d(k) + Bf\left(Hx(k), H_d x_d(k)\right) \quad (1a)$$

$$y(k) = Cx(k) \quad (1b)$$

$$x(k) = x^0(k), \text{ for } k = -d, \dots, 0 \quad (1c)$$

where the constant matrices A, A_d, B, C, H and H_d are of appropriate dimensions.

The function $f : \mathbb{R}^{s_1} \times \mathbb{R}^{s_2} \rightarrow \mathbb{R}^q$ satisfies the Lipschitz condition with Lipschitz constant γ_f , i.e :

$$\left\| f\left(z_1, z_2\right) - f\left(\hat{z}_1, \hat{z}_2\right) \right\| \leq \gamma_f \left\| \begin{bmatrix} z_1 - \hat{z}_1 \\ z_2 - \hat{z}_2 \end{bmatrix} \right\|, \forall z_1, z_2, \hat{z}_1, \hat{z}_2. \quad (2)$$

Now, consider the following new structure of the proposed observer defined by the equations (78) :

$$\begin{aligned} \hat{x}(k+1) = & A\hat{x}(k) + A_d \hat{x}_d(k) + Bf\left(v(k), w(k)\right) \\ & + L\left(y(k) - C\hat{x}(k)\right) + L_d\left(y_d(k) - C\hat{x}_d(k)\right) \end{aligned} \quad (3a)$$

$$v(k) = H\hat{x}(k) + K^1\left(y(k) - C\hat{x}(k)\right) + K_d^1\left(y_d(k) - C\hat{x}_d(k)\right) \quad (3b)$$

$$w(k) = H_d \hat{x}_d(k) + K^2\left(y(k) - C\hat{x}(k)\right) + K_d^2\left(y_d(k) - C\hat{x}_d(k)\right). \quad (3c)$$

The dynamic of the estimation error is :

$$\varepsilon(k+1) = (A - LC)\varepsilon(k) + (A_d - L_dC)\varepsilon_d(k) + B\delta f_k \quad (4)$$

with

$$\delta f_k = f(Hx(k), H_d x_d(k)) - f(v(k), w(k)).$$

From (35), we obtain

$$\|\delta f_k\| \leq \gamma_f \left\| \begin{bmatrix} (H - K^1C)\varepsilon(k) - K_d^1C\varepsilon_d(k) \\ (H_d - K_d^2C)\varepsilon_d(k) - K^2C\varepsilon(k) \end{bmatrix} \right\|. \quad (5)$$

2.2 Observer synthesis conditions

This subsection is devoted to the observer synthesis method that provides a sufficient condition ensuring the asymptotic convergence of the estimation error towards zero. The synthesis conditions, expressed in term of LMI, are given in the following theorem.

Theorem 2.1. *The estimation error is asymptotically stable if there exist a scalar $\alpha > 0$ and matrices $P = P^T > 0$, $Q = Q^T > 0$, $R, R_d, \bar{K}^1, \bar{K}^2, \bar{K}_d^1$ and \bar{K}_d^2 of appropriate dimensions such that the following LMI is feasible :*

$$\begin{bmatrix} -P + Q & 0 & M_{13} & M_{14} & M_{15}^T & M_{16}^T \\ (*) & -Q & M_{23} & M_{24} & M_{25}^T & M_{26}^T \\ (*) & (*) & M_{33} & 0 & 0 & 0 \\ (*) & (*) & (*) & -P & 0 & 0 \\ (*) & (*) & (*) & (*) & -\alpha\gamma_f^2 I_{s_1} & 0 \\ (*) & (*) & (*) & (*) & (*) & -\alpha\gamma_f^2 I_{s_2} \end{bmatrix} < 0 \quad (6)$$

where

$$M_{13} = A^T P B - C^T R B \quad (7a)$$

$$M_{14} = A^T P - C^T R \quad (7b)$$

$$M_{15} = \gamma_f^2 (\alpha H - \bar{K}^1 C) \quad (7c)$$

$$M_{16} = \gamma_f^2 \bar{K}^2 C \quad (7d)$$

$$M_{23} = A_d^T P B - C^T R_d B \quad (7e)$$

$$M_{24} = A_d^T P - C^T R_d \quad (7f)$$

$$M_{25} = \gamma_f^2 \bar{K}_d^1 C \quad (7g)$$

$$M_{26} = \gamma_f^2 (\alpha H_d - \bar{K}_d^2 C) \quad (7h)$$

$$M_{33} = B^T P B - \alpha I_q \quad (7i)$$

The gains L and L_d, K^1, K^2, K_d^1 and K_d^2 are given respectively by

$$L = P^{-1}R^T, \quad L_d = P^{-1}R_d^T$$

$$K^1 = \frac{1}{\alpha}\bar{K}^1, \quad K^2 = \frac{1}{\alpha}\bar{K}^2,$$

$$K_d^1 = \frac{1}{\alpha}\bar{K}_d^1, \quad K_d^2 = \frac{1}{\alpha}\bar{K}_d^2.$$

Proof. Consider the following Lyapunov-Krasovskii functional :

$$V_k = \varepsilon^T(k)P\varepsilon(k) + \sum_{i=1}^{i=d} \left(\varepsilon_i^T(k)Q\varepsilon_i(k) \right). \quad (8)$$

Using the dynamics (4), we obtain

$$V_{k+1} - V_k = \zeta_k^T \mathbb{M}_1 \zeta_k$$

where

$$\mathbb{M}_1 = \begin{bmatrix} \tilde{A}^T P \tilde{A} - P + Q & \tilde{A}^T P \tilde{A}_d & \tilde{A}^T P B \\ (\star) & \tilde{A}_d^T P \tilde{A}_d - Q & \tilde{A}_d^T P B \\ (\star) & (\star) & B^T P B \end{bmatrix}, \quad (9a)$$

$$\zeta_k^T = [\varepsilon^T(k) \quad \varepsilon_d^T(k) \quad \delta f_k^T], \quad (9b)$$

$$\tilde{A} = A - LC, \quad (9c)$$

$$\tilde{A}_d = A_d - L_d C. \quad (9d)$$

Using the notations $\bar{K}^1 = \alpha K^1, \bar{K}^2 = \alpha K^2, \bar{K}_d^1 = \alpha K_d^1$ and $\bar{K}_d^2 = \alpha K_d^2$, the condition (5) can be rewritten as follows :

$$\zeta_k^T \mathbb{M}_2 \zeta_k \geq 0 \quad (10)$$

with

$$\mathbb{M}_2 = \begin{bmatrix} \frac{1}{\alpha \gamma_f^2} \mathbb{M}_3 & 0 \\ 0 & -\alpha I_q \end{bmatrix}, \quad (11a)$$

$$\mathbb{M}_3 = \begin{bmatrix} \mathbb{M}_{15}^T \mathbb{M}_{15} + \mathbb{M}_{16}^T \mathbb{M}_{16} & \mathbb{M}_{15}^T \mathbb{M}_{25} + \mathbb{M}_{16}^T \mathbb{M}_{26} \\ (\star) & \mathbb{M}_{26}^T \mathbb{M}_{26} + \mathbb{M}_{25}^T \mathbb{M}_{25} \end{bmatrix}, \quad (11b)$$

and $\mathbb{M}_{15}, \mathbb{M}_{16}, \mathbb{M}_{25}, \mathbb{M}_{26}$ are defined in (7).

