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Homogenization of Nonlocal Electrostatic Problems by Means of the Two-Scale Fourier Transform

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1. Introduction

Multiple scales phenomena are ubiquitous, ranging from mechanical properties of wood, turbulent flow in gases and fluids, combustion, remote sensing of earth to wave propagation or heat conduction in composite materials. The obstacle with multi-scale problems is that they, due to limited primary memory even in the largest computational clusters, can not easily be modeled in standard numerical algorithms. Usually we are not even interested in the fine scale information in the processes. However, the fine scale properties are important for the macroscopic, effective, properties of for example a fiber composite. Attempts to find effective properties of composites dates back more than hundred years, e.g. see Faraday (1965); Maxwell (1954a;b); Rayleigh (1892). One way to find effective properties is to introduce a fine scale parameter, $\varepsilon > 0$, in the corresponding governing equations (modeling fast oscillating coefficients) and then study the asymptotic behavior of the sequence of solutions, and equations, when the fine scale parameter tends to zero. The limit yields the homogenized equations, that have constant coefficients (corresponding to homogeneous material properties). The discipline of partial differential equations dealing with such issues is called homogenization theory.

The foundation of homogenization theory was started by Spagnolo (1967) who introduced *G-convergence*, followed by Γ*-convergence* by Dal Maso (1993); De Giorgi (1975); De Giorgi & Franzoni (1975); De Giorgi & Spagnolo (1973), and *H-convergence* Tartar (1977). The *two-scale convergence* concept introduced by Nguetseng (1989) and developed by Allaire (1992); Allaire & Briane (1996) simplified many proofs. Floquet-Bloch expansion Bloch (1928); Floquet (1883) provides a method to find dispersion relations in the case the fine scales are on the same order as for example the wavelength of a propagating wave. The technique of Floquet-Bloch expansion can also be used to find the classical homogenized properties Allaire & Conca (1996); Bensoussan et al. (1978); Conca et al. (2002); Conca & Vanninathan (1997; 2002). *Two-scale transforms* have been introduced in different settings, Arbogast et al. (1990); Brouder & Rossano (2002); Cioranescu et al. (2002); Griso (2002); Laptev (2005); Nechvátal (2004). The general idea with the two-scale transform is to map bounded sequences of functions defined on $L^2(\Omega)$ to sequences defined on the product space $L^2(\Omega \times T^n)$ and then taking the weak limit in $L^2(\Omega \times T^n)$. Besides finding the effective material properties, one can also establish easily computed bounds of these. The bounds may be as simple as the arithmetic and harmonic averages, or more complex. For further reading we recommend the monograph by Milton (2002) as an introduction to the theory of composites.

In this paper we return to a two-scale Fourier transform, which belongs to the class of two-scale transforms, presented in Wellander (2004; 2007; 2009). The transform is applied to nonlocal constitutive relations in electrostatic applications for periodic composites. The current density is given as a spatial convolution of the electric field with a conductivity kernel. It turns out that the homogenized equation also posse's a nonlocal constitutive relation if we do not scale the non-localness. However, if we decrease the neighborhood which influence the current density simultaneously as we make the fine structure finer and finer then we are ending up with a constitutive relation which is local. To be strict, this is a three-scale problem. The finest scale is the variation of material properties. The second scale is the non-localness in the constitutive relation, and the third scale is the global equation, containing only the scales of the domain, boundary conditions and internal body forces.
The paper is organized in the following way. In Section 2 we give some basic definitions, mainly to do with two-scale convergence. In Section 3 we define and explore the two-scale Fourier transform and its application to homogenization of PDEs. In Section 4 we present the main assumptions and give some basic existence, uniqueness and a priori estimates. Section 5 is devoted to the main homogenization results. Some concluding remarks are given in Section 6.

2. Preliminaries

We begin to state the weak and two-scale convergence concepts. A bounded sequence $\{u^\varepsilon\}$ in $L^2(\Omega)$, where Ω is an open bounded set in $\mathbb{R}^n$, $n \geq 1$, with a Lipschitz continuous boundary $\partial\Omega$, has a subsequence which converges weakly in $L^2(\Omega)$, still denoted $\{u^\varepsilon\}$. That is,

$$\int_\Omega u^\varepsilon(x)\varphi(x)\,\mathrm{d}x \to \int_\Omega u(x)\varphi(x)\,\mathrm{d}x, \tag{1}$$

for all test functions $\varphi \in L^2(\Omega)$. We call u the weak limit of $\{u^\varepsilon\}$. Bounded sequences in $L^2(\Omega)$ does not imply strong convergence, i.e.,

$$\|u^\varepsilon - u\|_{L^2(\Omega)} \to 0$$

To study convergence of sequences with fast oscillations Nguetseng (1989) extended the class of test functions to functions with two scales, $\varphi \in C_0^\infty(\Omega; C^\infty(T^n))$, where T^n is the unit torus in $\mathbb{R}^n$. We will refer to two-scale convergence using smooth test functions as *distributional two-scale convergence*.

Definition 1. *A sequence $\{u^\varepsilon\}$ in $L^2(\Omega)$ is said to two-scale converge in a distributional sense to a function $u_0 = u_0(x,y)$ in $L^2(\Omega \times T^n)$ if*

$$\lim_{\varepsilon\to 0}\int_\Omega u^\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right)\,\mathrm{d}x = \int_\Omega\int_{T^n} u_0(x,y)\varphi(x,y)\,\mathrm{d}y\mathrm{d}x, \tag{2}$$

for all test functions $\varphi \in C_0^\infty(\Omega; C^\infty(T^n))$.

The extension of weak convergence to *weak two-scale convergence* reads,

Definition 2. *A sequence $\{u^\varepsilon\}$ in $L^2(\Omega)$ is said to weakly two-scale converge to a function $u_0 = u_0(x,y)$ in $L^2(\Omega \times T^n)$ if*

$$\lim_{\varepsilon\to 0}\int_\Omega u^\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right)\,\mathrm{d}x = \int_\Omega\int_{T^n} u_0(x,y)\varphi(x,y)\,\mathrm{d}y\mathrm{d}x, \tag{3}$$

for all test functions $\varphi \in L^2(\Omega; C(T^n))$.

A more general class of *admissible test functions* are those that two-scale converge strongly, *i.e.*, functions defined as

Definition 3. *If a sequence $\{u^\varepsilon\}$ in $L^2(\Omega)$ weakly two-scale converge to $u_0 \in L^2(\Omega \times T^n)$ and*

$$\lim_{\varepsilon \to 0} \|u^\varepsilon\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega \times T^n)}, \tag{4}$$

then it is said to two-scale converge strongly to $u_0 \in L^2(\Omega \times T^n)$.

