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Unscented Filtering Algorithm for Discrete-Time Systems with Uncertain Observations and State-Dependent Noise

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1. Introduction

Most conventional filtering algorithms address situations in which the signal to be estimated is always present in the observations. However, in many real situations, usually the measurement device or the transmission of such measurements can be subject to random failures, generating observations which may consist of noise only. More specifically, there is a positive probability (*false alarm probability*) that the signal to be estimated is not present in the corresponding observation; that is, the observations may be only noise (*uncertain observations*). Since it is not generally known whether the observation used for estimation contains the signal or it is only noise, and only the probabilities of occurrence of such cases are available to the estimation, the observation equation is designed by including a random multiplicative noise described by a sequence of Bernoulli random variables, whose values - one or zero - indicate the presence or absence of the signal in the observations, respectively.

The least-squares optimal estimation problem in systems with uncertain observations is not easily treatable in general, due to the fact that the multiplicative noise perturbing the observations causes that the joint distribution of the signal and the observations is not gaussian (even if the signal and additive noises are gaussian processes). For this reason, the research on the estimation problem in these systems has been focused on the search of suboptimal estimators for the signal that can be easily derived. Nahi (1969) was the first who described this observation model and analyzed the linear least-squares estimation problem in linear systems with independent uncertainty. After that, numerous studies have been developed in this context, assuming different hypotheses on the Bernoulli random variables modelling the uncertainty when the state-space model is known and, also, when only covariance information is available (see Nakamori et al. (2005) and references therein).

On the other hand, there are many practical applications in communication theory (phase modulation of analog communication systems, object tracking in video sequences, robot navigation, location tracking, navigation sensors, etc.) where the observations are not linear function of the signal to be estimated. Although the estimation problem in discrete-time systems from uncertain observations has been extensively studied in linear systems, the literature on nonlinear filtering with uncertainty, which is the focus of this chapter, is fairly

limited, with the exception of a few results such as those reported in NaNacara & Yaz (1997) and, more recently, in Hermoso & Linares (2007) and Nakamori et al. (2009).

Nonlinear filtering is an interesting research area in which many approaches have been developed, the most popular being the extended Kalman filter (see e.g. Simon (2006), among others), which approximates the optimal estimator by linearizing the nonlinear system equations around the last state estimate to generate a linear system to which the Kalman filter equations can be applied. This technique provides approximations of the mean and covariance of the signal which are accurate, at least, up to the first terms of their Taylor series expansions. Assuming full knowledge of the state-space model of the signal to be estimated, the extended Kalman filter has been widely applied by different authors. For example, in Angrisani et al. (2006) the discrete extended Kalman filter is used to estimate the shape factors of ultrasonic echo envelopes. Boussak (2005) addressed the speed and rotor position estimation problem of interior permanent magnet synchronous motor drive through an extended Kalman filter algorithm. The node localization problem in a delay-tolerant sensor network is studied in Pathirana et al. (2005) using an estimation technique based on the robust extended Kalman filter. Routray et al. (2002) applied an extended Kalman filter to the frequency estimation problem of distorted signals in power systems. When the state equations are unknown and only the covariance functions of the processes involved in the observation equation are available, Nakamori (1999) derived filtering and fixed-point smoothing algorithms for discrete-time systems with nonlinear observation mechanism, by using a similar idea to the extended Kalman filter.

Although the extended Kalman filter has been successfully applied to numerous nonlinear discrete systems, the use of truncated Taylor expansion yields some important drawbacks involving, on the one hand, the evaluation of the Jacobian matrices and, on the other, its instability. Among other nonlinear techniques, the unscented Kalman filtering (see e.g. Julier & Uhlmann (2004)), which does not require the calculation of Jacobian matrices, is a relatively new one that improves the extended one, providing approximations of the mean which are accurate up to the second term of its Taylor expansion.

Different generalizations of the extended and the unscented Kalman filters have been proposed in Hermoso & Linares (2007) for a class of nonlinear discrete-time systems with additive noises, using uncertain observations; from comparison between both techniques, superior performance of the unscented filter is also found for this class of systems.

The current chapter is concerned with the state estimation problem for nonlinear discrete-time systems with uncertain observations, when the evolution of the state is governed by nonlinear functions of the state and noise, and the additive noise of the observation is correlated with that of the state. The random interruptions in the observation process are modelled by a binary white noise taking either the value one (when the measurement is the current system output) or the value zero (when only noise is available). A filtering algorithm is designed using the scaled unscented transformation, which provides approximations of the first and second-order statistics of a nonlinear transformation of a random vector. This algorithm extends to that proposed in Hermoso & Linares (2007) in two directions. On the one hand, we consider a more general state transition model in which the noise is not necessarily additive and, on the other, the independence between the state and observation noises is removed, thus addressing those situations in which the observation noise is correlated with the state.

The chapter is organized as follows: in Section 2 the system model is described; more specifically, we introduce the nonlinear state transition model, perturbed by a white noise, and the observation model, governed by nonlinear functions of the state affected by an

additive white noise correlated with that of the state and a multiplicative noise describing the uncertainty. In Section 3 the least-squares estimation problem from uncertain observation is formulated and a brief review of the unscented transformation and the scaled unscented transformation is presented. Next, in Section 4, the estimation algorithm is derived using the unscented filtering procedure, which acts in the prediction and update steps. The filter update is accomplished by the Kalman filter equations, which require the conditional statistics of the observation; hence, the correlation between the state and observation noise must be taken into account in this phase. Finally, the performance of the proposed unscented filter is illustrated in Section 5 by a numerical simulation example, where a first order ARCH model is considered to describe the state evolution.

Keywords

Nonlinear stochastic systems, Uncertain observations, Unscented Kalman filter.

2. Nonlinear model: system description and assumptions

In some practical situations, there exist random failures in the observation mechanism, accidental loss of some measurements, or data inaccessibility during certain times; this causes that the measurements may be either the current system output or only noise. This occurs, for instance, in tracking systems where the observations may either contain actual output contaminated with noise or be noise alone, and only the probabilities of occurrence of such cases are available to the estimation.