Consequently

$$V_{k+1} - V_k \leq \zeta_k^T (\mathbb{M}_1 + \mathbb{M}_2) \zeta_k. \quad (12)$$

By using the *Schur lemma* (see the Appendix), we deduce that the inequality

$$\mathbb{M}_1 + \mathbb{M}_2 < 0$$

is equivalent to

$$\mathbb{M}_4 < 0$$

where

$$\mathbb{M}_4 = \begin{bmatrix} -P + Q & 0 & \tilde{A}^T P B & \tilde{A}^T P & \mathbb{M}_{15}^T & \mathbb{M}_{16}^T \\ (\star) & -Q & \tilde{A}_d^T P B & \tilde{A}_d^T P & \mathbb{M}_{25}^T & \mathbb{M}_{26}^T \\ (\star) & (\star) & \mathbb{M}_{33} & 0 & 0 & 0 \\ (\star) & (\star) & (\star) & -P & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & -\alpha \gamma_f^2 I_{s_1} & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & -\alpha \gamma_f^2 I_{s_2} \end{bmatrix}. \tag{13}$$

Using the notations $R = L^T P$ and $R_d = L_d^T P$, we deduce that the inequality $\mathbb{M}_4 < 0$ is identical to (6). This means that under the condition (6) of Theorem 2.1, the function V_k is strictly decreasing and therefore the estimation error is asymptotically stable. This ends the proof of Theorem 2.1. \square

Remark 2.2. The Schur lemma and its application in the proof of Theorem 2.1 are detailed in the Appendix of this paper.

2.3 Illustrative example

In this section, we present a numerical example in order to valid the proposed results. Consider an example of an instable system under the form (1) described by the following parameters :

$$A = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0.5 & 0.3 \\ 0.5 & 0 & 0.3 \\ 0.3 & 0.3 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \\ 0 & 0 \end{bmatrix}, \quad H = [1 \ 0 \ 1],$$

$$H_d = [1 \ 0 \ 0], \quad C = [1 \ 0 \ 0]$$

and

$$f(Hx, H_d x_d, y) = \gamma_f \begin{bmatrix} \sin(x^1(k) + x^3(k)) \\ \cos(x^2(k-1)) \end{bmatrix}$$

where

$$x = [x^1 \ x^2 \ x^3]^T$$

and $\gamma_f = 10$ is the Lipschitz constant of the function f . Applying the proposed method (condition (6)), we obtain the following gains :

$$L = [0.0701 \ 1.8682 \ 2.9925]^T,$$

$$L_d = [0.3035 \ 0.2942 \ 0.0308]^T,$$

$$K^1 = 0.9961, K^2 = -2.8074 \times 10^{-5},$$

$$K_d^1 = -9.0820 \times 10^{-4}, K_d^2 = -0.0075$$

and

$$\alpha = 10^{-7}.$$

3. Extension to H_∞ performance analysis

In this section, we propose an extension of the previous result to H_∞ robust observer design problem. In this case, we give an observer synthesis method which takes into account the noises affecting the system.

Consider the disturbed system described by the equations :

$$x(k+1) = Ax(k) + A_d x_d(k) + E_\omega \omega(k) + Bf(Hx(k), H_d x_d(k)) \quad (14a)$$

$$y(k) = Cx(k) + D_\omega \omega(k) \quad (14b)$$

$$x(k) = x^0(k), \text{ for } k = -d, \dots, 0 \quad (14c)$$

where $\omega(k) \in \ell_2^s$ is the vector of bounded disturbances. The matrices E_ω and D_ω are constants with appropriate dimensions.

The corresponding observer has the same structure as in (3). We recall it hereafter with some different notations.

$$\begin{aligned} \hat{x}(k+1) = & A\hat{x}(k) + A_d \hat{x}_d(k) + Bf(v_1(k), v_2(k)) \\ & + L(y(k) - C\hat{x}(k)) + L_d(y_d(k) - C\hat{x}_d(k)) \end{aligned} \quad (15a)$$

$$v_1(k) = H\hat{x}(k) + K^1(y(k) - C\hat{x}(k)) + K_d^1(y_d(k) - C\hat{x}_d(k)) \quad (15b)$$

$$v_2(k) = H_d \hat{x}_d(k) + K^2(y(k) - C\hat{x}(k)) + K_d^2(y_d(k) - C\hat{x}_d(k)). \quad (15c)$$

Our aim is to design the matrices L, L_d, K^1, K^2, K_d^1 and K_d^2 such that (15) is an asymptotic observer for the system (14). The dynamics of the estimation error

$$\varepsilon(k) = x(k) - \hat{x}(k)$$

is given by the equation :

$$\begin{aligned} \varepsilon(k+1) = & (A - LC)\varepsilon(k) + (A_d - L_d C)\varepsilon_d(k) + B\delta f_k \\ & + (E_\omega - LD_\omega)\omega(k) - L_d D_\omega \omega_d(k) \end{aligned} \quad (16)$$

with

$$\delta f_k = f\left(Hx(k), H_d x_d(k)\right) - f\left(v_1(k), v_2(k)\right)$$

satisfies (5).

The objective is to find the gains L, L_d, K^1, K^2, K_d^1 and K_d^2 such that the estimation error converges robustly asymptotically to zero, i.e :

$$\|\varepsilon\|_{\ell_2^s} \leq \lambda \|\omega\|_{\ell_2^s} \quad (17)$$

where $\lambda > 0$ is the disturbance attenuation level to be minimized under some conditions that we will determined later.

The inequality (17) is equivalent to

$$\|\varepsilon\|_{\ell_2^s} \leq \frac{\lambda}{\sqrt{2}} \left(\|\omega\|_{\ell_2^s}^2 + \|\omega_d\|_{\ell_2^s}^2 - \sum_{k=-d}^{-1} \omega^2(k) \right)^{\frac{1}{2}}. \quad (18)$$

Without loss of generality, we assume that

$$\omega(k) = 0 \text{ for } k = -d, \dots, -1.$$

Then, (18) becomes

$$\|\varepsilon\|_{\ell_2^s} \leq \frac{\lambda}{\sqrt{2}} \left(\|\omega\|_{\ell_2^s}^2 + \|\omega_d\|_{\ell_2^s}^2 \right)^{\frac{1}{2}}. \quad (19)$$

Remark 3.1. In fact, if $\omega(k) \neq 0$ for $k = -d, \dots, -1$, we must replace the inequality (17) by

$$\|\varepsilon\|_{\ell_2^s} \leq \lambda \left(\|\omega\|_{\ell_2^s}^2 + \frac{1}{2} \sum_{k=-d}^{-1} \omega^2(k) \right)^{\frac{1}{2}} \quad (20)$$

in order to obtain (19).

Robust H_∞ observer design problem Li & Fu (1997) : Given the system (14) and the observer (15), then the problem of robust H_∞ observer design is to determine the matrices L, L_d, K^1, K^2, K_d^1 and K_d^2 so that

$$\lim_{k \rightarrow \infty} \varepsilon(k) = 0 \text{ for } \omega(k) = 0; \quad (21)$$

$$\|\varepsilon\|_{\ell_2^s} \leq \lambda \|\omega\|_{\ell_2^s} \quad \forall \omega(k) \neq 0; \quad \varepsilon(k) = 0, k = -d, \dots, 0. \quad (22)$$

From the equivalence between (17) and (19), the problem of robust H_∞ observer design (see the Appendix) is reduced to find a *Lyapunov* function V_k such that

$$W_k = \Delta V + \varepsilon^T(k) \varepsilon(k) - \frac{\lambda^2}{2} \omega^T(k) \omega(k) - \frac{\lambda^2}{2} \omega_d^T(k) \omega_d(k) < 0 \quad (23)$$

where

$$\Delta V = V_{k+1} - V_k.$$

At this stage, we can state the following theorem, which provides a sufficient condition ensuring (23).