Strongly two-scale converging functions are called admissible test functions. Some examples are functions in $L^2(\Omega; C(T^n))$, or for Ω bounded, $C(\overline{\Omega}; C(T^n))$ or $L^2(T^n; C(\overline{\Omega}))$. See Allaire (1992) for more details regarding this issue. The basic compactness results, Nguetseng (1989), reads

Theorem 1. *For every bounded sequence $\{u^\varepsilon\}$ in $L^2(\Omega)$ there exists a subsequence and a function u_0 in $L^2(\Omega \times T^n)$ such that u^ε two-scale converges weakly to u_0.*

Theorem 2. *Assume that $\{u^\varepsilon\}$ is a bounded sequence in $H^1(\Omega)$. Then there exists a subsequence, still denoted $\{u^\varepsilon\}$, which two-scale converges weakly to $u_0 = u$, and ∇u^ε two-scale converges weakly to $\nabla_x u + \nabla_y u_1$. Here u is the weak $L^2(\Omega)$-limit in (1) and $u_1 \in L^2(\Omega; H^1(T^n))$.*

By the Rellich theorem, u is the strong L^2-limit of the sequence $\{u^\varepsilon\}$. We close this section by definition of some nonstandard function spaces.

$$
\begin{aligned}
&H(\mathrm{div}, \Omega) := \{\boldsymbol{u} \in L^2(\Omega; \mathbb{R}^n) : \mathrm{div}\, \boldsymbol{u} \in L^2(\Omega)\}\\
&H(\mathrm{curl}, \Omega) := \{\boldsymbol{u} \in L^2(\Omega; \mathbb{R}^3) : \mathrm{curl}\, \boldsymbol{u} \in L^2(\Omega; \mathbb{R}^3)\}\\
&L^p(\mathrm{div}, \Omega) := \{\boldsymbol{u} \in L^p(\Omega; \mathbb{R}^n) : \mathrm{div}\, \boldsymbol{u} \in L^p(\Omega)\}\\
&L^p(\mathrm{curl}, \Omega) := \{\boldsymbol{u} \in L^p(\Omega; \mathbb{R}^3) : \mathrm{curl}\, \boldsymbol{u} \in L^p(\Omega; \mathbb{R}^3)\}\\
&l^{1,2}(\mathbb{Z}^n) := \{\phi \in l^2(\mathbb{Z}^n) : 2\pi i \boldsymbol{m} \phi(\boldsymbol{m}) \in l^2(\mathbb{Z}^n; \mathbb{C}^n) \forall \boldsymbol{m} \in \mathbb{Z}^n\}\\
&l^2(\mathrm{div}, \mathbb{Z}^n; \mathbb{C}^n) := \{\boldsymbol{\phi} \in l^2(\mathbb{Z}^n; \mathbb{C}^n) : 2\pi i \boldsymbol{m} \cdot \boldsymbol{\phi}(\boldsymbol{m}) \in l^2(\mathbb{Z}^n) \forall \boldsymbol{m} \in \mathbb{Z}^n\}\\
&l^2(\mathrm{curl}, \mathbb{Z}^3; \mathbb{C}^3) := \{\boldsymbol{\phi} \in l^2(\mathbb{Z}^3; \mathbb{C}^3) : 2\pi i \boldsymbol{m} \times \boldsymbol{\phi}(\boldsymbol{m}) \in l^2(\mathbb{Z}^3; \mathbb{C}^3) \forall \boldsymbol{m} \in \mathbb{Z}^3\}
\end{aligned}
$$

3. The two-scale fourier transform

We define the *two-scale Fourier transform*, which is nothing but the standard Fourier transform evaluated at $\boldsymbol{\xi} + \varepsilon^{-1}\boldsymbol{m}$ where $\boldsymbol{\xi}$ is restricted to a cube in $\mathbb{R}^n$ with sidelength $1/\varepsilon$, Wellander (2009).

Definition 4 (Two-scale Fourier transform)**.** *For any function f in $L^1(\mathbb{R}^n)$ and every $0 < \varepsilon$ the* two-scale Fourier transform at the ε-scale *of f is defined by*

$$\mathcal{F}_\varepsilon\{f\}(\boldsymbol{\xi}, \boldsymbol{m}) = \widehat{f}_\varepsilon(\boldsymbol{\xi}, \boldsymbol{m}) = \int_{\mathbb{R}^n} f(\boldsymbol{x}) e^{-2\pi i \boldsymbol{x} \cdot \left(\boldsymbol{\xi} + \frac{\boldsymbol{m}}{\varepsilon}\right)} \, \mathrm{d}\boldsymbol{x},$$

for all $\boldsymbol{\xi} \in \left]-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right[^n$, $\boldsymbol{m} \in \mathbb{Z}^n$. The inverse is given by

$$\mathcal{F}_\varepsilon^{-1}\{\widehat{f}_\varepsilon\}(\boldsymbol{x}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^n} \int_{\boldsymbol{\xi} \in \left]-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right[^n} \widehat{f}_\varepsilon(\boldsymbol{\xi}, \boldsymbol{m}) e^{2\pi i \boldsymbol{x} \cdot \left(\boldsymbol{\xi} + \frac{\boldsymbol{m}}{\varepsilon}\right)} \, \mathrm{d}\boldsymbol{\xi}.$$

The forward transform is well defined for any $\boldsymbol{\xi}$ in $\mathbb{R}^n$. For the inverse we only need the ones in the cube $]-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}[^n$. For fixed ε, the transform is the usual Fourier transform, where we for each $\boldsymbol{m}$ integrate over the cube $]-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}[^n$ with respect to $\boldsymbol{\xi}$ in the inner loop, see Figure 1 for the one dimensional case. It is a question of cutting the frequency space into n-dimensional cubes of side length $1/\varepsilon$ centered at the points $\boldsymbol{m}/\varepsilon$ and summing up the contribution from each cube. When $\varepsilon \to 0$ then $\boldsymbol{\xi}$ belongs to the whole real space, $\mathbb{R}^n$. The standard Fourier transform is recovered if we let $\boldsymbol{m} = \mathbf{0}$ and permit $\boldsymbol{\xi}$ to take any value in $\mathbb{R}^n$ for all $\varepsilon > 0$. The cube $]-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}[^n$ corresponds precisely to the first Brillouin zone appearing in the Floquet-Bloch theory Bloch (1928); Floquet (1883) which is extensively used in solid state physics.

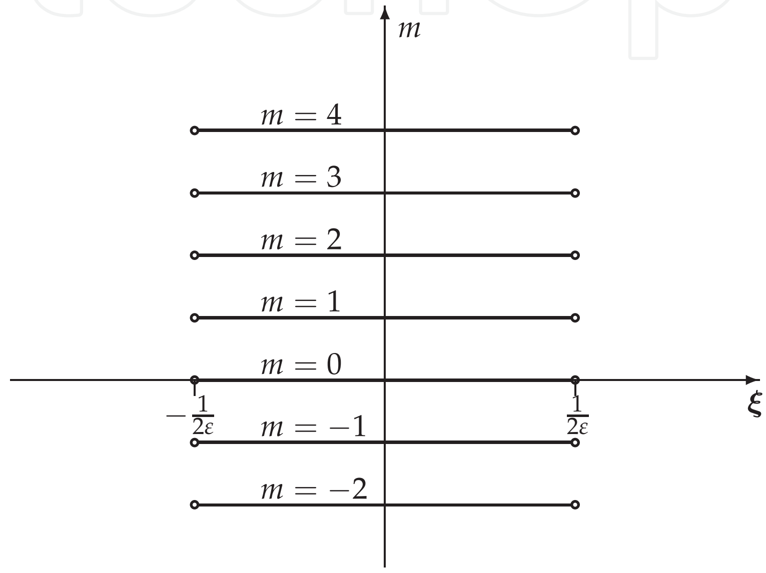

Fig. 1. The Fourier indices used in the inverse transform. Pieces of length $1/\varepsilon$ centered at m/ε are cut from the ξ-axis and stacked along the m-axis. Hence, the pieces labeled $m=0$ and $m=1$ corresponds to the intervals $]-1/2\varepsilon, 1/2\varepsilon[$ and $]1/2\varepsilon, 3/2\varepsilon[$ on the $\boldsymbol{\xi}-axis$, respectively.

The transform can be defied as a mapping $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n; l^2(\mathbb{Z}^n))$, and then by interpolation to $L^p(\mathbb{R}^n)$, $p \in [1,2]$ with values in $L^q(\mathbb{R}^n; l^q(\mathbb{Z}^n))$, $\frac{1}{p}+\frac{1}{q}=1$. Here $l^p(\mathbb{Z}^n)$ is the space of all sequences indexed by the n-tuple of integers, equipped with the usual p-norm. The transform extends to the Fourier theory for tempered distributions Taylor (1996), but in this case we have to include one of the boundaries of the Brillouin zone in the definition of the inverse transform. For example, the semi open cube $[-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}[^n$ would be suitable. By doing this modification we will not exclude Dirac functions with support on the boundary of the Brillouin zone. Keeping the same notation as in the L^1-case we define the L^p-version of the two-scale Fourier transform.