Our aim is to estimate an n -dimensional discrete-time state process, $\{x_k; k \geq 0\}$, whose evolution is perturbed by a q -dimensional white noise, $\{w_k; k \geq 0\}$, and governed by known functions of the state and noise; that is:

$$x_{k+1} = f_k(x_k, w_k), \quad k \geq 0, \quad (1)$$

where $f_k : \mathbb{R}^{n+q} \rightarrow \mathbb{R}^n$ is assumed to be continuously differentiable with respect to x_k and w_k . Consider that the nonlinear observation, y_k , is either the current system output (with probability p_k) or only noise (with probability $1 - p_k$), and assume that this occurs independently at different sampling times. So, considering independent random variables $\gamma_k \in \{0, 1\}$, $k \geq 1$, with the understanding that $\gamma_k = 1$ means that the measurement at time k is the current system output and $\gamma_k = 0$ means that only noise is available, and assuming that $P[\gamma_k = 1] = p_k$, the observation model is specified by nonlinear functions of the state perturbed by additive white noise, $\{v_k; k \geq 1\}$, and multiplicative noise, $\{\gamma_k; k \geq 1\}$, describing the uncertainty; that is:

$$y_k = \gamma_k h_k(x_k) + v_k, \quad k \geq 1, \quad (2)$$

where $h_k : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are continuously differentiable functions.

The first and second-order moments of the processes determining the evolution of the state and describing the observations are specified by the following hypotheses.

(H1) The initial state, x_0 , is a random vector with mean \bar{x}_0 and covariance P_0 .

(H2) The state and observation white noises, $\{w_k; k \geq 0\}$ and $\{v_k; k \geq 1\}$, respectively, are correlated zero-mean processes with covariance matrices

$$\begin{aligned} E[w_k w_k^T] &= Q_k, \quad k \geq 0 \\ E[v_k v_k^T] &= R_k, \quad k \geq 1 \\ E[w_j v_k^T] &= S_k \delta_{j,k-1}, \quad k \geq 1, \end{aligned}$$

where δ denotes the Kronecker delta function.

(H3) The multiplicative noise $\{\gamma_k; k \geq 1\}$ describes the uncertainty in the observations and is a sequence of independent Bernoulli random variables with known probabilities $P[\gamma_k = 1] = p_k$. The probability $1 - p_k$, named *false alarm probability*, represents the probability that the observed value at time k does not contain the signal.

(H4) $x_0, (\{w_k; k \geq 0\}, \{v_k; k \geq 1\})$ and $\{\gamma_k; k \geq 1\}$ are mutually independent.

3. Least-squares estimation problem from uncertain observations

The least-squares estimator of the state x_k from the observations $Y^k = \{y_1, \dots, y_k\}$ is the conditional expectation of x_k given Y^k ,

$$E[x_k/Y^k] = \int x_k g(x_k/Y^k) dx_k,$$

and, hence, the computation of this conditional mean requires the knowledge of $g(x_k/Y^k)$, the conditional density function of x_k given Y^k .

Due to the uncertainty in the observations, this conditional density function is a mixture or weighted sum of 2^k conditional density functions (corresponding to the different values of $\gamma_1, \dots, \gamma_k$) and, moreover, computation of these conditional densities is generally difficult (even if the distributions of the processes involved in the system are known) due to the nonlinearity of the functions f_k and h_k . These severe drawbacks have motivated the search of suboptimal estimators based on approximations of the conditional mean to address the estimation problem in systems with uncertain observations and, more generally, in nonlinear systems.

One of the most frequently used methods to address the estimation problem in nonlinear systems without uncertainty in the observations (i.e. system models like (1)-(2) with $\gamma_k = 1, \forall k$) is the well-known *extended Kalman filter* (Simon (2006)), based on the linearization of the state and observation equations. However, as indicated in Julier & Uhlmann (2004), the extended Kalman filter has serious handicaps, which should be kept in mind when it is used; in particular, the performance of this filter can be very poor if the functions f_k and h_k present intense nonlinearities. In such cases, alternative approximations improving the estimation must be used. Among others, the *unscented Kalman filter* is a superior alternative to the extended one in a great variety of application domains, including state estimation.

Among other advantages, the unscented Kalman filter overcomes the deficiencies of linearization of the extended Kalman filter by providing an algorithm based on a direct, explicit mechanism for transforming the mean and covariance information when a nonlinear function is considered (Julier & Uhlmann (2004)). Unlike the linearization operation of the extended Kalman filter, the unscented Kalman filter uses the nonlinear models directly; it captures the posterior mean and covariance accurately up to the terms corresponding to the third-order moments in the Taylor series expansions, for the Gaussian distribution, and at least up to second-order for an arbitrary distribution.

Various generalizations of the extended and unscented Kalman filters were proposed in Hermoso & Linares (2007) for a class of nonlinear discrete-time systems with uncertain observations, when the state and observation noises are additive and the Bernoulli variables modelling the uncertainty are independent; from comparison of both techniques, superior performance of the unscented filter is also found in this uncertainty case.

We propose a modification of the unscented Kalman filter for estimating the state of the nonlinear system model with uncertain observations described in Section 2. This filter provides an approximation to the conditional mean $E[x_k/Y^k]$ based on the use of the unscented transformation; more precisely, we will use an extension of this transformation, called scaled unscented transformation. Both transformations are briefly described below (see Julier & Uhlmann (2004) for details).

3.1 Unscented Transformation (UT)

This is a method for approximating the mean and covariance matrix of a random vector, $Y = g(X)$, from the mean and covariance of X . The idea is to choose deterministically a fixed number of points and weights which capture the mean and covariance of X exactly; then, the mean and covariance of $g(X)$ are approximated by the weighted sample mean and covariance of the transformed points.

Specifically, if X is an N -dimensional random vector with mean \hat{X} and covariance P_X , the UT considers $2N + 1$ points and weights, $\{(\xi_i, \psi_i), i = 0, \dots, 2N\}$, called *sigma points*, which are defined as follows:

$$\begin{aligned}\xi_0 &= \hat{X} \\ \xi_i &= \hat{X} + \left(\sqrt{(N + \kappa)P_X} \right)_i, \quad i = 1, \dots, N \\ \xi_i &= \hat{X} - \left(\sqrt{(N + \kappa)P_X} \right)_{i-N}, \quad i = N + 1, \dots, 2N \\ \psi_0 &= \frac{\kappa}{N + \kappa} \\ \psi_i &= \frac{1}{2(N + \kappa)}, \quad i = 1, \dots, 2N,\end{aligned}\tag{3}$$

where κ is a scaling parameter which can be used to capture additional information on the distribution of X , and $(A)_j$ denotes the j -th column of a matrix A .