Theorem 3.2. The robust H_∞ observer design problem corresponding to the system (14) and the observer (15) is solvable if there exist a scalar $\alpha > 0$ matrices $P = P^T > 0$, $Q = Q^T > 0$, $R, R_d, \bar{K}^1, \bar{K}^2, \bar{K}_d^1$ and \bar{K}_d^2 of appropriate dimensions so that the following convex optimization problem is feasible :

$$\min(\gamma) \text{ subject to } \Gamma < 0 \quad (24)$$

where

$$\Gamma = \begin{bmatrix} \begin{bmatrix} -P + Q + I_n & 0 & \mathbb{M}_{13} & 0 & 0 \\ (\star) & -Q & \mathbb{M}_{23} & 0 & 0 \\ (\star) & (\star) & \mathbb{M}_{33} & \mathbb{M}_{34} & \mathbb{M}_{35} \\ (\star) & (\star) & (\star) & -\gamma I_s & 0 \\ (\star) & (\star) & (\star) & (\star) & -\gamma I_s \end{bmatrix} & \begin{bmatrix} \mathbb{M}_{14} & \mathbb{M}_{15}^T & \mathbb{M}_{16}^T \\ \mathbb{M}_{24}^T & \mathbb{M}_{25}^T & \mathbb{M}_{26}^T \\ 0 & 0 & 0 \\ E_\omega^T P - C^T R & 0 & 0 \\ -D_\omega R_d & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \mathbb{M}_{14} & \mathbb{M}_{15}^T & \mathbb{M}_{16}^T \\ \mathbb{M}_{24}^T & \mathbb{M}_{25}^T & \mathbb{M}_{26}^T \\ 0 & 0 & 0 \\ E_\omega^T P - C^T R & 0 & 0 \\ -D_\omega R_d & 0 & 0 \end{bmatrix}^T & \begin{bmatrix} -P & 0 & 0 \\ (\star) & -\alpha \gamma_f^2 I_{s_1} & 0 \\ (\star) & (\star) & -\alpha \gamma_f^2 I_{s_2} \end{bmatrix} \end{bmatrix} \quad (25)$$

with

$$\mathbb{M}_{34} = B^T P E_\omega - B^T R^T C, \quad (26a)$$

$$\mathbb{M}_{35} = -B^T R_d^T D_\omega, \quad (26b)$$

and $\mathbb{M}_{13}, \mathbb{M}_{14}, \mathbb{M}_{15}, \mathbb{M}_{16}, \mathbb{M}_{24}, \mathbb{M}_{25}, \mathbb{M}_{26}, \mathbb{M}_{33}$ are defined in (7).

The gains L and $L_d, K^1, K^2, K_d^1, K_d^2$ and the minimum disturbance attenuation level λ are given respectively by

$$L = P^{-1} R^T, \quad L_d = P^{-1} R_d^T$$

$$K^1 = \frac{1}{\alpha} \bar{K}^1, \quad K^2 = \frac{1}{\alpha} \bar{K}^2,$$

$$K_d^1 = \frac{1}{\alpha} \bar{K}_d^1, \quad K_d^2 = \frac{1}{\alpha} \bar{K}_d^2,$$

$$\lambda = \sqrt{2\gamma}.$$

Proof. The proof of this theorem is an extension of that of Theorem 2.1.

Let us consider the same *Lyapunov-Krasovskii* functional defined in (8). We show that if the convex optimization problem (24) is solvable, we have $W_k < 0$. Using the dynamics (16), we obtain

$$W_k = \eta^T S_1 \eta \quad (27)$$

where

$$S_1 = \begin{bmatrix} \mathbb{M}_1 + \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} \tilde{A}^T P \tilde{E}_\omega - \tilde{A}^T P \tilde{D}_\omega \\ \tilde{A}_d^T P \tilde{E}_\omega - \tilde{A}_d^T P \tilde{D}_\omega \\ B^T P \tilde{E}_\omega - B^T P \tilde{D}_\omega \end{bmatrix} \\ \begin{bmatrix} \tilde{A}^T P \tilde{E}_\omega - \tilde{A}^T P \tilde{D}_\omega \\ \tilde{A}_d^T P \tilde{E}_\omega - \tilde{A}_d^T P \tilde{D}_\omega \\ B^T P \tilde{E}_\omega - B^T P \tilde{D}_\omega \end{bmatrix}^T & \begin{bmatrix} \tilde{E}_\omega^T P \tilde{E}_\omega - \gamma I_s & \tilde{E}_\omega^T P \tilde{D}_\omega \\ \tilde{D}_\omega^T P \tilde{E}_\omega & \tilde{D}_\omega^T P \tilde{D}_\omega - \gamma I_s \end{bmatrix} \end{bmatrix}, \quad (28)$$

where

$$\tilde{E}_\omega = E_\omega - LC \quad (29a)$$

$$\tilde{D}_\omega = L_d D_\omega \quad (29b)$$

$$\eta^T = [\varepsilon^T \quad \varepsilon_d^T \quad \delta f_k \quad \omega^T \quad \omega_d^T], \quad (29c)$$

$$\gamma = \frac{\lambda^2}{2}. \quad (29d)$$

The matrices \mathbb{M}_1 , \tilde{A} and \tilde{A}_d are defined in (9).

As in the proof of Theorem 2.1, since δf_k satisfies (5), we deduce, after multiplying by a scalar $\alpha > 0$, that

$$\eta^T \mathbb{S}_2 \eta \geq 0 \quad (30)$$

where

$$\mathbb{S}_2 = \begin{bmatrix} \frac{1}{\alpha \gamma_f^2} \mathbb{M}_3 & 0 & 0 & 0 \\ 0 & -\alpha I_q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (31)$$

and \mathbb{M}_3 is defined in (11b).

The inequality (31) implies that

$$W_k = \eta^T (\mathbb{S}_1 + \mathbb{S}_2) \eta. \quad (32)$$

Now, using the *Schur Lemma* and the notations $R = L^T P$ and $R_d = L_d^T P$, we deduce that the inequality $\mathbb{S}_1 + \mathbb{S}_2 < 0$ is equivalent to $\Gamma < 0$. The estimation error converges robustly asymptotically to zero with a *minimum* value of the disturbance attenuation level $\lambda = \sqrt{2\gamma}$ if the convex optimization problem (24) is solvable. This ends the proof of Theorem 3.2. \square

Remark 3.3. We can obtain a synthesis condition which contains more degree of freedom than the LMI (6) by using a more general design of the observer. This new design of the observer can take the following structure :

$$\begin{aligned} \hat{x}(k+1) = & A\hat{x}(k) + A_d\hat{x}_d(k) + Bf(v(k), w(k)) \\ & + L(y(k) - C\hat{x}(k)) + \sum_{i=1}^d L_i(y_i(k) - C\hat{x}_i(k)) \end{aligned} \quad (33a)$$

$$v(k) = H\hat{x}(k) + K^1(y(k) - C\hat{x}(k)) + \sum_{i=1}^d K_i^1(y_i(k) - C\hat{x}_i(k)) \quad (33b)$$

$$w(k) = H_d\hat{x}_d(k) + K^2(y(k) - C\hat{x}(k)) + \sum_{i=1}^d K_i^2(y_i(k) - C\hat{x}_i(k)). \quad (33c)$$