Definition 5. *For any function f in $L^p(\mathbb{R}^n)$, $p \in [1,2]$, and every $0 < \varepsilon$ the* two-scale Fourier transform at the ε-scale *of f is defined by*

$$\mathcal{F}_\varepsilon\{f\}(\boldsymbol{\xi},\boldsymbol{m}) = \widehat{f}_\varepsilon(\boldsymbol{\xi},\boldsymbol{m}) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \boldsymbol{x}\cdot\left(\boldsymbol{\xi}+\frac{\boldsymbol{m}}{\varepsilon}\right)}\,\mathrm{d}\boldsymbol{x}$$

for all $\boldsymbol{\xi} \in \left]-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right[^n$, $\boldsymbol{m} \in \mathbb{Z}^n$. *The inverse is given by*

$$\mathcal{F}_\varepsilon^{-1}\{\widehat{f}_\varepsilon\}(\boldsymbol{x}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^n} \int_{\boldsymbol{\xi}\in\left]-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right[^n} \widehat{f}_\varepsilon(\boldsymbol{\xi}, \boldsymbol{m}) e^{2\pi i \boldsymbol{x}\cdot\left(\boldsymbol{\xi}+\frac{\boldsymbol{m}}{\varepsilon}\right)}\,\mathrm{d}\boldsymbol{\xi}.$$

We have Parseval-Plancherel's relations, which holds because of the corresponding identities for the usual Fourier transform.

Theorem 3. *(Parseval-Plancherel) Suppose that $\boldsymbol{f}$ and $\boldsymbol{g}$ belong to $L^2(\mathbb{R}^n; \mathbb{R}^p)$. Then for every $\varepsilon > 0$*

$$\int_{\mathbb{R}^n} \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{g}(\boldsymbol{x})\, dx = \sum_{\boldsymbol{m}\in\mathbb{Z}^n} \int_{\boldsymbol{\xi}\in\left]-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right[^n} \widehat{\boldsymbol{f}}_\varepsilon(\boldsymbol{\xi}, \boldsymbol{m}) \cdot \overline{\widehat{\boldsymbol{g}}_\varepsilon(\boldsymbol{\xi}, \boldsymbol{m})}\,\mathrm{d}\boldsymbol{\xi},$$

$$\|\boldsymbol{f}\|_{L^2(\mathbb{R}^n;\mathbb{R}^p)} = \left(\sum_{\boldsymbol{m}\in\mathbb{Z}^n} \int_{\boldsymbol{\xi}\in\left]-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right[^n} |\widehat{\boldsymbol{f}}_\varepsilon(\boldsymbol{\xi}, \boldsymbol{m})|^2\,\mathrm{d}\boldsymbol{\xi}\right)^{1/2} = \|\widehat{\boldsymbol{f}}_\varepsilon\|_{L^2\left(\left]-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right[^n \times \mathbb{Z}^n;\mathbb{R}^p\right)}.$$

The following properties of the two-scale Fourier transform follow at once from the usual Fourier transform.

Proposition 1. *The two-scale Fourier transform has the following properties,*

(i) $\mathcal{F}_\varepsilon\{fg\} = \mathcal{F}_\varepsilon\{f\} * \mathcal{F}_\varepsilon\{g\}$, *for* $f, g \in L^2(\mathbb{R}^n)$.

(ii) $\mathcal{F}_\varepsilon\{f * g\} = \mathcal{F}_\varepsilon\{f\}\mathcal{F}_\varepsilon\{g\}$ *for* $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$, $p \in [1, 2]$.

(iii) $\mathcal{F}_\varepsilon\{\nabla \boldsymbol{u}\}(\boldsymbol{\xi}, \boldsymbol{m}) = 2\pi i(\boldsymbol{\xi} + \varepsilon^{-1}\boldsymbol{m})\mathcal{F}_\varepsilon\{\boldsymbol{u}\}(\boldsymbol{\xi}, \boldsymbol{m})$ *for* $\boldsymbol{u} \in W^{1,p}(\mathbb{R}^n)$, $p \in [1, 2]$.

(iv) $\mathcal{F}_\varepsilon\{\nabla \cdot \boldsymbol{u}\}(\boldsymbol{\xi}, \boldsymbol{m}) = 2\pi i(\boldsymbol{\xi} + \varepsilon^{-1}\boldsymbol{m}) \cdot \mathcal{F}_\varepsilon\{\boldsymbol{u}\}(\boldsymbol{\xi}, \boldsymbol{m})$ *for* $\boldsymbol{u} \in L^p(\mathrm{div}, \mathbb{R}^n)$, $p \in [1, 2]$.

(v) $\mathcal{F}_\varepsilon\{\nabla \times \boldsymbol{u}\}(\boldsymbol{\xi}, \boldsymbol{m}) = 2\pi i(\boldsymbol{\xi} + \varepsilon^{-1}\boldsymbol{m}) \times \mathcal{F}_\varepsilon\{\boldsymbol{u}\}(\boldsymbol{\xi}, \boldsymbol{m})$ *for* $\boldsymbol{u} \in L^p(\mathrm{curl}, \mathbb{R}^3)$, $p \in [1, 2]$.

(vi) $\mathcal{F}_\varepsilon\{ue^{-2\pi i \boldsymbol{x}\cdot\left(\boldsymbol{\eta}+\frac{\boldsymbol{s}}{\varepsilon}\right)}\}(\boldsymbol{\xi}, \boldsymbol{m}) = \mathcal{F}_\varepsilon\{u\}(\boldsymbol{\xi} + \boldsymbol{\eta}, \boldsymbol{m} + \boldsymbol{s})$ *for* $u \in L^p(\mathbb{R}^n)$, $p \in [1, 2]$, $\boldsymbol{\xi} \in]-1/2\varepsilon, 1/2\varepsilon[^n$.

The convolution in Fourier space ($*$) is defined as

$$\mathcal{F}_\varepsilon\{f\} * \mathcal{F}_\varepsilon\{g\}(\boldsymbol{\xi}, \boldsymbol{m}) = \sum_{\boldsymbol{s}\in\mathbb{Z}^n} \int_{\boldsymbol{\eta}\in\left]-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right[^n} \mathcal{F}_\varepsilon\{f\}(\boldsymbol{\eta}, \boldsymbol{s})\mathcal{F}_\varepsilon\{g\}(\boldsymbol{\xi} - \boldsymbol{\eta}, \boldsymbol{m} - \boldsymbol{s})\,\mathrm{d}\boldsymbol{\eta},$$

for $\boldsymbol{\xi} \in]-1/2\varepsilon, 1/2\varepsilon[^n$. Translated functions like $\mathcal{F}_\varepsilon\{g\}(\cdot, \boldsymbol{m} - \boldsymbol{s})$ are extended by zero outside $]-1/2\varepsilon, 1/2\varepsilon[^n$ for all $\boldsymbol{m}$ and $\boldsymbol{s}$ in $\mathbb{Z}^n$.
The admissible test functions (as in Definition 3) converge strongly in Fourier space.