Although $\sum_{i=0}^{2N} \psi_i = 1$, the sigma points $\{(\xi_i, \psi_i), i = 0, \dots, 2N\}$ do not necessarily define a probability distribution since κ can be a negative number (the only condition is $N + \kappa > 0$); however, its moments can be defined as in a discrete probability distribution, and it is easy to prove that the first and second-order moments of the sigma points are equal to those of X . To approximate the statistics of a transformation $Y = g(X)$, each point ξ_i is propagated through the function g , and the first and second order moments of Y are approximated by those of the transformed sigma points $g(\xi_i)$, $i = 0, \dots, 2N$. Therefore, the mean of Y is approximated by the weighted average of the transformed points,

$$\hat{Y} \approx \sum_{i=0}^{2N} \psi_i g(\xi_i),$$

the covariance of Y is approximated by

$$P_Y \approx \sum_{i=0}^{2N} \psi_i \left(g(\xi_i) - \hat{Y} \right) \left(g(\xi_i) - \hat{Y} \right)^T$$

and the cross-covariance of X and Y is approximated by

$$P_{XY} \approx \sum_{i=0}^{2N} \psi_i \left(\xi_i - \hat{X} \right) \left(g(\xi_i) - \hat{Y} \right)^T.$$

If g is an analytic function, the approximations of the mean and covariance of $g(X)$ are accurate up to the second and first term of their Taylor expansion series, respectively. However, the approximations are inaccurate if the higher-order sample moments have a great effect on the Taylor expansions; this can occur, for example, if the dimension, N , of vector X is very large, since the radius of the sphere containing the sigma points increases with N . This drawback can be avoided by using the scaled unscented transformation.

3.2 Scaled Unscented Transformation (SUT)

The SUT considers a set of sigma points, $\chi_i = \xi_0 + \alpha(\xi_i - \xi_0)$, $i = 0, \dots, 2N$, where α is a scaling parameter which can be arbitrarily small. The points χ_i have basically the same form as in (3), just replacing κ by $\lambda = \alpha^2(N + \kappa) - N$; the associated weights, calculated in order to capture the mean and covariance of X , are now

$$W_0 = \frac{\psi_0}{\alpha^2} + (1 - 1/\alpha^2)$$

$$W_i = \frac{\psi_i}{\alpha^2}, \quad i = 1, \dots, 2N.$$

Besides reducing the dispersion of the sigma points considered, the SUT allows to modify them in order to prevent nonpositive semidefinite approximated covariances (which can occur if $W_0 < 0$), as well as to incorporate additional information on the fourth-order moments of X ; this is achieved by modifying the weight of χ_0 in the approximation of the covariance, which improves the precision in this approximation. Thus, the sigma points and weights in the SUT are specified as follows:

$$\begin{aligned} \chi_0 &= \hat{X} \\ \chi_i &= \hat{X} + \left(\sqrt{(N + \lambda)P_X} \right)_i, \quad i = 1, \dots, N \\ \chi_i &= \hat{X} - \left(\sqrt{(N + \lambda)P_X} \right)_{i-N}, \quad i = N + 1, \dots, 2N \\ W_0^{(m)} &= \frac{\lambda}{N + \lambda} \\ W_0^{(c)} &= W_0^{(m)} + (1 - \alpha^2 + \beta) \\ W_i^{(m)} &= W_i^{(c)} = \frac{1}{2(N + \lambda)}, \quad i = 1, \dots, 2N \\ \lambda &= \alpha^2(N + \kappa) - N, \end{aligned}$$

where α is the scaling parameter (usually a small value), and κ and β are used to incorporate prior information on the distribution of X ($\kappa = 3 - N$ and $\beta = 2$ provide the optimal values if X has a Gaussian distribution).

The mean and covariance of a transformation $g(X)$ are approximated, respectively, by the sample mean and covariance of the transformed values, $g(\chi_i)$, $i = 0, \dots, 2N$, with weights $W_i^{(m)}$ for the mean and $W_i^{(c)}$ for the covariance. The cross-covariance of X and $Y = g(X)$ is approximated by the sample cross-covariance of χ_i , $i = 0, \dots, 2N$ and the transformed values, $g(\chi_i)$, $i = 0, \dots, 2N$, with weights $W_i^{(c)}$.

4. Unscented filtering algorithm

The aim is to obtain an estimator of x_k , the system state at time k described in (1), based on the observations given in (2) up to that time, $Y^k = \{y_1, \dots, y_k\}$; for this purpose we compute an approximation, $\hat{x}_{k/k}$, of the conditional mean $E[x_k/Y^k]$. As usual, the estimator $\hat{x}_{k/k}$ will be obtained from the estimator at the previous time, $\hat{x}_{k-1/k-1}$, through the following prediction and update steps:

- (i) *Prediction:* Taking into account the relationship (1), approximations $\hat{x}_{k/k-1}$ and $P_{k,k/k-1}^{xx}$ of $E[x_k/Y^{k-1}]$ and $Cov[x_k/Y^{k-1}]$, respectively, are obtained by applying a SUT to the nonlinear transformation $x_k = f_{k-1}(x_{k-1}, w_{k-1})$; then, this step requires us to work jointly with the state and noise vectors x_{k-1} and w_{k-1} .
- (ii) *Update:* When the predictor $\hat{x}_{k/k-1}$ is available, it is updated with the new observation y_k to obtain an approximation of $E[x_k/Y^k]$ and $Cov[x_k/Y^k]$; this is achieved using the following expression, with a similar structure to those of the Kalman filter:

$$E[x_k/Y^k] \approx \hat{x}_{k/k} = \hat{x}_{k/k-1} + Cov[x_k, y_k/Y^{k-1}] \left(Cov[y_k/Y^{k-1}] \right)^{-1} (y_k - E[y_k/Y^{k-1}]).$$

This expression requires the conditional statistics of y_k ; specifically, it is necessary to approximate the conditional mean and covariance, $E[y_k/Y^{k-1}]$ and $Cov[y_k/Y^{k-1}]$, as well as the conditional cross-covariance $Cov[x_k, y_k/Y^{k-1}]$.