If such an observer is used, the adequate Lyapunov-Krasovskii functional that we propose is under the following form :

$$V_k = \varepsilon^T(k) P \varepsilon(k) + \sum_{j=1}^d \sum_{i=1}^j \left(\varepsilon_i^T(k) Q_j \varepsilon_i(k) \right). \quad (34)$$

4. Systems with differentiable nonlinearities

4.1 Reformulation of the problem

In this section, we need to assume that the function f is differentiable with respect to x . Rewrite also f under the detailed form :

$$f(Hx, H_d z) = \begin{bmatrix} f_1(H_1 x, H_1^d z) \\ \vdots \\ f_q(H_q x, H_q^d z) \end{bmatrix}. \quad (35)$$

where $H_i \in \mathbb{R}^{s_i \times n}$ and $H_i^d \in \mathbb{R}^{r_i \times n}$ for all $i \in \{1, \dots, q\}$. Here, we use the following reformulation of the Lipschitz condition :

$$-\infty < a_{ij} \leq \frac{\partial f_i}{\partial \zeta_j^i}(\zeta^i, z^i) \leq b_{ij} < +\infty, \quad \forall \zeta^i \in \mathbb{R}^{s_i}, \quad \forall z^i \in \mathbb{R}^{r_i} \quad (36)$$

$$-\infty < a_{ij}^d \leq \frac{\partial f_i}{\partial \zeta_j^i}(x^i, \zeta^i) \leq b_{ij}^d < +\infty, \quad \forall \zeta^i \in \mathbb{R}^{r_i}, \quad \forall x^i \in \mathbb{R}^{s_i} \quad (37)$$

where $x^i = H_i x$ and $z^i = H_i^d z$.

The conditions (36)-(37) imply that the differentiable function f is γ_f -Lipschitz where

$$\gamma_f = \sqrt{\sum_{i=1}^q \max \left(\sum_{j=1}^{j=s_i} \max (|a_{ij}|^2, |b_{ij}|^2), \sum_{j=1}^{j=r_i} \max (|a_{ij}^d|^2, |b_{ij}^d|^2) \right)}$$

The reformulation of the Lipschitz condition for differentiable functions as in (36) and (37) plays an important role on the feasibility of the synthesis conditions and avoids high gain as shown in Zemouche et al. (2008). In addition, it is shown in Alessandri (2004) that the use of the classical Lipschitz property leads to restrictive synthesis conditions.

Remark 4.1. For simplicity of the presentation, we assume, without loss of generality, that f satisfies (36) and (37) with $a_{ij} = 0$ and $a_{lm}^d = 0$ for all $i, l = 1, \dots, q, j = 1, \dots, s$ and $m = 1, \dots, r$, where $s = \max_{1 \leq i \leq q} (s_i)$ and $r = \max_{1 \leq i \leq q} (r_i)$. Indeed, if there exist subsets $S_1, S_1^d \subset \{1, \dots, q\}$, $S_2 \subset \{1, \dots, s\}$ and $S_2^d \subset \{1, \dots, r\}$ such that $a_{ij} \neq 0$ for all $(i, j) \in S_1 \times S_2$ and $a_{lm}^d \neq 0$ for all $(l, m) \in S_1^d \times S_2^d$, we can

consider the nonlinear function

$$\begin{aligned} \tilde{f}(x_k, x_{k-d}) = & f(Hx_k, H_d x_{k-d}) - \left(\sum_{(i,j) \in S_1 \times S_2} a_{ij} H_{ij} H_i \right) x_k \\ & - \left(\sum_{(l,m) \in S_1^d \times S_2^d} a_{lm}^d H_{lm}^d H_l^d \right) x_{k-d} \end{aligned} \quad (38)$$

where

$$H_{ij} = e_q(i) e_{s_i}^T(j) \text{ and } H_{lm}^d = e_q(l) e_{r_l}^T(m).$$

Therefore, \tilde{f} satisfies (36) and (37) with $\tilde{a}_{ij} = 0$, $\tilde{a}_{ij}^d = 0$, $\tilde{b}_{ij} = b_{ij} - a_{ij}$ and $\tilde{b}_{ij}^d = b_{ij}^d - a_{ij}^d$, and then we rewrite (1a) as

$$x_{k+1} = \tilde{A}x_k + \tilde{A}_d x_{k-d} + B\tilde{f}(x_k, x_{k-d})$$

with

$$\tilde{A} = A + B \sum_{(i,j) \in S_1 \times S_2} a_{ij} H_{ij} H_i, \quad \tilde{A}_d = A_d + B \sum_{(i,j) \in S_1^d \times S_2^d} a_{ij}^d H_{ij}^d H_i^d$$

Inspired by Fan & Arcak (2003), we consider the following state observer :

$$\hat{x}_{k+1} = A\hat{x}_k + A_d \hat{x}_{k-d} + \sum_{i=1}^{i=q} B e_q(i) f_i(v_k^i, w_k^i) \quad (39a)$$

$$+ L(y_k - C\hat{x}_k) + L^d(y_{k-d} - C\hat{x}_{k-d})$$

$$v_k^i = H_i \hat{x}_k + K_i(y_k - C\hat{x}_k) \quad (39b)$$

$$w_k^i = H_i^d \hat{x}_{k-d} + K_i^d(y_{k-d} - C\hat{x}_{k-d}) \quad (39c)$$

$$\hat{x}_k = \hat{x}_0, \forall k \in \{-d, \dots, 0\} \quad (39d)$$

Therefore, the aim is to find the gains $L \in \mathbb{R}^{n \times p}$, $L^d \in \mathbb{R}^{n \times p}$, $K_i \in \mathbb{R}^{s_i \times p}$ and $K_i^d \in \mathbb{R}^{r_i \times p}$, for $i = 1, \dots, q$, such that the estimation error

$$\varepsilon_k = x_k - \hat{x}_k \quad (40)$$

converges asymptotically towards zero.

The dynamics of the estimation error is given by :

$$\varepsilon_{k+1} = (A - LC)\varepsilon_k + (A_d - L^d C)\varepsilon_{k-d} + \sum_{i=1}^{i=q} B e_q(i) \delta f_i \quad (41)$$

where

$$\delta f_i = f_i(H_i x_k, H_i^d \hat{x}_k) - f_i(v_k^i, w_k^i).$$

Using the DMVT-based approach given firstly in Zemouche et al. (2008), there exist $z_i \in \text{Co}(H_i x, v^i), z_i^d \in \text{Co}(H_i^d x_{k-d}, w^i)$ for all $i = 1, \dots, q$ such that :

$$\delta f_i = \sum_{j=1}^{j=s_i} h_{ij}(k) e_{s_i}^T(j) \chi_i + \sum_{j=1}^{j=r_i} h_{ij}^d(k) e_{r_i}^T(j) \chi_i^d \quad (42)$$

where

$$\chi_i = (H_i - K_i C) \varepsilon_k \quad (43)$$

$$\chi_i^d = (H_i^d - K_i^d C) \varepsilon_{k-d} \quad (44)$$

$$h_{ij}(k) = \frac{\partial f_i}{\partial v_j^i} (z_i(k), H_i^d x_{k-d}) \quad (45)$$

$$h_{ij}^d(k) = \frac{\partial f_i}{\partial v_j^i} (v_k^i, z_i^d(k)) \quad (46)$$

Hence, the estimation error dynamics (41) becomes :

$$\begin{aligned} \varepsilon_{k+1} &= (A - LC) \varepsilon_k + (A_d - L^d C) \varepsilon_{k-d} \\ &\quad + \sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} h_{ij}(k) B H_{ij} \chi_i \\ &\quad + \sum_{i=1}^{i=q} \sum_{j=1}^{j=r_i} h_{ij}^d(k) B H_{ij}^d \chi_i^d \end{aligned} \quad (47)$$

4.2 New synthesis method

The content of this section consists in a new observer synthesis method. A novel sufficient stability condition ensuring the asymptotic convergence of the estimation error towards zero is provided. This condition is expressed in term of LMI easily tractable.