Proposition 2. *Assume sequence $\{\phi^\varepsilon\}$ two-scale converges strongly to ϕ. Extend $\widehat{\phi}^\varepsilon_\varepsilon(\cdot, \boldsymbol{m})$ by zero outside $\left]-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right[^n$ for all $\boldsymbol{m} \in \mathbb{Z}^n$ then*

$$\widehat{\phi}^\varepsilon_\varepsilon(\boldsymbol{\xi}, \boldsymbol{m}) \to \widehat{\phi}(\boldsymbol{\xi}, \boldsymbol{m}) \ \textit{strongly in } L^2(\mathbb{R}^n \times \mathbb{Z}^n).$$

Proof: By assumption $\{\phi^\varepsilon\}$ is bounded in $L^2(\Omega)$. It follows that the two-scale Fourier transformed sequence $\widehat{\phi}^\varepsilon_\varepsilon(\boldsymbol{\xi}, \boldsymbol{m})$ is bounded in $L^2\left(\left]-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right[^n \times \mathbb{Z}^n\right)$. The extended function is bounded in $L^2(\mathbb{R}^n \times \mathbb{Z}^n)$ and converges weakly in $L^2(\mathbb{R}^n \times \mathbb{Z}^n)$. Further, Parseval-Plancherel (Theorem 3) yields

$$\lim_{\varepsilon\to 0} \|\widehat{\phi}^{\varepsilon}_{\varepsilon}\|_{L^2(]-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}[^n\times\mathbb{Z}^n)} = \lim_{\varepsilon\to 0} \|\widehat{\phi}^{\varepsilon}_{\varepsilon}\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^n)} =$$
$$\lim_{\varepsilon\to 0} \|\phi^{\varepsilon}_{\varepsilon}\|_{L^2(\Omega)} = \|\phi\|_{L^2(\Omega\times T^n)} = \|\widehat{\phi}\|_{L^2(\mathbb{R}^n\times\mathbb{Z}^n)}.$$

The statement follows since the sequence converges weakly and in norm. □

We continue by restating Nguetseng's two-scale compactness theorem (Theorems 1 and 2) in Fourier space.

Proposition 3. *Let $\{u^\varepsilon\}$ be a uniformly bounded sequence in $L^2(\mathbb{R}^n)$ and $\boldsymbol{\xi} \in \mathbb{R}^n$, $\boldsymbol{m} \in \mathbb{Z}^n$ arbitrary.*

(i) *If ϕ^ε two-scale converge strongly to ϕ (as in Proposition 2) then there exists a subsequence and $u^0 \in L^2(\Omega \times T^n)$ such that*

$$\lim_{\varepsilon\to 0} \widehat{(u^\varepsilon\phi^\varepsilon)}_\varepsilon(\boldsymbol{\xi},\boldsymbol{m}) = \widehat{(u^0\phi)}(\boldsymbol{\xi},\boldsymbol{m}).$$

(ii) *The sequence $\widehat{u}^{\varepsilon}_{\varepsilon}(\,\cdot\,,\boldsymbol{m})$ extended by zero outside $\left]-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}\right[^n$*

$$\widehat{u}^{\varepsilon}_{\varepsilon} \rightharpoonup \widehat{u}^0$$

weakly in $L^2(\mathbb{R}^n \times \mathbb{Z}^n)$.

(iii) *If there exists a compact set K in $\mathbb{R}^n$ and a positive number ε_0 such that* $\operatorname{supp} u^\varepsilon \subset K$, *for all $\varepsilon < \varepsilon_0$, then*

$$\lim_{\varepsilon\to 0} \widehat{u}^{\varepsilon}_{\varepsilon}(\boldsymbol{\xi},\boldsymbol{m}) = \widehat{u}^0(\boldsymbol{\xi},\boldsymbol{m}).$$

pointwise in $\mathbb{R}^n \times \mathbb{Z}^n$.

Remark 1. *If $\{u^\varepsilon\}$ in Proposition 3 (iii) is uniformly bounded in $L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ (or just bounded in $L^1(\mathbb{R}^n)$) then the convergence is pointwise in Fourier space. That is due to the fact that a subsequence of $\{u^\varepsilon\}$ converges weakly in $L^1(\mathbb{R}^n)$ and $e^{-2\pi i(x\cdot\boldsymbol{\xi}+y\cdot m)}$ is a function in $L^\infty(\mathbb{R}^n \times T^n)$.*

We have the following corollary which follows from Proposition 3

Corollary 1. *If $\{u^\varepsilon\}$ is a bounded sequence in $L^2(\mathbb{R}^n)$ and if there exists a compact set K in $\mathbb{R}^n$ and a positive number ε_0 such that* $\operatorname{supp} u^\varepsilon \subset K$, *for all $\varepsilon < \varepsilon_0$ (or if $\{u^\varepsilon\}$ is bounded in $L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$), then there exists a subsequence such that,*

$$\mathcal{F}_\varepsilon\{u^\varepsilon\}(\boldsymbol{\xi},0) \to \widehat{u}^0(\boldsymbol{\xi},0),$$

as $\varepsilon \to 0$, for all $\boldsymbol{\xi} \in \mathbb{R}^n$. Here $\widehat{u}^0(\boldsymbol{\xi},0)$ is the Fourier transform of the weak limit of $\{u^\varepsilon\}$ in $L^2(\Omega)$.

Remark 2. *Proposition 3 (i) can be illustrated by the following commutative diagram*

$$\begin{array}{ccc}
u^\varepsilon\phi^\varepsilon & \xrightarrow{2-s} & u^0\phi \\
\big\downarrow \mathcal{F}_\varepsilon & & \big\downarrow \mathcal{F} \\
\widehat{(u^\varepsilon\phi^\varepsilon)}_\varepsilon(\boldsymbol{\xi},\boldsymbol{m}) & \xrightarrow{\text{pointwise}} & \widehat{(u^0\phi)}(\boldsymbol{\xi},\boldsymbol{m})
\end{array}$$

Assertions (ii) and (iii) are illustrated by

$$\begin{array}{ccc} u^{\varepsilon} & \xrightarrow{2-s} & u^{0} \\ \Big\downarrow \mathcal{F}_{\varepsilon} & & \Big\downarrow \mathcal{F} \\ \widehat{u}^{\varepsilon}_{\varepsilon}(\boldsymbol{\xi},\boldsymbol{m}) & \xrightarrow{weakly/pointwise} & \widehat{u}^{0}(\boldsymbol{\xi},\boldsymbol{m}) \end{array}$$

which indicates that for sequences defined on bounded domains the two-scale convergence becomes (by considering the exponential function as a test function) pointwise convergence in Fourier space.

Next we give some compactness results for the two-scale Fourier transform. The first one asserts that we recover the standard Fourier transform of any function in L^2 as the limit of the two-scale Fourier transformed function.

Proposition 4. *Let $u \in L^2(\mathbb{R}^n)$ and $\widehat{u}$ be the standard Fourier transform of u. Then,*

$$\lim_{\varepsilon\to 0} \mathcal{F}_{\varepsilon}\{u\}(\boldsymbol{\xi},\boldsymbol{m}) = \widehat{u}(\boldsymbol{\xi})\delta_{\boldsymbol{m}\boldsymbol{0}},$$

pointwise for all $\boldsymbol{\xi} \in \mathbb{R}^n, \boldsymbol{m} \in \mathbb{Z}^n$.

Here $\mathbf{0}$ is the n-dimensional null vector and $\delta_{\boldsymbol{kl}}$ is the Kronecker delta defined by

$$\delta_{\boldsymbol{kl}} = \begin{cases} 1, & \boldsymbol{k} = \boldsymbol{l}, \\ 0, & \boldsymbol{k} \neq \boldsymbol{l} \end{cases}$$

We find that a sequence of scaled periodic functions are recovered as the Fourier transform of the unscaled function.