Hence, in view of (2), the correlation between x_k and v_k must be taken into account in this step. More specifically, since $x_k = f_{k-1}(x_{k-1}, w_{k-1})$, the correlation between w_{k-1} and v_k must be taken into account.

These reasons lead us to work jointly with the vectors x_{k-1} , w_{k-1} and v_k and hence, we define the following $(n + q + m)$ -dimensional augmented vectors:

$$X_k = \begin{pmatrix} x_k \\ w_k \\ v_{k+1} \end{pmatrix}, \quad k \geq 0.$$

The problem is then reformulated as that of finding the filter of this augmented vector, $\hat{X}_{k/k}$, whose first n -dimensional block-component provides the filter for the original state.

The prediction and update steps are detailed in the following subsections.

4.1 Unscented algorithm: prediction step

The starting points of the proposed algorithm are the filter and the covariance matrix at the initial state X_0 which, from the model hypotheses, are given by:

$$\hat{X}_{0/0} = E[X_0] = \begin{pmatrix} \bar{x}_0 \\ 0 \\ 0 \end{pmatrix}, \quad P_{0,0/0}^{XX} = Cov[X_0] = \begin{pmatrix} P_0 & 0 & 0 \\ 0 & Q_0 & S_1 \\ 0 & S_1^T & R_1 \end{pmatrix}.$$

For each $k > 1$, we start with approximations $\hat{X}_{k-1/k-1}$ and $P_{k-1,k-1/k-1}^{XX}$ of the conditional mean and covariance of X_{k-1} given Y^{k-1} which, from the independence between (w_{k-1}, v_k)

and Y^{k-1} , the conditional independence between x_{k-1} and (w_{k-1}, v_k) and hypothesis (H2), are given by:

$$\hat{X}_{k-1/k-1} = \begin{pmatrix} \hat{x}_{k-1/k-1} \\ 0 \\ 0 \end{pmatrix}, \quad P_{k-1,k-1/k-1}^{XX} = \begin{pmatrix} P_{k-1,k-1/k-1}^{xx} & 0 & 0 \\ 0 & Q_{k-1} & S_k \\ 0 & S_k^T & R_k \end{pmatrix}.$$

The aim is to find approximations $\hat{X}_{k/k-1}$ and $P_{k,k/k-1}^{XX}$ for the conditional mean and covariance of X_k given Y^{k-1} which, reasoning similarly, are

$$\hat{X}_{k/k-1} = \begin{pmatrix} \hat{x}_{k/k-1} \\ 0 \\ 0 \end{pmatrix}, \quad P_{k,k/k-1}^{XX} = \begin{pmatrix} P_{k,k/k-1}^{xx} & 0 & 0 \\ 0 & Q_k & S_{k+1} \\ 0 & S_{k+1}^T & R_{k+1} \end{pmatrix}. \quad (4)$$

Hence, we only need the conditional statistics $\hat{x}_{k/k-1}$ and $P_{k,k/k-1}^{xx}$ of $x_k = f_{k-1}(x_{k-1}, w_{k-1})$, which are approximated from $\hat{X}_{k-1/k-1}$ and $P_{k-1,k-1/k-1}^{XX}$ using the SUT, as follows:

- We consider a set of sigma-points

$$\left\{ \chi_{i,k-1/k-1} = \left(\chi_{i,k-1/k-1}^x \mid \chi_{i,k-1/k-1}^w \mid \chi_{i,k-1/k-1}^v \right)^T, i = 0, \dots, 2N \right\} \quad (N = n + q + m),$$

whose mean and covariance are exactly $\hat{X}_{k-1/k-1}$ and $P_{k-1,k-1/k-1}^{XX}$:

$$\begin{aligned} \chi_{0,k-1/k-1} &= \hat{X}_{k-1/k-1} \\ \chi_{i,k-1/k-1} &= \hat{X}_{k-1/k-1} + \left(\sqrt{(N + \lambda) P_{k-1,k-1/k-1}^{XX}} \right)_i, \quad i = 1, \dots, N \\ \chi_{i,k-1/k-1} &= \hat{X}_{k-1/k-1} - \left(\sqrt{(N + \lambda) P_{k-1,k-1/k-1}^{XX}} \right)_{i-N}, \quad i = N + 1, \dots, 2N \end{aligned} \quad (5)$$

and their associated weights, $W_i^{(m)}$ for the mean and $W_i^{(c)}$ for the covariance:

$$\begin{aligned} W_0^{(m)} &= \frac{\lambda}{N + \lambda} \\ W_0^{(c)} &= \frac{\lambda}{N + \lambda} + (1 - \alpha^2 + \beta) \\ W_i^{(m)} &= W_i^{(c)} = \frac{1}{2(N + \lambda)}, \quad i = 1, \dots, 2N \\ \lambda &= \alpha^2(N + \kappa) - N \end{aligned}$$

where α is a scaling parameter determining the spread of the sigma-points around $\hat{X}_{k-1/k-1}$, and κ and β are tuning parameters.

- Then, by defining $f_{k-1}^a(X_{k-1}) = f_{k-1}(x_{k-1}, w_{k-1}) = x_k$, the mean and covariance of x_k given Y^{k-1} are approximated by the corresponding sample statistics of the transformed sigma-points, $f_{k-1}^a(\chi_{i,k-1/k-1}) = f_{k-1} \left(\chi_{i,k-1/k-1}^x \mid \chi_{i,k-1/k-1}^w \right)$:

$$\begin{aligned}\hat{x}_{k/k-1} &= \sum_{i=0}^{2N} W_i^{(m)} f_{k-1}^a(\chi_{i,k-1/k-1}) \\ P_{k,k/k-1}^{xx} &= \sum_{i=0}^{2N} W_i^{(c)} \left(f_{k-1}^a(\chi_{i,k-1/k-1}) - \hat{x}_{k/k-1} \right) \left(f_{k-1}^a(\chi_{i,k-1/k-1}) - \hat{x}_{k/k-1} \right)^T.\end{aligned}\quad (6)$$

The conditional mean and covariance of X_k given Y^{k-1} are then approximated by (4) with $\hat{x}_{k/k-1}$ and $P_{k,k/k-1}^{xx}$ given in (6).