Theorem 4.2. *The estimation error (40) converges asymptotically towards zero if there exist matrices $P = P^T > 0$, $Q = Q^T > 0$, R , R^d , K_i and K_i^d , for $i = 1, \dots, q$, of adequate dimensions so that the following LMI is feasible :*

$$\begin{bmatrix} -P + Q & 0 & \mathbb{M} & 0 & A^T P - C^T R \\ (*) & -Q & 0 & \mathbb{N} & A_d^T P - C^T R^d \\ (*) & (*) & -Y & 0 & \Sigma^T P \\ (*) & (*) & (*) & -Y^d & (\Sigma^d)^T P \\ (*) & (*) & (*) & (*) & -P \end{bmatrix} < 0 \quad (48)$$

where

$$\mathbb{M} = [\mathbb{M}_1(K_1) \cdots \mathbb{M}_q(K_q)] \quad (49)$$

$$\mathbb{M}_i(K_i) = \left[\underbrace{(H_i - K_i C)^T \cdots (H_i - K_i C)^T}_{s_i \text{ times}} \right] \quad (50)$$

$$\mathbb{N} = [\mathbb{N}_1(K_1^d) \cdots \mathbb{N}_q(K_q^d)] \quad (51)$$

$$\mathbb{N}_i(K_i^d) = \left[\underbrace{(H_i^d - K_i^d C)^T \cdots (H_i^d - K_i^d C)^T}_{r_i \text{ times}} \right] \quad (52)$$

$$\Sigma = B \begin{bmatrix} H_{11} \cdots H_{1s_1} & H_{21} \cdots H_{qs_q} \end{bmatrix} \quad (53)$$

$$\Sigma^d = B \begin{bmatrix} H_{11}^d \cdots H_{1r_1}^d & H_{21} \cdots H_{qr_q} \end{bmatrix} \quad (54)$$

$$Y = \text{diag}(\beta_{11} I_{s_1}, \dots, \beta_{1s_1} I_{s_1}, \beta_{21} I_{s_2}, \dots, \beta_{qs_q} I_{s_q}) \quad (55)$$

$$Y^d = \text{diag}(\beta_{11}^d I_{r_1}, \dots, \beta_{1r_1}^d I_{r_1}, \beta_{21}^d I_{r_2}, \dots, \beta_{qr_q}^d I_{r_q}) \quad (56)$$

$$\beta_{ij} = \frac{2}{b_{ij}}, \quad \beta_{ij}^d = \frac{2}{b_{ij}^d} \quad (57)$$

Hence, the gains L , L^d are given, respectively, by $L = P^{-1}R^T$, $L^d = P^{-1}(R^d)^T$ and the matrices K_i , K_i^d are free solutions of the LMI (48).

Proof. For the proof, we use the following Lyapunov-Krasovskii functional candidate :

$$V_k = \varepsilon_k^T P \varepsilon_k + \sum_{i=1}^{i=d} \varepsilon_{k-i}^T Q \varepsilon_{k-i}$$

Considering the difference $\Delta V = V_{k+1} - V_k$ along the system (1), we have

$$\begin{aligned} \Delta V = & \varepsilon_k^T \left[(A - LC)^T P (A - LC) - P + Q \right] \varepsilon_k + \varepsilon_{k-d}^T \left[(A_d - L^d C)^T P (A_d - L^d C) - Q \right] \varepsilon_{k-d} \\ & + 2\varepsilon_k^T (A - LC)^T P (A_d - L^d C) \varepsilon_{k-d} + 2\varepsilon_k^T (A - LC)^T P \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} B H_{ij} \zeta_{ij} \right) \\ & + 2\varepsilon_k^T (A - LC)^T P \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=r_i} B H_{ij}^d \zeta_{ij}^d \right) + 2\varepsilon_{k-d}^T (A_d - L^d C)^T P \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} B H_{ij} \zeta_{ij} \right) \\ & + 2\varepsilon_{k-d}^T (A_d - L^d C)^T P \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=r_i} B H_{ij}^d \zeta_{ij}^d \right) + \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} B H_{ij} \zeta_{ij} \right)^T P \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} B H_{ij} \zeta_{ij} \right) \\ & + \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=r_i} B H_{ij}^d \zeta_{ij}^d \right)^T P \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=r_i} B H_{ij}^d \zeta_{ij}^d \right) \end{aligned} \quad (58)$$

where

$$\zeta_{ij} = h_{ij}(k) \chi_i, \quad \zeta_{ij}^d = h_{ij}^d(k) \chi_i^d. \quad (59)$$

From (36) and (37), we have

$$\sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} \zeta_{ij}^T \left(\frac{1}{h_{ij}} - \frac{1}{b_{ij}} \right) \zeta_{ij} \geq 0 \quad (60)$$

$$\sum_{i=1}^{i=q} \sum_{j=1}^{j=r_i} (\zeta_{ij}^d)^T \left(\frac{1}{h_{ij}^d} - \frac{1}{b_{ij}^d} \right) \zeta_{ij}^d \geq 0 \quad (61)$$

Using (43) and (59), the inequalities (60) and (61) become, respectively,

$$\sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} \varepsilon^T (H_i - K_i C)^T \zeta_{ij} - \sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} \frac{1}{b_{ij}} \zeta_{ij}^T \zeta_{ij} \geq 0 \quad (62)$$

$$\sum_{i=1}^{i=q} \sum_{j=1}^{j=r_i} \varepsilon_{k-d}^T (H_i^d - K_i^d C)^T \zeta_{ij}^d - \sum_{i=1}^{i=q} \sum_{j=1}^{j=r_i} \frac{1}{b_{ij}^d} (\zeta_{ij}^d)^T \zeta_{ij}^d \geq 0 \quad (63)$$