Proposition 5. *Let $u \in L^2(T^n)$, and define $u^{\varepsilon}(\boldsymbol{x}) = u(\boldsymbol{x}/\varepsilon)$. Then,*

$$\mathcal{F}_{\varepsilon}\{u^{\varepsilon}\}(\mathbf{0},\boldsymbol{m}) = \widehat{u}(\boldsymbol{m})$$

for all $0 < \varepsilon$ such that $1/\varepsilon$ is an integer, $\boldsymbol{m} \in \mathbb{Z}^n$, and

$$\lim_{\varepsilon\to 0} \mathcal{F}_{\varepsilon}\{u^{\varepsilon}\}(\boldsymbol{\xi},\boldsymbol{m}) = \widehat{u}(\boldsymbol{m})$$

for all $\boldsymbol{\xi} \in \mathbb{R}^n, \boldsymbol{m} \in \mathbb{Z}^n$. Here,

$$\widehat{u}(\boldsymbol{m}) = \int_{T^n} u(\boldsymbol{x}) e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{m}} \,\mathrm{d}\boldsymbol{x},$$

Proof: The definition of the two-scale Fourier transform, Definition 5, yields

$$\mathcal{F}_{\varepsilon}\{u^{\varepsilon}\}(\mathbf{0},\boldsymbol{m}) = \int_{T^n} u(\boldsymbol{x}/\varepsilon) e^{-2\pi i \boldsymbol{x}\cdot(\mathbf{0}+\frac{\boldsymbol{m}}{\varepsilon})} \,\mathrm{d}\boldsymbol{x} = \varepsilon^n \int_{T^n_{1/\varepsilon}} u(\boldsymbol{s}) e^{-2\pi i \boldsymbol{s}\cdot(\mathbf{0}+\boldsymbol{m})} \,\mathrm{d}\boldsymbol{s} =$$

$$= \varepsilon^n \sum_{1}^{\varepsilon^{-n}} \int_{T^n} u(\boldsymbol{s}) e^{-2\pi i \boldsymbol{s}\cdot(\mathbf{0}+\boldsymbol{m})} \,\mathrm{d}\boldsymbol{s} = \int_{T^n} u(\boldsymbol{s}) e^{-2\pi i \boldsymbol{s}\cdot(\mathbf{0}+\boldsymbol{m})} \,\mathrm{d}\boldsymbol{s} = \widehat{u}(\boldsymbol{m})$$

for all $0 < \varepsilon$ such that $1/\varepsilon$ is an integer, $\boldsymbol{\xi} \in \mathbb{R}^n, \boldsymbol{m} \in \mathbb{Z}^n$. Here, $T^n_{1/\varepsilon}$ is the $1/\varepsilon$-torus in $\mathbb{R}^n$. The second statement follows by similar arguments. □

In the next three propositions we will assume that there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ the support of all sequences are contained in a compact set K in $\mathbb{R}^n$. It follows that $e^{-2\pi i \boldsymbol{x}\cdot(\boldsymbol{\xi}+\frac{\boldsymbol{m}}{\varepsilon})}$ belongs to $L^2(K)$ and is an admissible test function in the two-scale convergence sense. If the support is not compact then the convergence in Fourier space will be weak in L^2, as in Proposition 3 (iii). The proofs will be similar in these cases, just multiply with test functions in $L^2(\mathbb{R}^n \times \mathbb{Z}^n)$ before taking the limits. Alternatively, we can localize the sequence first by multiplying with functions $\phi \in C_0(\Omega)$.

Proposition 6. *If $\{u^\varepsilon\}$ is a bounded sequence in $H^1(\mathbb{R}^n)$ then there exists a subsequence such that,*

(i) $\mathcal{F}_\varepsilon\{u^\varepsilon\}(\boldsymbol{\xi},\boldsymbol{m}) \to \widehat{u}(\boldsymbol{\xi})\delta_{\boldsymbol{m}\mathbf{0}}$,

(ii) $\mathcal{F}_\varepsilon\{\nabla u^\varepsilon\}(\boldsymbol{\xi},\boldsymbol{m}) \to 2\pi i\boldsymbol{\xi}\widehat{u}(\boldsymbol{\xi})\delta_{\boldsymbol{m}\mathbf{0}} + 2\pi i\boldsymbol{m}\widehat{u}^1(\boldsymbol{\xi},\boldsymbol{m})$

as $\varepsilon \to 0$, for all $\boldsymbol{\xi} \in \mathbb{R}^n, \boldsymbol{m} \in \mathbb{Z}^n$. Here $\widehat{u}$ is the standard Fourier transform of u which is the weak limit of u^ε in $L^2(\mathbb{R}^n)$, and $\widehat{u}^1 \in L^2(\mathbb{R}^n; l^{1,2}(\mathbb{Z}^n))$ is the Fourier transform of a function $u^1 \in L^2(\mathbb{R}^n; H^1(T^n))$.

Proposition 7. *If $\{\boldsymbol{u}^\varepsilon\}$ is a bounded sequence in $H(\mathrm{div}, \mathbb{R}^n)$ then there exists a subsequence such that,*

(i) $\mathcal{F}_\varepsilon\{\boldsymbol{u}^\varepsilon\}(\boldsymbol{\xi},\boldsymbol{m}) \to \widehat{\boldsymbol{u}}^0(\boldsymbol{\xi},\boldsymbol{m})$, *with* $2\pi i\boldsymbol{m}\cdot\widehat{\boldsymbol{u}}^0(\boldsymbol{\xi},\boldsymbol{m}) = 0$

(ii) $\mathcal{F}_\varepsilon\{\nabla\cdot\boldsymbol{u}^\varepsilon\}(\boldsymbol{\xi},\boldsymbol{m}) \to 2\pi i\boldsymbol{\xi}\cdot\widehat{\boldsymbol{u}}(\boldsymbol{\xi})\delta_{\boldsymbol{m}\mathbf{0}} + 2\pi i\boldsymbol{m}\cdot\widehat{\boldsymbol{u}}^1(\boldsymbol{\xi},\boldsymbol{m})$

as $\varepsilon \to 0$, for all $\boldsymbol{\xi} \in \mathbb{R}^n, \boldsymbol{m} \in \mathbb{Z}^n$, where $\widehat{\boldsymbol{u}}(\boldsymbol{\xi}) = \widehat{\boldsymbol{u}}^0(\boldsymbol{\xi},0)$ is the standard Fourier transform of $\boldsymbol{u} \in H(\mathrm{div}, \mathbb{R}^n)$, $\boldsymbol{u}(x) = \int_{T^n} u^0(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}$ and $\widehat{\boldsymbol{u}}^1 \in L^2(\mathbb{R}^3; l^2(\mathrm{div}, \mathbb{Z}^3; \mathbb{C}^n))$ is the Fourier transform of a function $\boldsymbol{u}^1 \in L^2(\mathbb{R}^n; H(\mathrm{div}, T^n))$.