4.2 Unscented algorithm: update step

The approximations $\hat{X}_{k/k-1}$ and $P_{k,k/k-1}^{XX}$ given in (4) and (6) are now updated with the new observation, y_k , by using the Kalman filter equations

$$\begin{aligned}\hat{X}_{k/k} &= \hat{X}_{k/k-1} + Cov[X_k, y_k/Y^{k-1}] \left(Cov[y_k/Y^{k-1}] \right)^{-1} \left(y_k - E[y_k/Y^{k-1}] \right) \\ P_{k,k/k}^{XX} &= P_{k,k/k-1}^{XX} - Cov[X_k, y_k/Y^{k-1}] \left(Cov[y_k/Y^{k-1}] \right)^{-1} Cov[y_k, X_k/Y^{k-1}].\end{aligned}$$

For this purpose, we need to approximate the conditional mean, $E[y_k/Y^{k-1}]$, and covariance, $Cov[y_k/Y^{k-1}]$, of y_k given Y^{k-1} , as well as the conditional cross-covariance of X_k and y_k given Y^{k-1} , $Cov[X_k, y_k/Y^{k-1}]$.

In systems with uncertain observations, the conditional distribution of $\gamma_k h_k(x_k)$ given Y^{k-1} has a mixture type whose components are the conditional distributions corresponding to $\gamma_k = 1$ and $\gamma_k = 0$, with mixture parameters $P[\gamma_k = 1/Y^{k-1}]$ and $P[\gamma_k = 0/Y^{k-1}]$, respectively. Since $P[\gamma_k = 1/Y^{k-1}] = p_k$ (which follows from (H3) and (H4)), the approximations of the conditional statistics of $\gamma_k h_k(x_k)$ are directly obtained using this mixture type, and, taking into account (2), the statistics of y_k given Y^{k-1} are expressed in terms of those corresponding to $z_k = h_k(x_k)$ and v_k as follows:

$$\begin{aligned}E[y_k/Y^{k-1}] &= p_k E[z_k/Y^{k-1}] \\ Cov[y_k/Y^{k-1}] &= p_k Cov[z_k/Y^{k-1}] + p_k(1-p_k)E[z_k/Y^{k-1}]E[z_k^T/Y^{k-1}] \\ &\quad + p_k Cov[z_k, v_k/Y^{k-1}] + p_k Cov[v_k, z_k/Y^{k-1}] + R_k \\ Cov[X_k, y_k/Y^{k-1}] &= p_k Cov[X_k, z_k/Y^{k-1}] + Cov[X_k, v_k/Y^{k-1}].\end{aligned}\quad (7)$$

Moreover, since z_k and v_k are conditionally independent of w_k and v_{k+1} , the conditional cross-covariances $Cov[X_k, z_k/Y^{k-1}]$ and $Cov[X_k, v_k/Y^{k-1}]$ require only the conditional cross-covariances of x_k with z_k and v_k , respectively; that is:

$$Cov[X_k, y_k/Y^{k-1}] = \begin{pmatrix} p_k Cov[x_k, z_k/Y^{k-1}] + Cov[x_k, v_k/Y^{k-1}] \\ 0 \\ 0 \end{pmatrix}\quad (8)$$

Then, we proceed to approximate the conditional statistics appearing in (7) and (8), which correspond to the vectors z_k , x_k and v_k .

- The first two vectors, z_k and x_k , are both functions of x_k and, consequently, their conditional statistics can be approximated from $\hat{x}_{k/k-1}$ and $P_{k,k/k-1}^{xx}$ by considering a set of sigma-points, $\{\chi_{i,k/k-1}^x, i = 0, \dots, 2n\}$, whose mean and covariance are exactly $\hat{x}_{k/k-1}$ and $P_{k,k/k-1}^{xx}$:

$$\begin{aligned}\chi_{0,k/k-1}^x &= \hat{x}_{k/k-1} \\ \chi_{i,k/k-1}^x &= \hat{x}_{k/k-1} + \left(\sqrt{(n+\lambda)P_{k,k/k-1}^{xx}} \right)_i, \quad i = 1, \dots, n \\ \chi_{i,k-1/k-1}^x &= \hat{x}_{k/k-1} - \left(\sqrt{(n+\lambda)P_{k,k/k-1}^{xx}} \right)_{i-n}, \quad i = n+1, \dots, 2n\end{aligned}\quad (9)$$

and their associated weights, $\omega_i^{(m)}$ for the mean and $\omega_i^{(c)}$ for the covariance:

$$\begin{aligned}\omega_0^{(m)} &= \frac{\lambda}{n+\lambda} \\ \omega_0^{(c)} &= \frac{\lambda}{n+\lambda} + (1 - \alpha^2 + \beta) \\ \omega_i^{(m)} &= \omega_i^{(c)} = \frac{1}{2(n+\lambda)}, \quad i = 1, \dots, 2n \\ \lambda &= \alpha^2(n + \kappa) - n.\end{aligned}$$

Then the statistics of $z_k = h_k(x_k)$ are approximated by those of the transformed sigma-points, $h_k(\chi_{i,k/k-1}^x)$:

$$\begin{aligned}E[z_k/Y^{k-1}] &\approx \hat{z}_{k/k-1} = \sum_{i=0}^{2n} \omega_i^{(m)} h_k(\chi_{i,k/k-1}^x) \\ Cov[z_k/Y^{k-1}] &\approx P_{k,k/k-1}^{zz} = \sum_{i=0}^{2n} \omega_i^{(c)} \left(h_k(\chi_{i,k/k-1}^x) - \hat{z}_{k/k-1} \right) \left(h_k(\chi_{i,k/k-1}^x) - \hat{z}_{k/k-1} \right)^T \\ Cov[x_k, z_k/Y^{k-1}] &\approx P_{k,k/k-1}^{xz} = \sum_{i=0}^{2n} \omega_i^{(c)} \left(\chi_{i,k/k-1}^x - \hat{x}_{k/k-1} \right) \left(h_k(\chi_{i,k/k-1}^x) - \hat{z}_{k/k-1} \right)^T.\end{aligned}\quad (10)$$