Consequently,

$$\Delta V \leq \begin{bmatrix} \varepsilon_k \\ \varepsilon_{k-d} \\ \zeta_k \\ \zeta_k^d \end{bmatrix}^T \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ (*) & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ (*) & (*) & \Gamma_{33} & \Gamma_{34} \\ (*) & (*) & (*) & \Gamma_{44} \end{bmatrix} \begin{bmatrix} \varepsilon_k \\ \varepsilon_{k-d} \\ \zeta_k \\ \zeta_k^d \end{bmatrix} \quad (64)$$

where

$$\Gamma_{11} = (A - LC)^T P (A - LC) - P + Q \quad (65)$$

$$\Gamma_{12} = (A - LC)^T P (A_d - L^d C) \quad (66)$$

$$\Gamma_{13} = M^T (K_1, \dots, K_q) + (A - LC)^T P \Sigma \quad (67)$$

$$\Gamma_{14} = (A - LC)^T P \Sigma^d \quad (68)$$

$$\Gamma_{22} = (A_d - L^d C)^T P (A_d - L^d C) - Q \quad (69)$$

$$\Gamma_{23} = (A_d - L^d C)^T P \Sigma \quad (70)$$

$$\Gamma_{24} = N^T (K_1^d, \dots, K_q^d) + (A_d - L^d C)^T P \Sigma^d \quad (71)$$

$$\Gamma_{33} = \Sigma^T P \Sigma - Y \quad (72)$$

$$\Gamma_{34} = \Sigma^T P \Sigma^d \quad (73)$$

$$\Gamma_{44} = (\Sigma^d)^T P \Sigma^d - Y^d \quad (74)$$

$$\zeta_k = [\zeta_{11}^T, \dots, \zeta_{1s_1}^T, \zeta_{21}^T, \dots, \zeta_{qs_q}^T]^T \quad (75)$$

$$\zeta_k^d = [(\zeta_{11}^d)^T, \dots, (\zeta_{1r_1}^d)^T, (\zeta_{21}^d)^T, \dots, (\zeta_{qr_q}^d)^T]^T \quad (76)$$

and $M(K_1, \dots, K_q)$, Σ , Y are defined in (49), (53) and (55) respectively.

Using the Schur Lemma and the notation $R = L^T P$, the inequality (48) is equivalent to

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ (\star) & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ (\star) & (\star) & \Gamma_{33} & \Gamma_{34} \\ (\star) & (\star) & (\star) & \Gamma_{44} \end{bmatrix} < 0. \quad (77)$$

Consequently, we deduce that under the condition (48), the estimation error converges asymptotically towards zero. This ends the proof of Theorem 4.2. \square

Remark 4.3. Note that we can consider a more general observer with more degree of freedoms as follows :

$$\hat{x}_{k+1} = A\hat{x}_k + A_d x_{k-d} + \sum_{i=1}^{i=q} B e_q(i) f_i(v_k^i, w_k^i) + \sum_{l=0}^{l=d} L_l (y_{k-l} - C\hat{x}_{k-l}) \quad (78a)$$

$$v_k^i = H_i \hat{x}_k + \sum_{l=0}^{l=d} K_{i,l} (y_{k-l} - C\hat{x}_{k-l}) \quad (78b)$$

$$w_k^i = H_i^d \hat{x}_{k-d} + \sum_{l=0}^{l=d} K_{i,l}^d (y_{k-d} - C\hat{x}_{k-d}) \quad (78c)$$

This leads to a more general LMI using the general Lyapunov-Krasovskii functional :

$$V_k = \varepsilon_k^T P \varepsilon_k + \sum_{j=1}^{j=d} \sum_{i=1}^{i=j} \varepsilon_{k-i}^T Q_j \varepsilon_{k-i}$$

4.3 Numerical example

Now, we present a numerical example to show the performances of the proposed method. We consider the modified chaotic system introduced in Cherrier et al. (2006), and described by :

$$\dot{x} = Gx + F(x(t), x(t - \tau)) \quad (79)$$

where

$$G = \begin{bmatrix} -\alpha & -\alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{bmatrix}, \quad F(x(t), x(t - \tau)) = \begin{bmatrix} -\alpha \delta \tanh(x_1(t)) \\ 0 \\ \epsilon \sin(\sigma x_1(t - \tau)) \end{bmatrix}$$

Since the proposed method concerns discrete-time systems, then we consider the discrete-time version of (79) obtained from the Euler discretization with sampling period $T = 0.01$. Hence, we obtain a system under the form (1a) with the following parameters :

$$A = I_3 + TG, \quad A_d = 0_{\mathbb{R}^{3 \times 3}}, \quad B = \begin{bmatrix} -\alpha \delta T & 0 \\ 0 & 0 \\ 0 & \epsilon T \end{bmatrix}$$

and

$$f(x_k, x_{k-d}) = \begin{bmatrix} \tanh(x_1(k)) \\ \sin(\sigma x_1(k-d)) \end{bmatrix}$$

that we can write under the form (35) with

$$H_1 = [1 \ 0 \ 0], \ H_1^d = [0 \ 0 \ 0]$$

$$H_2 = [0 \ 0 \ 0], \ H_2^d = [\sigma \ 0 \ 0]$$

Assume that the first component of the state x is measured, *i.e.* : $C = [1 \ 0 \ 0]$.

The system exhibits a chaotic behavior for the following numerical values :

$$\alpha = 9, \ \beta = 14, \ \gamma = 5, \ d = 2$$

$$\delta = 5, \ \epsilon = 1000, \ \sigma = 100$$

as can be shown in the figure 1.

The bounds of the partial derivatives of f are

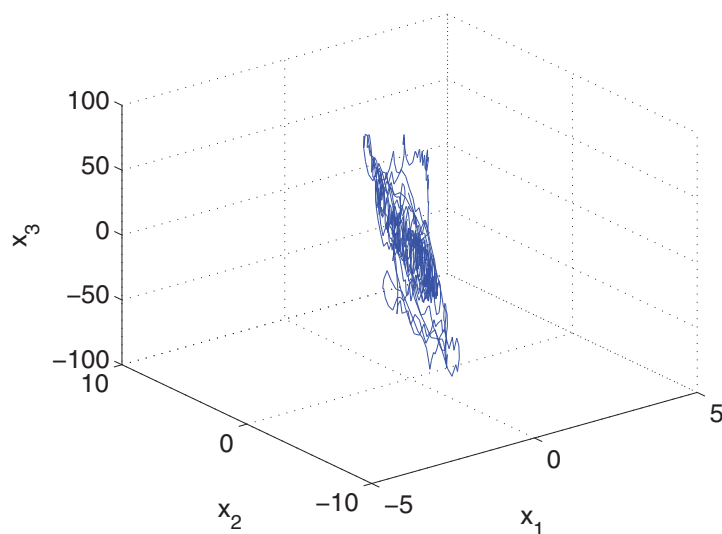


Fig. 1. Phase plot of the system

$$a_{11} = 1, \ b_{11} = 1, \ a_{21}^d = -1, \ b_{21}^d = 1$$

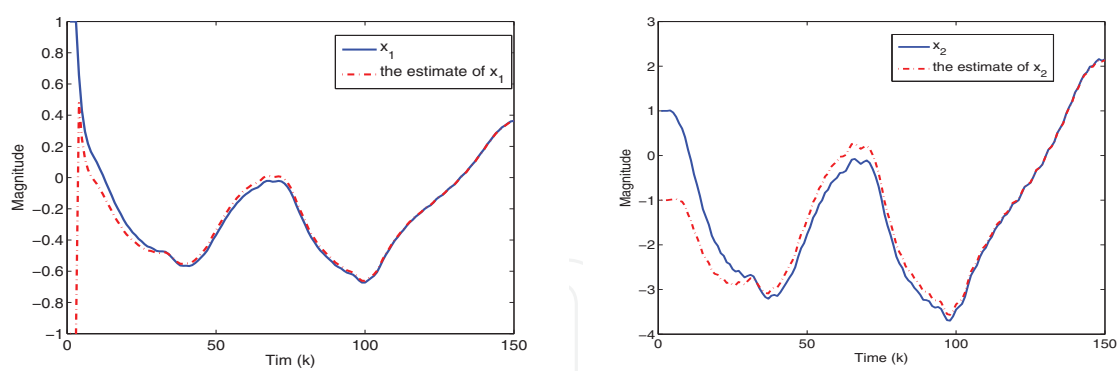
According to the remark 4.1, we must solve the LMI (48) with

$$\tilde{b}_{21}^d = b_{21}^d - a_{21}^d = 2, \ \tilde{A}_d = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -T\epsilon\sigma \end{bmatrix}$$