Proposition 8. *If $\{\boldsymbol{u}^\varepsilon\}$ is a bounded sequence in $H(\mathrm{curl}, \mathbb{R}^3)$ then there exists a subsequence such that,*

(i) $\mathcal{F}_\varepsilon\{\boldsymbol{u}^\varepsilon\}(\boldsymbol{\xi},\boldsymbol{m}) \to \widehat{\boldsymbol{u}}^0(\boldsymbol{\xi},\boldsymbol{m}) = \widehat{\boldsymbol{u}}(\boldsymbol{\xi})\delta_{\boldsymbol{m}\mathbf{0}} + 2\pi i\boldsymbol{m}\widehat{\phi}(\boldsymbol{\xi},\boldsymbol{m})$, *with* $2\pi i\boldsymbol{m}\times\widehat{\boldsymbol{u}}^0(\boldsymbol{\xi},\boldsymbol{m}) = 0$

(ii) $\mathcal{F}_\varepsilon\{\nabla\times\boldsymbol{u}^\varepsilon\}(\boldsymbol{\xi},\boldsymbol{m}) \to 2\pi i\boldsymbol{\xi}\times\widehat{\boldsymbol{u}}(\boldsymbol{\xi})\delta_{\boldsymbol{m}\mathbf{0}} + 2\pi i\boldsymbol{m}\times\widehat{\boldsymbol{u}}^1(\boldsymbol{\xi},\boldsymbol{m})$

as $\varepsilon \to 0$, for all $\boldsymbol{\xi} \in \mathbb{R}^3, \boldsymbol{m} \in \mathbb{Z}^3$. Here $\widehat{\boldsymbol{u}}(\boldsymbol{\xi}) = \widehat{\boldsymbol{u}}^0(\boldsymbol{\xi},0)$ is the Fourier transform of $\boldsymbol{u}(\boldsymbol{x}) = \int_{T^n} u^0(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}$, $\boldsymbol{u} \in H(\mathrm{curl}, \mathbb{R}^3)$, $\widehat{\phi} \in L^2(\mathbb{R}^3; l^2(\mathbb{Z}^3))$ is the Fourier transform of a function $\phi \in L^2(\mathbb{R}^3; H^1(T^3))$ and $\widehat{\boldsymbol{u}}^1 \in L^2(\mathbb{R}^3; l^2(\mathrm{curl}, \mathbb{Z}^3; \mathbb{C}^3))$ is the Fourier transform of a function $\boldsymbol{u}^1 \in L^2(\mathbb{R}^3; H(\mathrm{curl}, T^3))$.

4. The non-local homogenization problems

We will consider two non-local elliptic problems. The physical problem in mind is a nonlocal electrostatic equation for a periodic composite. This is an elliptic problem with spatial convolution of the electric field with a conductivity, which consists of a periodic part multiplied with a localizing function. The localizer gives a finite contribution to the current density when convoluted with the electric fields in the neighborhood of the observation point.

4.1 Assumptions and weak formulation

The domain, Ω, is assumed to be a bounded subset of $\mathbb{R}^n$, $n \in \mathbb{N}$ with a Lipshitz boundary $\partial\Omega$. We assume the current density is given by a spatial convolution of the electric field with a nonlocal kernel $\mathbf{K}$ which gives the current density contribution at a point due to the electric field in the neighborhood of $\boldsymbol{x}$,

$$\boldsymbol{J}(\boldsymbol{x}, \nabla\phi) = \boldsymbol{J}(\boldsymbol{x}) = \int_\Omega \mathbf{K}(\boldsymbol{x}-\boldsymbol{\xi})\nabla\phi(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}. \tag{5}$$

The kernel maps electric fields to current densities ($\mathbb{R}^n \to \mathbb{R}^n$) and decays monotonically for large arguments. To model the fine scale structure in a heterogeneous material we introduce the fine scale parameter $\varepsilon > 0$. The scaled current density is given by

$$\boldsymbol{J}^\varepsilon(\boldsymbol{x}) = \int_\Omega \mathbf{K}^\varepsilon(\boldsymbol{x} - \boldsymbol{\xi}) \nabla \phi^\varepsilon(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi} \tag{6}$$

where ϕ^ε is the electric potential. We integrate over the support of $\mathbf{K}^\varepsilon$ which overlaps Ω, which has to be taken into account close to the boundary $\partial\Omega$. The static equation reads

$$\begin{cases} -\nabla \cdot \boldsymbol{J}^\varepsilon(\boldsymbol{x}) = f^\varepsilon(\boldsymbol{x}) & \boldsymbol{x} \in \Omega \\ \phi^\varepsilon|_{\partial\Omega} = 0 \end{cases} \tag{7}$$

where f^ε is some given current density source bounded in $L^2(\Omega)$ which converges strongly to f in $H^{-1}(\Omega)$ when $\varepsilon \to 0$. Equation (7) is to be understood in the weak sense, *i.e.*,

$$\int_\Omega \boldsymbol{J}^\varepsilon(\boldsymbol{x}) \cdot \nabla \psi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \int_\Omega f^\varepsilon(\boldsymbol{x}) \psi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \qquad \forall \psi \in H_0^1(\Omega) \tag{8}$$

We introduce the scaled bilinear form

$$a^\varepsilon(\phi, \psi) = \int_\Omega \int_\Omega \mathbf{K}^\varepsilon(\boldsymbol{x} - \boldsymbol{\xi}) \nabla \phi(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi} \cdot \nabla \psi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \tag{9}$$

Equation (8) can now be restated in the following weak formulation. Find $\phi^\varepsilon \in H_0^1(\Omega)$ such that

$$a^\varepsilon(\phi^\varepsilon, \psi) = \int_\Omega f^\varepsilon(\boldsymbol{x}) \psi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \qquad \forall \psi \in H_0^1(\Omega) \tag{10}$$

We will assume that the kernel $\mathbf{K}$ is such that the following boundedness and coercivity properties follows

Theorem 4. *There exist constants $C_1, C_2 > 0$ such that*

$$|a^\varepsilon(\phi, \psi)| \leq C_1 \|\nabla \phi\|_{L^2(\Omega;\mathbb{R}^n)} \|\nabla \psi\|_{L^2(\Omega;\mathbb{R}^n)} \tag{11}$$

$$C_2 \|\nabla \phi\|^2_{L^2(\Omega;\mathbb{R}^n)} \leq a^\varepsilon(\phi, \phi) \tag{12}$$

for all $\phi, \psi \in H_0^1(\Omega)$

The precise form of the kernel $\mathbf{K}$ will be given in the next sections.

4.2 Existence of unique solution

For the existence of solution we need the Lax-Milgram theorem (e.g. see Evans (1998))

Theorem 5 (Lax-Milgram)**.** *Assume that*

$$B : H \times H \to \mathbb{R}$$

is a bilinear mapping, for which there exist constants $\alpha, \beta > 0$ such that

$$|B[u, v]| \leq \alpha \|u\| \|v\| \quad (u, v \in H)$$

and

$$\beta \|u\|^2 \leq |B[u, u]| \quad (u \in H).$$

Finally, let $f : H \to \mathbb{R}$ be a bounded linear functional on H. Then there exists a unique element $u \in H$ such that

$$B[u, v] = \langle f, v \rangle$$

for all $v \in H$.

Here, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between H and its dual H'.

Theorem 6 (Existence and uniqueness)**.** *Equation* (10) *has a unique solution $\phi^\varepsilon \in H_0^1(\Omega)$ for each $\varepsilon > 0$.*

Proof: The result follows from Theorems 4 and 5. □

The main question to be answered is: Which equation with constand coefficients has a solution that is the best possible approximation of the solution of equation (10) when ε is small? To be able to answer this question we need to find the limit of the bilinear form when $\varepsilon \to 0$. The first step is to establish a priori estimates of the sequence of solutions.

4.3 A priori estimates

We have the standard a priori estimate

Theorem 7 (A priori estimate)**.** *The solutions of* (10) *satisfies*

$$\|\phi^\varepsilon\|_{H_0^1(\Omega)} \leq C \tag{13}$$

uniformly with respect to $\varepsilon > 0$.