- The vector v_k , however, cannot be expressed in terms of $X_k = (x_k^T | w_k^T | v_{k+1}^T)^T$, but it is a function of X_{k-1} ; so its conditional statistics must be approximated from those of X_{k-1} . Thus, expressing $z_k = h_k(f_{k-1}^a(X_{k-1})) = h_k(f_{k-1}(x_{k-1}, w_{k-1}))$ and using the sigma-points

$$\chi_{i,k-1/k-1} = \left(\chi_{i,k-1/k-1}^{xT} | \chi_{i,k-1/k-1}^{wT} | \chi_{i,k-1/k-1}^{vT} \right)^T, \quad i = 0, \dots, 2N$$

associated to $\hat{X}_{k-1/k-1}$ and $P_{k-1,k-1/k-1}^{XX}$, the following approximations are used:

$$\text{Cov}[z_k, v_k/Y^{k-1}] \approx P_{k,k/k-1}^{zv} = \sum_{i=0}^{2N} W_i^{(c)} h_k \left(f_{k-1}(\chi_{i,k-1/k-1}^x, \chi_{i,k-1/k-1}^w) \right) \chi_{i,k/k-1}^{vT} \quad (11)$$

$$\text{Cov}[x_k, v_k/Y^{k-1}] \approx P_{k,k/k-1}^{xv} = \sum_{i=0}^{2N} W_i^{(c)} f_{k-1}(\chi_{i,k-1/k-1}^x, \chi_{i,k-1/k-1}^w) \chi_{i,k/k-1}^{vT}.$$

Finally, these statistics are substituted in (7) and (8) to obtain approximations $\hat{y}_{k/k-1}$, $P_{k,k/k-1}^{yy}$ and $P_{k,k/k-1}^{Xy}$ of the conditional statistics of y_k ,

$$\begin{aligned} \hat{y}_{k/k-1} &= p_k \hat{z}_{k/k-1} \\ P_{k,k/k-1}^{yy} &= p_k P_{k,k/k-1}^{zz} + p_k(1-p_k) \hat{z}_{k/k-1} \hat{z}_{k/k-1}^T + p_k P_{k,k/k-1}^{zv} + p_k P_{k,k/k-1}^{vz} + R_k, \\ P_{k,k/k-1}^{Xy} &= \begin{pmatrix} p_k P_{k,k/k-1}^{xz} + P_{k,k/k-1}^{xv} \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \quad (12)$$

These approximations are used in the following equations providing the filter of X_k and the corresponding filtering error covariance matrix:

$$\begin{aligned} \hat{X}_{k/k} &= \hat{X}_{k/k-1} + P_{k,k/k-1}^{Xy} \left(P_{k,k/k-1}^{yy} \right)^{-1} (y_k - \hat{y}_{k/k-1}), \quad k \geq 1 \\ P_{k,k/k}^{XX} &= P_{k,k/k-1}^{XX} - P_{k,k/k-1}^{Xy} \left(P_{k,k/k-1}^{yy} \right)^{-1} P_{k,k/k-1}^{yX}, \quad k \geq 1, \end{aligned} \quad (13)$$

with initial conditions

$$\hat{X}_{0/0} = \begin{pmatrix} \bar{x}_0 \\ 0 \\ 0 \end{pmatrix}, \quad P_{0,0/0}^{XX} = \begin{pmatrix} P_0 & 0 & 0 \\ 0 & Q_0 & S_1 \\ 0 & S_1^T & R_1 \end{pmatrix}. \quad (14)$$

Computational summary

In summary, given $\hat{X}_{k-1/k-1}$ and $P_{k-1,k-1/k-1}^{XX}$, the above results suggest the following recursive computational procedure to obtain the proposed unscented filter:

- (I) Compute the sigma-points given in (5), whose mean and covariance are $\hat{X}_{k-1/k-1}$ and $P_{k-1,k-1/k-1}^{XX}$, respectively, and, with them:
 - (Ia) Compute $\hat{x}_{k/k-1}$ and $P_{k,k/k-1}^{xx}$ by (6).
 - (Ib) Compute $P_{k,k/k-1}^{zv}$ and $P_{k,k/k-1}^{xv}$ by (11).
- (II) Compute the sigma-points given in (9), whose mean and covariance are $\hat{x}_{k/k-1}$ and $P_{k,k/k-1}^{xx}$, respectively, and, with them, compute $\hat{z}_{k/k-1}$, $P_{k,k/k-1}^{zz}$, and $P_{k,k/k-1}^{xz}$ by (10).
- (III) From (Ia), compute $\hat{X}_{k/k-1}$ and $P_{k,k/k-1}^{XX}$ by (4).
- (IV) From (Ib) and (II), compute $\hat{y}_{k/k-1}$, $P_{k,k/k-1}^{yy}$ and $P_{k,k/k-1}^{Xy}$ by (12).
- (V) From (III) and (IV), compute $\hat{X}_{k/k}$ and $P_{k,k/k}^{XX}$ by (13).

The initial conditions of the proposed algorithm, $\hat{X}_{0/0}$ and $P_{0,0/0}^{XX}$, are given in (14).

Finally, by extracting the first n -dimensional block-components of $\hat{X}_{k/k}$ and $P_{k,k/k}^{XX}$, the filter of the original state vector, x_k , and the filtering error covariance matrix are obtained, respectively.

Remark 1: Although the derivation of the algorithm does not require that the functions f_k and h_k are continuously differentiable, these hypotheses guarantee that the approximations of the conditional mean and covariances are accurate, at least, up to the first and second terms of their Taylor series expansions, respectively.

Remark 2: The proposed algorithm reduces to that in Hermoso & Linares (2007) when the functions f_k are linear in the noise, and the state and observation noises are uncorrelated. Moreover, the unscented filter agrees with the optimal linear one when the functions f_k and h_k are linear.

5. Numerical simulation results

In this section, a numerical simulation example is presented to illustrate the application of the proposed unscented filter. The application deals with a first order autoregressive conditional heteroscedastic model (ARCH (1)); these models, introduced by Engle in 1982 and widely known in volatility modelling in finance (Peiris & Thavaneswaran (2007)), have been considered in Tanizaki (2000) as an example to compare the performance of various nonlinear filters when the observed variables consist of a sum of the ARCH (1) process and an independent error term.