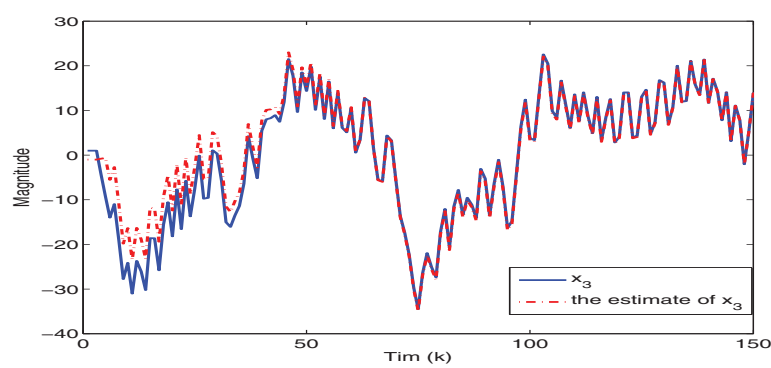
Hence, we obtain the following solutions :

$$L = \begin{bmatrix} 1.3394 \\ 4.9503 \\ 40.8525 \end{bmatrix}, \ L^d = \begin{bmatrix} 0 \\ 0 \\ -1000 \end{bmatrix}, \ K_1 = 0.9999, \ K_2 = -0.0425, \ K_1^d = -1.792 \times 10^{-13}, \ K_2^d = 100$$

The simulation results are shown in figure 2.



(a) The first component x_1 and its estimate \hat{x}_1 (b) The second component x_2 and its estimate \hat{x}_2



(c) The third component x_3 and its estimate \hat{x}_3

Fig. 2. Estimation error behavior

5. Conclusion

This chapter investigates the problem of observer design for a class of Lipschitz nonlinear time-delay systems in the discrete-time case. A new observer synthesis method is proposed, which leads to a less restrictive synthesis condition. Indeed, the obtained synthesis condition, expressed in term of LMI, contains more degree of freedom because of the general structure of the proposed observer. In order to take into account the noise (if it exists) which affects the considered system, a section is devoted to the study of H_∞ robustness. A dilated LMI condition is established particularly for systems with differentiable nonlinearities. Numerical examples are given in order to show the effectiveness of the proposed results.

A. Schur Lemma

In this section, we recall the *Schur lemma* and how it is used in the proof of Theorem 2.1.

Lemma A.1. *Boyd et al. (1994) Let Q_1, Q_2 and Q_3 be three matrices of appropriate dimensions such that $Q_1 = Q_1^T$ and $Q_3 = Q_3^T$. Then, the two following inequalities are equivalent :*

$$\begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} < 0, \quad (80)$$

$$Q_3 < 0 \text{ and } Q_1 - Q_2 Q_3^{-1} Q_2^T < 0. \quad (81)$$

Now, we use the *Lemma A.1* to demonstrate the equivalence between $\mathbb{M}_1 + \mathbb{M}_2 < 0$ and $\mathbb{M}_4 < 0$.
We have

$$\begin{aligned} \mathbb{M}_1 + \mathbb{M}_2 = & \begin{bmatrix} -P + Q & 0 & \tilde{A}^T P B \\ (*) & -Q & \tilde{A}_d^T P B \\ (*) & (*) & B^T P B - \alpha I_q \end{bmatrix} + \begin{bmatrix} \tilde{A}^T P \tilde{A} & \tilde{A}^T P \tilde{A}_d & 0 \\ (*) & \tilde{A}_d^T P \tilde{A}_d & 0 \\ (*) & (*) & 0 \end{bmatrix} \\ & + \frac{1}{\alpha \gamma_f^2} \begin{bmatrix} \mathbb{M}_{15}^T \mathbb{M}_{15} + \mathbb{M}_{16}^T \mathbb{M}_{16} & \mathbb{M}_{15}^T \mathbb{M}_{25} + \mathbb{M}_{16}^T \mathbb{M}_{26} & 0 \\ (*) & \mathbb{M}_{26}^T \mathbb{M}_{26} + \mathbb{M}_{25}^T \mathbb{M}_{25} & 0 \\ (*) & (*) & 0 \end{bmatrix}. \end{aligned} \quad (82)$$

By isolating the matrix

$$\Lambda = \begin{bmatrix} P & 0 & 0 \\ 0 & \alpha \gamma_f^2 I_{s_1} & 0 \\ 0 & 0 & \alpha \gamma_f^2 I_{s_2} \end{bmatrix}$$

we obtain

$$\mathbb{M}_1 + \mathbb{M}_2 = \begin{bmatrix} -P + Q & 0 & \tilde{A}^T P B \\ (*) & -Q & \tilde{A}_d^T P B \\ (*) & (*) & B^T P B - \alpha I_q \end{bmatrix} - \begin{bmatrix} \tilde{A}^T & \mathbb{M}_{15}^T & \mathbb{M}_{16}^T \\ \tilde{A}_d^T & \mathbb{M}_{25}^T & \mathbb{M}_{26}^T \\ 0 & 0 & 0 \end{bmatrix} Y (-\Lambda)^{-1} Y \begin{bmatrix} \tilde{A} & \tilde{A}_d & 0 \\ \mathbb{M}_{15} & \mathbb{M}_{25} & 0 \\ \mathbb{M}_{16} & \mathbb{M}_{26} & 0 \end{bmatrix} \quad (83)$$

where

$$Y = \begin{bmatrix} P & 0 & 0 \\ 0 & I_{s_1} & 0 \\ 0 & 0 & I_{s_2} \end{bmatrix}.$$

By setting

$$Q_1 = \begin{bmatrix} -P + Q & 0 & \tilde{A}^T P B \\ (*) & -Q & \tilde{A}_d^T P B \\ (*) & (*) & B^T P B - \alpha I_q \end{bmatrix}, \quad Q_2 = \begin{bmatrix} \tilde{A}^T & \mathbb{M}_{15}^T & \mathbb{M}_{16}^T \\ \tilde{A}_d^T & \mathbb{M}_{25}^T & \mathbb{M}_{26}^T \\ 0 & 0 & 0 \end{bmatrix} Y \text{ and } Q_3 = -\Lambda$$

we have

$$\mathbb{M}_1 + \mathbb{M}_2 = Q_1 - Q_2 Q_3^{-1} Q_2^T. \quad (84)$$

Since $Q_3 < 0$, we deduce from the *Lemma A.1* that

$$\mathbb{M}_1 + \mathbb{M}_2 < 0$$

is equivalent to (80), which is equivalent to

$$\mathbb{M}_4 < 0$$

where \mathbb{M}_4 is defined in (13). This ends the proof of equivalence between $\mathbb{M}_1 + \mathbb{M}_2 < 0$ and $\mathbb{M}_4 < 0$. The *Lemma A.1* is used of the same manner in theorem 3.2.