Proof: Letting $\psi = \phi^\varepsilon$ in (10), the coercivity property in equation (12) and Hölder's inequality yields

$$C\|\nabla\phi^\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^n)} \leq \|f^\varepsilon\|_{L^2(\Omega)}\|\phi^\varepsilon\|_{L^2(\Omega)} \tag{14}$$

The Poincare inequality and the boundedness of $\|f^\varepsilon\|_{L^2(\Omega)}$ gives

$$\|\nabla\phi^\varepsilon\|_{L^2(\Omega;\mathbb{R}^n)} \leq C \tag{15}$$

$$\|\phi^\varepsilon\|_{L^2(\Omega)} \leq C \tag{16}$$

The assertion is proved. □

5. Homogenization

5.1 Case I, Non-vanishing non-localness

Let us consider a non-vanishing convolution kernel. Assume that $\mathbf{K}$ is an admissible test function in the two-scale sense, as in Definition 3, *i.e.*, satisfying Proposition 2. As a model let us use

$$\mathbf{K}(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} C\boldsymbol{\sigma}(\boldsymbol{y}) \exp\left(\frac{1}{\left|\frac{\boldsymbol{x}}{r}\right|^2 - 1}\right) & , \quad |\boldsymbol{x}| < r. \\ 0 & , \quad |\boldsymbol{x}| \geq r \end{cases} \tag{17}$$

where r is the radius of the non-local influence zone, $\boldsymbol{\sigma}$ is the conductivity associated with the non-locality, it is assumed to be Y-periodic, *i.e.*, $\boldsymbol{\sigma}(\boldsymbol{y} + \boldsymbol{e}) = \boldsymbol{\sigma}(\boldsymbol{y})$ for all $\boldsymbol{y} \in]0, 1[^n$, and $C > 0$ is a constant.

The scaled kernel reads

$$\mathbf{K}^\varepsilon(\boldsymbol{x}) = \mathbf{K}\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\right) = \begin{cases} C\boldsymbol{\sigma}\left(\frac{\boldsymbol{x}}{\varepsilon}\right) \exp\left(\frac{1}{\left|\frac{\boldsymbol{x}}{r}\right|^2 - 1}\right) & , \quad |\boldsymbol{x}| < r. \\ 0 & , \quad |\boldsymbol{x}| \geq r \end{cases} \tag{18}$$

The conductivity $\boldsymbol{\sigma}$ satisfies the coercivity condition

$$\boldsymbol{\sigma}\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geq c_1|\boldsymbol{\xi}|^2 \tag{19}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^n$, $\boldsymbol{x} \in \Omega$ a.e., and is bounded, *i.e.*, $\boldsymbol{\sigma} \in L^\infty(\Omega; \mathbb{R}^{n\times n})$

Theorem 8 (Homogenization, non-vanishing non-localness)**.** *Let $\{\phi^\varepsilon\}$ be a sequence of solutions to* (10) *where the kernel in the bilinear form* (9) *is given by* (18)*. The sequence $\{\phi^\varepsilon\}$ converges weakly in $H_0^1(\Omega)$ to $\phi \in H_0^1(\Omega)$, the unique solution of the Homogenized Problem*

$$-\nabla\cdot\int_{\Omega\cap\operatorname{supp}\boldsymbol{\sigma}_h(\boldsymbol{x}-\,\cdot\,)} \boldsymbol{\sigma}_h(\boldsymbol{x}-\boldsymbol{z})\,\nabla\phi(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} = f(\boldsymbol{x}), \tag{20}$$

a.e. in Ω, where the homogenized conductivity is given by

$$\boldsymbol{\sigma}_h(\boldsymbol{x}) = \int_{T^n} \mathbf{K}(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \int_{T^n} C\boldsymbol{\sigma}(\boldsymbol{y})\exp\left(\frac{1}{\left|\frac{\boldsymbol{x}}{r}\right|^2-1}\right)\mathrm{d}\boldsymbol{y} \tag{21}$$

Proof: Since $\phi^\varepsilon \in H^1(\mathbb{R}^n)$ and $f^\varepsilon \in L^2(\mathbb{R}^n)$ we can apply the two-scale Fourier transform to (7). The a priori estimate in Theorem 7 and Definition 5 gives

$$2\pi i(\boldsymbol{\xi}+\varepsilon^{-1}\boldsymbol{m})\cdot\widehat{\mathbf{K}}_\varepsilon^\varepsilon(\boldsymbol{\xi},\boldsymbol{m})2\pi i(\boldsymbol{\xi}+\varepsilon^{-1}\boldsymbol{m})\widehat{\phi}_\varepsilon^\varepsilon(\boldsymbol{\xi},\boldsymbol{m}) = \widehat{f}_\varepsilon^\varepsilon(\boldsymbol{\xi},\boldsymbol{m}), \tag{22}$$

for all $\varepsilon < 0, \boldsymbol{\xi} \in \mathbb{R}^n, \boldsymbol{m} \in \mathbb{Z}^n$. Next we multiply with ε and send a subsequence (still denoted by ε) to zero. Taking Propositions 6 and 2 into account will give us the Fourier transform of the local problem as the L^2-weak limit in Fourier space,

$$2\pi i\boldsymbol{m}\cdot\widehat{\mathbf{K}}(\boldsymbol{\xi},\boldsymbol{m})2\pi i\left(\boldsymbol{\xi}\widehat{\phi}(\boldsymbol{\xi})\delta_{\boldsymbol{m}\mathbf{0}} + \boldsymbol{m}\widehat{\phi}^1(\boldsymbol{\xi},\boldsymbol{m})\right) = 0, \tag{23}$$

for a.e. $\boldsymbol{\xi} \in \mathbb{R}^n$, and all $\boldsymbol{m} \in \mathbb{R}^n$. It has a trivial solution $\widehat{\phi}^1(\boldsymbol{\xi},\boldsymbol{m}) = 0$ for all $\boldsymbol{m} \neq \mathbf{0}$. To get the homogenized problem we let $\boldsymbol{m} = \mathbf{0}$ in (22), extract another subsequence and send $\varepsilon \to 0$ which yields the standard Fourier transform of the weak $L^2(\Omega)$-limit,

$$2\pi i\boldsymbol{\xi}\cdot\widehat{\mathbf{K}}(\boldsymbol{\xi},0)2\pi i\boldsymbol{\xi}\widehat{\phi}(\boldsymbol{\xi}) = \widehat{f}(\boldsymbol{\xi}), \tag{24}$$

for a.e. $\boldsymbol{\xi} \in \mathbb{R}^n$. Apparently we do not need $\widehat{\phi}^1$ in the homogenized equation. The homogenized equation (24) is the Fourier transform of

$$-\nabla\cdot\int_{\Omega\cap\operatorname{supp}\boldsymbol{\sigma}_h(\boldsymbol{x}-\,\cdot\,)}\int_{T^n} \mathbf{K}(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\nabla\phi(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} = f(\boldsymbol{x}), \tag{25}$$

Here $\boldsymbol{\sigma}_h$, is the mean value of $\mathbf{K}$ with respect to the local variable. Indeed, it is the homogenized conductivity

$$\boldsymbol{\sigma}_h(\boldsymbol{x}) = \int_{T^n}\mathbf{K}(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \tag{26}$$

The homogenized equation has a unique solution (see Theorem 10 below) which implies that the whole sequence converges. □

5.2 Case II, Vanishing non-localness

In this case we will use the same kernel, but we will scale both variables, *i.e.*, let

$$\mathbf{K}(\boldsymbol{y}) = \begin{cases} C\boldsymbol{\sigma}(\boldsymbol{y})\exp\left(\frac{1}{\left|\frac{\boldsymbol{y}}{r}\right|^2-1}\right) & , \quad |\boldsymbol{y}| < r. \\ 0 & , \quad |\boldsymbol{y}| \geq r \end{cases} \tag{27}$$

where $r > 1$ is the radius of the non-local influence zone, and $C > 0$ is a constant and scale the kernel as

$$\mathbf{K}^{\varepsilon}(\boldsymbol{x}) = \varepsilon^{-n}\mathbf{K}\left(\frac{\boldsymbol{x}}{\varepsilon}\right) = \begin{cases} \varepsilon^{-n}C\boldsymbol{\sigma}\left(\frac{\boldsymbol{x}}{\varepsilon}\right)\exp\left(\frac{1}{\left|\frac{\boldsymbol{x}}{\varepsilon r}\right|^2-1}\right) & , \quad |\boldsymbol{x}| < \varepsilon r. \\ 0 & , \quad |\boldsymbol{x}| \geq \varepsilon r \end{cases} \tag{28}$$