Here, according to the theoretic study, we assume that the measurements can be only the error term with a known probability, and that the noise process is correlated with the ARCH (1) process.

Let us consider that the evolution of the state is described by the following discrete-time multiplicative transition equation

$$x_{k+1} = \sqrt{a + bx_k^2} w_k, \quad k \geq 0,$$

where the initial state x_0 is a Gaussian variable with zero mean and unity variance, the noise $\{w_k; k \geq 0\}$ is a zero-mean Gaussian process with variance $Q_k = 1$ and $a = 1 - b$ is taken to normalize the unconditional variance of x_k to be one.

Uncertain observations of the state with additive noise are considered for the estimation:

$$y_k = \gamma_k x_k + v_k, \quad k \geq 1,$$

where $\{v_k; k \geq 1\}$ is a zero-mean white process with variance $R_k = 1$, and the multiplicative noise, $\{\gamma_k; k \geq 1\}$, is a sequence of independent Bernoulli variables with constant known probability $P[\gamma_k = 1] = p$.

The state and additive observation noises $\{w_k; k \geq 0\}$ and $\{v_k; k \geq 1\}$ are assumed to be joint Gaussian processes with known and constant cross-covariance $S_k = S, \forall k$.

We have implemented a MATLAB program that simulates the state x_k for $b = 0.5$, and the uncertain measurements, y_k , for $k = 1, \dots, 50$, for different values of S and p , and provides the unscented filtering estimates of x_k .

The root mean square error (RMSE) criterion was used to quantify the performance of the estimates. Considering 1000 independent simulations and denoting by $\{x_k^{(s)}, k = 1, \dots, 50\}$

the s -th set of the artificially simulated states and by $\hat{x}_{k/k}^{(s)}$ the filtering estimate at time k in the s -th simulation run, the RMSE of the filter at time k is calculated by

$$\text{RMSE}_k = \left(\frac{1}{1000} \sum_{s=1}^{1000} \left(x_k^{(s)} - \hat{x}_{k/k}^{(s)} \right)^2 \right)^{1/2}.$$

Let us first examine the performance of the algorithm for different values of S ; since analogous results are obtained for opposite correlations S and $-S$, only nonnegative values are considered in the simulations shown here. Figure 1 displays the RMSE_k when the uncertainty probability is $p = 0.5$ and different values of S are considered; specifically, $S = 0, 0.3, 0.5, 0.7$ and $S = 0.9$; this figure shows, as expected, that the higher the value of S (which means that the correlation between the state and the observations increases) the smaller that of RMSE_k and, consequently, the performance of the estimators is better. Analogous results are obtained for other different values of p and S .

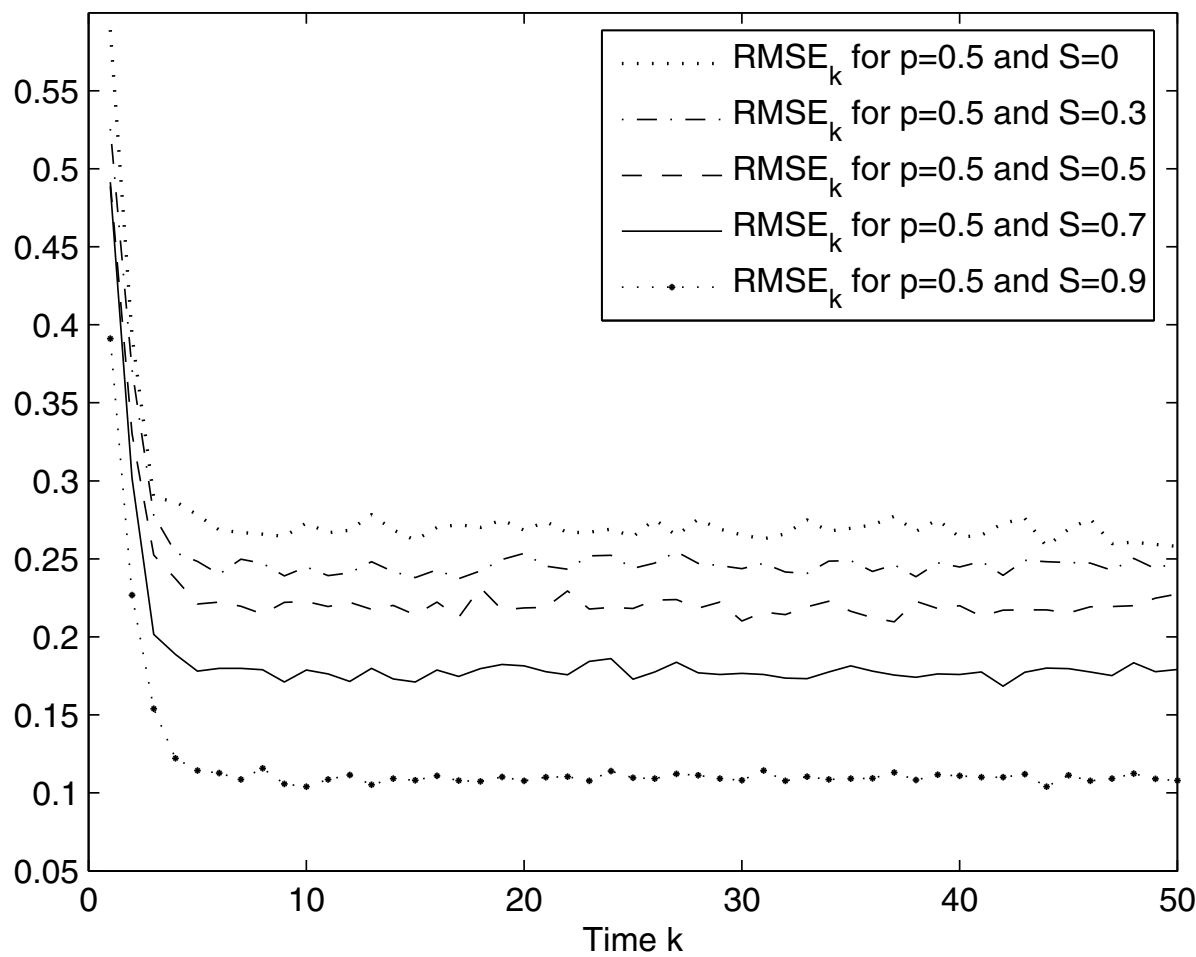


Fig. 1. RMSE_k for the unscented filtering estimates when $p = 0.5$ and $S = 0, 0.3, 0.5, 0.7, 0.9$. Moreover, in order to compare the performance of the estimators as a function of the uncertainty probability p , the means of RMSE_k corresponding to the 50 iterations were

calculated for the different values of S considered in Figure 1 and $p = 0.1, 0.2, \dots, 0.9$. The results are shown in Figure 2, from which it is apparent that the means decrease when p increases (that is, when the probability that the observations contain the state is greater) and consequently, as expected, the performance of the estimators deteriorates as the probability p falls. From this figure, it is also inferred that, for each fixed value of p , the means decrease as S increases, which extends the result in Figure 1 to different values of p .