B. Some Details on Robust H_∞ Observer Design Problem

Hereafter, we show why the problem of robust H_∞ observer design is reduced to find a *Lyapunov* function V_k so that $W_k < 0$, where W_k is defined in (23). In other words, we show that $W_k < 0$ implies that the inequalities (21) and (22) are satisfied.

If $\omega(k) = 0$, we have $W_k < 0$ implies that $\Delta V < 0$. Then, from the *Lyapunov* theory, we deduce that the estimation error converges asymptotically towards zero, and then we have (21).

Now, if $\omega(k) \neq 0$; $\varepsilon(k) = 0, k = -d, \dots, 0$, we obtain $W_k < 0$ implies that

$$\sum_{k=0}^N \|\varepsilon(k)\|^2 < \frac{\lambda^2}{2} \sum_{k=0}^N \|\omega(k)\|^2 + \frac{\lambda^2}{2} \sum_{k=0}^N \|\omega_d(k)\|^2 - \sum_{k=0}^N (V_{k+1} - V_k) \quad (85)$$

Since without loss of generality, we have assumed that $\omega(k) = 0$ for $k = -d, \dots, -1$ and $\varepsilon(k) = 0, k = -d, \dots, 0$, we deduce that

$$\sum_{k=0}^N \|\varepsilon(k)\|^2 < \frac{\lambda^2}{2} \sum_{k=0}^N \|\omega(k)\|^2 + \frac{\lambda^2}{2} \sum_{k=0}^{N-d} \|\omega(k)\|^2 - V_N < \frac{\lambda^2}{2} \sum_{k=0}^N \|\omega(k)\|^2 + \frac{\lambda^2}{2} \sum_{k=0}^{N-d} \|\omega(k)\|^2. \quad (86)$$

When N tends toward infinity, we obtain

$$\sum_{k=0}^{\infty} \|\varepsilon(k)\|^2 \leq \frac{\lambda^2}{2} \sum_{k=0}^{\infty} \|\omega(k)\|^2 + \frac{\lambda^2}{2} \sum_{k=0}^{\infty-d} \|\omega(k)\|^2 \leq \frac{\lambda^2}{2} \sum_{k=0}^{\infty} \|\omega(k)\|^2 + \frac{\lambda^2}{2} \sum_{k=0}^{N-d} \|\omega(k)\|^2. \quad (87)$$

As

$$\sum_{k=0}^{\infty} \|\omega(k)\|^2 = \sum_{k=0}^{\infty-d} \|\omega(k)\|^2 = \|\omega\|_{\ell_2^2}^2$$

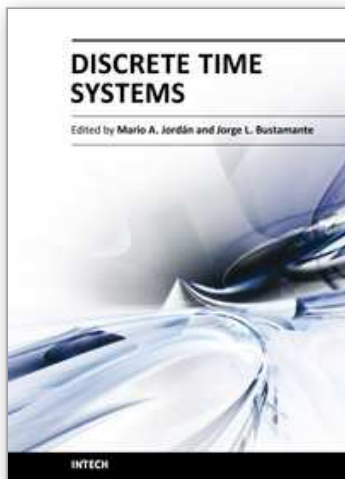
then the final relation (22) is inferred.

C. References

- Aggoune, W., Boutayeb, M. & Darouach, M. (1999). Observers design for a class of nonlinear systems with time-varying delay, *CDC'1999, Phoenix, Arizona USA. December*.
- Alessandri, A. (2004). Design of observers for Lipschitz nonlinear systems using LMI, *NOLCOS, IFAC Symposium on Nonlinear Control Systems, Stuttgart, Germany*.
- Boyd, S., El Ghaoui, L., Feron, E. & Balakrishnan, V. (1994). Linear matrix inequalities in system and control theory, *SIAM Studies in Applied Mathematics*, Philadelphia, USA.
- Cherrier, E., Boutayeb, M. & Ragot, J. (2005). Observers based synchronization and input recovery for a class of chaotic models, *Proceedings of the 44th IEEE Conference on Decision and Control and European Control Conference, Seville, Spain*.
- Cherrier, E., Boutayeb, M. & Ragot, J. (2006). Observers-based synchronization and input recovery for a class of nonlinear chaotic models, *IEEE Trans. Circuits Syst. I* 53(9): 1977–1988.
- Fan, X. & Arcak, M. (2003). Observer design for systems with multivariable monotone nonlinearities, *Systems and Control Letters* 50: 319–330.

- Germani, A., Manes, C. & Pepe, P. (2002). A new approach to state observation of nonlinear systems with delayed output, *IEEE Trans. Autom. Control* **47**(1): 96–101.
- Germani, A. & Pepe, P. (2004). An observer for a class of nonlinear systems with multiple discrete and distributed time delays, *16th MTNS, Leuven, Belgium*.
- Li, H. & Fu, M. (1997). A linear matrix inequality approach to robust H_∞ filtering, *IEEE Trans. on Signal Processing* **45**(9): 2338–2350.
- Lu, G. & Ho, D. W. C. (2004a). Robust H_∞ observer for a class of nonlinear discrete systems, *Proceedings of the 5th Asian Control Conference, ASCC2004, Melbourne, Australia*.
- Lu, G. & Ho, D. W. C. (2004b). Robust H_∞ observer for a class of nonlinear discrete systems with time delay and parameter uncertainties, *IEE Control Theory Application* **151**(4).
- Raff, T. & Allgöwer, F. (2006). An EKF-based observer for nonlinear time-delay systems, *2006 American Control Conference ACC'06, Minneapolis, Minnesota, USA*.
- Trinh, H., Aldeen, M. & Nahavandi, S. (2004). An observer design procedure for a class of nonlinear time-delay systems, *Computers & Electrical Engineering* **30**: 61–71.
- Xu, S., Lu, J., Zhou, S. & Yang, C. (2004). Design of observers for a class of discrete-time uncertain nonlinear systems with time delay, *Journal of the Franklin Institute* **341**: 295–308.
- Zemouche, A., Boutayeb, M. & Bara, G. I. (2006). On observers design for nonlinear time-delay systems, *2006 American Control Conference ACC'06, Minneapolis, Minnesota, USA*.
- Zemouche, A., Boutayeb, M. & Bara, G. I. (2007). Observer design for a class of nonlinear time-delay systems, *2007 American Control Conference ACC'07, New York, USA*.
- Zemouche, A., Boutayeb, M. & Bara, G. I. (2008). Observers for a class of Lipschitz systems with extension to \mathcal{H}_∞ performance analysis, *Systems & Control Letters* **57**(1): 18–27.

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Discrete-Time Systems comprehend an important and broad research field. The consolidation of digital-based computational means in the present, pushes a technological tool into the field with a tremendous impact in areas like Control, Signal Processing, Communications, System Modelling and related Applications. This book attempts to give a scope in the wide area of Discrete-Time Systems. Their contents are grouped conveniently in sections according to significant areas, namely Filtering, Fixed and Adaptive Control Systems, Stability Problems and Miscellaneous Applications. We think that the contribution of the book enlarges the field of the Discrete-Time Systems with signification in the present state-of-the-art. Despite the vertiginous advance in the field, we also believe that the topics described here allow us also to look through some main tendencies in the next years in the research area.

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