The assumptions for Case I applies. Note that the scaling implies

$$\lim_{\varepsilon\to 0}\widehat{\mathbf{K}}^{\varepsilon}_{\varepsilon}(\boldsymbol{\xi},\boldsymbol{m}) = \widehat{\mathbf{K}}(\boldsymbol{m}) \tag{29}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^n$. The support of $\widehat{\mathbf{K}}(\boldsymbol{m})$ is continuous ($\mathbb{R}^n$) in contrast to what is indicated by the argument $\boldsymbol{m}$. The reason is that the mollifier has a compact support and larger than the unit cell $]0,1[^n$, since $r > 1$. Actually, this case asks for a modified definition of the two-scale Fourier transform, e.g. in one dimension we let $|\boldsymbol{\xi}| < \delta/2\epsilon$, and $m = \pm\delta, \pm 2\delta, \ldots$. Then we send $\delta \to 0$, but slower than ε, e.g. $\delta = \sqrt{\varepsilon}$ should work.

Theorem 9 (Homogenization, vanishing non-localness)**.** *Let $\{\phi^{\varepsilon}\}$ be a sequence of solutions to* (10) *where the kernel in the bilinear form* (9) *is given by* (28). *The sequence $\{\phi^{\varepsilon}\}$ converges weakly in $H^1_0(\Omega)$ to $\phi \in H^1_0(\Omega)$, the unique solution of the Homogenized Problem*

$$-\nabla\cdot\boldsymbol{\sigma}_h\nabla\phi(\boldsymbol{x}) = f(\boldsymbol{x}), \tag{30}$$

a.e. in Ω, where the homogenized conductivity is given as the mean value

$$\boldsymbol{\sigma}_h = \int_{|\boldsymbol{y}|<r}\mathbf{K}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \int_{|\boldsymbol{y}|<r}C\boldsymbol{\sigma}(\boldsymbol{y})\exp\left(\frac{1}{\left|\frac{\boldsymbol{y}}{r}\right|^2-1}\right)\mathrm{d}\boldsymbol{y} \tag{31}$$

Proof: Since ϕ^{ε} is bounded in $\in H^1(\mathbb{R}^n)$ and $f^{\varepsilon} \in L^2(\mathbb{R}^n)$ we can apply the (modified) two-scale Fourier transform in Definition 5 to (10),

$$2\pi i(\boldsymbol{\xi}+\varepsilon^{-1}\boldsymbol{m})\cdot\widehat{\mathbf{K}}^{\varepsilon}_{\varepsilon}(\boldsymbol{\xi},\boldsymbol{m})2\pi i(\boldsymbol{\xi}+\varepsilon^{-1}\boldsymbol{m})\widehat{\phi}^{\varepsilon}_{\varepsilon}(\boldsymbol{\xi},\boldsymbol{m}) = \widehat{f}^{\varepsilon}_{\varepsilon}(\boldsymbol{\xi},\boldsymbol{m}), \tag{32}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^n, \boldsymbol{m} \in \mathbb{Z}^n$. Next we multiply with ε and send a subsequence (still denoted by ε) to zero. Taking Proposition 6 and limit (29) into account will give us the Fourier transform of the local problem as the L^2-weak limit in Fourier space,

$$2\pi i\boldsymbol{m}\cdot\widehat{\mathbf{K}}(\boldsymbol{m})2\pi i\left(\boldsymbol{\xi}\widehat{\phi}(\boldsymbol{\xi})\delta_{\boldsymbol{m}\mathbf{0}} + \boldsymbol{m}\widehat{\phi}^1(\boldsymbol{\xi},\boldsymbol{m})\right) = 0, \tag{33}$$

for a.e. $\boldsymbol{\xi} \in \mathbb{R}^n$, and all $\boldsymbol{m} \in \mathbb{R}^n$. It has a trivial solution $\widehat{\phi}^1(\boldsymbol{\xi},\boldsymbol{m}) = 0$ for all $\boldsymbol{m} \neq \mathbf{0}$. To get the homogenized problem we let $\boldsymbol{m} = \mathbf{0}$ in (32) and send another subsequence $\varepsilon \to 0$ which yields the usual Fourier transform of the weak $L^2(\Omega)$-limit in (30), once again using Proposition 6,

$$2\pi i\boldsymbol{\xi}\cdot\widehat{\mathbf{K}}(\mathbf{0})2\pi i\boldsymbol{\xi}\widehat{\phi}(\boldsymbol{\xi}) = \widehat{f}(\boldsymbol{\xi}), \tag{34}$$

for a.e. $\boldsymbol{\xi} \in \mathbb{R}^n$. Applying (29) gives the homogenized equation. The homogenized equation reads in real space

$$-\nabla \cdot \int_{|\boldsymbol{y}|<r} \mathbf{K}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}\, \nabla\phi(\boldsymbol{x}) = f(\boldsymbol{x}), \tag{35}$$

Inspection of equation (35) yields the homogenized conductivity as

$$\boldsymbol{\sigma}_h = \int_{|\boldsymbol{y}|<r} \mathbf{K}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}, \tag{36}$$

The whole sequence converges since the homogenized equation has a unique solution in $H_0^1(\Omega)$, see Theorem 10. □

Theorem 10 (Existence of solution)**.** *The homogenized problems* (20) *and* (30) *has each a unique solution in* $H_0^1(\Omega)$.

Proof: It follows from the assumptions that the homogenized conductivity $\boldsymbol{\sigma}_h$ inherits the properties of $\mathbf{K}$. The statement follows from Theorem 5. □

6. Remarks and conclusions

The localization of the constitutive relation for Case II in (30) can equally be obtained by multiplying the kernel in (17) with r^{-1} and sending $r \to 0$ either before sending $\varepsilon \to 0$ or after. Introducing a spatially local contribution in the constitutive relations will somewhat complicate the analysis, but it is doable. An effect that we have not taken into account is the influence of the boundary $\partial\Omega$. In real life, e.g. for wave propagation in cases the wavelength is on the same order as the material periodicity, we expect the nonlocal constitutive relation to depend on the distance to the boundary. We will return to these issues in forthcoming papers. We conclude that spatially nonlocal constitutive relations are particularly easy to homogenize since we need only to integrate the kernel over the fast variable. In retrospective, this is to some degree expected since spatial convolution is an averaging procedure which smoothers fast oscillations.

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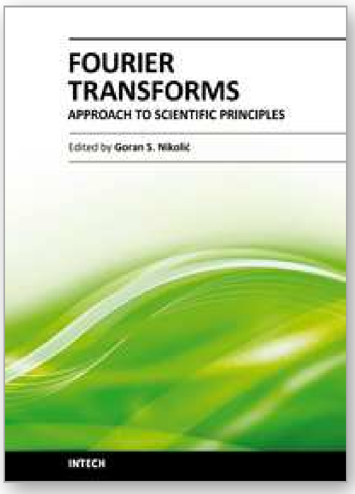

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This book aims to provide information about Fourier transform to those needing to use infrared spectroscopy, by explaining the fundamental aspects of the Fourier transform, and techniques for analyzing infrared data obtained for a wide number of materials. It summarizes the theory, instrumentation, methodology, techniques and application of FTIR spectroscopy, and improves the performance and quality of FTIR spectrophotometers.

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