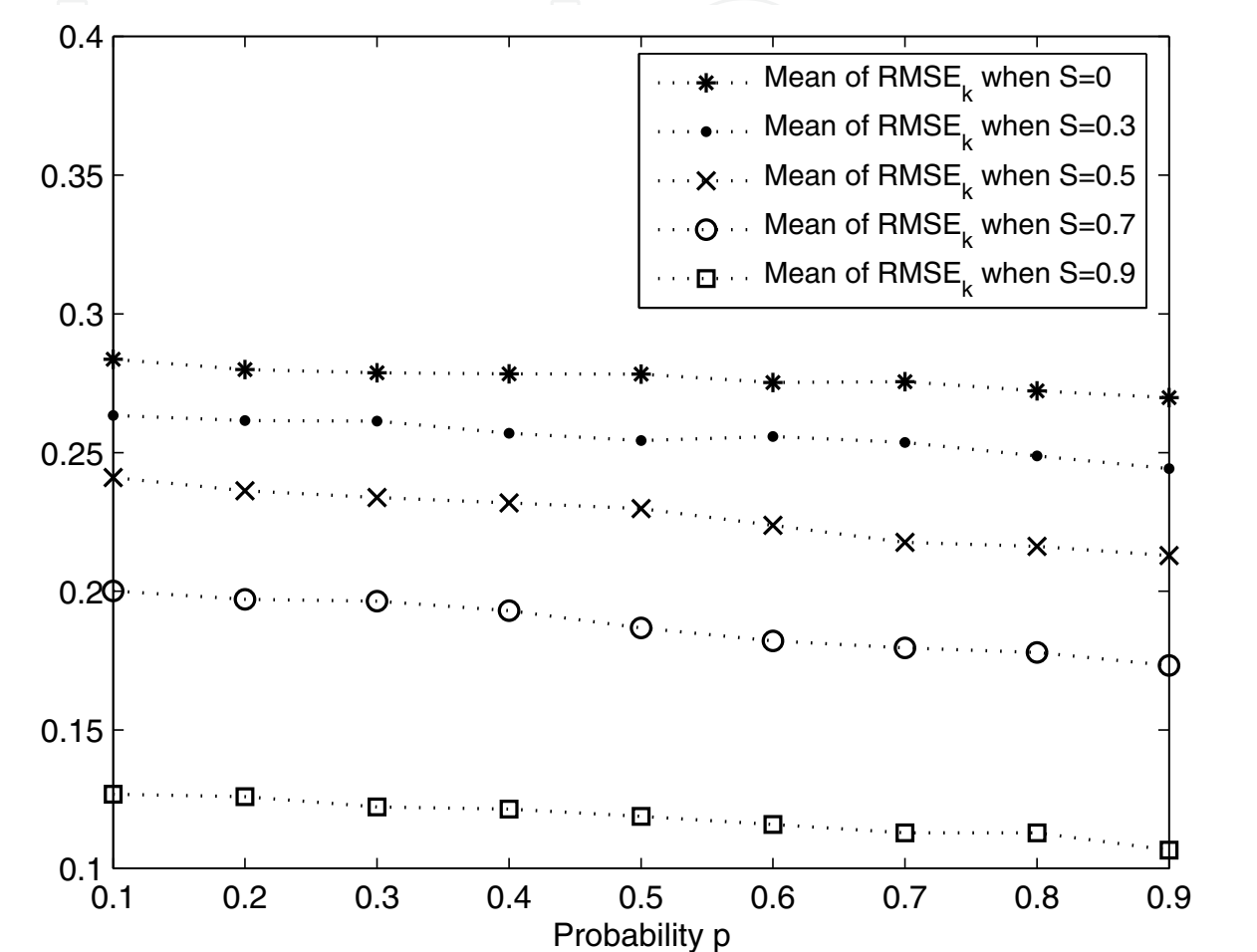


Fig. 2. Mean of $RMSE_k$ for the unscented filtering estimates when $S = 0, 0.3, 0.5, 0.7, 0.9$, versus p .

6. Conclusion

In this chapter, a recursive unscented filtering algorithm for state estimation in a class of nonlinear discrete-time stochastic systems with uncertain observations is obtained. The uncertainty is modelled by a binary white noise taking the value one (when the measurement is the current system output) or zero (when only noise is observed), and the additive noise of the observation is correlated with that of the state. We propose a filtering algorithm based on the scaled unscented transformation, which provides approximations of the first and second-order statistics of a nonlinear transformation of a random vector. This algorithm extends to that in Hermoso & Linares (2007) in two directions. On the one hand, we consider a more general state model in which the noise is not necessarily additive

and, on the other, the independence between the state and observation noises is removed, thus covering those situations in which the observation noise is correlated with the state. The algorithm performance is illustrated with a simulation example in which a first-order ARCH model is considered to describe the state evolution.

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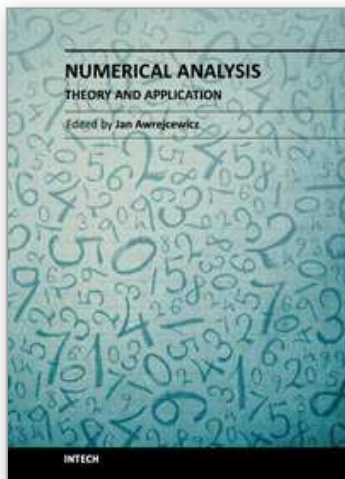
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